UNIVERSITÀ DEGLI STUDI DI NAPOLI “FEDERICO II”

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Laurea Magistrale in Fisica

Top Yukawa contribution to the radiative decay of the Higgs boson into bottom quarks

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A. A. 2017/2018

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Introduction

The discovery of the Higgs boson [1] in 2012, by the ATLAS [2] and CMS [3] experiments at the Large Hadron Collider (LHC) at CERN, constitutes a breakthrough in particle Physics and the confirmation of the predictions of the Higgs mechanism about the Electroweak Symmetry Breaking (EWSB). Higher precision studies are required to test further predictions of the Standard Model on the couplings of the Higgs boson with itself and with all the other elementary particles [4].

The Higgs boson decays mainly into $b\bar{b}$ quarks, with a Branching Ratio of 58% for $m_H = 125$ GeV. Because the next-to-leading order (NLO) correction to the decay process is significant [5], we may expect a non negligible contribution also from the next-to-next-leading-order (NNLO) correction. In this thesis work we recompute the NLO correction and outline a subtraction formalism in order to obtain differential distributions of jet observables in the Higgs boson decay into massive quarks. We then compute analytically the NNLO Yukawa correction via top triangle to the Higgs decay into a $b\bar{b}$ pair. We build an analytic form for this amplitude exploiting the technique presented in [6]: first, we reduce the amplitude to a linear combination of a minimal set of integrals, then we derive a system of partial differential equations (PDEs) with the set of integrals as unknowns and finally, we find a solution to the system of PDEs in terms of Goncharov Polylogarithms (GPLs) [7]. Because there are fast libraries to numerically evaluate up to the desired precision the GPLs [8], they are a good candidate of special functions to be used into our simulation programs. In the literature we can find a full massless NNLO computation of $H \to b\bar{b}$ [9] and an approximated result for our specific contribution [10]. A full analytical computation of the NNLO top

Yukawa contribution with massive b is presented here for the first time.

The thesis is structured as follows. In the first chapter we present a brief introduction to the mechanism of EWSB and an overview of the Higgs boson phenomenology at LHC. Next we calculate the fully differential partial decay width of the Higgs boson into bottom quarks up to the first order in QCD. In particular, we first compute the fully inclusive result and then build a subtraction scheme needed to regularize the divergent integrations. To perform the integrals over the phase space numerically and reconstruct the distributions of exclusive observables, we use Monte Carlo techniques [11].

In the second chapter we present the techniques we have employed to address our two loop calculation starting from the method to find a minimal set of integrals to be computed, the master integrals (MI), exploiting Lorentz and integration by parts identities [12]. As previously mentioned, these integrals can be brought to unknowns of a system of partial differential equations [6] and analytically solved in terms of Goncharov Polylogarithms (GPLs) [7]. At this purpose, the system needs to be transformed to a *canonical form* [13], that we obtain using the Magnus series expansion [14].

In the third chapter we apply the methods discussed above to the computation of the top Yukawa contribution to the Higgs decay into a $b\bar{b}$ pair and compare our result with an approximated formula derived by Chetyrkin and Kwiatkwoski [10]. Finally, we evaluate the impact of our calculation on the distributions of exclusive observables.

The exact computation of the top Yukawa contribution to the Higgs boson decay into bottom quarks at both inclusive and differential level is the original part of the present work.

We remark that the techniques employed in the present thesis work are applicable for a large class of computations with massive internal and external particles, relevant for the phenomenology of the experiments at the LHC.

Chapter 1

The Higgs Boson and the $b\bar{b}$ decay mode

1.1 Higgs and the Standard Model

In the Standard Model [15, 16, 17] (SM) the electroweak interactions are described by a gauge filed theory invariant under $SU(2)_L \times U(1)_Y$ symmetry group. The mechanism of electroweak symmetry breaking (EWSB) [1] provides a general framework to preserve the gauge structure of this interactions at high energies and still generate the observed masses of W and Z bosons. The scale of EWSB is set by the vacuum expectation value (VEV) of an $SU(2)$ scalar doublet, the Higgs field. The masses of all fermions are postulated to exist because of a Yukawa coupling with the Higgs field.

The EWSB mechanism breaks the weak gauge $SU(2)_L \times U(1)_Y$ into $U(1)_{em}$, and the generators satisfy the well known Gell-Mann-Nishijima formula

$$Q = T_3 + \frac{Y}{2} \tag{1.1}$$

where Q is the electric charge, T_3 the third component of the Isospin and Y the Hypercharge. We introduce a complex $SU(2)$ doublet Φ of hypercharge $Y = 1$, the Higgs field

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{1.2}$$

where ϕ^+ is the positive charged component and ϕ^0 the neutral one. In the SM the scalar potential reads

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2 \tag{1.3}$$

In order the EWSB to occur we need $\mu^2 < 0$ and $\lambda > 0$ and imposing the spontaneous symmetry breaking with residual symmetry $U(1)_{em}$ we have

$$\langle \phi \rangle_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \quad \text{with } v = \frac{|\mu|}{\lambda} \tag{1.4}$$

Among the four generators of the $SU(2)_L \times U(1)_Y$ symmetry, three are spontaneously broken, implying the existence of three Goldstone bosons. The Higgs field couples to the gauge bosons through the covariant derivative term

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) \tag{1.5}$$

where

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}}(t_+ W_\mu^+ + t_- W_\mu^-) + eQA_\mu + \frac{g}{cos\theta_w}(t_3 - x_w)Z_\mu \tag{1.6}$$

Here e is the electric charge and $x_w = \sin^2\theta_w$ with θ_w the Weinberg angle. Expanding the Higgs field about $\langle \phi \rangle_0$ we obtain the mass terms for the gauge fields $W^\pm$ and Z and the masses

$$m_W = \frac{gv}{2} \quad m_Z = \frac{m_W}{cos\theta_w} \tag{1.7}$$

The fourth generators remains unbroken since $Q\,\langle \phi \rangle_0 = 0$.

From the initial four degrees of freedom of the Higgs field, three are absorbed as longitudinal components of the massive gauge bosons, while the untouched degree of freedom is the physical Higgs boson, a new scalar particle. The Higgs boson is neutral under Q and colorless, hence does not couple at tree level with photons and gluons.

The fermions of the SM acquire mass trough renormalizable Yukawa interactions between the Higgs field and the fermions

$$\mathcal{L}_{\text{Yukawa}} = -h_{d_{ij}} \bar{q}_{L_i} \Phi d_{R_j} - h_{u_{ij}} \bar{q}_{L_i} \widetilde{\Phi} u_{R_j} - h_{\ell_{ij}} \bar{\ell}_{L_i} \Phi e_{R_j} + h.c. \tag{1.8}$$

where $\widetilde{\Phi} = it_2\Phi^*$ and q_L (ℓ_L) and u_R, d_R (e_R) are the quark (lepton) $SU(2)_L$ doublets and singlets, respectively. Each term is parametrized by a 3×3 matrix in family space. Once the Higgs acquires a VEV and performing a rotation in the basis of the mass eigenstates where the interaction matrix becomes diagonal $h_{f_{ij}} \to h_{f_i}\delta_{ij}$, the fermions will acquire masses

$$m_{f_i} = h_{f_i}\frac{v}{2} \tag{1.9}$$

1.2 Higgs phenomenology

The main production mechanisms of the Higgs boson at the LHC, in order of decreasing cross section, are: gluon fusion, weak-boson fusion, associated production with a gauge boson and associated production with a pair of $t\bar{t}$ quarks. In figure 1.1 the lowest-order diagrams for these processes are showed.

The proton proton center of mass energy dependence of the different production cross sections is reported in figure 1.2 [18, 19, 20, 21]. The knowledge of the production mechanisms is not sufficient to effectively detect the Higgs boson since we also need a distinct signature for the events related to its successive decay. Despite the large branching ratio of the $b\bar{b}$ production (figure 1.3), the main production channel, gluon fusion, is not the preferred one to detect this decay due to the presence of a large source of background from the direct production of a $b\bar{b}$ pairs via the QCD interaction.

Indeed the dominant decay modes are $H \to b\bar{b}$, being the heaviest particle accessible, and $H \to WW$, followed by $H \to gg$ and $H \to \tau^+\tau^-$. In order

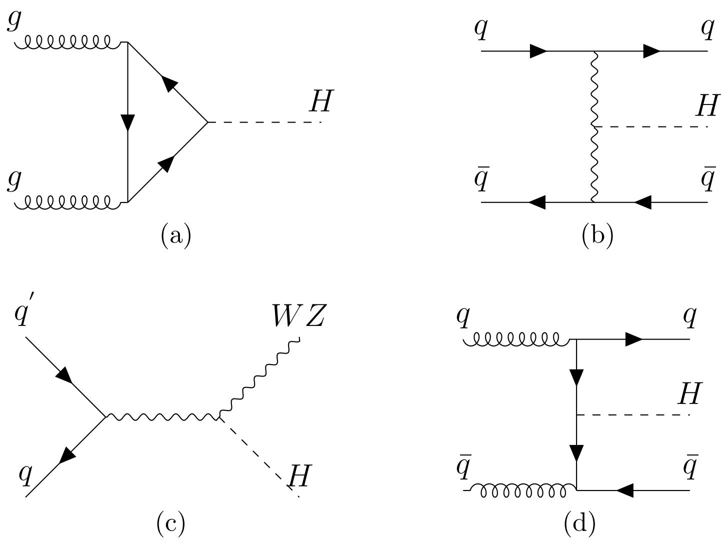


Figure 1.1: Leading order Feynamn diagrams for the main contribution to Higgs production at hadron collider. (a) Gluon fusion. (b) Vector bosons fusion. (c) Associated production with vector bosons. (d) Associated production with $t\bar{t}$.

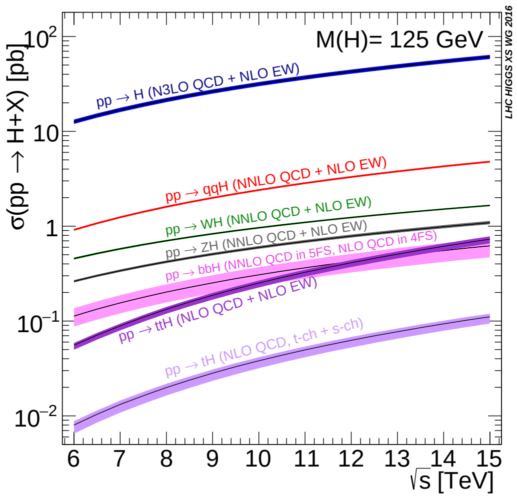


Figure 1.2: The SM Higgs boson production cross sections as function of the c.o.m. energy $\sqrt{s}$ in pp collision. In the whole range of energy considered the main production mechanism is the gluon fusion via top triangle.

to measure the mass of the observed state, the ATLAS and CMS experiments combine results for the two high resolution decay channels, $h \to \gamma\gamma$ and $h \to ZZ \to 4\ell$. In figure 1.4 [22] is reported the summary of the mass measurements for the ATLAS and CMS experiments.

Only during this year both the CMS and ATLAS collaborations, have reported about the direct measurement of the Higgs coupling to the bottom quark [23, 24]. The search channel used is the VH production that, with the successive decay into

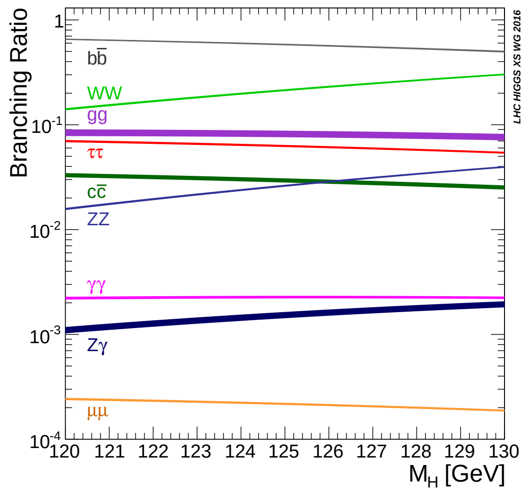


Figure 1.3: In figure the Brancing Ratios of the Higgs boson decay. As shown the main one is the $b\bar{b}$ production.

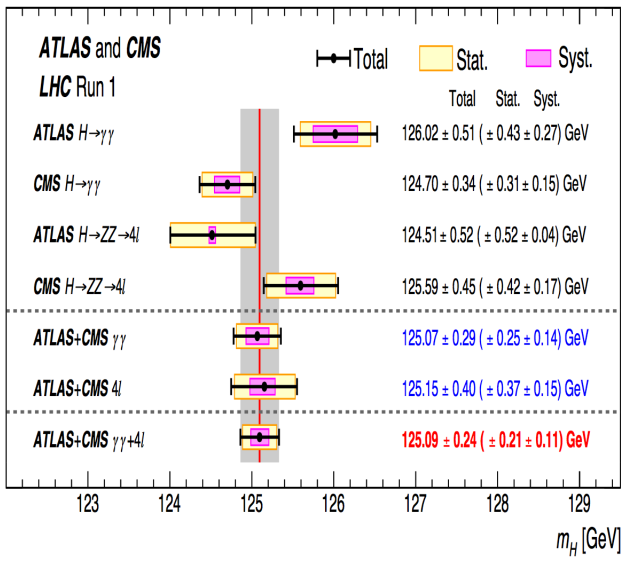


Figure 1.4: A sum up of the Higgs boson mass measurements by the CMS and ATLAS experiments.

leptons of the vector boson, allows to purify the signal. The recent measurements are still consistent with the SM predictions and don't show signals of new physics (figure 1.5).

1.3 Higgs into massive quarks at NLO

The $H \to b\bar{b}$ decay mode offers a direct measurement of the Higgs coupling with fermions and with its BR of 58% at $m_H = 125$ Gev constitutes the prevalent decay channel, so its detection allows to make a fundamental test of the SM.

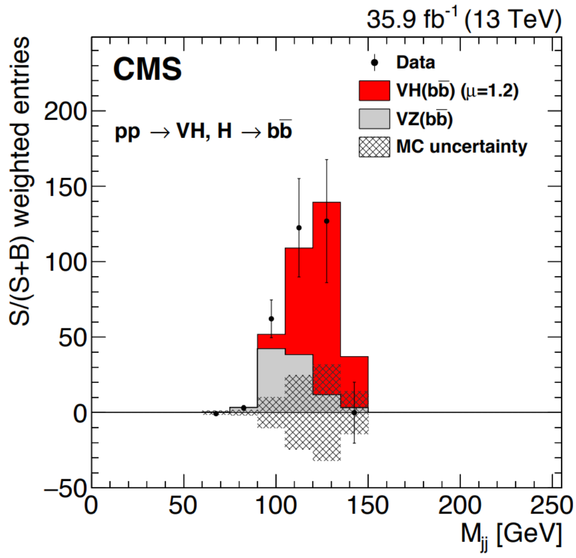


Figure 1.5: The measurements for the $b\bar{b}$ mode at CMS and ATLAS. They are consistent with the SM predictions.

Moreover this channel can also be used to search for signal of new physics that can manifest as distortion of the observed spectra with respect to the predictions.

In this section we derive a well known result, the massive NLO correction to $H \to b\bar{b}$ [5, 25], re-obtaining the analytical expression for the full inclusive decay rate.

We start evaluating the Born amplitude that can be directly calculated as squared modulus of the diagram in figure 1.6.

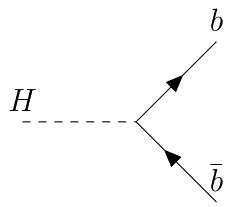


Figure 1.6: Born diagram

Its expression is given by

$$\mathcal{B} = 2C_A y_b^2 \left(s - 4m_b^2\right) \tag{1.10}$$

where C_A is the number of colors and $y_b = -im_b e/2M_W \sin(\theta_w)$ is the Yukawa coupling of Higgs to fermion. From this expression we easily get the decay rate at LO

$$\Gamma_0 = \frac{1}{8\pi} C_A y_b^2 \sqrt{s} \left(1 - 4\frac{m_b^2}{s}\right)^{3/2} \tag{1.11}$$

Going to the next order, the Kinoshita-Lee-Nauenberg (KLN) theorem [26, 27] requires that we need to sum both the real and virtual contribution to the decay rate. Using the on shell renormalization scheme [28] the virtual contribution to consider will be only the vertex correction (figure 1.7) to be interfered with the the Born diagram (figure 1.6).

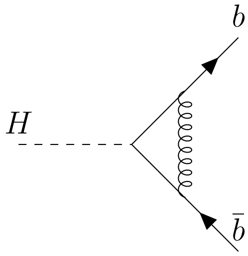


Figure 1.7: Vertex correction, the virtual NLO contribution

Concerning the real contributions, the NLO correction, g^2 order, is given by the emission of an extra gluon in the final state. The amplitude to compute will be given by the squared modulus of the sum of the two diagrams in figure 1.8

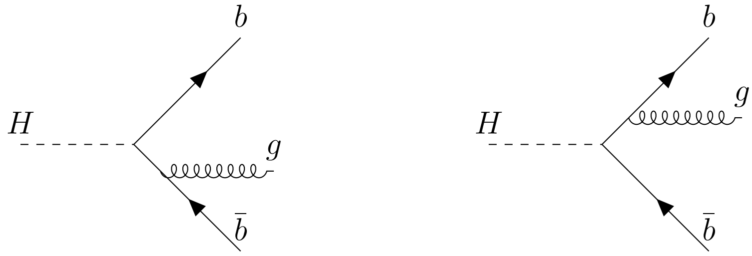
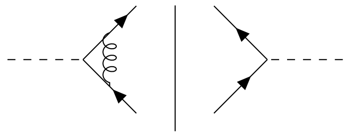


Figure 1.8: Gluon emission, real contribution.

1.3.1 Virtual contribution

The expression of the unrenormalized amplitude of the vertex correction (figure 1.7) in the kinematic $H(p) \to b(b)\bar{b}(a)$ can be easily obtained performing basic Dirac algebra, we can then reduce to the computation of just three scalar integrals simply expressing the terms in the numerator in terms of denominators. In terms of the three scalar integrals, the virtual amplitude is:

$\mathcal{M}_v =$ 

$$= 4iC_AC_Fy_b^2g^2\left((8m_b^4 - 6m_b^2 s + s^2)I_3 + (-4(-1+\epsilon)m_b^2 + \epsilon s)I_2 + (4m_b^2 - 2s)I_1\right) \tag{1.12}$$

where ϵ is the parameter of the dimensional regularization (that we use throughout this thesis with $d = 4 - 2\epsilon$), $C_F = 4/3$ is the Casimir of the fundamental representation of the color group, $C_A = 3$ is the dimension of the fundamental representation. The integrals I_1, I_2 and I_3, in our region of interest $4m_b^2 < s = m_H^2$, are given by

$$\begin{aligned} I_1 = &\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{[\ell^2][(\ell+b)^2 - m_b^2]} \\ &= \frac{i}{(16\pi^2)}\left(\frac{4\pi\mu^2}{m_b^2}\right)^\epsilon\left(\frac{1}{\epsilon} + 2 - \gamma\right) \end{aligned} \tag{1.13}$$

$$\begin{aligned} I_2 = &\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{[(\ell-a)^2 - m_b^2][(\ell+b)^2 - m_b^2]} \\ &= \frac{i}{(16\pi^2)}\left(\frac{4\pi\mu^2}{m_b^2}\right)^\epsilon\left(\frac{1}{\epsilon} + 2 - \gamma + \beta\left(\log\frac{1-\beta}{1+\beta} + i\pi\right)\right) \end{aligned} \tag{1.14}$$

$$\begin{aligned} I_3 = &\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{[(\ell-a)^2 - m_b^2][(\ell+b)^2 - m_b^2][\ell^2]} \\ &= \frac{\Gamma(1+\epsilon)}{s\beta}\left(\frac{4\pi\mu^2}{m_b^2}\right)^\epsilon\left(\frac{\log(x)}{\epsilon} + \log^2\left(\frac{1-\beta}{2\beta}\right) - \frac{1}{2}\log^2(x) - \frac{2}{3}\pi^2\right. \\ &\left. + 2\mathrm{Li}_2\left(\frac{\beta-1}{\beta+1}\right) - i\pi\left(1 + 2\log\left(\frac{1-\beta}{2\beta}\right) - \log(x)\right)\right) \end{aligned} \tag{1.15}$$

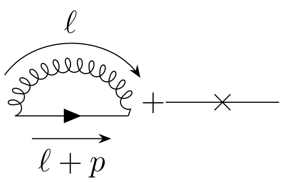
with

$$\beta = \sqrt{1 - \frac{4m_b^2}{s}} \quad x = \frac{1-\beta}{1+\beta} \tag{1.16}$$

and μ the renormalization scale. To deal with Ultra-Violet (UV) divergences we need to use a renormalization scheme. We choose the on shell scheme [28]. The renormalized lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_b)\psi - \frac{m_b}{v}H\bar{\psi}\psi + \bar{\psi}(i\partial\!\!\!/\delta_\psi - m_b\delta_{m_b})\psi - m_b\frac{\delta_{m_b}}{v}H\bar{\psi}\psi \tag{1.17}$$

where we have used that at the lowest order in g no renormalization of the Higgs field is required. So the counter term of the Yukawa coupling depends only on δ_{m_b}. The 1PI contribution to the propagator corrections $\Sigma_\psi(p\!\!\!/)$ is given by

$-i\Sigma_\psi(p\!\!\!/) =$ 

$$=(-ig)^2\tau_a^2 \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2}\gamma_\mu \frac{1}{(\ell\!\!\!/ + p\!\!\!/ - m_b)}\gamma^\mu + i(p\!\!\!/\delta_\psi - m_b\delta_{m_b}) \tag{1.18}$$

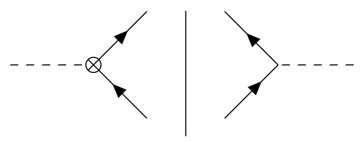
The parameters δ_ψ and δ_{m_b} can be determined fixing the renormalization prescription. We use

$$\begin{aligned} \Sigma_\psi(p\!\!\!/)\Big|_{p\!\!\!/=m_b} &= 0 \\ \frac{d}{dp\!\!\!/}\Sigma_\psi(p\!\!\!/)\Bigg|_{p\!\!\!/=m_b} &= 0 \end{aligned} \tag{1.19}$$

Performing the integrals in dimensional regularization to deal with both IR and UV divergences, up to order $o(\epsilon^0)$, we get

$$\begin{aligned} \delta_\psi &= \frac{C_F g^2}{16\pi^2}\left(-\frac{3}{\epsilon} - 4 + 3\gamma + 3\log\left(\frac{m_b^2}{4\pi\mu^2}\right)\right) + o(\epsilon^0) \\ \delta_{m_b} &= \frac{C_F g^2}{8\pi^2}\left(-\frac{3}{\epsilon} - 4 + 3\gamma + 3\log\left(\frac{m_b^2}{4\pi\mu^2}\right)\right) + o(\epsilon^0) \end{aligned} \tag{1.20}$$

We can then write the vertex counter event as

$\mathcal{M}_c =$  $= 2\delta_{m_b}\mathcal{B}$ (1.21)

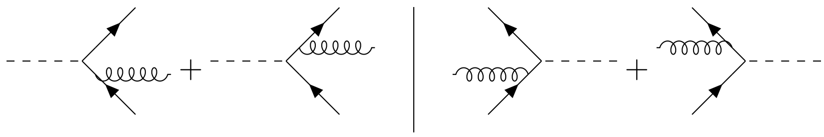
The renormalized virtual contribution, defined as the sum of the virtual correction and the vertex counter term, is then

$$
\begin{aligned}
\frac{\mathcal{M}_v + \mathcal{M}_c}{\mathcal{N}} = & \\
= & \frac{1}{\epsilon}\left(\log\left(\frac{2}{\beta+1}-1\right)\left(2m_b^2 - s\right) - \beta s\right) \\
& + 4m_b^2\left(\log\left(\frac{\beta-1}{\beta}\right) - \log\left(\frac{1}{\beta}+1\right)\right) \\
& + \frac{1}{\beta}\left(12m_b^2 + \log(m_b)(-16m_b^2 - 6\beta^2 s + 4s) + 4\beta^2 s - 3s\right) \\
& + \frac{1}{3s\beta^2}\left(\left(8m_b^4 - 6m_b^2 s + s^2\right)\left(6\mathrm{Li}_2\left(\frac{1}{2}\left(\frac{s\beta}{4m_b^2 - s}+1\right)\right)\right.\right. \\
& \left.\left. +3\log^2\left(-\frac{\beta s}{8m_b^2 - 2s} - \frac{1}{2}\right) - 6\log\left(\frac{2}{\beta+1}-1\right)\log(m_b) + \pi^2\right)\right)
\end{aligned}
\tag{1.22}
$$

where the normalization factor is $\mathcal{N} = \Gamma(1+\epsilon)\mathcal{B}C_F g_s^2(4\pi\mu^2)^\epsilon(4\pi^2 s\beta)$. Because the counter event cancels the UV divergence, the remaining ϵ pole is related to the IR divergence.

1.3.2 Real contribution

The real contribution can be computed straightforwardly:

$\mathcal{M}_r =$ 

$$
= C_F g_s^2 \mu^{2\epsilon}\left(\mathcal{B}\left(-\frac{4m_b^2}{\mathrm{sag}^2} - \frac{4m_b^2}{\mathrm{sbg}^2} + \frac{4\mathrm{sab}}{\mathrm{sagsbg}}\right) + C_A y_b^2(\epsilon - 1)\left(-\frac{4\mathrm{sag}}{\mathrm{sbg}} - \frac{4\mathrm{sbg}}{\mathrm{sag}} - 8\right)\right)
\tag{1.23}
$$

where

$$
\mathrm{sab} = 2a\cdot b \quad \mathrm{sag} = 2a\cdot g \quad \mathrm{sbg} = 2b\cdot g
\tag{1.24}
$$

In the expression above the factor that multiplies $\mathcal{B}$ is divergent in the limit of g soft. When computing the inclusive decay rate, its pole will cancel the IR divergence of the virtual contribution.

1.3.3 Inclusive decay rate

Now that we have calculated both the real and the virtual contributions, we can perform the integration over the phace space and obtain the inclusive decay rate.

First of all we need the expressions of the phace spaces in 2 and 3 particles, ϕ_2 and ϕ_3, in dimensional regularization. Remembering that we are dealing with a decay $1 \to 2$ or $1 \to 3$, ϕ_2 can be integrated freely over the entire volume, while the real contribution will have just 2 independent parameters. ϕ_2 is then

$$\phi_2 = \frac{\pi^\epsilon \Gamma(1-\epsilon)\sqrt{1-\frac{4m_b^2}{s}}\left(\frac{4}{s-4m_b^2}\right)^\epsilon}{(8\pi)\Gamma(2-2\epsilon)} \tag{1.25}$$

Considering the decay of a particle of momentum P into three particles of momenta p_1, p_2 and p_3, we can define the variables

$$x_i = \frac{2P \cdot p_i}{P^2} \tag{1.26}$$

so that their sum $x_1 + x_2 + x_3 = 2$ by momentum conservation. Specifying the momenta to our case: $p_1^2 = p_2^2 = m_b^2$ and $p_3^2 = 0$, the phase space ϕ_3 is then:

$$\begin{aligned}\phi_3 &= \prod_{i=1}^{3} \int \frac{d^d p_i}{(2\pi)^d} (2\pi)^d \delta^{(d)}\left(P - \sum_{j=1}^{3} p_j\right) \delta(p_1^2 - m_b^2)\delta(p_2^2 - m_b^2)\delta(p_3^2) \\ &= \frac{1}{(2\pi)^{2d-3}} \prod_{i=1}^{2} \int \frac{d^{d-1} p_i}{2E_i} \delta((P - p_1 - p_2)^2)\end{aligned} \tag{1.27}$$

We can then express each $d^{d-1}p_i$ as

$$d^{d-1}p_i = d^{d-3}\, |p_i|\, |p_i|^2\, d\Omega_{d-2}^{(i)} \tag{1.28}$$

where the solid angle in n dimension is such that

$$\Omega_n = 2^n \pi^{n/2} \frac{\Gamma(\frac{1}{2}m)}{\Gamma(m)} \tag{1.29}$$

The delta function fixes a constraint on the angle between the particles 1 and 2 trough the relation

$$\begin{aligned} 0 = (P{-}p_1 - p_2)^2 & \\ &= s + 2m_b^2 - 2P\cdot p_1 - 2P\cdot p_2 + 2p_1\cdot p_2 \\ &= s + 2m_b^2 - 2P\cdot p_1 - 2P\cdot p_2 + 2E_1E_2 - 2\,|p_1|\,|p_1|\,cos\theta_{12} \end{aligned} \tag{1.30}$$

So we can change the angle variables and integrate completely one of the $d\Omega_{d-2}^{(i)}$ with 1.29. The integration over θ_{12} can be performed using that

$$\int d\Omega_n = 2\pi^{n/2}\frac{1}{\Gamma(\frac{1}{2}m)}\int_0^\pi d\theta_i(\sin\theta_i)^{n-1} \tag{1.31}$$

Expressing everything in terms of the parameters x_1 and x_2 we obtain the following result, in accordance with reference [29]

$$\phi_3 = H\int_{\sqrt{r}}^1 dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 \left(4\left(x_1^2 - r\right)\left(x_2^2 - r\right) - \left(2r - x_1^2 + (-x_1 - x_2 + 2)^2 - x_2^2\right)^2\right)^{-\epsilon} \tag{1.32}$$

where

$$r = \frac{4m_b^2}{s} \qquad H = \frac{s\left(\frac{16\pi}{s}\right)^{2\epsilon}}{(2^7\pi^3)\,\Gamma(2-2\epsilon)} \tag{1.33}$$

and the bounds of integration are specified by

$$x_{2\pm} = \frac{(x_1 - 2)\,(r - 2(x_1 - 1)) \pm 2(x_1 - 1)\sqrt{x_1^2 - r}}{-r + 4x_1 - 4} \tag{1.34}$$

We can know perform the integration of the real amplitude $\mathcal{M}_r$ 1.23 over x_1 and x_2. In terms of our integration variables we have

$$\begin{aligned} \mathcal{M}_r = C_F g_s^2\mu^2 \Bigg(\mathcal{B}\left(\frac{4\left(s(x_1 + x_2 - 1) - 2m_b^2\right)}{s^2(1-x_1)(1-x_2)} - \frac{4m_b^2}{s^2(1-x_1)^2} - \frac{4m_b^2}{s^2(1-x_2)^2}\right) & \\ +C_A(\epsilon - 1)y_b^2\left(-\frac{4(1-x_1)}{1-x_2} - \frac{4(1-x_2)}{1-x_1} - 8\right)\Bigg) & \end{aligned} \tag{1.35}$$

Referring to equation 1.32 we define

$$J = 4\left(x_1^2 - r\right)\left(x_2^2 - r\right) - \left(2r - x_1^2 + (-x_1 - x_2 + 2)^2 - x_2^2\right)^2 \tag{1.36}$$

Because the problem is symmetric in the exchange of x_1 and x_2, when performing the integration of $\mathcal{M}_r$ over ϕ_3, we can reduce to the calculation of 5 integrals of which only 2 with IR divergences. We then define the following integrals

$$\mathcal{I}_1 = \int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 J^{-\epsilon} \frac{1}{(1-x_1)^2} \tag{1.37}$$
$$\mathcal{I}_2 = \int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 J^{-\epsilon} \frac{1}{(1-x_1)(1-x_2)} \tag{1.38}$$
$$\mathcal{I}_3 = \int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 \frac{1-x_2}{1-x_1} \tag{1.39}$$
$$\mathcal{I}_4 = \int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 \tag{1.40}$$
$$\mathcal{I}_5 = \int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 \frac{1}{1-x_1} \tag{1.41}$$

where in the last three integrals we already set $\epsilon = 0$ being finite. Their value is given in appendix A. In terms of them the real contribution to the decay rate $\Delta\Gamma_r$ can be written as

$$\begin{aligned}\Delta\Gamma_r = C_F g_s^2 (\mu^2)^\epsilon \left(\frac{\mathcal{B}}{s}\left(4\left(\mathcal{I}_2\left(1 - \frac{2m_b^2}{s}\right) - 2\mathcal{I}_5\right) - \frac{8\mathcal{I}_1 m_b^2}{s^2}\right)\right. \\ \left. + C_A(\epsilon - 1) y_b^2 (-8\mathcal{I}_3 - 8\mathcal{I}_4)\right);\end{aligned} \tag{1.42}$$

The virtual contribution $\Delta\Gamma_v$ is easily obtained multiplying $\mathcal{M}_r + \mathcal{M}_c$ of equation 1.22 by the ϕ_2 volume 1.25

$$\Delta\Gamma_v = (\mathcal{M}_r + \mathcal{M}_c)\phi_2 \tag{1.43}$$

Summing real and virtual contributions and expanding in ϵ we observe the cancellation of the ϵ pole. The NLO correction to the decay of Higgs into $b\bar{b}$, $\Delta\Gamma_{\rm nlo} = \Delta\Gamma_r + \Delta\Gamma_v$, is then:

$$
\begin{aligned}
\Delta\Gamma_{\text{nlo}} = \Gamma_0 & \frac{1}{384\pi^2\sqrt{s}\left(s-4m_b^2\right)^{3/2}} \\
& \left(-3s^2\left(12\left(\beta^3+\beta\right)+\left(-3\beta^4-2\beta^2+5\right)\log\left(\frac{(\beta+1)^2}{(\beta-1)^2}\right)-16\log\left(-\frac{\beta+1}{\beta-1}\right)\right)\right. \\
& +16\left(4m_b^2-s\right)\left(-12m_b^2\log^2\left(\frac{1}{2}\left(\frac{1}{\beta}-1\right)\right)+6m_b^2\log^2\left(\frac{1-\beta}{\beta+1}\right)\right. \\
& -24m_b^2\log\left(\frac{\beta+1}{1-\beta}\right)-24m_b^2\log(2)\log\left(\frac{1-\beta}{\beta+1}\right)-24m_b^2\log\left(\frac{1-\beta}{\beta+1}\right) \\
& +12\left(s-2m_b^2\right)\mathrm{Li}_2\left(\frac{\beta-1}{2\beta}\right)+12(1+\log(8))\left(\log\left(\frac{1-\beta}{\beta+1}\right)\left(2m_b^2-s\right)-\beta s\right) \\
& +6\left(\log\left(\frac{s}{m_b^2}\right)-2\log(\beta)\right)\left(\log\left(\frac{1-\beta}{\beta+1}\right)\left(s-2m_b^2\right)+\beta s\right) \\
& +8\pi^2m_b^2+18\beta s+6s\log^2\left(\frac{1}{2}\left(\frac{1}{\beta}-1\right)\right)-3s\log^2\left(\frac{1-\beta}{\beta+1}\right)+12\beta s\log(2) \\
& +12s\log\left(\frac{1-\beta}{\beta+1}\right)+12s\log(2)\log\left(\frac{1-\beta}{\beta+1}\right)-4\pi^2s \\
& +\frac{6m_b^2}{1-\beta^2}\left(2\left(\beta^2+3\right)\log\left(-(\beta-1)^3\right)+8\beta\log\left(2\beta^2\right)-2\left(\beta^2-1\right)\log(\beta+1)\right. \\
& +\left(\beta^2-1\right)\log\left(1-\beta^2\right)-2\left(\beta^2+3\right)\log\left(1-\beta^2\right) \\
& +4\beta\left(\log\left(-\frac{64}{\beta^2-1}\right)+2\log\left(1-\sqrt{1-\beta^2}\right)+2\log(\beta)-2\right) \\
& +2\left(\beta^2+3\right)\log\left(\left(\sqrt{1-\beta^2}-2\right)^2\right)-2\left(\beta^2-1\right)\log\left(\sqrt{1-\beta^2}\left(\sqrt{1-\beta^2}-2\right)^2\right) \\
& \left.-8\log\left(\sqrt{1-\beta^2}\left(\sqrt{1-\beta^2}-2\right)^2\right)-8\beta\log\left(\beta^2+\sqrt{1-\beta^2}-1\right)\right) \\
& +6\left(\beta^2-1\right)s\tanh^{-1}\left(\frac{2\beta}{\beta^2+1}\right)-12\beta s+12s\log\left(\frac{\beta+1}{1-\beta}\right). \\
& -\frac{1}{2}\left(2m_b^2-s\right)\left(24\log^2\left(\sqrt{1-\beta^2}-\beta+1\right)-24\log^2\left(\sqrt{1-\beta^2}+\beta+1\right)\right. \\
& -12\log(16)\log\left(\sqrt{1-\beta^2}-\beta+1\right)-36\log\left(\sqrt{1-\beta^2}+1\right)\log\left(\sqrt{1-\beta^2}-\beta+1\right) \\
& +6\log\left(\frac{(\beta-1)^2}{(\beta+1)^2}\right)\log\left(4-4\beta^2\right)+12\log\left(\frac{(\beta-1)^2}{(\beta+1)^2}\right)\log\left(1-\sqrt{1-\beta^2}\right) \\
& +48\log(2)\log\left(\sqrt{1-\beta^2}+\beta+1\right)+36\log\left(\sqrt{1-\beta^2}+1\right)\log\left(\sqrt{1-\beta^2}+\beta+1\right) \\
& -6\log\left(\frac{4\beta}{(\beta-1)^2}\right)\log\left(\frac{(\beta+1)^2}{(\beta-1)^2}\right)-12\mathrm{Li}_2\left(\frac{1}{2}\left(\beta-\sqrt{1-\beta^2}+1\right)\right) \\
& +132\mathrm{Li}_2\left(\frac{\beta}{\sqrt{1-\beta^2}+1}\right)-132\mathrm{Li}_2\left(-\frac{\beta}{\sqrt{1-\beta^2}+1}\right)
\end{aligned}
$$

$$+ 12\mathrm{Li}_2\left(-\frac{\beta\left(-\beta+\sqrt{1-\beta^2}+1\right)}{2\left(\sqrt{1-\beta^2}+1\right)}\right)$$
$$+ 132\mathrm{Li}_2\left(\frac{-\beta+\sqrt{1-\beta^2}+1}{2\sqrt{1-\beta^2}+2}\right) - 132\mathrm{Li}_2\left(\frac{\beta+\sqrt{1-\beta^2}+1}{2\sqrt{1-\beta^2}+2}\right)$$
$$-6\mathrm{Li}_2\left(-\frac{4\beta}{(\beta-1)^2}\right) + 12\mathrm{Li}_2(-\beta) - 12\mathrm{Li}_2(\beta) - 6\mathrm{Li}_2\left(\frac{(\beta-1)^2}{(\beta+1)^2}\right) + \pi^2\Bigg)\Bigg)\Bigg) \tag{1.44}$$

The result has been compared to the one of Braaten et al [25] finding a perfect numerical agreement up to any number of digits and it determines a correction of about 20% on the LO decay rate.

1.4 Subtraction scheme

In this section we outline the features of a QCD differential calculation in the subtraction formalism [30].

Suppose we have n partons in the final state with on shell momenta $\Phi_n = \{k_i\}_{i=1}^n$ constrained by the momentum conservation

$$q = k_1 + \dots k_n \tag{1.45}$$

where q is the initial momentum. With $d\Phi_n$ we denote the n particles phase space

$$d\Phi_n = \prod_{i=1}^n \frac{d^3k_i}{(2\pi)^3 2E_i}(2\pi)^4\delta\left(q - \sum_{i=1}^n k_i\right) \tag{1.46}$$

At NLO the total cross section is given by the sum of the virtual and real contributions, the first integrated over $d\Phi_n$ phase space, the latter over $d\Phi_{n+1}$ because it concerns the emission of another parton. According to the KLN theorem, the real contribution is required to cancel the IR divergences. Denoting with $\mathcal{B}$ the LO amplitude, with $\mathcal{V}$ and $\mathcal{R}$ the virtual and real ones, we have

$$\sigma_{\mathrm{NLO}} = \int d\Phi_n[\mathcal{B}(\Phi_n) + \mathcal{V}(\Phi_n)] + \int d\Phi_{n+1}\mathcal{R}(\Phi_{n+1}) \tag{1.47}$$

More in general we can consider an observable $\mathcal{O}$, function of the kinematic configuration Φ_n. We only refer to infrared-safe observables in the sense that their expectation values does not show IR divergences. The expectation value of an observable $\mathcal{O}$ is then

$$\langle\mathcal{O}\rangle = \int d\Phi_n \mathcal{O}(\Phi_n)[\mathcal{B}(\Phi_n) + \mathcal{V}(\Phi_n)] + \int d\Phi_{n+1}\mathcal{O}(\Phi_{n+1})\mathcal{R}(\Phi_{n+1}) \tag{1.48}$$

The integrals above can not in general be performed analytically so a numerical approach needs indeed to be used. We can think that our events, specified by the final momenta configuration, are weighted by $\mathcal{B} + \mathcal{V}$ for Φ_n and by $\mathcal{R}$ for Φ_{n+1}. The KLN theorem ensures that the inclusive cross section is finite, but because the real and virtual terms live in different phase spaces the numerical integration can be performed only if the integrands are both made finite in a consistent way.

From the KLN theorem we deduce that in the IR divergent regions a real configuration is not distinguishable from a Born one. We can then imagine to subtract the singular events from the real configuration and add them back to the Born one achieving, in this way, the cancellation the IR divergences. This can be realized introducing, in the singular regions, configurations with negative weights, called real counter events. The whole procedure that follows constitutes the subtraction formalism [30].

For each singular region labeled with α we introduce a function $\mathcal{C}^\alpha$, the counter event, and a mapping $M^{(\alpha)}$

$$M^{(\alpha)}\Phi_{n+1} = \overline{\Phi}_n^{(\alpha)} \tag{1.49}$$

that maps a real configuration into a Born one. The mapping needs to be smooth near the singular limit and performing the limit it must reduce to the identity. The counter events $\mathcal{C}^\alpha$ have to be chosen in such a way the function

$$\mathcal{O}(\Phi_{n+1})\mathcal{R}(\Phi_{n+1}) - \mathcal{O}(M^{(\alpha)}\Phi_{n+1})\mathcal{C}^{(\alpha)}(\Phi_{n+1}) \tag{1.50}$$

has at most integrable singularities in the real phase space.

In our case, being the b quark considered as massive, there is only a soft singularity.

If we add and subtract the counter event to the expectation value of the observable $\mathcal{O}$ we get

$$\begin{aligned}\langle\mathcal{O}\rangle = \int d\Phi_n \mathcal{O}(\Phi_n)[\mathcal{B}(\Phi_n) + \mathcal{V}(\Phi_n)] + \sum_\alpha \int d\Phi_{n+1}\mathcal{O}(\bar{\Phi}_n)\mathcal{C}^{(\alpha)}(\Phi_{n+1}) \\ + \int d\Phi_{n+1}\left(\mathcal{O}(\Phi_{n+1})\mathcal{R}(\Phi_{n+1}) - \sum_\alpha \mathcal{O}(\bar{\Phi}_n)\mathcal{C}^{(\alpha)}(\Phi_{n+1})\right)\end{aligned} \tag{1.51}$$

The last integral is then well defined and suitable for numerical computation being completely free of divergences in $d = 4$ dimension. The integral over the Born configuration is instead still divergent, to make it finite we need to properly factorize the Born phase space of the counter event and then add it to the Born integrand. With obvious meaning of the subscript, we can write

$$d\Phi_{n+1} = d\bar{\Phi}_n^{(\alpha)} d\Phi_{\text{rad}}^{(\alpha)} \tag{1.52}$$

Demanding that the counter events $\mathcal{C}^\alpha$ and the mapping are chosen such that $\mathcal{C}^{(\alpha)}(\Phi_{n+1})$ is integrable over $d\Phi_{\text{rad}}^{(\alpha)}$ in $D = 4 - 2\epsilon$ dimension, we can add it coherently to the Born integrand canceling the IR divergences of the virtual matrix element. Defining

$$\overline{\mathcal{C}}^{(\alpha)}(\overline{\Phi}) = \int d\Phi_{\text{rad}}^{(\alpha)}\mathcal{C}^{(\alpha)}(\Phi_{n+1}) \tag{1.53}$$

indeed, we can make the virtual term finite

$$V(\Phi_n) = \mathcal{V}(\Phi_n) + \sum_\alpha \overline{\mathcal{C}}^{(\alpha)}(\overline{\Phi})\bigg|_{\overline{\Phi}_n = \Phi_n} \tag{1.54}$$

If we define the real integrand as

$$R(\Phi_{n+1}) = \mathcal{O}(\Phi_{n+1})\mathcal{R}(\Phi_{n+1}) - \sum_\alpha \mathcal{O}(\bar{\Phi}_n)\mathcal{C}^{(\alpha)}(\Phi_{n+1}) \tag{1.55}$$

the expected value of an observable will be given by

$$\langle \mathcal{O} \rangle = \int d\Phi_n \mathcal{O}(\Phi_n)[\mathcal{B}(\Phi_n) + V(\Phi_n)] + \int d\Phi_{n+1} R(\Phi_{n+1}) \tag{1.56}$$

where thanks to the subtraction we made both terms finite and numerically computable.

1.4.1 Application to the Higgs decay at NLO

To apply the formalism outlined above to our NLO calculation we have to choose a valid mapping from the 3 partons kinematic to the Born one of 2 particles. We recall that our transformation needs to reduce to the identity for the soft gluon and that has to preserve the on-shell condition over the new momenta. We start by considering a Lorentz boost. Fixing the real kinematic configuration as

$$p = a + b + g \tag{1.57}$$

where a and b are the momenta of $\bar{b}$ and b quarks and g is the gluon momentum, in the c.o.m. frame we have

$$\vec{a} + \vec{b} + \vec{g} = 0 \tag{1.58}$$

where the arrows indicate three momenta. Because we want to map our configuration into a Born one, the resulting four-momenta $\widetilde{a}$ and $\widetilde{b}$ will have to satisfy the following conditions

$$p = \widetilde{a} + \widetilde{b} \quad \widetilde{a}^2 = \widetilde{b}^2 = m_b^2 \tag{1.59}$$

The boost can be performed imagining to “absorb” the g momentum into the b one, in view of this we can define the momenta $k_{\text{rec}} = a$ and $k = b + g$. We can then perform the boost $\Lambda(\beta)$ of a along the $\vec{k}_{\text{rec}} = -\vec{k}$ direction, so that $\widetilde{a} = \Lambda(\beta)a$ and $\widetilde{a}^2 = a^2 = m_b^2$. The β value can be determined demanding that the momentum, that absorbs the gluon radiation, satisfies the on-shell condition of the Born configuration

$$\widetilde{b}^2 = (p - \Lambda(\beta)k_{\text{rec}})^2 = m_b^2 \tag{1.60}$$

This mapping trivially reduces to the identity in the limit of a soft gluon radiation. Performing the boost we obtain the following equation that fixes the value of β

$$\left(\frac{\beta|\vec{k}|}{\sqrt{1-\beta^2}} - \frac{k^0_{\text{rec}}}{\sqrt{1-\beta^2}} + m_H\right)^2 - \left(\frac{|\vec{k}|}{\sqrt{1-\beta^2}} - \frac{\beta k^0_{\text{rec}}}{\sqrt{1-\beta^2}}\right)^2 = m_b^2 \tag{1.61}$$

And choosing the positive root we get

$$\beta = \frac{4\left|\vec{k}_{\text{rec}}\right| k^0_{\text{rec}} p^2 + \sqrt{p^8 - 4m_b^2 p^6}}{p^2\left(4{k^0_{\text{rec}}}^2 - 4m_b^2 + p^2\right)} \tag{1.62}$$

Furthermore, we need to choose a counter event to cancel the soft divergence. We subtract the whole soft divergent factor:

$$\mathcal{C}(\Phi_3) = C_F g_s^2 \mu^{2\epsilon} \mathcal{B}\left(-\frac{4m_b^2}{\text{sag}^2} - \frac{4m_b^2}{\text{sbg}^2} + \frac{4\text{sab}}{\text{sagsbg}}\right) \tag{1.63}$$

As explained in the previous section, the counter event will be subtracted to the real contribution and its integral over the gluon radiation variables added to the virtual contribution.

Now we need the expression of the integrated counter event over the radiation variables. To proper factor the 2 particles phase space we can just take the expression of $d\Phi_3$ 1.32 and divide it by the Φ_2 volume 1.25.

Recalling that $r = 4m_b^2/s$ and defining $N = 16\pi^2\Gamma(1+\epsilon)(\pi\mu^2)^\epsilon/(\sqrt{(1-r)})$, the integrated counter event is then

$$\begin{aligned}\frac{\overline{\mathcal{C}}^{(\alpha)}(\overline{\Phi})}{N} &= \frac{1}{\epsilon r s}\left(4(r(s-2m_b^2)\log\left(\frac{r}{-r+2\sqrt{1-r}+2}\right) + 4m_b^2\sqrt{1-r})\right)\\ &\quad - \frac{1}{rs}4\left(r(2m_b^2-s)(-\text{Li}_2\left(-\sqrt{1-r}\right) + \text{Li}_2\left(\sqrt{1-r}\right)\right.\\ &\quad + \text{Li}_2\left(\frac{1}{2}(\sqrt{1-r}-\sqrt{r}+1)\right)\end{aligned}$$

$$
\begin{aligned}
& -11\mathrm{Li}_2\left(\frac{\sqrt{1-r}}{\sqrt{r}+1}\right)+11\mathrm{Li}_2\left(-\frac{\sqrt{1-r}}{\sqrt{r}+1}\right) \\
& -\mathrm{Li}_2\left(\frac{(\sqrt{1-r}-\sqrt{r}-1)\sqrt{1-r}}{2(\sqrt{r}+1)}\right) \\
& -11\mathrm{Li}_2\left(\frac{-\sqrt{1-r}+\sqrt{r}+1}{2\sqrt{r}+2}\right)+11\mathrm{Li}_2\left(\frac{\sqrt{1-r}+\sqrt{r}+1}{2\sqrt{r}+2}\right) \\
& +\frac{1}{2}\left(\mathrm{Li}_2\left(-\frac{r+2\sqrt{1-r}-2}{-r+2\sqrt{1-r}+2}\right)+\mathrm{Li}_2\left(\frac{4\sqrt{1-r}}{r+2\sqrt{1-r}-2}\right)\right. \\
& -\log\left(-\frac{r+2\sqrt{1-r}-2}{-r+2\sqrt{1-r}+2}\right)(\log(4r)+2\log(1-\sqrt{r})) \\
& \left.+\log\left(\frac{r-2(\sqrt{1-r}+1)}{r+2\sqrt{1-r}-2}\right)\log\left(-\frac{4\sqrt{1-r}}{r+2\sqrt{1-r}-2}\right)-\frac{\pi^2}{6}\right) \\
& -2\log^2(-\sqrt{1-r}+\sqrt{r}+1)+2\log^2(\sqrt{1-r}+\sqrt{r}+1) \\
& +\log(16)\log(-\sqrt{1-r}+\sqrt{r}+1)+3\log(\sqrt{r}+1)\log(-\sqrt{1-r}+\sqrt{r}+1) \\
& -4\log(2)\log(\sqrt{1-r}+\sqrt{r}+1)-3\log(\sqrt{r}+1)\log(\sqrt{1-r}+\sqrt{r}+1)) \\
& -(-\log\left(\frac{1}{s-4m_b^2}\right)+2\log\left(\frac{1}{s}\right)+\log(64))(r(s-2m_b^2)\log\left(\frac{r}{-r+2\sqrt{1-r}+2}\right) \\
& +4m_b^2\sqrt{1-r})+m_b^2\left(2(r-4)\log(r)-r\log(r)-8\sqrt{1-r}\log(\sqrt{r}-r)\right. \\
& -2(r-4)\log(r(\sqrt{1-r}-3)-4\sqrt{1-r}+4)+2r\log(\sqrt{1-r}+1) \\
& -2(r-4)\log((\sqrt{r}-2)^2)+2r\log((\sqrt{r}-2)^2\sqrt{r})-8\log((\sqrt{r}-2)^2\sqrt{r}) \\
& +4\sqrt{1-r}(\log(-64(r-1))-\log(r)+2\log(1-\sqrt{r})-2) \\
& \left.+8\sqrt{1-r}\log(2-2r)\right) \\
& +rs\left(-2\sqrt{1-r}+2\log\left(-\frac{r}{r+2\sqrt{1-r}-2}\right)+r\tanh^{-1}\left(\frac{2\sqrt{1-r}}{r-2}\right)\right)\Bigg)
\end{aligned}
\tag{1.64}
$$

We can know perform the numerical integration of the decay rate $\Delta\Gamma_{\mathrm{nlo}}$, using the vegas algorithm [11]. As a test of the implementation we have verified that the sum of the real and virtual integration perfectly agrees with the inclusive formula in equation 1.44.

1.4.2 JADE algorithm

In the following we will reconstruct jets using the JADE algorithm [31, 32, 33]. The first step of the jet algorithm is to define a distance function between two clusters i, j in terms of their four-momenta p_i, p_j. The distance function of the JADE algorithm d_{ij} is defined as

$$d_{ij}^2 = 2E_iE_j(1 - \cos\theta_{ij}) \tag{1.65}$$

where θ_{ij} is the angle between the three-momenta $\vec{p}_i$, $\vec{p}_j$. More often used is the scaled expression y_{ij} of the distance

$$y_{ij} = \frac{d_{ij}^2}{E_{CM}^2} = \frac{2E_iE_j(1 - \cos\theta_{ij})}{E_{CM}^2} \tag{1.66}$$

with E_{CM} the energy in center of mass frame.

The algorithm starts from a list of particles, that we consider the initial set of clusters. If the two clusters with minimum distance y_{ij} have a distance that is below a chosen cut-off scale y_{cut}, they are merged together. The algorithms is looped until the remaining clusters have a distance above the cut-off scale.

The role of the jet algorithm is very important to our simulations; the distance defined in equation 1.66 collects, in the same jet event, the real contribution in the unresolved (soft) limit and the corresponding virtual contribution obtained removing the soft radiation. For this reason the algorithm is said to be infrared safe.

It is worth mentioning that a central role is played by the choice of the cut-off scale y_{cut}: if we choose to be too exclusive, with a tiny cut-off scale, we end up with unbalanced cancellation among the real and virtual contributions; this may result even into the unphysical prediction of a negative cross sections for the lowest (2 in our case) jet rate.

1.5 Phenomenological study

Thanks to the subtraction we can now make differential predictions of jet observables. For illustrative purposes we consider an Higgs factory, as could be a muon collider operating at a center of mass energy of 125GeV. Further, we do not include any Electroweak corrections that would be mandatory for a precise estimate of the event yield. In figure 1.9 we present the percentage of 2 and 3 particle events selected by the JADE jet-algorithm: the y_{cut} parameter chooses the minimum "distance" between two jets to be considered as distinguishable, the more the y_{cut} the more the 2 jets events.

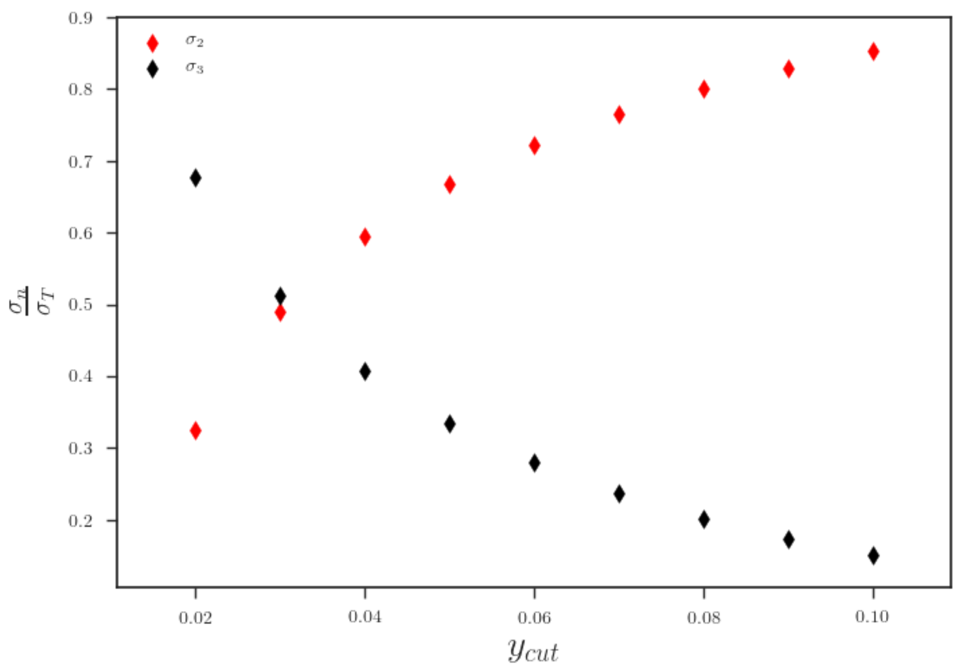


Figure 1.9: Percentage of 2 and 3 particle events selected by the JADE jet-algorithm. The distributions are normalized dividing by the total decay rate σ_T.

For the following plots we fix the value of $y_{\mathrm{cut}} = 0.1$. In figure 1.10 we plot the distribution of the jet with maximum energy for two jets events divided by the Higgs mass . This is an interesting observable both from an experimental and theoretical point of view. In fact it can be measured indeed and it is an index of the reliability of the subtraction method: near the $E_{\max} = 0.5$ region both virtual and real events contribute to the weights then, thanks to the subtraction method, we can make a finite prediction and be exclusive on the 2 particles events.

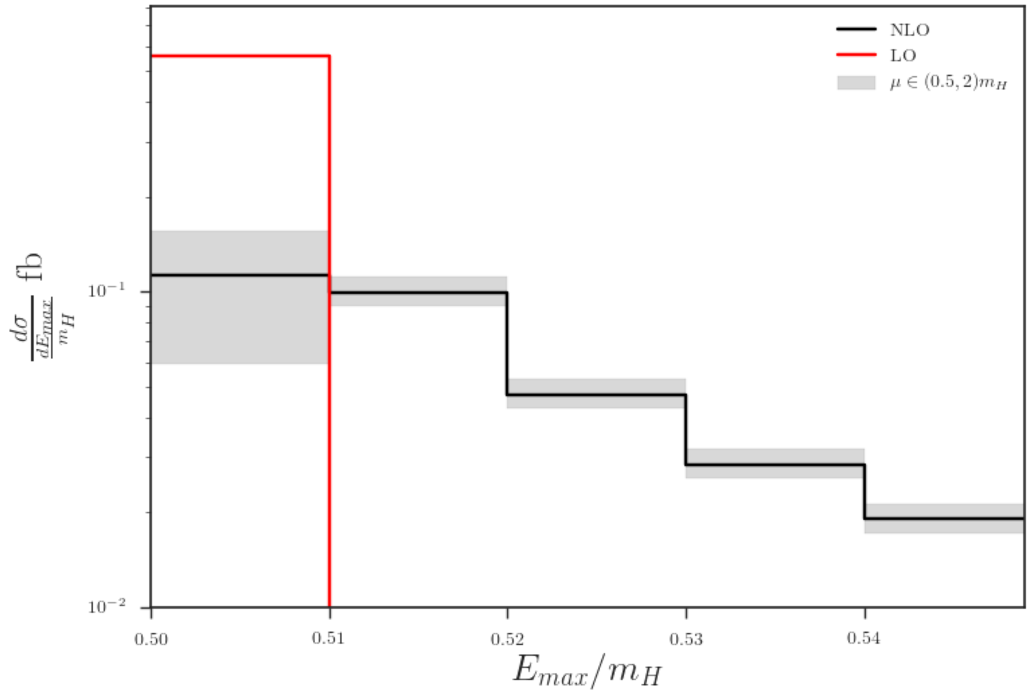


Figure 1.10: $E_{\max}$ distribution with NLO correction, the band is determined plotting the observable at renormalization scale $\mu \in (0.5, 2)m_H$. The value of y_{cut} is fixed at 0.1 to ensure a non-instantly vanishing distribution. Near 0.5 the dependence on the QCD scale is quite large, this is caused by the presence of very large virtual and real weights in the region of interest.

With the same spirit we plot in figure 1.11 the absolute value of y rapidity of the jet with highest energy. Here the absolute value is necessary because near $y_h = 0$ the numerical precision might create instability. In fact, because the kinematic $1 \to 2$ has two jets with equal energy $m_H/2$, due to the numerical precision, jets with a fixed sign rapidity might prevail and be selected as the ones of maximum energy. The result would be an asymmetry in the distribution. The rapidity is moreover an observable of great theoretical interest; in fact, because the Born kinematic has a non trivial distribution, the effect of the subtraction is manifest in the whole plot and not only in a specific region like the $E_{\max}$ case.

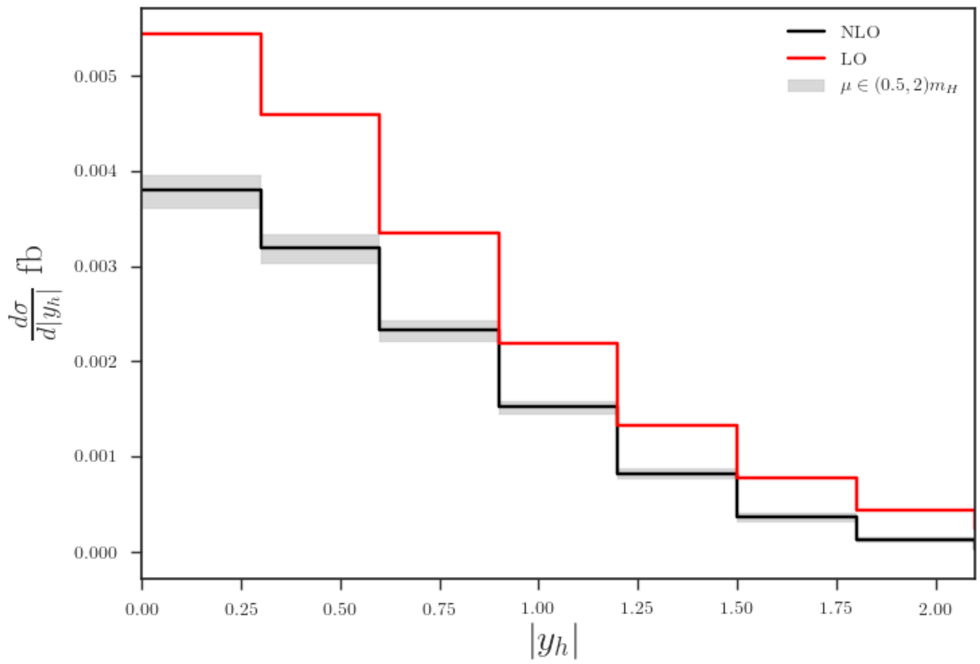


Figure 1.11: Absolute value of y rapidity for the jet with highest energy. The band is determined plotting the observable at renormalization scale $\mu \in (0.5, 2)m_H$ and $y_{\text{cut}} = 0.1$.

Chapter 2

Multiloop calculation

When computing a multi-loop process, the best strategy (due to the complexity and the great amount) is to reduce at minimum the number of integrals necessary for the calculation. We refer to this minimal set as Master Integrals (MI). In this chapter we show which are the identities that can be exploited to reconstruct such a basis and one of the most useful methods to try and find an analytical solution to these integrals.

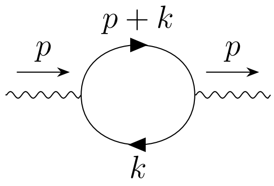
2.1 Setting up the environment

Our first attempt is to obtain a general form for a multi-loop integral in terms of the involved momenta, in order to have a uniform notation to work with. Suppose we have E independent external momenta (p_i) and L momenta of integration (k_j), we call each propagator $D_i = (q_i^2 - m_i^2)$, where q_i is its momentum. By definition, a scalar multi-loop integral has the following general structure:

$$I(n) = \int \prod_{i=1}^{L} d^d k_i \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_N^{\alpha_N}} \tag{2.1}$$

where α_j states for the power of the j-th denominator (positive or negative) and n represents the vector of the dependencies (external momenta and scalar parameters). Dealing with a Feynman diagram, we would like to express any term contributing as a scalar integral. At one-loop level our aim is easily obtained using just the denominators already appearing in the amplitude.

Example. Consider the one loop correction to the photon propagator



$$\propto \int d^d k \,\mathrm{Tr}\, \frac{\gamma^\alpha (\not{p} + \not{k} + m)\gamma^\beta (\not{k} + m)}{((p+k)^2 - m^2)(k^2 - m^2)} \tag{2.2}$$

where we can identify $D_1 = (p+k)^2 - m^2$ and $D_2 = k^2 - m^2$. The scalar product $p \cdot k$ can be expressed as

$$\begin{aligned} p \cdot k &= \frac{1}{2}\left((p+k)^2 - p^2 - k^2\right) \\ &= \frac{1}{2}\left(D_1 - D_2 - p^2\right) \end{aligned} \tag{2.3}$$

When the number of loops is greater than one, it is never possible to express all the scalar products, appearing in the amplitude, in terms of the physical denominators. As we will see in the further calculation, the problem can be solved by the introduction of auxiliary topologies: to restore the form 2.1, we enlarge the number of propagators so that all the scalar products can be expressed in terms of them. A set of propagators which span all the scalar products is called *integral family*. A *topology* instead is defined as a set of propagators that occur in the integral with only positive powers.

2.2 Identities

Once we have put all integrals in the scalar form, we can go on reducing the expression in terms of MI. The procedure involves the use of two kinds of identities, that we will analyze in detail in this paragraph. As it will be clear soon, the number of identities that can be written is very large, so we always deal with a redundant system of dependent equations. As expected the choice of MI is not unique but can become quite relevant when it comes to the methods used to compute the integrals. We now focus simply on the identities and leave the choice of the most convenient basis to the following sections.

2.2.1 Lorentz Identities

The first kind of identities is known as *Lorentz Identities* LI and it is based on symmetry properties of the integrals. A general Feynman Integral FI is invariant under Lorentz transformations of its external momenta. So considering the infinitesimal transformation

$$p_i^\mu \to \omega_{\mu\nu} p_{i\nu} \tag{2.4}$$

where $\omega_{\mu\nu}$ is a general antisymmetric tensor with $|\omega_{\mu\nu}| \ll 1 \quad \forall \mu, \nu$, at the first order in $\omega_{\mu\nu}$ a scalar multi-loop integral transforms as

$$I(n) \to \left(1 + \omega_{\mu\nu} \sum_i p_{i\nu} \frac{\partial}{\partial p_i^\mu}\right) I(n) \tag{2.5}$$

Using the antisymmetry of $\omega_{\mu\nu}$ we get the identity

$$\sum_i \left(p_{i\nu} \frac{\partial}{\partial p_i^\mu} - p_{i\mu} \frac{\partial}{\partial p_i^\nu}\right) I(n) = 0 \tag{2.6}$$

Contracting with antisymmetric tensors built with external momenta, for instance $p_k p_h - p_h p_k$, we get all possible Lorentz Identities (LI). If we consider a Feynman diagram with Z external legs, in d dimensions, the number of independent external momenta is $N_{ind} = \min(d, Z-1)$, because of the overall momenta conservation joined with the maximum number of linearly independent momenta in the space. The number of independent LI is then $N_{ind}(N_{ind}-1)/2$ that is the number of antysymetric rank 2 tensors that we can build in the space $N_{ind} \times N_{ind}$.

2.2.2 Integration by parts identities

We refer to the second kind of identities as the *integration by parts identities* IBP. Following [34, 35], a FI is invariant under the substitution of its integral momenta

$$k_i \to A_{ij} k_j + B_{ij} p_j \tag{2.7}$$

Considering an infinitesimal transformation $k_i \to k_i + \beta_{ij} q_j$, where $q = (k_1 \dots k_L, p_1 \dots p_E)$, the integral measure changes as

$$d^d k_i \to d^d k_i + \beta_{ij} \delta_{ij} d \tag{2.8}$$

where δ_{ij} is the Kronecker delta, d the dimension of integration; the Einstein convention of summations over repeated index is meant only for the j index. So, imposing the invariance, we have that

$$\begin{aligned} I(n) = \int \prod_{i=1}^{L} \left(1 + \beta_{ij} \left(d\delta_{ij} + q_j^\mu \frac{\partial}{\partial q_i^\mu}\right)\right) \frac{d^d k_i}{D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_N}} = \\ \int \left(1 + \frac{\partial}{\partial q_i^\mu} q_j^\mu\right) \frac{d^d k_i}{D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_N}} \end{aligned} \tag{2.9}$$

It is possible to associate the structure of a Lie Algebra to the operator $O_{ij} = \frac{\partial}{\partial q_i^\mu} q_j^\mu$, in fact with direct computation we find:

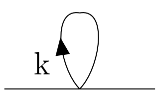
$$[O_{ij}, O_{kl}] = \delta_{il} O_{kj} - \delta_{kj} O_{il} \tag{2.10}$$

The shift invariance can be thus expressed with the formula:

$$\int \frac{\partial}{\partial q_i^\mu} q_j^\mu \frac{d^d k_i}{D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_N}} = 0 \tag{2.11}$$

So performing the derivatives it is clear how to obtain all possible sets of Integration by Parts Identities (IBP). It is also instructive to understand that these identities is consistent with the fact that, in dimensional regularization, the integral of the derivative vanishes. Using Gauss theorem, we're left with the flux on the boundaries of integration region; in dimensional regularization, we can modify the dimension of the surface on which we are performing the flux obtaining 0.

Example: 1-loop massive tadpole.



$$\propto \int d^d k \frac{1}{k^2 - m^2} \tag{2.12}$$

It is clear that dealing with only one momentum among the q_i previously defined, we can write only the IBP with respect to this momentum, k; performing all the steps:

$$\begin{aligned} 0 &= \int d^d k \frac{\partial}{\partial k^\mu} \left(k^\mu \frac{1}{k^2 - m^2} \right) \\ &= \int d^d k \left(\frac{d}{k^2 - m^2} - \frac{2k^2}{(k^2 - m^2)^2} \right) \\ &= \int d^d k \left(\frac{d-2}{k^2 - m^2} - \frac{2m^2}{(k^2 - m^2)^2} \right) \end{aligned} \tag{2.13}$$

We finally obtain

$$\int \frac{d^d k}{(k^2 - m^2)^2} = \frac{d-2}{2m^2} \int \frac{d^d k}{k^2 - m^2} \tag{2.14}$$

This simple example shows how, using the IBP, we can express integrals belonging to the same family in terms of each other. Once evaluated the simple tadpole, multiplying by a constant factor, we obtain the integral with the second power.

Example: 1-loop massive bubble:

We consider the family integral with 2 propagators $D_1 = k^2 - m^2$ $D_2 = (k+p)^2 - m^2$, which includes all the 1-loop bubble diagrams:

$$I_{\alpha_1 \alpha_2} = \int \frac{d^d k}{D_1^{\alpha_1} D_2^{\alpha_2}} \tag{2.15}$$

Of the two possible identities (w.r.t. k or p) let us compute the IBP with respect to k, for the bubble integral ($\alpha_1 = \alpha_2 = 1$). We get the following identity:

$$I_{11}(d-3) - 2m^2 I_{21} - I_{02} - (2m^2 - p^2) = 0 \tag{2.16}$$

Using the equality $I_{\alpha_1 \alpha_2} = I_{\alpha_2 \alpha_1}$, due to the shift $k \to k - p$, and the previous IBP (2.14) for which

$$I_{20} = \frac{d-2}{2m^2} I_{10} \tag{2.17}$$

we can choose the master integrals of our family to be I_{11} and I_{10}. From (2.16). We can finally express I_{21} in terms of them:

$$I_{21} = \frac{d-3}{4m^2 - p^2} I_{11} - \frac{d-2}{2m^2(4m^2 - p^2)} I_{10} \tag{2.18}$$

2.3 Get the Master Integrals

We now have a very large number of identities, a subset of them can be considered to constitute a homogeneous system with the integrals as unknowns. Unfortunately, as we previously said, the identities are not independent, so there exists a minimal set of integrals that have to be computed. The great amount of identities, growing vary fast with the number of propagators and loops, requires automated procedures to get the master integrals. One of the most applied algorithms is the Laporta's one [36]. This algorithm is distinguishable from the others mainly because of two aspects:

- It starts from e pre-defined subset of IBP and LI
- It presents great efficiency in time due to an optimal ordering of the integrals

We limit the number of denominators with positive and negative powers, choosing two integers r and s. This first step allows us to write down a finite number of IBP and LI that constitute our pre-defined subset. The algorithm then proceeds computing the identities and using Gauss method of substitution to get the MI. At each step of iteration, we start from a new identity $\sum_i c_i W_i = 0$ and we rewrite it in terms of the other integrals $W_i^{'}$, already selected from the previous identities. To each integral $W_i^{'}$, we assign a weight based on, in order of importance:

- number of denominators
- combination of denominators
- exponents

and the integral with greatest weight $W_l^{'}$ is selected. The identity is then written in the form $W_l^{'} = \sum_{i \neq l} c_i^{'} W_i^{'}$ allowing further substitutions and it is added to database. Whilst the extraction of the integrals is performed top bottom with the weights order, the identities (IBP and LI) are computed bottom up. These allows minus computing time due to the lower number of substitutions required. It is worth mentioning that selecting a finite number of identities does not guarantee that we reach the minimum number of integrals, so it is important to always give a certain margin to the r and s values.

Laporta's algorithm has been implemented in software like Reduze2 [37] for parallel computing e speed up performances.

2.4 Differential equations for master integrals

We now have reduced the problem to the solution of a limited number of integrals (the Master Integrals). However a direct analytical solution is often not applicable. One of the most successful and current methods is the method of *differential equations*; in recent years this method has been applied with success to a vast class of problems. Firstly proposed by Kotikov [38] and the extended to all invariants by Gehrmann and Remiddi [39], this method consists in deriving an homogeneous system of differential equations in which the unknowns are the master integrals. With a particular choice of the basis (as we'll see in the following sections) the problem can be greatly simplified, making the method of differential equation really effective.

2.4.1 Derive the differential equations

We can derive the differential equations for MI simply differentiating the integrals with respect to kinematics invariants. The effect of such differentiation will be to change the power of denominators of the integrals. For an internal mass m_i we simply have:

$$\partial_{m_i^2} \int \prod_{j=1}^{L} d^d k_i \frac{1}{D_1^{\alpha_1} \cdots D_i^{\alpha_i} \cdots D_N^{\alpha_N}} = \alpha_i \int \prod_{j=1}^{L} d^d k_i \frac{1}{D_1^{\alpha_1} \cdots D_i^{\alpha_i+1} \cdots D_N^{\alpha_N}} \tag{2.19}$$

where $D_i = q_i - m_i^2$. If we derive with respect to a general kinematic invariant X_j, we can express the differential operator in terms of derivatives w.r.t. external momenta, that appear in the integrals, using the chain rule:

$$p_k^\mu \partial_{p_{\mu,i}} = \sum_j p_k^\mu \frac{\partial X_j}{p_{\mu,i}} \partial_{X_j} \tag{2.20}$$

The left hand operators are not all independent because they are related by Lorentz Identities. The number of independent equations we can write is given by:

$$N_{ind}^2 - \frac{N_{ind}(N_{ind}-1)}{2} = \frac{(N_{ind}-1)(N_{ind}-2)}{2} \tag{2.21}$$

where N_{ind} is the number of independent external momenta. In fact, we have to subtract to the total number of possible contraction $p_k^\mu \partial_{p_{\mu,i}}$ the number of constraints given by the LI. The result corresponds to the number of independent kinematics invariants.

The system 2.20 can be solved and all derivatives with respect to X_i can be expressed in terms of p_i. We now recall the fact that we can always express all the remaining factors, which appear after performing the derivatives, in terms of denominators. In these differential equations for the master integrals, the integrals appearing to the right hand side belong to the same sector or sub-sectors of the ones to the left hand side: this leads to a differential relation among our MIs and other integrals that can be expressed again in terms of the MIs. Performing the derivatives of each MI w.r.t a set of independent kinematic invarinats, we finally end up with an homogeneous system of partial differential equations in which the unknowns are the MIs.

Example: 1-loop bubble. We want to compute the differential equation for the integral I_{11} we already introduced (equation 2.15). The first step is computing the derivative of I_{11} with respect to the invariant p^2. Using that

$$p^\mu \frac{\partial}{\partial p^\mu} = 2p^2 \frac{\partial}{\partial p^2} \tag{2.22}$$

we obtain:

$$\begin{aligned}\frac{dI_{11}}{dp^2} &= \frac{1}{2p^2}\int d^dk\left(p^\mu\frac{\partial}{\partial p^\mu}\frac{1}{D_1D_2}\right)\\ &= -\frac{1}{2p^2}I_{11} - \frac{1}{2}I_{12} + \frac{1}{2p^2}I_{02}\end{aligned} \tag{2.23}$$

With (2.18) and (2.17), we can express everything in terms of the two MI I_{11} and I_{10} :

$$\frac{d}{dp^2}I_{11} = -\frac{4p^2 - 4m^2 - p^2d}{2p^2(p^2-4m)}I_{11} - \frac{d-2}{p^2(p^2-4m^2)}I_{10} \tag{2.24}$$

We obtained this way a differential equation that, solved, gives the expression for I_{11} when I_{10} is known. The problem of computing amplitudes can so be turned into the solution of a system of differential equations. In the following section we present a method to solve a typical system that appears when dealing with physical integrals.

2.4.2 Canonical basis

In the previous sections we have seen that, using IBPs and LIs it is possible to find a basis for our integrals (the MIs); then differentiating by kinematic invariants it is possible to write a system of differential equations with MIs as unknowns. As we already said, our aim is to determine the solution by finding a primitive for these functions; moreover we will need to fix the constants of integration by imposing appropriate boundary conditions. In order to simplify the problem, we can exploit the arbitrariness of the basis choice to rewrite the system in a more convenient form. In these section we define the *canonical form* that, if found, always lets us perform the first step of integration and find the primitive.

Consider a fixed complete set of independent kinematic invariants X_i and choose one invariant X_Q. We notice that the dependence on it can be reconstructed with the mass dimension of the integrals. If we call $\widetilde{f}_i$ the N master integrals, referring to the dimensionless quantities

$$f_i = (X_Q)^{-\dim(\widetilde{f}_i)/\dim(X_Q)}\widetilde{f}_i \tag{2.25}$$

we can drop the dependence on X_Q and reduce by one the number of parameters. At this point we need to define a new set of independent kinematic invariants and the easiest choice could be to build it from the previous one introducing for $i \neq Q$ the dimensionless ratios

$$x_i = \frac{X_i}{X_Q^{\dim(X_i)/\dim(X_Q)}} \tag{2.26}$$

Working in $d = 4 - 2\epsilon$ dimensions, the set of equations takes the form

$$\partial_m f(\epsilon, \vec{x}) = A_m(\epsilon, \vec{x}) f(\epsilon, \vec{x}) \tag{2.27}$$

where $\vec{x}$ denotes the vector of kinematic invariants, $\partial_m = \frac{\partial}{\partial x_m}$ and A_m is an $N \times N$ matrix. The following integrability conditions have to be satisfied:

$$\partial_n A_m - \partial_m A_n + [A_n, A_m] = 0 \tag{2.28}$$

In practice we are always interested in an Laurent expansion of the MIs near $\epsilon = 0$. The problem is easily solved if we can find a suitable basis in which the system assumes a canonical form, firstly proposed by Henn [13]. Under a change of basis $f = Bg$, A_m transforms as

$$A_m \to B^{-1} A_m B - B^{-1} \partial_m B \tag{2.29}$$

The system is said to be in the canonical form if, in the new basis, A_m has a factorized dependence on ϵ

$$\partial_m g(\epsilon, \vec{x}) = \epsilon A_m(\vec{x}) g(\epsilon, \vec{x}) \tag{2.30}$$

Now the integrability conditions are

$$\partial_n A_m - \partial_m A_n = 0, \quad [A_n, A_m] = 0 \tag{2.31}$$

The system can be expressed in a complete differential form in the following way

$$dg(\epsilon, \vec{x}) = \epsilon dA(\vec{x}) g(\epsilon, \vec{x}) \tag{2.32}$$

where dA is the differential form

$$dA = \sum_{m=1}^{M} A_m(\vec{x}) dx_m \tag{2.33}$$

and it is exact because of the relation 2.31. Furthermore, having factorized the dependence on ϵ, we can perform the integration recursively order by order. If we rescale each g_i by an appropriate ϵ factor, we can always write:

$$g(\epsilon, \vec{x}) = g^{(0)}(\vec{x}) + \epsilon g^{(1)}(\vec{x}) + \epsilon^2 g^{(2)}(\vec{x}) + \dots \tag{2.34}$$

The system 2.32 can be rewritten in terms of the $g^{(n)}$ in the following form:

$$\begin{aligned} dg^{(0)}(\vec{x}) &= 0 \\ dg^{(1)}(\vec{x}) &= dA(\vec{x}) g^{(0)}(\vec{x}) \\ dg^{(2)}(\vec{x}) &= dA(\vec{x}) g^{(1)}(\vec{x}) \\ &\vdots \end{aligned} \tag{2.35}$$

All the $g_i^{(0)}$ will be constants (to be fixed); then known $g^{(n-1)}$ we can integrate and find $g^{(n)}$. The general solution of the canonical system of differential equations can be given in terms of *Chen's iterated integrals* [40]

$$g(\epsilon, \vec{x}) = \left(1 + \int_\gamma dA + \int_\gamma dA dA + \dots\right) g(\epsilon, \vec{x}_0) \tag{2.36}$$

where $g(\epsilon, \vec{x}_0)$ is a vector of arbitrary constants and γ is a path, going from $\vec{x}_0$ to $\vec{x}$, in the domain of A. The solution is then completely iterative and from a formal point of view completely determined. In particular, we will see that the symbolic integration of the system is easily performed in terms of **Goncharov polylogarithms** GPL [7] if the system is in a *d-log* canonical form which occurs, by definition, if the differential form dA can be written as

$$dA(\vec{x}) = \sum_{i=1}^{K} \widetilde{A}_i d\log(L_i(\vec{x})) \tag{2.37}$$

where $\widetilde{A}_i$ are $N \times N$ constant matrices and $L_i(\vec{x})$ are K rational functions of the invariants $\vec{x}$. We commonly call this set of functions the *alphabet* of the system and $L_i(\vec{x})$ the *letters*.

2.4.3 Goncharov polylogarithms

From a practical point of view, we want our solution to be possibly expressed in terms of functions that easily can be numerically evaluated. For the Goncharov polylogarithms there exists a library in the GiNac framework [8], so they are a very good candidate for our purpose. A GPL is defined in terns of iterated integrals in the following way:

$$G(z_1, \ldots, z_k; x) = \int_0^x dt \frac{1}{t - z_1} G(z_2, \ldots, z_k; t) \tag{2.38}$$

for $(z_1, \ldots, z_k) \neq \vec{0}_k$. The variables z_i are called *indices* while x is the *argument* of the GPL. The number of iterated integration defines the *weight* of the GPL. The empty index is defined as

$$G(; x) = 1 \tag{2.39}$$

Instead when all the k indices are zero we have:

$$G(0, \ldots, 0; x) = \frac{1}{k!} \log^k(x) \tag{2.40}$$

It can be easily checked that a GPL, $G(z1, \ldots, z_k; x)$, is divergent whenever $z_1 = x$. Similarly $G(z1, \ldots, z_k; x)$ is analytic at $x = 0$ if $z_k \neq 0$. If the rightmost index is $z_k \neq 0$ then the GPL satisfies a scaling propriety for each $w \in \mathcal{C}$

$$G(z_1, \ldots, z_k; x) = G(wz_1, \ldots, wz_k; wx) \tag{2.41}$$

One of the most useful proprieties of GPL is that we can write the product of two GPLs with equal integration limit as a linear combination of GPL.

If we consider two GPL of weight one we have:

$$G(a;x)G(b;x) = \int_0^x \frac{dt_1}{t_1-a} \int_0^x \frac{dt_2}{t_2-b} = \iint_S \frac{dt_1 dt_2}{(t_1-a)(t_2-b)} \tag{2.42}$$

where in the last step S is the square of corners $(0,0)$,$(0,x)$,$(x,0)$ and (x,x). If then we split the integration over the square in two integration one over above the diagonal the other below , we have:

$$\begin{aligned} G(a;x)&G(b;x) \\ &= \iint_{0\leq t_2 \leq t_1 \leq x} \frac{dt_1 dt_2}{(t_1-a)(t_2-b)} + \iint_{0\leq t_1 \leq t_2 \leq x} \frac{dt_1 dt_2}{(t_1-a)(t_2-b)} \\ &= \int_0^x \frac{dt_1}{t_1-a} \int_0^{t_1} \frac{dt_2}{t_2-b} + \int_0^x \frac{dt_2}{t_2-a} \int_0^{t_2} \frac{dt_1}{t_1-a} \\ &= G(a,b;x) + G(b,a;x) \end{aligned} \tag{2.43}$$

We can extend the same argument for GPL of higher weights, we end up with a propriety common to all nested integrals. The product of two GPL of weight k_1 and k_2 with equal integration limit can be written as linear combination of GPLs of weight $k_1 + k_2$, their indices are obtained as *shuffle* product of the starting indices

$$G(\vec{m},x)G(\vec{n};x) = \sum_{\vec{s}=\vec{m}\sqcup\!\sqcup\vec{n}} G(\vec{s};x) \tag{2.44}$$

where $\vec{m}\sqcup\!\sqcup\vec{n}$ denotes the shuffle product of m and n. It is defined as all possible combinations that can be obtained from the indices m and n that separately preserve the order of the two sets. To clarify we can imagine to shuffle together two decks of cards. The result will preserve the starting order of each deck.

2.4.4 Integrating in terms of polylogarithms

We present here how to integrate in terms of polylogarithms with an instructive example. Consider the one-loop-bubble and the differential equation 2.24. We remark that in this case we have a single independent invariant p^2 or s and two master integrals I_{11} ans I_{10}. I_{10} being completely independent on s has derivative $\partial_s I_{10} = 0$. So we can't extract any further information on the integral from the differential equation and we need to compute it analytically. Luckily it is very easy and using measure

$$\widetilde{d^d k} = -\left(\frac{m^2}{\mu^2}\right)^{\epsilon} \frac{d^d k}{\Gamma(1+\epsilon) i\pi^d} \tag{2.45}$$

it is given by

$$\widetilde{f_1} = I_{10} = m^2 \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} \tag{2.46}$$

We can completely eliminate the dependence on m as seen above, then we choose the following scaling factors to remove the ϵ poles.

$$\begin{aligned} f_1 &= \widetilde{f_1}\epsilon(-2-2\epsilon) \\ f_2 &= \widetilde{f_2}\epsilon(-1-2\epsilon) \end{aligned} \tag{2.47}$$

The system of two equation is then

$$\begin{aligned} \partial_s f_1 &= 0 \\ \partial_s f_2 &= \frac{-1+2\epsilon}{(-4+s)s} f_1 + \frac{2-\epsilon s}{(-4+s)s} f_2 \end{aligned} \tag{2.48}$$

we recognize the matrix $A(\epsilon, s)$

$$A(\epsilon, s) = \begin{pmatrix} 0 & 0 \\ \frac{-1+2\epsilon}{(-4+s)s} & \frac{2-\epsilon s}{(-4+s)s} \end{pmatrix} \tag{2.49}$$

Using the invariant s, the equations in canonical form will present squareroots. These can be eliminated with the substitution

$$s \to -\frac{(1-x)^2}{x} \tag{2.50}$$

and w.r.t. x the derivatives are

$$
\begin{aligned}
\partial_x f_1 &= 0 \\
\partial_x f_2 &= \frac{(1-2\epsilon)f_1}{(x^2-1)s} + \left(\frac{-2}{x^2-1} + \frac{\epsilon(1-x)}{x+x^2}\right) f_2
\end{aligned}
\tag{2.51}
$$

The canonical form is achieved thanks to equation 2.29 where the matrix B is

$$
B = \begin{pmatrix} 1 & 0 \\ \frac{1}{1-x} & \frac{1+x}{1-x} \end{pmatrix}
\tag{2.52}
$$

Next we will see how to compute the matrix B, the main purpose of this example is to show how to perform the integration in terms of GPL. The basic ideas will apply with slight generalization to more complex problems.

Performing the transformation, the system is in canonical form and the matrix $A(x)$ will be

$$
A_x(x) = \begin{pmatrix} 0 & 0 \\ \frac{1}{x} - \frac{1}{1+x} & \frac{1}{x} - \frac{2}{1+x} \end{pmatrix}
\tag{2.53}
$$

We can put everything in full differential form as 2.32

$$
\begin{aligned}
&d \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \\
&\epsilon \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} d\log(x) + \epsilon \begin{pmatrix} 0 & 0 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} d\log(1+x)
\end{aligned}
\tag{2.54}
$$

We notice that this is a d-log canonical form with alphabet $(x, 1+x)$. The integration can now be made in terms of GPL. We work in the unphysical region $0 < x < 1$ or $s < 0$ where the GPL are real. The solution can be then analytically continued with the substitution $s \to s + i\epsilon$.

Expanding g in ϵ we get, as previously seen $dg^{(0)} = 0$. These constants need to be determined. We already know g_1 from the direct integration of f_1 and the equation $g = B^{-1} f$:

$$
g_1 = 2
\tag{2.55}
$$

We have then $g_1^{(0)} = 2$. The other constant $g_2^{(0)}$ could be determined if we knew g_2 in a particular x limit. We can exploit the fact that both f_1 and f_2 are finite in the limit $x \to 1$ or $s \to 0$. In fact f_1 is it self a constant, while f_2 is the massive 1 loop bubble, so performing an s cut, we immediately understand that it has a brunch cut at $s \to 4m^2$ so it is finite at $s \to 0$. With this information in mind and the equation $g = Bf$, we write g_2 in terms of f_i

$$g_2 = -\frac{1}{1+x} f_1 - \frac{-1+x}{1+x} f_2 \tag{2.56}$$

Both f_1 and f_2 are finite at $x \to 1$ so, evaluating 2.56 in this limit, we get

$$g_2\big|_{x=1} = -\left.\frac{f_1}{2}\right|_{x=1} \tag{2.57}$$

Equation 2.57 is valid at **each order** in ϵ. So in particular we have $g_2^{(0)} = -1$. We can now write equation 2.54 at the first order in ϵ

$$\begin{aligned} d \begin{pmatrix} g_1^{(1)} \\ g_2^{(1)} \end{pmatrix} &= \\ \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} d\log(x) &+ \epsilon \begin{pmatrix} 0 & 0 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} d\log(1+x) \end{aligned} \tag{2.58}$$

The integration in terms of GPL is performed according to the their definition. We simply get

$$\begin{aligned} g_1^{(1)} &= C_1 \\ g_2^{(2)} &= C_2 + G(0;x) \end{aligned} \tag{2.59}$$

Again C_1 can be determined from equation 2.55 and C_2 from equation 2.57. We get

$$C_1 = 0 \quad C_2 = 0 \tag{2.60}$$

With the same technique we can evaluate next orders and thanks to equation $f = Bg$ we can go back to physical integrals. Up to the second order in ϵ we get

$$f_2 = -\frac{1}{(x-1)(2\epsilon-1)}$$
$$\Bigg((x+1)\Bigg(\epsilon^2\Bigg(\frac{1}{3}\pi^2 G(-1;,x) - \frac{1}{6}\pi^2 G(0;,x) +$$
$$4G(-1,-1,0;,x) - 2G(-1,0,0;,x) - 2G(0,-1,0;,x) + G(0,0,0;,x) - 2\zeta(3)\Bigg)$$
$$+\epsilon\left(-2G(-1,0;,x) + G(0,0;,x) - \frac{\pi^2}{6}\right) + G(0;,x) - 1\Bigg) + \frac{2}{\epsilon}\Bigg) \tag{2.61}$$

2.4.5 Find a canonical form

If we are able to find a canonical form, chances are we can find a solution to our problem. There might still be difficulties in determining the boundaries, but at least the symbolic integration is easily performed. At the actual state of art, nothing guarantees that we can always find a canonical form. Several algorithms exist and all request a precise previous manipulation of the basis to be applied. There are different implementation of these algorithms, for instance Fuchsia [41] based on Roman Lee algorithm [35] or Canonica [42] by C.Meyer. In this work we applied a method based on the Magnus series, proposed by Argeri et al [14] [43].

This method relies on a basis of master integrals obeying a liner system of differential equation in ϵ parameter. Following [14], suppose we have a system

$$\partial_x f(\epsilon, x) = A(\epsilon, x) f(\epsilon, x) \tag{2.62}$$

where f is the vector of MI, x a variable depending on kinematic invariants and A is a matrix linear in ϵ

$$A(\epsilon, x) = A_0(x) + \epsilon A_1(x) \tag{2.63}$$

The basic idea is to remove the dependence on A_0 by a "rotation" of the basis of MI, the way we eliminate the dependence on the known kernel in quantum mechanics with the *interaction picture*. Let's start considering the Schrödinegr equation with an Hamiltonian $H(t) = H_0(t) + \epsilon H_1(t)$ linear in ϵ

$$i\hbar\partial_t |\Psi(t)\rangle = H(t) |\Psi(t)\rangle \tag{2.64}$$

we can make disappear the solvable kernel H_0 from the equation with the interaction picture. Defining the operator U so that $|\Psi(t)\rangle = U |\Psi(t)\rangle_I$, if U obeys the following equation

$$i\hbar\partial_t U = H_0(t)U \tag{2.65}$$

than in the basis $|\Psi(t)\rangle$ the Schrödinegr equation becomes

$$i\hbar\partial_t |\Psi(t)\rangle_I = \epsilon H_{1I}(t) |\Psi(t)\rangle_I \tag{2.66}$$

where $H_{1I} = U^\dagger H_1 U$ is the operator in the interaction picture. Following the same concept, we can change variable in equation 2.62, according to $f = Bg$, where B satisfies

$$\partial_x B = A_0(x)B \tag{2.67}$$

and we end up with the following equation

$$\partial_x g(\epsilon, x) = \epsilon \widetilde{A}_1(x) g(\epsilon, x) \tag{2.68}$$

where $\widetilde{A}_1(x) = B^{-1}A_1(x)B$. So if we are able to provide a solution for equation 2.67 and compute the matrix B we can put our system in canonical form.

The solution of equation 2.67 can be given as Magnus Series expansion according to the Magnus theorem [44]. Given a linear system of differential equation [45]

$$\partial_x Y(x) = A(x)Y(x), \quad Y(x_0) = Y_0 \tag{2.69}$$

in the general case of $A(x)$ not commuting with its integral $\int_{x_0}^x d\tau A(\tau)$ we can use the Magnus Theorem to give a solution to equation 2.69

$$Y(x) = e^{\Omega(x,x_0)}Y(x_0) \equiv e^{\Omega(x)}Y_0 \tag{2.70}$$

where $\Omega(x)$ is a series called *Magnus Expansion*

$$\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \tag{2.71}$$

The first terms of the Magnus Expansion are listed below

$$\begin{aligned}
\Omega_1(x) &= \int_{x_0}^{x} d\tau_1 A(\tau_1) \\
\Omega_2(x) &= \frac{1}{2}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{x} d\tau_2 \left[A(\tau_1), A(\tau_2)\right] \\
\Omega_3(x) &= \frac{1}{6}\int_{x_0}^{x} d\tau_1 \int_{x_0}^{x} d\tau_2 \int_{x_0}^{x} d\tau_3 \left[A(\tau_1), \left[A(\tau_2), A(\tau_3)\right]\right] + \left[A(\tau_3), \left[A(\tau_2), A(\tau_1)\right]\right]
\end{aligned} \tag{2.72}$$

We notice that if the integrals commutes, then only the first term is not 0. Of course in general that's not the case, but we can equivalently give a solution to equation 2.69 in terms of Dyson series. In reference [45] is given a proof of the equivalence between Magnus and Dyson expansion.

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x), \quad Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \cdots \int_{x_0}^{\tau_{n-1}} d\tau_n A(\tau_1) \cdots A(\tau_n) Y_0 \tag{2.73}$$

The Dyson series then terminates whenever the repeated product of matrix A by itself is null. We can exploit the proprieties of the two representations of the solution to try and compute matrix B of equation 2.67.

First of all we split matrix $A_0(x)$ in diagonal and off-diagonal part

$$A_0(x) = D_0(x) + N_0(x) \tag{2.74}$$

Using the fact that diagonal matrices commutes, we can at first perform a transformation that eliminates D_0 from equation 2.62. In fact in the basis g_1, so that $f = B_1 g_1$, equation 2.62 assumes the form (see 2.29)

$$\partial_x g_1(\epsilon, x) = B_1^{-1} \left(N_0(x) + \epsilon A_1(x) B_1 - \partial_x B\right) g_1(\epsilon, x) \tag{2.75}$$

If B_1 satisfies

$$\partial_x B_1 = D_0(x) B_1 \tag{2.76}$$

then

$$\partial_x g_1(\epsilon, x) = \left(N_{0,1}(x) + \epsilon A_{1,1}(x)\right) g_1(\epsilon, x) \tag{2.77}$$

where $N_{0,1} = B_1^{-1} N_0 B$ and $A_{1,1} = B_1^{-1} A_1 B$. Because D_0 is diagonal, using the Magnus Expansion 2.72, B_1 will be given by

$$B_1 = e^{\int_{x_0}^{x} d\tau D_0(\tau)} \tag{2.78}$$

where only the first term of the expansion appears and the computation is extremely simplified. To remove N_0 from equation 2.75 we can make use of the Dyson representation. In fact matrix N_0 is usually a sparse matrix in x, so it is very likely that its Dyson series will be finite. In the basis g_2, with $g_1 = B_2 g_2$ and matrix B_2 computed as

$$B_2 = N_{0,1}(x_0) + \sum_{n=1}^{\infty} \int_{x_0}^{x} d\tau_1 \cdots \int_{x_0}^{\tau_{n-1}} d\tau_n N_{0,1}(\tau_1) \cdots N_{0,1}(\tau_n) \tag{2.79}$$

equation 2.75 will be in canonical form

$$\partial_x g_2(\epsilon, x) = \epsilon \widetilde{A}(x) g_2(\epsilon, x) \tag{2.80}$$

where $\widetilde{A}(x) = B^{-1} A(x) B$ and matrix B is given by

$$B = B_1 B_2 = e^{\Omega([D_0])} e^{\Omega([N_{0,1}])} \tag{2.81}$$

The Magnus Series expansion is a very effective method whenever we are able to manipulate our base of MI and end up with a linear system of differential equation in ϵ. We remark that, despite the example given for one variable, the method is valid also for multivariate problems without any limitation.

For instance, if we have two variables x and y, all we have to do is first compute matrix B_1 and B_2 that eliminate A_{x0} from $A_x = A_{x0} + \epsilon A_{x1}$. The x derivative is now in canonical form. We transform A_y according to 2.29 where $B = B_1 B_2$ and then repeat the process to compute B_3 and B_4 on the transformed A_y to eliminate A_{y0}. In formulae: suppose we have two variables x and y and a system

$$\begin{aligned} \partial_x f(x,y,\epsilon) &= A_x(x,y,\epsilon) f(x,y,\epsilon) \\ \partial_y f(x,y,\epsilon) &= A_y(x,y,\epsilon) f(x,y,\epsilon) \end{aligned} \tag{2.82}$$

where both A_x and A_y are linear in ϵ, $A_i = A_{0i} + \epsilon A_{i0}$. We can at first work on A_x and compute matrices B_1 and B_2 to eliminate A_{0x}. Defining $\widetilde{B} = B_1 B_2$ the system will be in the form

$$\begin{aligned} \partial_x g_2(x,y,\epsilon) &= \epsilon \widetilde{A}_x(x,y) g_2(x,y,\epsilon) \\ \partial_y g_2(x,y,\epsilon) &= \widetilde{A}_y(x,y,\epsilon) g_2(x,y,\epsilon) \end{aligned} \tag{2.83}$$

with $\widetilde{A}_x(x,y) = \widetilde{B}^{-1} A_x(x,y) \widetilde{B}$, $\widetilde{A}_y(x,y,\epsilon) = \widetilde{B}^{-1} A_y(x,y,\epsilon) \widetilde{B} - \partial_y \widetilde{B}$ and $f = \widetilde{B} g_2$. Then we compute B_3 and B_4 to eliminate $\widetilde{A}_{y0}$ from $\widetilde{A}_y = \widetilde{A}_{y0} + \epsilon \widetilde{A}_{y1}$. If $\hat{B} = B_3 B_4$, the system, in the base g_4, will be in canonical form

$$\begin{aligned} \partial_x g_4(x,y,\epsilon) &= \epsilon \hat{A}_x(x,y) g_4(x,y,\epsilon) \\ \partial_y g_4(x,y,\epsilon) &= \epsilon \hat{A}_y(x,y) g_4(x,y,\epsilon) \end{aligned} \tag{2.84}$$

with $\hat{A}_x(x,y) = \hat{B}^{-1} \widetilde{A}_x \hat{B} - \partial_x \hat{B}$, $\hat{A}_y(x,y) = \hat{B}^{-1} \widetilde{A}_{y1} \hat{B}$. The canonical base g_4 is then defined by the transformation $B = B_1 B_2 B_3 B_4$ with $f = B g_4$.

2.4.6 Uniform Trascendentality and Boundary Conditions

Now that the symbolic integration is performed, we are left with the problem of determining the Boundary Conditions (BCs); we will show, in this section, how the properties of integrals in the canonical form can be helpful at this purpose. Suppose we can express the solution in terms of nested integrals (e.g. GPLs), we may have powerful tools to numerically evaluate them, but, to determinate analytically the constant of integration, an analytical value of the nested integrals has to be known in a point. The direct integration is quite demanding when dealing with higher and higher weights, but it is possible to proceed alternatively.

We recall that we call weight of a function the number of nested integrals of its definition. Dealing with integrals in canonical form, we immediately get, as solution, a Laurent expansion of them. If we assign to ϵ a weight $W(\epsilon) = -1$ all terms of the expansion will have weight 0. Because we still have the constant of integration to fix, this is the only term that could break the uniformity.

A function $f(\epsilon, x)$ is ϵ *uniform trascendental* (UT) if all the terms of its Laurent expansion in ϵ have equal weight (also total weight), provided that $W(\epsilon) = -1$. A function is *pure* UT if its total weight is 0.

To establish the total weight of a function we notice that the product of two UT function $f_1 f_2$ of weights w_1 and w_2 is an UT function of weight $w_1 + w_2$. The sum and the invertion of UT function are still UT.

With this information in mind, we need to give the weight associated to the typical trascendental constants we deal with when computing FIs.

- π. Because π is the trascendental constant that appears when dealing with analytical continuation of logarithms, it is reasonable to assign

$$W(\pi) = 1 \tag{2.85}$$

- $\zeta(n)$. The values of $\zeta(2n)$, where $n \in N$, are related to π with the following

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{2(2n)!} \tag{2.86}$$

  where B_{2n} is $2n$-th Bernoulli number. Because B_{2n} is rational $\forall n$ we can associate a weight $W(\zeta(2n)) = 2n$. This definition can be extended to the n odd values and we have

$$W(\zeta(n)) = n \tag{2.87}$$

- $\Gamma(1+\epsilon)$. Before we discuss the UT properties of the Γ function we need to associate a weight to the Euler-Mascheroni constant γ. Recalling two possible definitions of γ, we have

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\\ \gamma &= -\int_0^\infty \log(t)e^{-t}dt\end{aligned} \tag{2.88}$$

Because the harmonic series $\sum 1/k$ is made of rational numbers, from the first definition, it is reasonable to assign weight $W(\gamma) = 1$. The second definition reads instead as follows

$$\begin{gathered}-\gamma = \int_0^\infty \log(t)e^{-t}dt = \\ \int_{y=0}^{y=1}\frac{y}{1-y}\log\frac{y}{1-y}e^{-\frac{y}{1-y}}d\log\frac{y}{1-y}\end{gathered} \tag{2.89}$$

where the change of variable is performed in order to deal with a compact domain of integration. So if we suppose to assign $W(\gamma) = 1$, there has to be $W\left(\frac{y}{1-y}\log\frac{y}{1-y}e^{-\frac{y}{1-y}}\right) = 0$ because of the presence of $d\log$ that has already weight 1. So we finally have

$$W\left(\frac{y}{1-y}e^{-\frac{y}{1-y}}\right) = -1 \tag{2.90}$$

To prove UT of the Γ function, we need to analyze its expansion.

$$\begin{gathered}\Gamma(1+\epsilon) = \int_0^\infty t^{\epsilon}e^{-t}dt = \\ \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\int_0^\infty \log^n(t)e^{-t}dt = \\ \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\int_{y=0}^{y=1}\frac{y}{1-y}\log^n\frac{y}{1-y}e^{-\frac{y}{1-y}}d\log\frac{y}{1-y}\end{gathered} \tag{2.91}$$

So with equation 2.90 it is clear that each term of the expansion has weight 0. Recalling that $z\Gamma(z) = \Gamma(1+z)$, also $\Gamma(\epsilon)$ is a UT function but with weight one. In formula, provided the assignment $W(\gamma) = 1$, we have

$$W(\Gamma(1+\epsilon)) = 1, \qquad W(\Gamma(\epsilon)) = 0 \tag{2.92}$$

Now that all the notable weights have been defined, a priori criteria do not exist to in general determine if a graph is an UT function, except in some simple cases [6]. If we deal with solutions of d-log canonical forms, non UT could arise only because of the constant term, this in general isn't very luckily to happen. So how do we determine analytically the value of the BCs? We could make the ansatz that we're dealing with pure UT graphs, this lead us to make a guess on the possible constants that could appear when evaluating, order by order, the integrals. In fact, if the graph is a pure UT function and we are evaluating the n-th order, the constant needs to have weight n. This limits the possible choice to a linear combination of constants of weight n multiplied by rational factors; for instance at 5-th order we might have $k_1\pi^5 + k_2\zeta(5) + k_3\zeta(2)\pi^3 + \dots$ where $k_i \in Q$. This ansatz can be checked because we can evaluate with the desired precision (e.g. using GiNaC [8]) all the GPLs and this allows us to give an analytical solution to our integrals.

Chapter 3

Top Yukawa contribution

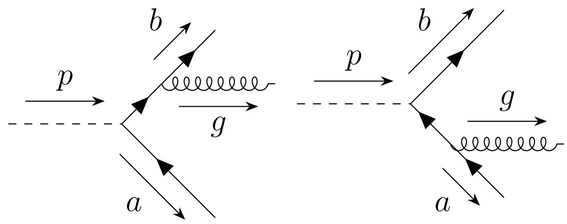
Now that all the techniques have been outlined, we can proceed with the computation of the top Yukawa contribution to the decay of the Higgs boson into $b\bar{b}$. Because the NLO correction is about 30% of the LO one, we may expect a significant contribution also from the NNLO corrections. In this chapter, we first compute real and virtual contribution and then evaluate the impact of our calculation on the distribution of some observables of interest.

3.1 Real contribution

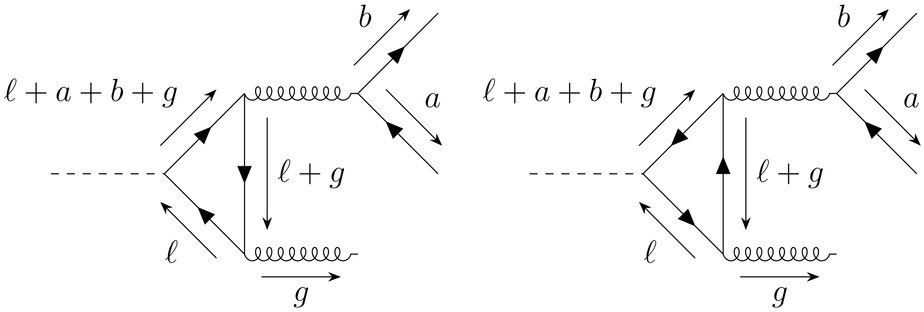
In this section we compute the real contribution to the decay of Higgs boson into a $b\bar{b}$ pair via top triangle. So we consider the process $H(s) \to b(b)\bar{b}(a)g(g)$. The diagrams involved are showed in fig. 3.2a at leading order and in fig. 3.2b at one-loop level. This contribution is both infrared and ultraviolet finite.

As in the first chapter we define momenta and momentum conservation as follows:

$$\begin{aligned} g^2 = 0, \quad a^2 = b^2 = m_b^2 \\ p^2 = (a+b+g)^2 = s \end{aligned} \tag{3.1}$$



(a) Born digrams



(b) Triangle diagrams

Figure 3.1: Diagrams of the Real contribution.

The contribution of the two Born diagrams, $\mathcal{M}_b^{(i)}$, $i = 1, 2$, is given by

$$\begin{aligned}
\mathcal{M}_b^{(1)} &= -igy_b\bar{u}(b,i)\gamma^\nu(\tau_c)_{ij}\frac{i}{\not{b}+\not{g}-m_b}v(a,j)\epsilon_\nu(g,c) \\
\mathcal{M}_b^{(2)} &= -igy_b\bar{u}(b,i)\frac{i}{-\not{a}-\not{g}-m_b}\gamma^\nu(\tau_c)_{ij}v(a,j)\epsilon_\nu(g,c)
\end{aligned} \tag{3.2}$$

where y_b is the Yukawa coupling, i, j the color indices and τ_c the fundamental representation of the color algebra.

The one loop-level diagrams, $\mathcal{M}_n^{(i)}$ (figure 3.2b), have the following expressions

$$\begin{aligned}
\mathcal{M}_n^{(i)} =& - ig^3y_t\epsilon^{\mu_2}(g,d)(\tau_d)_{ij}(\tau_c)_{kl}(\tau_d)_{lk}\bar{u}(b,i)\gamma^{\mu_1}v(a,j) \\
&\frac{1}{(a+b)^2}\int\frac{d^d\ell}{(2\pi)^d}\frac{N^{(i)}_{\mu_1\mu_2}}{[(\ell+a+b+g)^2-m_t^2][\ell^2-m_t^2][(\ell+g)^2-m_t^2]}
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
N^{(1)}_{\mu_1\mu_2} &= \mathrm{Tr}\left[\gamma_{\mu_1}(\not{\ell}+\not{a}+\not{b}+\not{g}+m_t)(\not{\ell}+m_t)\gamma_{\mu_2}(\not{\ell}+\not{g}+m_t)\right] \\
N^{(2)}_{\mu_1\mu_2} &= \mathrm{Tr}\left[\gamma_{\mu_1}(-\not{\ell}-\not{g}+m_t)\gamma_{\mu_2}(-\not{\ell}+m_t)(-\not{\ell}-\not{a}-\not{b}-\not{g}+m_t)\right]
\end{aligned} \tag{3.4}$$

The next to next leading order (NNLO) amplitude, g_s^4 order, is then

$$\mathcal{A}_r = 2\,\mathrm{Re}\left(\mathcal{M}_n^{(1)} + \mathcal{M}_n^{(2)}\right)\left(\mathcal{M}_b^{(1)} + \mathcal{M}_b^{(2)}\right)^* \tag{3.5}$$

The color factor is easily computed and we have:

$$(\tau_d)_{ij}(\tau_c)_{kl}(\tau_d)_{lk}(\tau_c)_{ji} = \frac{C_A C_F}{2} \tag{3.6}$$

where C_A is the number of colors and C_F the Casimir fundamental representation of the $SU(C_A)$ algebra associated to the colors.

We are now dealing with a process $1 \to 3$ particles, so to reduce ourselves to a minimum number of Master Integrals and express all the scalar products in terms of denominators, we need of an auxiliary topology. So we add the forth denominator to the three of equation 3.3.

$$\begin{aligned} D_1 &= (\ell + a + b + g)^2 - m_t^2 \quad & D_2 &= \ell^2 - m_t^2 \\ D_3 &= (\ell + g)^2 - m_t^2 & D_4 &= (\ell + a)^2 - m_t^2 \end{aligned} \tag{3.7}$$

Then we can express all the scalar products in terms of denominators and kinematic invariants

$$\begin{aligned} \ell \cdot \ell &= D_2 + m_t^2 \\ \ell \cdot g &= \frac{1}{2}\left(D_3 - D_2\right) \\ \ell \cdot b &= \frac{1}{2}\left(D_1 - D_3 - D_4 + D_2 - m_b^2 - 2a\cdot b - 2a\cdot g - 2b\cdot g\right) \\ \ell \cdot a &= \frac{1}{2}\left(D_4 - D_2 - m_b^2\right); \end{aligned} \tag{3.8}$$

With the help of software Reduze2 [37], we find 4 master integrals listed in figure 3.2. We have 3 bubbles and 1 triangle integral. While $f_B(p)$ and $f_B(a+b)$ will have the same analytical expression, $f_B(g)$ will be different because it depends on $g^2 = 0$. The scalar integrals we found are all well known; in this case is then not necessary to compute them via partial differential equation. We have then

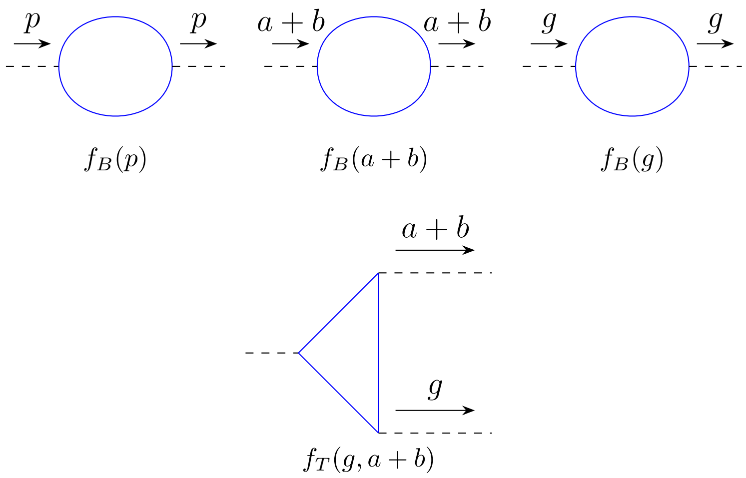


Figure 3.2: The Master Integrals: blue lines stand for denominators with m_t. On the dashed lines, we have instead the momenta.

[46, 47]:

$$
\begin{aligned}
f_B(p) &= \frac{\mu^{2\epsilon}}{i\pi^{2-\epsilon}\Gamma(1+\epsilon)} \int d^d\ell \frac{1}{[(\ell+p)^2 - m_t^2][\ell^2 - m_t^2]} = \\
&\mu^{2\epsilon}\left[\frac{1}{\epsilon} + 2 - \log\left(p^2 - i0\right) + \sum_{i=1}^{2} \lambda_i \log\left(\frac{\lambda_i - 1}{\lambda_i}\right) - \log(\lambda_i - 1) + o(\epsilon)\right] \\
&\text{with } \lambda_{1,2} = \frac{p^2 \pm \sqrt{(p^2)^2 - 4p^2(m_t^2 - i0)}}{2p^2} \\
f_B(g) &= \mu^{2\epsilon}\left[\frac{1}{\epsilon} - \log\left(m_t^2 - i0\right) + o(\epsilon)\right] \\
f_T(g, a+b) &= \frac{\mu^{2\epsilon}}{i\pi^{2-\epsilon}\Gamma(1+\epsilon)} \int d^d\ell \frac{1}{[(\ell+p)^2 - m_t^2][\ell^2 - m_t^2][(\ell+g)^2 - m_t^2]} = \\
&\frac{1}{2p((a+b)^2 - 1)}\left[\log^2\left(\frac{\sqrt{-\frac{4i0+(a+b)^2-4m_t^2}{(a+b)^2}} + i}{\sqrt{-\frac{4i0+(a+b)^2-4m_t^2}{(a+b)^2}} - i}\right) - \log^2\left(\frac{\sqrt{-\frac{4i0-4m_t^2+p}{p}} + i}{\sqrt{-\frac{4i0-4m_t^2+p}{p}} - i}\right) + o(\epsilon)\right]
\end{aligned}
\tag{3.9}
$$

We now have the expressions of the master integrals in terms of the Feynman prescription. In the decay region, $m_b^2 < p^2 < m_t^2$, we have then

$$
f_1 = \mu^{2\epsilon}\left[\frac{1}{\epsilon} - \log\left(\frac{m_t^2}{p^2}\right) - \sqrt{\frac{4m_t^2}{p^2} - 1}\,\tan^{-1}\left(\frac{\sqrt{\frac{4m_t^2}{p^2} - 1}}{\frac{2m_t^2}{p^2} - 1}\right) - \log\left(p^2\right) + 2 + o(\epsilon)\right]
$$

$$f_3 = \mu^{2\epsilon}\left[\frac{1}{\epsilon} - \log\left(m_t^2\right) + o(\epsilon)\right]$$
$$f_4 = \frac{\mu^{2\epsilon}}{2(p^2-(a+b)^2)}\left[\tan^{-1}\left(\frac{\sqrt{\frac{4m}{k}-1}}{\frac{2m_t^2}{(a+b)^2}-1}\right)^2 - \tan^{-1}\left(\frac{\sqrt{\frac{4m_t^2}{p^2}-1}}{\frac{2m}{p^2}-1}\right)^2 + o(\epsilon)\right] \tag{3.10}$$

Performing now basic Dirac algebra and writing all the integrals in terms of the MIs we obtain the finite expression for the amplitude $\mathcal{A}_r$.

$$\begin{aligned}\mathcal{A}_r = C_A C_F y_b y_t g_s^4 \Bigg(-16 m_b m_t &\left(\frac{4m_b^2 - 2a\cdot g - 2b\cdot g}{2a\cdot g 2b\cdot g} - \frac{4\left(2m_b^2 + 2a\cdot b\right)}{(2a\cdot g + 2b\cdot g)^2}\right)(f_1 - f_2)\\ - 16 m_b m_t &\left(\frac{4m_b^2(2a\cdot g + 2b\cdot g) - (2a\cdot g + 2b\cdot g)^2}{2a\cdot g 2b\cdot g\left(2m_b^2 + 2a\cdot b\right)}\right.\\ &\left. - \frac{4}{2a\cdot g + 2b\cdot g}\right)\left(\frac{1}{2}\left(4m_t^2 - 2a\cdot g - 2b\cdot g\right) f_4 + 1\right)\Bigg)\end{aligned} \tag{3.11}$$

As expected from a power analysis of the amplitude, we observe the cancellation of the ϵ pole. Performing the limit $g \to 0$, the amplitude is finite, so no soft divergences appear in this calculation. According to KLN theorem we should find no soft divergences in the virtual contribution, calculated in the next section.

3.2 Two loops contribution

In this section we apply the methods presented in the previous chapter to the computation of the two loop diagrams of the top Yukawa contribution to the Higgs decay into bottom quarks, figure 3.3.

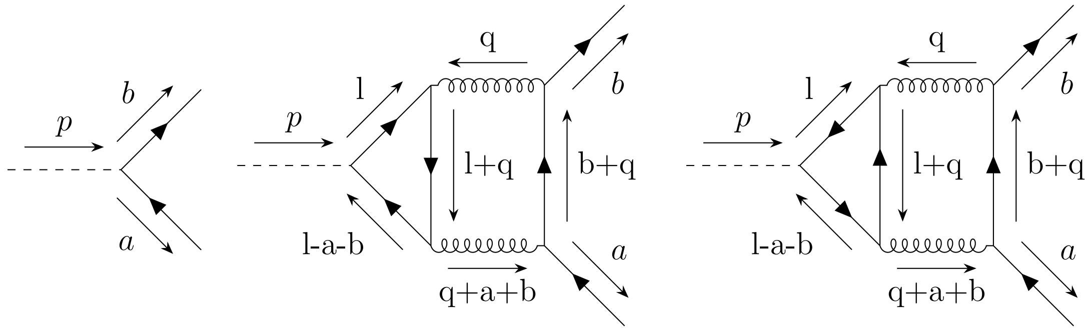


Figure 3.3: Here we represent the diagrams computed with the notation used for the momenta.

Fixing the kinematics, we have $H(p) \to b(b)\bar{b}(a)$, and the following relations

$$s = p^2 = (a+b)^2, \quad a^2 = b^2 = m_b^2 \tag{3.12}$$

The Born contribution is given by the simple expression

$$\mathcal{M}_0 = y_b \bar{u}(b) v(a) \tag{3.13}$$

where y_b is the Yukawa coupling Higgs to fermions and, for a generic fermion f, it is given by [48] $y_f = -im_f/v = -im_f e/2M_W \sin(\theta_w)$. The amplitude to calculate is

$$\mathcal{A} = 2\,\mathrm{Re}(\mathcal{M}_{tb}^{(1)} + \mathcal{M}_{tb}^{(2)})\mathcal{M}_0^* \tag{3.14}$$

where $\mathcal{M}_{tb}^{(i)}$, with $i = 1, 2$, are the contributions of the two two-loops diagrams:

$$\begin{gathered}\mathcal{M}_{tb}^{(i)} = g^4 y_t (\tau_a)_{ij} (\tau_b)_{jk} (\tau_a)_{nm} (\tau_b)_{mn} g^{\mu_1\mu_2} g^{\mu_3\mu_4} \times \\ \int \frac{d^d\ell}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{N^{(i)}_{\mu_2\mu_3}}{D_1 D_2 D_3 D_4 D_5 D_6} \bar{u}(b,i) \gamma_{\mu_1} (\not{q} + \not{b} + m_b) \gamma_{\mu_4} v(a,k)\end{gathered} \tag{3.15}$$

We have for the two diagrams:

$$\begin{aligned} N^{(1)}_{\mu_2\mu_3} &= -\,\mathrm{Tr}\left[\gamma_{\mu_2}(\not{\ell} + mt)(\not{\ell} - \not{a} - \not{b} + mt)\gamma_{\mu_3}(\not{\ell} + \not{q} + m_t)\right] \\ N^{(2)}_{\mu_2\mu_3} &= -\,\mathrm{Tr}\left[\gamma_{\mu_2}(-\not{\ell} - \not{q} + mt)\gamma_{\mu_3}(-\not{\ell} + \not{a} + \not{b} + mt)(-\not{\ell} + m_t)\right] \end{aligned} \tag{3.16}$$

The denominators D_j are defined as follows

$$\begin{aligned} D_1 &= \ell^2 - mt^2 & D_2 &= (\ell+q)^2 - mt^2 \\ D_3 &= (\ell - a - b)^2 - mt^2 & D_4 &= q^2 - mt^2 \\ D_5 &= (b+q)^2 - mt^2 & D_6 &= (\ell - a)^2 - mt^2 \end{aligned} \tag{3.17}$$

As explained in section 2.1, to deal with only scalar integrals and express all the scalar products in temrs of denominators, we need to enlarge the number of denominators, relying on an auxiliary topology. So we complete the set of denominators adding D_7

$$D_7 = (\ell - a)^2 \tag{3.18}$$

All the scalar products appearing in the amplitude can be written in terms of the 7 denominators and kinematics invariants as:

$$
\begin{aligned}
&\ell \cdot \ell = D_1 + m_t^2 && q \cdot q = D_4 \\
&\ell \cdot q = \tfrac{1}{2}(D_2 - D_1 - D_4) && b \cdot q = \tfrac{1}{2}(D_6 - D_4) \\
&a \cdot q = \tfrac{1}{2}(D_5 - D_6 - s) && \ell \cdot a = \tfrac{1}{2}(D_1 - D_7 + m_t^2 + m_b^2) \\
&\ell \cdot b = \tfrac{1}{2}(D_7 - D_3 - m_t^2 - m_b^2)
\end{aligned}
\tag{3.19}
$$

The color factor is easily computed, we have:

$$
\sum_{i,k} (\tau_a)_{ij} (\tau_b)_{jk} (\tau_a)_{nm} (\tau_b)_{mn} = \frac{C_A C_F}{2} \tag{3.20}
$$

3.2.1 Getting the master integrals

Now that we have set up the calculation of the amplitude, a part from basic Dirac algebra, we have to compute all the scalar integrals appearing after performing the substitutions 3.19. Thanks to the software Reduze2 [37], we can reduce all of them to combinations of master integrals. Given the auxiliary topology of the seven denominators D_i and the kinematic relations 3.12, after performing the reduction and applying the Laporta's algorithm, we get 20 MI, divided in 14 subsectors.

To compute them and apply the Magus expansion (sec. 2.4.5), we want to choose a particular basis that satisfies a linear system of differential equations in the ϵ parameter. By mistakes and trials it is possible to find such a basis for our calculation. The master integrals are listed in fig 3.4. They satisfy a linear system of p.d.e in the form:

$$
\begin{aligned}
\partial_s f &= \left(A_{s0}(s, m_b) + \epsilon A_{s1}(s, m_b)\right) f \\
\partial_{m_b} f &= \left(A_{m_b 0}(s, m_b) + \epsilon A_{m_b 1}(s, m_b)\right) f
\end{aligned}
\tag{3.21}
$$

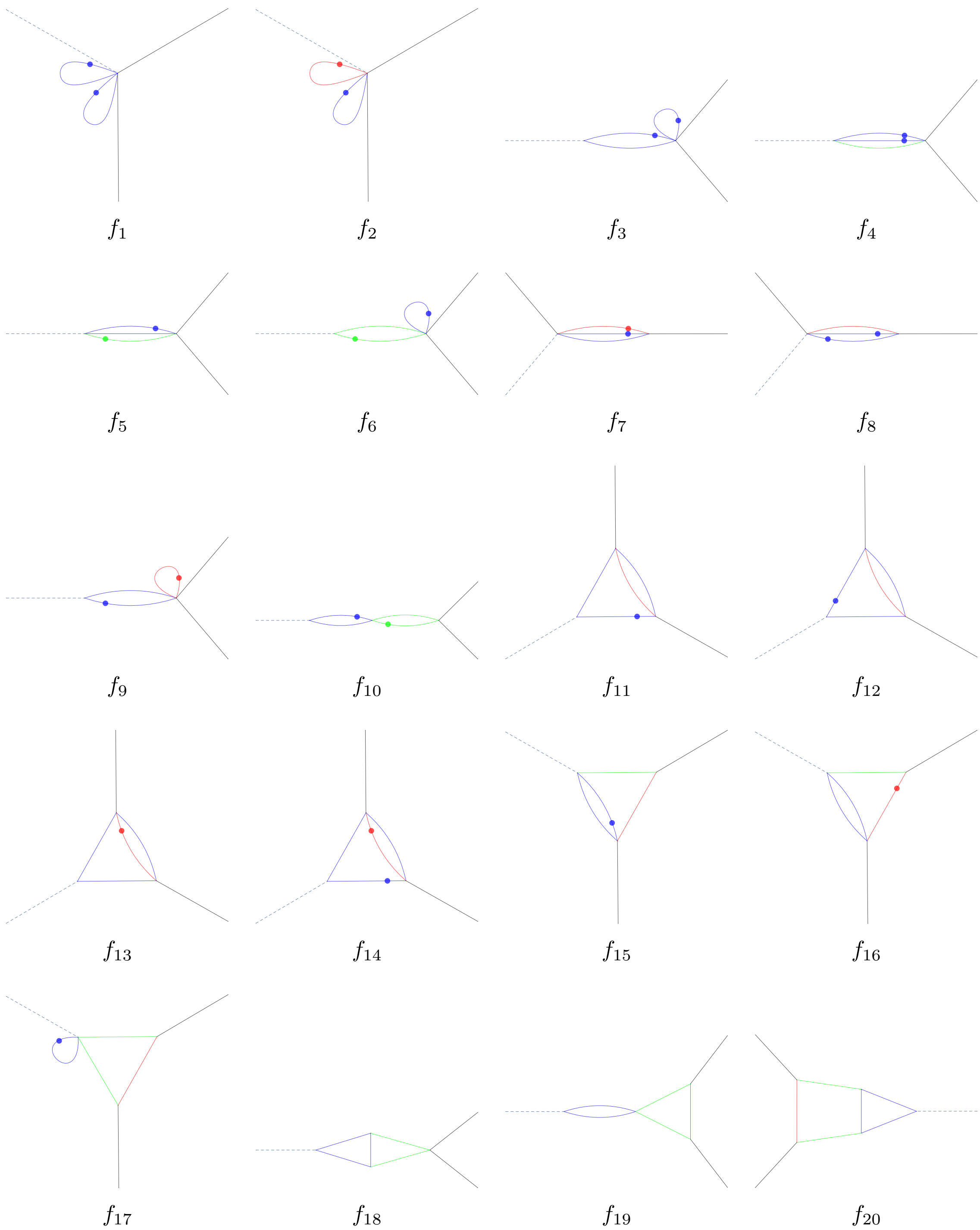


Figure 3.4: The chosen base of master integrals that gives a linear system of differential equation in the ϵ parameter. In each graph a red line stands for a quark bottom propagator, a blue line for a quark top propagator and a green line for a gluon propagator. A dot on a line stands for a square power of that propagator. The dashed line is the Higgs external leg while the continue black lines are quark bottom external legs.

3.2.2 Canonical Form

We have 3 kinematics invariants s, m_b, m_t with respect to we can compute the differential equations. As previously seen in sec 2.4.1, we can eliminate the dependence on a kinematic invariant because it can be reconstructed with a dimensional analysis of the integrals. We choose to eliminate the dependence on m_t and compute the differential equations w.r.t s and m_b.

With a view of getting a system in a d-log canonical form 2.37 that we rewrite here for brevity

$$dA(\vec{x}) = \sum_{i=1}^{K} \widetilde{A}_i d\log(L_i(\vec{x})) \tag{3.22}$$

we recall that the alphabet $L_i(\vec{x})$ needs to be made of rational functions in the kinematic invariants. By a direct computation, the invariants s and m_b lead to a system in which square roots appear. These can be eliminated by a clever change of variable (as we have seen in the example in section 2.4.4). We define the variables x and y so that

$$s = -m_t^2 \frac{(1-x^2)^2}{x^2}, \qquad m_b^2 = m_t^2 \frac{(1-x^2)^2 y^2}{(1-y^2)x^2} \tag{3.23}$$

and we will come up to a system with alphabet constituted by only rational functions of x and y.

At this point we have a system of p.d.e. in the variables x and y, that can be written in the following way

$$\begin{aligned} \partial_x f &= \left(A_{x0}(x,y) + \epsilon A_{x1}(x,y)\right) f \\ \partial_y f &= \left(A_{y0}(x,y) + \epsilon A_{y1}(x,y)\right) f \end{aligned} \tag{3.24}$$

We can apply the Maguns method to put it in the canonical form. The step that we need to perform are listed in section 2.4.5 in the 2 variables example. Since we are dealing with two 20×20 matrices, we do not report in detail all the steps of the calculation, but what we can say is that the method succeeds and the Dyson series terminates after the first two steps. The integrals we want now to compute are defined in equation 3.25, where $g = Bf$ and B is the transformation matrix of the Magnus algorithm.

$$
\begin{aligned}
g_1 &= \epsilon^2 f_1 \\
g_2 &= \epsilon^2 f_2 \\
g_3 &= -\epsilon^2 \frac{f_3 \left(x^4-1\right)}{x^2} \\
g_4 &= \epsilon^2 \frac{f_4 \left(x^2-1\right)^2}{x^2} \\
g_5 &= -\epsilon^2 \frac{\left(x^2-1\right)\left(f_5 \left(x^2+1\right)+f_4\right)}{x^2} \\
g_6 &= \epsilon^2 \frac{f_6 \left(x^2-1\right)^2}{x^2} \\
g_7 &= \epsilon^2 \frac{f_7 \left(x^2-1\right)^2 y^2}{x^2 \left(y^2-1\right)^2} \\
g_8 &= \epsilon^2 \frac{\left(2f_7+f_8\right)\left(x^2-1\right) y}{x \left(y^2-1\right)} \\
g_9 &= -\epsilon^2 \frac{f_9 \left(x^4-1\right)}{x^2} \\
g_{10} &= -\epsilon^2 \frac{f_{10} \left(x^2-1\right)^3 \left(x^2+1\right)}{x^4} \\
g_{11} &= \epsilon^2(1-2\epsilon) \frac{\left(x^2-1\right)\left(\left(y^2+1\right)\left(f_{11}\left(x^2+1\right)+2f_{12}\right)-2f_9\left(x^2+1\right)\right)}{x^2 \left(y^2-1\right)} \\
g_{12} &= -\epsilon^3 \frac{f_{12} \left(x^2-1\right)^2 \left(y^2+1\right)}{x^2 \left(y^2-1\right)} \\
g_{13} &= -\epsilon^3 \frac{f_{13} \left(x^2-1\right)^2 \left(y^2+1\right)}{x^2 \left(y^2-1\right)} \\
g_{14} &= \epsilon^2 \left(\frac{f_{13} \left(x^2-1\right)^2}{x^2 \left(y^2-1\right)} + \frac{f_{11} \left(x^2+1\right)(x-y)(x+y)}{x^2 \left(y^2-1\right)} - \frac{f_{12} \left(x^2 \left(y^2-3\right)+y^2+1\right)}{x^2 \left(y^2-1\right)} \right. \\
&\quad - \frac{f_3 \left(x^2-1\right)}{x^2} + \frac{f_{14} \left(x^2-1\right)^2 (x-y)(x+y)(xy-1)(xy+1)}{x^4 \left(y^2-1\right)^2} \\
&\quad \left. - \frac{f_9 \left(x^4-2x^2\left(y^2-1\right)-1\right)}{x^2 \left(y^2-1\right)} - 2f_7 - f_8 \right) \\
g_{15} &= -\epsilon^2 \frac{f_{15} \left(x^2-1\right)^2 \left(y^2+1\right)}{x^2 \left(y^2-1\right)} \\
g_{16} &= \epsilon^2(1-2\epsilon) \frac{1}{2\left(x^2+1\right)\left(y^2-1\right)^3} \left(\frac{2f_{16} \left(x^2-1\right)^3 y^2 \left(y^2+1\right)}{x^2} \right. \\
&\quad + \frac{4f_7 \left(x^2-1\right)^2 y^2 (x-y)(x+y)}{x^2} + \frac{4f_{15} \left(x^2-1\right)^2 \left(y^2-1\right)^2 \left(y^2+1\right)}{x^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2f_5\left(x^2+1\right)^2\left(x^2-1\right)\left(y^2-1\right)^2}{x^2}\\
&-\frac{f_4\left(x^2-1\right)\left(y^2-1\right)^2\left(x^4-2x^2\left(y^2+2\right)+2y^2-1\right)}{x^2}\\
&-2f_2\left(y^2-1\right)^2(x-y)(x+y)\Big)
\end{aligned}
$$

$$
\begin{aligned}
g_{17} &= -\epsilon^3\frac{f_{17}\left(x^2-1\right)^2\left(y^2+1\right)}{x^2\left(y^2-1\right)}\\
g_{18} &= \epsilon^3(1-2\epsilon)\frac{f_{18}\left(x^2-1\right)^2}{x^2}\\
g_{19} &= \epsilon^3(1-2\epsilon)\frac{\left(x^2-1\right)^2\left(y^2+1\right)\left(f_{19}\left(x^2-1\right)+2f_{17}\right)}{x^2\left(x^2+1\right)\left(y^2-1\right)}\\
g_{20} &= -\epsilon^4\frac{f_{20}\left(x^2-1\right)^4\left(y^2+1\right)}{x^4\left(y^2-1\right)}
\end{aligned}
\tag{3.25}
$$

Now that the we have applied the transformation of the MI, in the g basis the system is in canonical form and can be written as

$$
\begin{aligned}
\partial_x g(\epsilon,x,y) &= \epsilon A_x(x,y)g(\epsilon,x,y)\\
\partial_y g(\epsilon,x,y) &= \epsilon A_y(x,y)g(\epsilon,x,y)
\end{aligned}
\tag{3.26}
$$

Moreover thanks to the change of variable, previously made, the system is more specifically in a d-log canonical form.

$$
dg(x,y) = \sum_{i=1}^{12} A_i d\log(L_i(x,y)) \tag{3.27}
$$

where the constant matrices A_i can be read in the appendix B and the 12 letters of the alphabet are listed below

$$
\begin{array}{llll}
L_1 = 1-x & L_2 = x & L_3 = 1+x & L_4 = 1+x^2\\
L_5 = x-y & L_6 = 1-y & L_7 = y & L_8 = 1+y\\
L_9 = x+y & L_{10} = 1-xy & L_{11} = 1+xy & L_{12} = 1+y^2
\end{array}
\tag{3.28}
$$

3.2.3 Integrate the system

Now that we obtained a system in a d-log canonical form, as we already shown (sec. 2.4.4), the symbolic integration can be performed in terms of GPLs. What we have to do is to find a primitive of the differential form $g(x,y)$ in equation 3.27. With this purpose in mind, it is more convenient to work the with matrices A_x and A_y, so that the system is in the form

$$dg(x,y) = \epsilon\left[A_x(x,y)dx + A_y(x,y)dy\right]g(x,y) \tag{3.29}$$

With g integrals already scaled, we can Laurent expand each of them in ϵ, so that

$$g = g^{(0)} + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + \ldots \tag{3.30}$$

and solve the system recursively, order by order.

Because we have a two variables problem and at each step we deal with a differential form of the kind

$$dg^{(n)}(x,y) = \left[A_x(x,y)dx + A_y(x,y)dy\right]g^{(n-1)}(x,y) \tag{3.31}$$

a primitive of $dg^{(n)}$ is given by:

$$g^{(n)} = \int dy A_y g^{(n-1)} + \int dx \left[A_x g^{(n-1)} - \partial_x \int dy A_y g^{(n-1)}\right] + C^{(n)} \tag{3.32}$$

We work in the Euclidean region in which all the GPLs have real and positive arguments. Imposing all the letters 3.28 to be greater than 0, we get the boundaries of the Euclidean region:

$$0 < y < 1 \wedge y < x < 1 \tag{3.33}$$

In terms of the kinematic invariants s and m_b, the new variables can be written as

$$x = \frac{\sqrt{-\frac{s}{m_t^2} - \sqrt{4-\frac{s}{m_t^2}}\sqrt{-\frac{s}{m_t^2}} + 2}}{\sqrt{2}}, \quad y = \frac{\sqrt{-\frac{s}{m_b^2} - \sqrt{-\frac{s}{m_b^2}}\sqrt{4-\frac{s}{m_b^2}} + 2}}{\sqrt{2}} \tag{3.34}$$

so that the Euclidean region, corresponds to the kinematic configuration

$$s < 0 \wedge 0 < m_b^2 < m_t^2 \tag{3.35}$$

Once the solution is obtained we need to analytically continue it. This can be done restoring the Feynman prescription, that implies to perform the substitution $s \to s + i\epsilon$ and $m_b^2 \to m_b^2 - i\epsilon$. Assuming $m_b < m_t$, we can then distinguish three regions :

- **Unpyhsical region** $0 < s < 4m_b^2$. Both x and y are phases

  $$x = e^{i\frac{1}{2}\phi_t}, \quad y = e^{i\frac{1}{2}\phi_b} \tag{3.36}$$

  where

  $$\begin{aligned} \phi_i &= \arctan \frac{\sqrt{s}\sqrt{4m_i^2 - s}}{2m_i^2 - s}, \quad \text{with } s < 2m_i^2 \\ \phi_i &= \arctan \frac{\sqrt{s}\sqrt{4m_i^2 - s}}{2m_i^2 - s} + \pi, \quad \text{with } s > 2m_i^2 \end{aligned} \tag{3.37}$$

  The conditional definition of ϕ_i is needed if we use that the arctan has values in $[-\pi/2, \pi/2]$.

- **Decay region** $4m_b^2 < s < 4m_t^2$

  $$x = e^{i\frac{1}{2}\phi_t}, \quad y = \frac{i}{\sqrt{2}}\sqrt{\frac{s}{m_b^2} - \sqrt{\frac{s}{m_b^2}}\sqrt{\frac{s}{m_b^2} - 4} - 2} \tag{3.38}$$

- **High energy region** $s > 4m_t^2$

  $$\begin{aligned} x &= \frac{i}{\sqrt{2}}\sqrt{\frac{s}{m_t^2} - \sqrt{\frac{s}{m_t^2} - 4}\sqrt{\frac{s}{m_t^2}} - 2} \\ y &= \frac{i}{\sqrt{2}}\sqrt{\frac{s}{m_b^2} - \sqrt{\frac{s}{m_b^2}}\sqrt{\frac{s}{m_b^2} - 4} - 2} \end{aligned} \tag{3.39}$$

The computation is made in the Mathematica framework [49]. Referring to equation 3.32, to find a primitive of our differential form we need to deal with both integrals and derivatives of GPLs. While the integrals are easily performed simply applying their definition, the derivatives would require some work to be bring back to GPLs if performed directly on their definition. These difficulties can be overcome with the *coproduct* formalism [12] that allows an elegant way to software implement GPLs derivatives.

3.2.4 Coalgebras and Coproducts

In this section we give the necessary definitions and equations useful to compute the derivatives, we follow mainly [50, 12].

We recall that an *algebra* over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathcal{A}$ together with a map m (multiplication)

$$\begin{aligned} &m : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \\ &(a, b) \to (a, b) \equiv a \cdot b \end{aligned} \tag{3.40}$$

associative and with unit ϵ

$$\begin{aligned} &(a \cdot b) \cdot c = a \cdot (b \cdot c) \\ &\epsilon \cdot a = a \cdot \epsilon = a \end{aligned} \tag{3.41}$$

Moreover the multiplication is distributive and associative w.r.t. scalars

$$\begin{aligned} &a \cdot (b + c) = a \cdot b + a \cdot c \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c \\ &a \cdot (kb) = k(a \cdot b) \text{ and } k(\ell a) = (k\ell) \cdot a \end{aligned} \tag{3.42}$$

where $k, \ell \in \mathbb{K}$.

We can identify the multiple polylogarithms as an algebra where the multiplication is given by the shuffle product and scalars are (rational) numbers. More specifically we can talk of *graded algebra*. In fact, the shuffle product preserves the weight (the product of two GPLs of weights w_1 and w_2 is a linear combination af GPLs of weight $w_1 + w_2$). A graded algebra is, indeed, the direct sum of vector spaces of grade (or weight) n

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n \tag{3.43}$$

where the multiplication preserves the grading. Because m is a bilinear map, it is well known that there exists a unique map $\mu : \mathcal{A} \otimes \mathcal{A} \to A$, acting on the tensor product $\mathcal{A} \otimes \mathcal{A}$, such that

$$a \cdot b = m(a, b) = \mu(a \otimes b) \tag{3.44}$$

A *coalgebra* is, formally speaking, defined as the dual of an algebra. If $\mathcal{A}$ is an algebra with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, its dual $\mathcal{C} = \mathcal{A}^*$ is equipped with the linear map Δ

$$\Delta = \mu^\dagger : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \tag{3.45}$$

The linear map Δ is called *comultiplication.* The comultiplication is then an operation that allows us to split an object into the sum of more. The *coassociativity* expressed as

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta \tag{3.46}$$

ensures that the way we perform the decomposition into 3 or more elements is unique. In fact if $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$, we can further decompose into 3 elements both acting with $\mathrm{id} \otimes \Delta$ and $\Delta \otimes \mathrm{id}$. The coassociativity ensures that the results need to match.

A *bialgebra* is an algebra that is at the same time a coalgebra, i.e., a vector space equipped both with a multiplication μ and a comultiplication Δ. In this case they don't need to be one the induced dual operation of the other. We require the two operations to be compatible in the sense that

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \tag{3.47}$$

where the multiplication on the r.h.s is intended taken separately on each factor of the tensor product. Moreover an *Hopf algebra* $\mathcal{H}$ is a bialgebra equipped with another structure, the *antipode* $S : \mathcal{H} \to \mathcal{H}$, that we do not furtherly discuss.

As showed by Goncharov [51], the following definition of coproduct makes the GPLs an Hopf algebra. The definition is more transparent if we refer to a slightly generalization of GPLs in a different notation

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t) \tag{3.48}$$

Note that a G can easily be expressed as I with the following

$$G(a_n, \ldots, a_1; a_{n+1}) = I(0; a_1, \ldots, a_n; a_{n+1}) \tag{3.49}$$

where, by definition, the intermediate a_i are in reverse order.

On the new notation the coproduct of GPLs is defined, in the case in witch all the arguments are mutually different, as

$$\begin{aligned}\Delta(I(a_0; a_1, \ldots, a_n; a_{n+1})) = \\ &\sum_{0=i_1<i_2<\cdots<i_k<i_{k+1}=n} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \\ &\otimes \left[\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})\right]\end{aligned} \tag{3.50}$$

In [12] all the formal aspects of divergent and general GPLs are discussed.

Example. $I(a_0; a_1, a_2; a_3)$. By a direct application of equation 3.50 we can write

$$\begin{aligned}\Delta(I(a_0; a_1, a_2; a_3)) = \\ &1 \otimes I(a_0; a_1, a_2; a_3) + I(a_0; a_1, a_2; a_3) \otimes 1 + \\ &I(a_0; a_1; a_3) \otimes I(a_1; a_2; a_3) + I(a_0; a_2; a_3) \otimes I(a_0; a_1; a_2)\end{aligned} \tag{3.51}$$

As shown in [50], an analogous result in the *Symbol* formalism, allows us to conjecture that

$$\Delta\left(\frac{\partial}{\partial x_k} I_w\right) = \left(id \otimes \frac{\partial}{\partial x_k}\right) \Delta(I_w) \tag{3.52}$$

where I_w is a function of weight w. In other words the differential operator acts only on right element of the coproduct. This relation is very useful to compute derivatives of GPLs w.r.t. a generic argument. In fact, noticing that the vector space of different weights are in direct sum, the following has to hold

$$\frac{\partial}{\partial x_k} I_w = \mu \left(id \otimes \frac{\partial}{\partial x_k} \Delta_{n-1,1}(I_w) \right) \tag{3.53}$$

where μ is the multiplication and $\Delta_{n-i,i}$ denotes the elements of the coproduct split into weights $n-i, i$.

Example. Assume we want to compute the derivative w.r.t. y of $G(1, 1+y; z)$. Using equation 3.53 and the computed coproduct 3.51, we have

$$\begin{aligned} \frac{\partial}{\partial y} G(1, 1+y; z) &= \mu \left(id \otimes \frac{\partial}{\partial y} \right) \Delta_{1,1}(G(1, 1+y; z)) \\ &= G(1; z) \frac{\partial}{\partial y} G(1+y; 1) - G(1+y; z) \frac{\partial}{\partial y} G(1; 1+y) \\ &\qquad + G(1+y; z) \frac{\partial}{\partial y} G(1; z) \end{aligned} \tag{3.54}$$

where we applied that

$$I(a_0; a_1; a_2) = G(a_1; a_3) - G(a_1; a_0) \tag{3.55}$$

The derivatives now act on simply logarithms so we finally obtain

$$\frac{\partial}{\partial y} G(1, 1+y; z) = \frac{1}{y(1+y)} G(1; z) - \frac{1}{y} G(1+y; z) \tag{3.56}$$

The fact that, thanks to equation 3.53, the derivative acts always on weight one function or simply logarithms has simplified enormously the problem of computing the derivatives w.r.t. a general argument. Moreover the result is already expressed in terms of GPLs, requiring no further transformation to be applied.

3.2.5 Fix the Boundary Conditions

Now that a primitive can be found, we need to determine the boundary conditions of all the g_i MI.

The two double tadpoles g_1 and g_2 can be directly integrated, using the convenient mesure normalization

$$\widetilde{d^d\ell} = \left(\frac{m_t^2}{\mu^2}\right)^{\epsilon} \frac{d^d\ell}{i\pi^{d/2}\Gamma(1+\epsilon)} \tag{3.57}$$

their expression is given by

$$g_1 = 1, \quad g_2 = \left(\frac{\left(1-x^2\right)^2 y^2}{x^2\left(1-y^2\right)^2}\right)^{-\epsilon} \tag{3.58}$$

We can exploit the regularity of those f_i integrals that do not present a branch cut for $s \to 0$ or $x \to 1$. By performing an s cut on the diagrams in fig. 3.4 these are $f_3, f_4, f_5, f_7, f_8, f_9, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}$. Evaluating equations 3.25 in $x \to 1$ limit, we get the boundaries

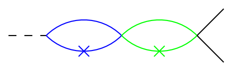
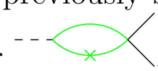
$$\begin{aligned} &\left[g_3 = g_4 = g_5 = g_7 = g_8 = g_9 = g_{11} = g_{12} = g_{13} = g_{14} = g_{15}\right]_{x=1} = 0 \\ &\left[g_{16}\right]_{x=1} = -\frac{\left[f_2\right]_{x=1}}{2} \end{aligned} \tag{3.59}$$

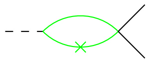
The integral f_6 needs to be directly computed and, from it, we get the expression of g_6

$$g_6 = \left(\frac{\left(1-x^2\right)^2}{x^2}\right)^{-\epsilon}\left(1 - \frac{\pi^2\epsilon^2}{6} - 2\zeta(3)\epsilon^3 - \frac{1}{40}\pi^4\epsilon^4 + \dots\right) \tag{3.60}$$

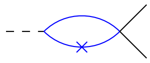
The integrals f_{17}, f_{19}, f_{20} are regular when $s \to 4m_b^2$ or $y \to i$, from equation 3.25 we get

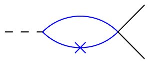
$$\left[g_{17} = g_{19} = g_{20}\right]_{y=i} = 0 \tag{3.61}$$

The boundary condition for $g_{10} \propto f_{10} =$  can be fixed at $x \to 1$ if we previously subtract its divergent part in the limit $s \to 0$ given by the subsector . We can use the expression of the subsector previously computed for f_6



$$
\begin{aligned}
&= \int \widetilde{d^d q} \frac{1}{q^2[(q+p)^2]^2} = (-s)^{-1-\epsilon} \int_0^1 dy \frac{y}{y(1-y)^{1+\epsilon}} \\
&= (-s)^{-1-\epsilon} \left(\frac{1}{\epsilon} - \frac{\pi^2 \epsilon}{6} - 2\zeta(3)\epsilon^2 - \frac{1}{40}\pi^3 \epsilon^4 + \dots \right)
\end{aligned}
\tag{3.62}
$$

Moreover the second subsector  is easily computed in the limit $s \to 0$

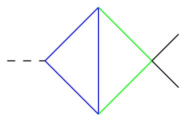


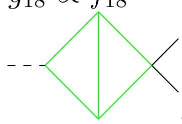
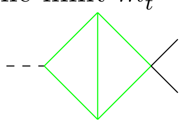
$$
\begin{aligned}
&= \int \widetilde{d^d \ell} \frac{1}{(\ell^2 - m_t^2)[(\ell - s)^2 - m_t^2]^2} \\
&= (m_t^2)^\epsilon \int_0^1 dy \frac{y}{[m_t^2 - sy(1-y)]^{1+\epsilon}} \\
\text{if } s \to 0 \ &= \frac{1}{2} \frac{1}{m_t^2}
\end{aligned}
\tag{3.63}
$$

Now that we know the analytically value of the finite ratio $\frac{f_{10}}{f_6}$ when $s \to 0$, using 3.25, we can write

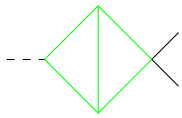
$$
\left. \frac{g_{10}}{g_6} \right|_{x=1} = 0 \tag{3.64}
$$

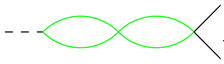
This equation allows us to compute the constant of integration of g_{10} provided that g_6 is known, as it is.

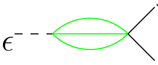
The last integral that needs fixed boundary conditions is $g_{18} \propto f_{18} =$ .

We can try and use the more easily integrated diagram , that, if known, allows us to match the twos in the limit $m_t^2 \to 0$ or $x \to 0$. To compute the massless five denominators topology , we can, at first, reduce to a set of MIs and then try to compute them directly. With the help of the software

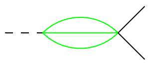
Reduze2 [37] we identify two MIs:

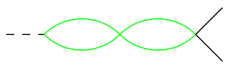
 $=$

$$\frac{1}{s^2\epsilon^2}\left(2\left(9\epsilon^2-9\epsilon+2\right)\right.$$  $+$ (3.65)

$$s(1-2\epsilon)\epsilon$$  $\left.\right)$

Luckily the two MIs can be directly integrated with no difficulties and their expressions are given by:

 $$= \frac{s(-s)^{-2\epsilon}\Gamma(1-\epsilon)^3\Gamma(2\epsilon-1)}{\Gamma(3-3\epsilon)\Gamma(\epsilon+1)^2}$$ (3.66)

 $$= \frac{(-s)^{-2\epsilon}\Gamma(1-\epsilon)^4\Gamma(\epsilon)^2}{\Gamma(2-2\epsilon)^2\Gamma(\epsilon+1)^2}$$

With equation 3.65 we know the expression of the massless 5 denominators topology and with equation 3.25 we can go back to g_{18}. We finally obtain

$$g_{18}|_{x=0} = (-s)^{-2\epsilon}\left(-6\zeta(3)\epsilon^3 - \frac{\pi^4}{10}\epsilon^4 + \dots\right) \tag{3.67}$$

3.2.6 Results

To perform the $\epsilon \to 0$ limit in the amplitude, we need to compute the g integrals up to the fourth order. Restoring the common normalization $d^dk/(2\pi)^d$ we get the following result for the amplitude:

$$\mathcal{A}_v = \frac{\alpha_s^2}{\pi^2} m_b m_t C_A C_F y_b y_t \mathcal{G}(x,y) \tag{3.68}$$

where the function $\mathcal{G}(x,y)$ of 256 GPLs is written in the appendix C.1. As expected from the real calculation, we observe the cancellation of the ϵ pole and the whole formula requires less then one second to be evaluated with GiNaC up to 30 digits.

3.3 Inclusive Decay

Once we obtained the analytical values of the real and virtual amplitudes $\mathcal{A}_r$ and $\mathcal{A}_v$, the correction to the inclusive decay rate will be given by:

$$\Delta\Gamma_{\text{yt}} = \int d\Phi_2 \mathcal{A}_v(\Phi_2) + \int d\Phi_3 \mathcal{A}_r(\Phi_3) \tag{3.69}$$

The 2 particle kinematic is trivial so we only need to multiply $\mathcal{A}_v$ by the 2 particle phase space volume 1.25, instead the integration over Φ_3 can be performed numerically with a Monte Carlo integration up to the desired precision. We now compare our exact result with the approximated formula computed by Chetyrkin and Kwiatkowski [10] that is valid for the case $m_b \ll m_H \ll m_t$:

$$\begin{aligned}\Delta\Gamma_{\text{CK}} = \frac{1}{8\pi} C_A y_b^2 m_H &\left(\frac{8}{3} - \frac{\pi^2}{9} - \frac{2}{3}\log\left(\frac{m_H^2}{m_t^2}\right) + \frac{1}{9}\log\left(\frac{m_b^2}{m_H^2}\right)^2\right. \\ &+ \frac{m_H^2}{m_t^2}\left(\frac{7}{1080}\log\left(\frac{m_b^2}{m_H^2}\right)^2 - \frac{41}{1620}\log\left(\frac{m_H^2}{m_t^2}\right) - \frac{7}{1080}\pi^2 + \frac{1381}{24300}\right) \\ &+ \frac{m_b^2}{m_H^2}\left(-\frac{4}{9}\log\left(\frac{m_b^2}{m_H^2}\right)^2 + \frac{16}{9}\log\left(\frac{m_b^2}{m_H^2}\right) + 4\log\left(\frac{m_H^2}{m_t^2}\right) + \frac{4}{9}\pi^2 - 10\right) \\ &\left.+ \frac{m_b^2}{m_t^2}\left(-\frac{7}{270}\log\left(\frac{m_b^2}{m_H^2}\right)^2 + \frac{1}{135}\log\left(\frac{m_b^2}{m_H^2}\right) - \frac{7}{270}\log\left(\frac{m_H^2}{m_t^2}\right) + \frac{7}{270}\pi^2 + \frac{713}{2700}\right)\right)\end{aligned} \tag{3.70}$$

In table 3.1 we show the discrepancy **d** between our result and the formula in [10], where **d** is defined as

$$\mathbf{d} = \left(1 - \frac{\Delta\Gamma_{\text{CK}}}{\Delta\Gamma_{\text{yt}}}\right) 100 \tag{3.71}$$

Because their expression is obtained in the limit $m_b \ll m_H \ll m_t$, we focus the comparison to the Decay region $4m_b^2 < m_H^2 < 4m_t^2$, fixing $m_b = 5$ GeV. We note that, for values of the masses in the range of validity of the approximated formula, the agreement is very good and excellent for the physical mass values. Such a level of agreement was not predictable a priori also becasuse in [10] the authors do not provide an estimate for the order of magnitude of the neglected terms in their expansion.

m_t \ m_H	20	75	125	180
100	2.123	0.075	1.025	6.704
125	2.329	0.011	0.335	2.107
175	2.452	-0.019	0.018	0.355
250	2.566	-0.024	-0.055	-0.035
350	2.656	-0.023	-0.069	-0.113

Table 3.1: The discrepancy **d** between our result and the approximated formula in [10], we fix $m_b = 5$ GeV.

3.4 Phenomenology

In this section, we include the top Yukawa contribution to the Higgs decay into a $b\bar{b}$ pair, extending the analysis of the first chapter on the impact of the radiative corrections. We start plotting $E_{\max}$ distribution for 2 jets events, figure 3.5. Unlike the NLO calculation, to show the impact of the yt contribution, we choose a fixed scale $\mu = m_H$ and $y_{\text{cut}} = 0.1$ for the JADE algorithm. In the lower panel we include also the ratio plot of the distributions. The correction adds about 1.5% to the NLO result and is rather flat.

We observe a similar impact of the correction also on the distribution of the absolute value of the rapidity of the jet with highest energy, figure 3.6.

In figure 3.7 we plot the transverse momentum P_T of the d-jet of bottom quarks. As expected we have a peak at $P_T = 0$ where the 2 particles kinematic dominates. The rest of the distribution is a pure real correction. The top Yukawa contribution is about 1% where the distribution has its second peak and rises to about 8% in the tail of the distribution.

Referring to the massless calculation [9], the inclusive decay rate at NNLO is

$$\Gamma_{\text{NNLO}}^{(\text{massless})} = \Gamma_{LO}\left[1 + \frac{\alpha_s}{\pi}\frac{17}{3} + \left(\frac{\alpha_s}{\pi}\right)^2 29.15\right] \tag{3.72}$$

Comparing this equation with our result for $\Delta\Gamma_{\text{yt}}$, we can estimate that our contribution to the total decay rate is about 30% of the total NNLO, resulting in

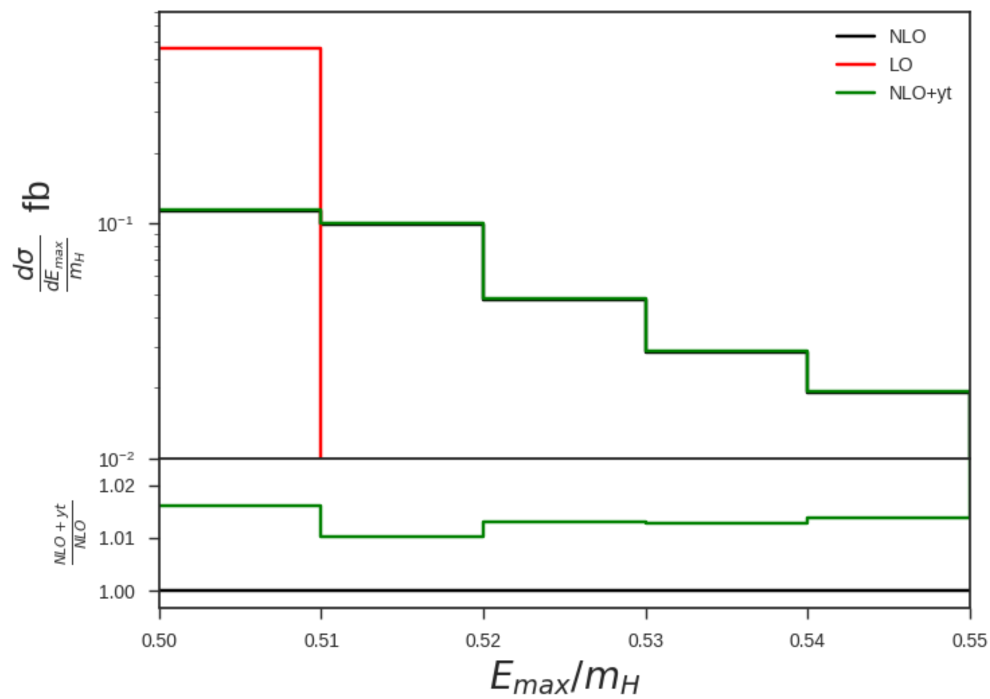


Figure 3.5: $E_{\max}$ distribution with top Yukawa corrections. The new contribution has an influence on the plot of about 1.5%. The error bars from the Monte Carlo simulation are to small to be seen. The QCD scale is fixed at $\mu = m_H$

a correction of about 1% with respect to the NLO correction.

Despite the fact that the top Yukawa contribution still remains quite relevant among the NNLO corrections, as shown in the plots, it gives a flat correction to the distributions with no remarkable impact on their shapes. The situation can be different at hadron colliders where both the process of production and decay of the Higgs boson receive radiative corrections and where different jet clustering algorithms are used. Moreover the kinematical regimes considered at CERN might emphasize specific contributions. In light of the relevance for the physics study of the Higgs boson at LHC, these studies will be the subject of further future investigations.

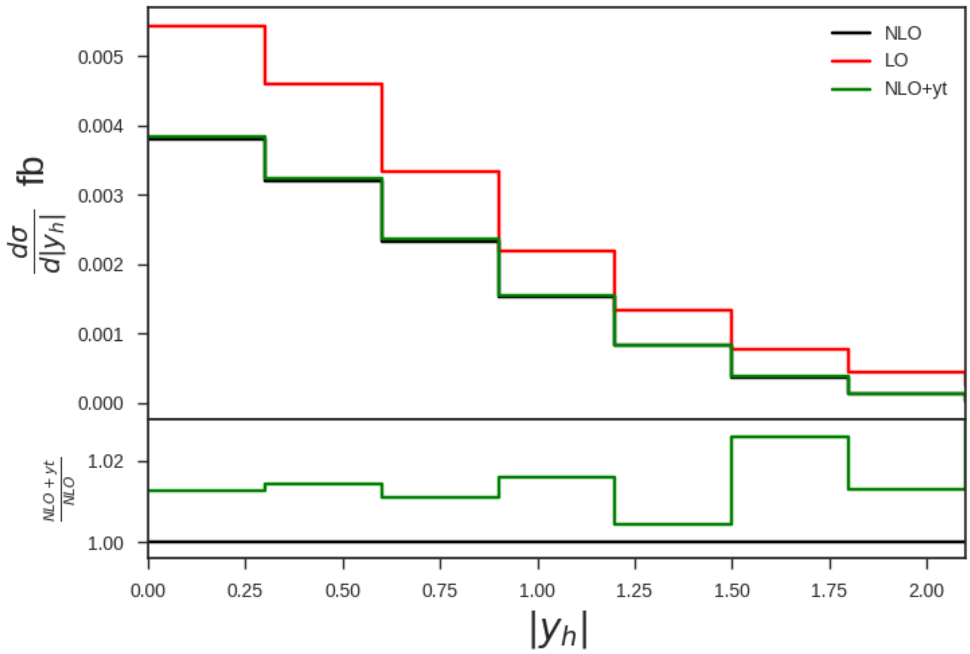


Figure 3.6: Absolute value of y rapidity of the jet with highest energy. Plot at QCD scale $\mu = m_H$. In the panel below the correction is about 2% of the NLO distribution.

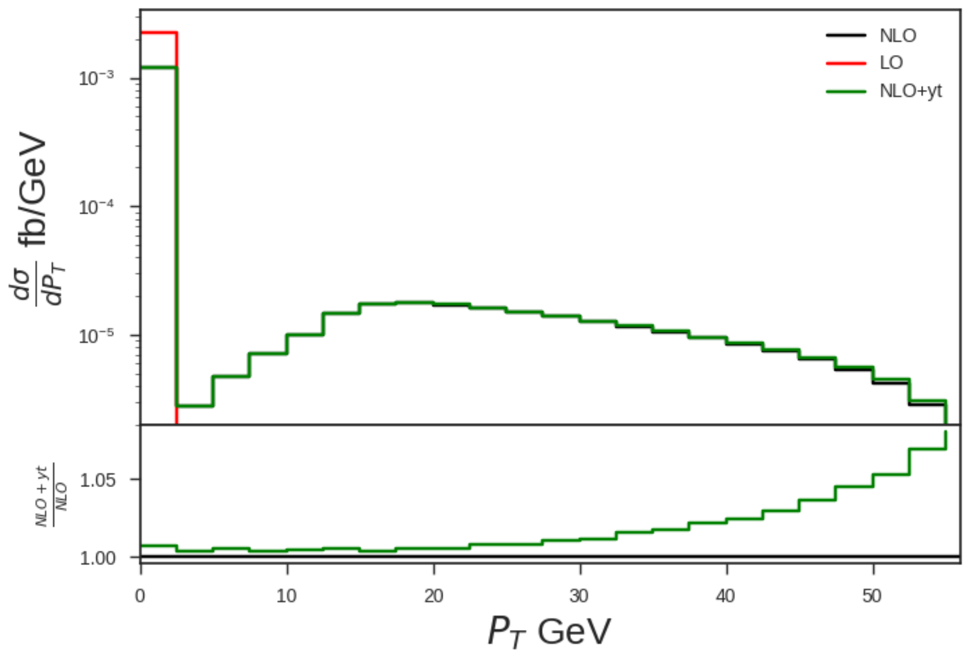


Figure 3.7: Transverse momentum of the d-jet of bottom quarks. Plot at QCD scale $\mu = m_H$. In the panel below the correction is about 1% of the NLO distribution, growing relatively fast in the high P_T region.

Conclusions

In this thesis work we have first re-obtained the NLO calculation of the Higgs boson decay into massive quarks and, using the subtraction method, we have built a Monte Carlo program able to produce binned histograms for any jet observables. We showed the relevance of the NLO correction, both on the inclusive decay rate and the shape of the distributions. The former gets a correction of about 20% while the latter are drastically modified.

We then focused on the NNLO contribution proportional to the top Yukawa coupling, presenting an exact calculation based on the method of partial differential equations with the Magnus series expansion. This led us to a complete analytical solution of the double virtual part of the computation written in terms of 256 GPLs. The numerical evaluation of the two loop amplitude can be performed, up to the desired precision, in a relatively short time (30 digits require less than one second).

An exact computation of this contribution was missing in the literature. We evaluated the discrepancy between our result and the approximated formula of Chetyrkin and Kwiatkowski [10], finding a remarkably good agreement for the physical values of the masses, $m_H = 125$ GeV, $m_b = 5$ GeV, $m_t = 175$ GeV. Our full result proves the validity of the approximated formula and establishes the size of its error that was not predicted. Finally, we have also computed the effect of the new contribution at the differential level, finding no specially privileged or suppressed region in phase space w.r.t. the NLO computation.

Next step of the present research work is the application of our results to the case of the Higgs boson production and decay at the hadron colliders. It is worth stressing that the techniques applied here are of general applicability. Indeed, they can be used to address the computation of higher order amplitudes for processes involving both internal and external masses that are needed to match the precision of present and future experiments at the LHC. This will be the subject of further studies.

Appendices

Appendix A

Inclusive integrals

We give here the analytical expressions, expanded in ϵ up to the zeroth order, of the integrals $\mathcal{I}_i$ 1.37 necessary to perform the NLO calculation. Recalling that $r = 4m_b^2/s$ and $J = 4\left(x_1^2 - r\right)\left(x_2^2 - r\right) - \left(2r - x_1^2 + (-x_1 - x_2 + 2)^2 - x_2^2\right)^2$, we have

$$\begin{aligned}
\mathcal{I}_1 = & \int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 J^{-\epsilon} \frac{1}{(1-x_1)^2} = \\
& - \frac{2\sqrt{1-r}}{\epsilon r} + \frac{1}{2r}\left(-8\sqrt{1-r} + (r-8)\log(r) - 8\sqrt{1-r}\log\left(\sqrt{r} - r\right)\right. \\
& - 4\sqrt{1-r}\log(2r) - 2(r-4)\left(\log\left(r\left(\sqrt{1-r} - 3\right) - 4\sqrt{1-r} + 4\right)\right. \\
& \left. + \log\left(\left(\sqrt{r} - 2\right)^2\right) - \log\left(\left(\sqrt{r} - 2\right)^2\sqrt{r}\right)\right) \\
& \left. + 2r\log\left(\sqrt{1-r} + 1\right) + 12\sqrt{1-r}\log(8-8r) + 8\sqrt{1-r}\log\left(1 - \sqrt{r}\right)\right)
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\mathcal{I}_2 = & \int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 J^{-\epsilon} \frac{1}{(1-x_1)(1-x_2)} = \frac{1}{\epsilon}\log\left(\frac{r}{-r + 2\sqrt{1-r} + 2}\right) \\
& \frac{1}{2}\left(\mathrm{Li}_2\left(-\frac{r + 2\sqrt{1-r} - 2}{-r + 2\sqrt{1-r} + 2}\right) + \mathrm{Li}_2\left(\frac{4\sqrt{1-r}}{r + 2\sqrt{1-r} - 2}\right)\right. \\
& - \log\left(-\frac{r + 2\sqrt{1-r} - 2}{-r + 2\sqrt{1-r} + 2}\right)\left(\log(4r) + 2\log\left(1 - \sqrt{r}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\log\left(\frac{r-2\left(\sqrt{1-r}+1\right)}{r+2\sqrt{1-r}-2}\right)\log\left(-\frac{4\sqrt{1-r}}{r+2\sqrt{1-r}-2}\right)-\frac{\pi^2}{6}\Bigg)\\
&-\mathrm{Li}_2\left(-\sqrt{1-r}\right)+\mathrm{Li}_2\left(\sqrt{1-r}\right)+\mathrm{Li}_2\left(\frac{1}{2}\left(\sqrt{1-r}-\sqrt{r}+1\right)\right)\\
&-11\mathrm{Li}_2\left(\frac{\sqrt{1-r}}{\sqrt{r}+1}\right)+11\mathrm{Li}_2\left(-\frac{\sqrt{1-r}}{\sqrt{r}+1}\right)-\mathrm{Li}_2\left(\frac{\left(\sqrt{1-r}-\sqrt{r}-1\right)\sqrt{1-r}}{2\left(\sqrt{r}+1\right)}\right)\\
&-11\mathrm{Li}_2\left(\frac{-\sqrt{1-r}+\sqrt{r}+1}{2\sqrt{r}+2}\right)+11\mathrm{Li}_2\left(\frac{\sqrt{1-r}+\sqrt{r}+1}{2\sqrt{r}+2}\right)\\
&-2\log^2\left(-\sqrt{1-r}+\sqrt{r}+1\right)+2\log^2\left(\sqrt{1-r}+\sqrt{r}+1\right)\\
&+\log(16)\log\left(-\sqrt{1-r}+\sqrt{r}+1\right)+3\log\left(\sqrt{r}+1\right)\log\left(-\sqrt{1-r}+\sqrt{r}+1\right)\\
&-4\log(2)\log\left(\sqrt{1-r}+\sqrt{r}+1\right)-3\log\left(\sqrt{r}+1\right)\log\left(\sqrt{1-r}+\sqrt{r}+1\right)
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\mathcal{I}_3 = &\int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 \frac{1-x_2}{1-x_1} =\\
&\frac{1}{32}\left(r^2\log\left(\frac{r^2-4\left(\sqrt{1-r}+2\right)r+8\left(\sqrt{1-r}+1\right)}{r^2}\right)+4(r-10)\sqrt{1-r}\right.\\
&\left.+16\log\left(-\frac{r}{r+2\sqrt{1-r}-2}\right)\right)
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\mathcal{I}_4 = &\int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 =\\
&\frac{1}{16}\left((r-4)r\log\left(\frac{r^2-4\left(\sqrt{1-r}+2\right)r+8\left(\sqrt{1-r}+1\right)}{r^2}\right)+4(r+2)\sqrt{1-r}\right)
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
\mathcal{I}_5 = &\int_{\sqrt{r}}^{1} dx_1 \int_{x_{2-}}^{x_{2+}} dx_2 \frac{1}{1-x_1} =\\
&-\sqrt{1-r}+\log\left(-\frac{r}{r+2\sqrt{1-r}-2}\right)+\frac{1}{2}r\tanh^{-1}\left(\frac{2\sqrt{1-r}}{r-2}\right)
\end{aligned}
\tag{A.5}
$$

Appendix B

Matrices of the Canonical Form

In this appendix we report the matrices of the canonical form obtained in 3.2.2, for the alphabet

$$\begin{array}{llll} L_1 = 1 - x & L_2 = x & L_3 = 1 + x & L_4 = 1 + x^2 \\ L_5 = x - y & L_6 = 1 - y & L_7 = y & L_8 = 1 + y \\ L_9 = x + y & L_{10} = 1 - xy & L_{11} = 1 + xy & L_{12} = 1 + y^2 \end{array}$$

We list below only the non 0 elements of the matrices.

$$\begin{array}{llll} (A_1)_{2,2} = -2 & (A_1)_{4,4} = 2 & (A_1)_{5,4} = 2 & (A_1)_{5,5} = -2 \\ (A_1)_{6,6} = -2 & (A_1)_{7,7} = 2 & (A_1)_{8,8} = -2 & (A_1)_{9,9} = -2 \\ (A_1)_{10,10} = -2 & (A_1)_{11,11} = -2 & (A_1)_{11,12} = 4 & (A_1)_{12,12} = 2 \\ (A_1)_{13,12} = -2 & (A_1)_{13,13} = -2 & (A_1)_{14,3} = 1 & (A_1)_{14,12} = 3 \\ (A_1)_{14,14} = -2 & (A_1)_{15,15} = 2 & (A_1)_{16,4} = 2 & (A_1)_{16,7} = -4 \\ (A_1)_{16,15} = -4 & (A_1)_{16,16} = -2 & (A_1)_{17,17} = -2 & (A_1)_{18,4} = 2 \\ (A_1)_{18,18} = -2 & (A_1)_{19,19} = -2 & (A_1)_{20,20} = -2 & \end{array}$$

$(A_2)_{3,1} = 2$ $(A_2)_{4,5} = 4$ $(A_2)_{5,1} = -2$ $(A_2)_{9,2} = 2$
$(A_2)_{10,6} = 2$ $(A_2)_{12,11} = 2$ $(A_2)_{15,2} = 2$ $(A_2)_{15,4} = 1$
$(A_2)_{15,5} = 2$ $(A_2)_{15,7} = -4$ $(A_2)_{15,16} = -4$ $(A_2)_{18,3} = 2$
$(A_2)_{18,10} = -2$ $(A_2)_{20,2} = -2$ $(A_2)_{20,4} = -1$ $(A_2)_{20,5} = -2$
$(A_2)_{20,7} = 4$ $(A_2)_{20,9} = 2$ $(A_2)_{20,11} = 2$ $(A_2)_{20,12} = -2$
$(A_2)_{20,15} = 4$ $(A_2)_{20,16} = 4$ $(A_2)_{20,17} = -2$ $(A_2)_{20,19} = -2$

$(A_3)_{2,2} = -2$ $(A_3)_{4,4} = 2$ $(A_3)_{5,4} = 2$ $(A_3)_{5,5} = -2$
$(A_3)_{6,6} = -2$ $(A_3)_{7,7} = 2$ $(A_3)_{8,8} = -2$ $(A_3)_{9,9} = -2$
$(A_3)_{10,10} = -2$ $(A_3)_{11,11} = -2$ $(A_3)_{11,12} = 4$ $(A_3)_{12,12} = 2$
$(A_3)_{13,12} = -2$ $(A_3)_{13,13} = -2$ $(A_3)_{14,3} = 1$ $(A_3)_{14,12} = 3$
$(A_3)_{14,14} = -2$ $(A_3)_{15,15} = 2$ $(A_3)_{16,4} = 2$ $(A_3)_{16,7} = -4$
$(A_3)_{16,15} = -4$ $(A_3)_{16,16} = -2$ $(A_3)_{17,17} = -2$ $(A_3)_{18,4} = 2$
$(A_3)_{18,18} = -2$ $(A_3)_{19,19} = -2$ $(A_3)_{20,20} = -2$

$(A_4)_{3,3} = -2$ $(A_4)_{5,4} = 3$ $(A_4)_{5,5} = -6$ $(A_4)_{9,9} = -2$
$(A_4)_{10,10} = -2$ $(A_4)_{11,11} = -2$ $(A_4)_{11,12} = 2$ $(A_4)_{14,3} = -1$
$(A_4)_{14,11} = -1$ $(A_4)_{14,12} = 1$ $(A_4)_{16,2} = 3$ $(A_4)_{16,4} = 3$
$(A_4)_{16,7} = -6$ $(A_4)_{16,15} = -6$ $(A_4)_{16,16} = -6$ $(A_4)_{19,17} = -2$
$(A_4)_{19,19} = -2$

$(A_5)_{7,1} = -1/2$ $(A_5)_{7,2} = 1/2$ $(A_5)_{7,7} = -3$ $(A_5)_{7,8} = 1$
$(A_5)_{8,1} = -1/2$ $(A_5)_{8,2} = 1/2$ $(A_5)_{8,7} = -3$ $(A_5)_{8,8} = 1$
$(A_5)_{11,1} = 1$ $(A_5)_{11,2} = -1$ $(A_5)_{11,7} = 6$ $(A_5)_{11,8} = -4$
$(A_5)_{11,14} = -2$ $(A_5)_{12,3} = -1$ $(A_5)_{12,9} = 1$ $(A_5)_{12,12} = -1$
$(A_5)_{12,13} = 1$ $(A_5)_{13,3} = 1$ $(A_5)_{13,9} = -1$ $(A_5)_{13,12} = 1$
$(A_5)_{13,13} = -1$ $(A_5)_{14,1} = 1/2$ $(A_5)_{14,2} = -1/2$ $(A_5)_{14,7} = 3$
$(A_5)_{14,8} = -3$ $(A_5)_{14,14} = -2$

$(A_6)_{2,2} = 2$ $(A_6)_{7,1} = 1$ $(A_6)_{7,2} = -1$ $(A_6)_{7,7} = 4$
$(A_6)_{9,9} = 2$ $(A_6)_{11,9} = -2$ $(A_6)_{11,13} = -2$ $(A_6)_{12,13} = -2$
$(A_6)_{13,13} = 4$ $(A_6)_{14,9} = -1$ $(A_6)_{14,13} = -3$ $(A_6)_{16,1} = -1$
$(A_6)_{16,2} = 2$ $(A_6)_{16,7} = -4$

$(A_7)_{12,7} = 8$ $(A_7)_{13,7} = -4$ $(A_7)_{13,11} = 2$ $(A_7)_{13,14} = -4$
$(A_7)_{15,4} = -1$ $(A_7)_{15,7} = 4$ $(A_7)_{17,2} = 2$ $(A_7)_{17,6} = -2$
$(A_7)_{19,2} = -2$ $(A_7)_{19,6} = 2$ $(A_7)_{19,9} = 2$ $(A_7)_{19,10} = -2$
$(A_7)_{20,3} = 2$ $(A_7)_{20,4} = 1$ $(A_7)_{20,11} = 2$ $(A_7)_{20,13} = -2$
$(A_7)_{20,14} = -4$ $(A_7)_{20,18} = -2$

$(A_8)_{12,7} = 8$ $(A_8)_{13,7} = -4$ $(A_8)_{13,11} = 2$ $(A_8)_{13,14} = -4$
$(A_8)_{15,4} = -1$ $(A_8)_{15,7} = 4$ $(A_8)_{17,2} = 2$ $(A_8)_{17,6} = -2$
$(A_8)_{19,2} = -2$ $(A_8)_{19,6} = 2$ $(A_8)_{19,9} = 2$ $(A_8)_{19,10} = -2$
$(A_8)_{20,3} = 2$ $(A_8)_{20,4} = 1$ $(A_8)_{20,11} = 2$ $(A_8)_{20,13} = -2$
$(A_8)_{20,14} = -4$ $(A_8)_{20,18} = -2$

$(A_9)_{7,1} = -1/2$	$(A_9)_{7,2} = 1/2$	$(A_9)_{7,7} = -3$	$(A_9)_{7,8} = -1$
$(A_9)_{8,1} = 1/2$	$(A_9)_{8,2} = -1/2$	$(A_9)_{8,7} = 3$	$(A_9)_{8,8} = 1$
$(A_9)_{11,1} = 1$	$(A_9)_{11,2} = -1$	$(A_9)_{11,7} = 6$	$(A_9)_{11,8} = 4$
$(A_9)_{11,14} = -2$	$(A_9)_{12,3} = -1$	$(A_9)_{12,9} = 1$	$(A_9)_{12,12} = -1$
$(A_9)_{12,13} = 1$	$(A_9)_{13,3} = 1$	$(A_9)_{13,9} = -1$	$(A_9)_{13,12} = 1$
$(A_9)_{13,13} = -1$	$(A_9)_{14,1} = 1/2$	$(A_9)_{14,2} = -1/2$	$(A_9)_{14,7} = 3$
$(A_9)_{14,8} = 3$	$(A_9)_{14,14} = -2$		

$(A_{10})_{7,1} = -1/2$	$(A_{10})_{7,2} = 1/2$	$(A_{10})_{7,7} = -3$	$(A_{10})_{7,8} = -1$
$(A_{10})_{8,1} = 1/2$	$(A_{10})_{8,2} = -1/2$	$(A_{10})_{8,7} = 3$	$(A_{10})_{8,8} = 1$
$(A_{10})_{11,1} = -1$	$(A_{10})_{11,2} = 1$	$(A_{10})_{11,7} = -6$	$(A_{10})_{11,8} = -4$
$(A_{10})_{11,9} = -2$	$(A_{10})_{11,11} = -2$	$(A_{10})_{11,13} = 2$	$(A_{10})_{11,14} = 2$
$(A_{10})_{12,3} = 1$	$(A_{10})_{12,9} = -1$	$(A_{10})_{12,12} = -1$	$(A_{10})_{12,13} = 1$
$(A_{10})_{13,3} = -1$	$(A_{10})_{13,9} = 1$	$(A_{10})_{13,12} = 1$	$(A_{10})_{13,13} = -1$
$(A_{10})_{14,1} = -1/2$	$(A_{10})_{14,2} = 1/2$	$(A_{10})_{14,3} = 1$	$(A_{10})_{14,7} = -3$
$(A_{10})_{14,8} = -1$	$(A_{10})_{14,9} = -1$	$(A_{10})_{14,12} = -1$	$(A_{10})_{14,13} = 1$
$(A_{10})_{16,1} = 1$	$(A_{10})_{16,2} = -1$	$(A_{10})_{16,7} = 6$	$(A_{10})_{16,8} = 2$

$(A_{11})_{7,1} = -1/2$	$(A_{11})_{7,2} = 1/2$	$(A_{11})_{7,7} = -3$	$(A_{11})_{7,8} = 1$
$(A_{11})_{8,1} = -1/2$	$(A_{11})_{8,2} = 1/2$	$(A_{11})_{8,7} = -3$	$(A_{11})_{8,8} = 1$
$(A_{11})_{11,1} = -1$	$(A_{11})_{11,2} = 1$	$(A_{11})_{11,7} = -6$	$(A_{11})_{11,8} = 4$
$(A_{11})_{11,9} = -2$	$(A_{11})_{11,11} = -2$	$(A_{11})_{11,13} = 2$	$(A_{11})_{11,14} = 2$
$(A_{11})_{12,3} = 1$	$(A_{11})_{12,9} = -1$	$(A_{11})_{12,12} = -1$	$(A_{11})_{12,13} = 1$
$(A_{11})_{13,3} = -1$	$(A_{11})_{13,9} = 1$	$(A_{11})_{13,12} = 1$	$(A_{11})_{13,13} = -1$
$(A_{11})_{14,1} = -1/2$	$(A_{11})_{14,2} = 1/2$	$(A_{11})_{14,3} = 1$	$(A_{11})_{14,7} = -3$
$(A_{11})_{14,8} = 1$	$(A_{11})_{14,9} = -1$	$(A_{11})_{14,12} = -1$	$(A_{11})_{14,13} = 1$
$(A_{11})_{16,1} = 1$	$(A_{11})_{16,2} = -1$	$(A_{11})_{16,7} = 6$	$(A_{11})_{16,8} = -2$

$$
\begin{array}{llll}
(A_{12})_{11,9} = 2 & (A_{12})_{11,11} = 2 & (A_{12})_{12,12} = 2 & (A_{12})_{13,13} = 2 \\
(A_{12})_{14,9} = 1 & (A_{12})_{14,11} = 1 & (A_{12})_{14,13} = -1 & (A_{12})_{15,15} = 2 \\
(A_{12})_{16,2} = -1 & (A_{12})_{16,4} = -1/2 & (A_{12})_{16,5} = -1 & (A_{12})_{16,7} = 2 \\
(A_{12})_{16,16} = 2 & (A_{12})_{17,17} = 2 & (A_{12})_{19,19} = 2 & (A_{12})_{20,20} = 2
\end{array}
$$

Appendix C

Virtual top Yukawa contribution

We report here the value of $\mathcal{G}(x,y)$

$$
\begin{aligned}
\mathcal{G}(x,y) = {} & \frac{1}{(x^2-1)^2\,(y^4-1)} \\
& \Big(\frac{1}{y^2-1} 8\,(x^2+1)^2\,(y^2+1)\,(y^4-1)\,\Big(\frac{2}{3}G(0;x)^3G(0;y) - \frac{1}{2}G(0;x)^2G(0;y)^2 + G(0;x)^2G(0,-x;y) \\
& +G(0;x)^2G(0,x;y) - \frac{1}{4}G(0;x)^4\Big) + \frac{4}{y^2}x^4y^2\,\Big(\big(y^4+8y^2-1\big) - 8x^3\,(y^5+y^3) - x^2\,(y^8-1) + 8x\,(y^5+y^3) \\
& +y^2\,(y^4-8y^2-1)\Big)\,\Big(-G(-1;x)G\Big(\frac{1}{x};y\Big) + G(0;x)G\Big(\frac{1}{x};y\Big) - G(0;y)G\Big(\frac{1}{x};y\Big) \\
& -G(1;x)G\Big(\frac{1}{x};y\Big) + G\Big(\frac{1}{x},-1;y\Big) + G\Big(\frac{1}{x},1;y\Big)\Big) \\
& + \frac{1}{y^2\,(y^2-1)}\,\Big(\frac{1}{3}(-32)y^2\,(y^2-1)\,\big(-3x^4\,(y^4-1) + 4\pi^2x^2y^2 + 3\,(y^4-1)\big)\,G(0,-i;x) + G(0,i;x) \\
& + 128\,(x^2-1)\,(x^2+1)\,y^4\,(y^2-1)\,(G(0,-i;x)G(0;y) + G(0,i;x)G(0;y)) \\
& + \frac{4}{3}y^2\,(y^2-1)\,\Big(12\,(x^4-1)\,(y^4+4y^2-1) + \pi^2\,\Big(x^4\,(y^2+1)^2 + 2x^2\,(y^4-6y^2+1) + (y^2+1)^2\Big)\Big)\,(G(0,-1;x) + G(0,1;x)) \\
& + 32y^2\,(y^2-1)\,\big(x^4y^2 + x^2\,(5-7y^4) + y^2\big)\,(G(0;x)G(0,-1;x) + G(0;x)G(0,1;x)) \\
& - 8\,(y^4-1)\,\big(2x^4y^4 - x^2\,(y^6+y^4+3y^2-1) + 2y^4\big)\,(G(0,-1;y) + G(0,1;y)) \\
& - 32y^4\,(y^2-1)\,(x^4+2x^2y^2-3)\,\Big(G(0;x)G\Big(0,\frac{1}{x};y\Big) + G(0;x)G\Big(0,-\frac{1}{x};y\Big)\Big) \\
& + 16y^2\,(y^2-1)\,\big(x^4y^2 + x^2\,(y^4+1) - 3y^2\big)\,\Big(G(-1;x)G\Big(0,\frac{1}{x};y\Big) + G(0;y)G\Big(0,\frac{1}{x};y\Big) + G(1;x)G\Big(0,\frac{1}{x};y\Big) \\
& +G(-1;x)G\Big(0,-\frac{1}{x};y\Big) + G(0;y)G\Big(0,-\frac{1}{x};y\Big) + G(1;x)G\Big(0,-\frac{1}{x};y\Big)\Big) \\
& + \frac{4}{3}\,(y^2-1)\,\Big(\Big(3x^4y^2\,(y^4+8y^2-1) - 24x^3\,(y^5+y^3) + x^2\,(-3y^8+8\pi^2y^4+3) + 24x\,(y^5+y^3) + 3y^2\,(y^4-8y^2-1)\Big)\,G\Big(0,\frac{1}{x};y\Big) \\
& + \Big(3x^4y^2\,(y^4+8y^2-1) + 24x^3\,(y^5+y^3) + x^2\,(-3y^8+8\pi^2y^4+3) - 24x\,(y^5+y^3) + 3y^2\,(y^4-8y^2-1)\Big)\,G\Big(0,-\frac{1}{x};y\Big) \\
& + \Big(3x^4y^2\,(y^4-8y^2-1) - 24x^3\,(y^5+y^3) + x^2\,(-3y^8-8\pi^2y^4+3) + 24x\,(y^5+y^3) + 3y^2\,(y^4+8y^2-1)\Big)\,G(0,-x;y)\Big) \\
& - \frac{4}{3}\,(y^2-1)\,\Big(-3x^4y^2\,(y^4-8y^2-1) - 24x^3\,(y^5+y^3) + x^2\,(3y^8+8\pi^2y^4-3) + 24x\,(y^5+y^3) - 3y^2\,(y^4+8y^2-1)\Big)\,G(0,x;y) \\
& + 32x^2y^2\,(y^2-1)\,(y^4-1)\,(G(0;x)G(0,-x;y) + G(0;x)G(0,x;y)) \\
& + 8\,(x^2+1)^2\,y^2\,(y^2+1)\,(y^4-1)\,\Big(G(0;x)G(0;y)G\Big(0,\frac{1}{x};y\Big) + G(0;x)G(0;y)G\Big(0,-\frac{1}{x};y\Big) \\
& -G(0;x)G(0;y)G(0,-x;y) - G(0;x)G(0;y)G(0,x;y)) \\
& + 16y^2\,(y^2-1)\,\big(-3x^4y^2 + x^2\,(y^4+1) + y^2\big)\,(2G(0;x)G(0,-1;y) + 2G(0;x)G(0,1;y) + G(-1;x)G(0,-x;y) \\
& + G(0;y)G(0,-x;y) + G(1;x)G(0,-x;y) + G(-1;x)G(0,x;y) + G(0;y)G(0,x;y) + G(1;x)G(0,x;y)) \\
& - 8\,(y^2-1)\,(y^4-1)\,\big(x^4y^2 - x^2\,(y^4+1) + y^2\big)\,(G(-1,-1;y) + G(-1,1;y) + G(1,-1;y) + G(1,1;y)) \\
& - 8y^2\,(y^2-1)\,\Big(x^4\,(y^2+1)^2 + 2x^2\,(y^4-6y^2+1) + (y^2+1)^2\Big)\,\Big(G(0;x)^2\,\Big(-G\Big(-1,\frac{1}{x};y\Big)\Big) \\
& + G(0;x)^2G(0;x)^2G(-1,-x;y)G(-1,x;y) + G(0;x)^2G(0,-1;y) + G(0;x)^2G(0,1;y) + G(0;x)^2G\Big(0,\frac{1}{x};y\Big)
\end{aligned}
$$

$$+ G(0;x)^2 G\left(0,-\frac{1}{x};y\right) - G(0;x)^2 G\left(1,\frac{1}{x};y\right) - G(0;x)^2 G\left(1,-\frac{1}{x};y\right) + G(0;x)^2 G(1,-x;y)$$
$$+G(0;x)^2 G(1,x;y) - G(0;x)^2 G\left(-1,-\frac{1}{x};y\right) + 3G(0,-1;x)G(0;x)^2 + 3G(0,1;x)G(0;x)^2\Big)$$
$$+ 4\left(y^2-1\right)\left(x^4y^2\left(y^4+8y^2-1\right)+8x^3\left(y^5+y^3\right)-x^2\left(y^8-1\right)-8x\left(y^5+y^3\right)+y^2\left(y^4-8y^2-1\right)\right)\left(G\left(-\frac{1}{x},-1;y\right)\right.$$
$$\left.+G\left(-\frac{1}{x},1;y\right)\right) + 4\left(y^2-1\right)\left(x^4y^2\left(y^4-8y^2-1\right)-8x^3\left(y^5+y^3\right)-x^2\left(y^8-1\right)+8x\left(y^5+y^3\right)\right.$$
$$\left.+y^2\left(y^4+8y^2-1\right)\right)\left(G(-x,-1;y)+G(-x,1;y)\right) + 4\left(y^2-1\right)\left(x^4y^2\left(y^4-8y^2-1\right)+8x^3\left(y^5+y^3\right)\right.$$
$$\left.-x^2\left(y^8-1\right)-8x\left(y^5+y^3\right)+y^2\left(y^4+8y^2-1\right)\right)\left(G(x,-1;y)+G(x,1;y)\right)\Big)$$
$$+ 32\left(x^4y^2-x^2\left(y^4-1\right)-y^2\right)\left(\frac{1}{12}\pi^2G(-x;y)+G(0,-1;x)G(-x;y)+G(0,1;x)G(-x;y)+\frac{1}{12}\pi^2G(x;y)\right.$$
$$+ G(0,-1;x)G(x;y) + G(0,1;x)G(x;y) - G(-1;x)G\left(-x,\frac{1}{x};y\right) + G(0;x)G\left(-x,\frac{1}{x};y\right)$$
$$- G(0;y)G\left(-x,\frac{1}{x};y\right) - G(1;x)G\left(-x,\frac{1}{x};y\right) + G(-1;x)G(-x,x;y) - G(0;x)G(-x,x;y)$$
$$+ G(0;y)G(-x,x;y) + G(1;x)G(-x,x;y) - G(-1;x)G\left(x,-\frac{1}{x};y\right)$$
$$+ G(0;x)G\left(x,-\frac{1}{x};y\right) - G(0;y)G\left(x,-\frac{1}{x};y\right) - G(1;x)G\left(x,-\frac{1}{x};y\right)$$
$$\left.+G(-1;x)G(x,-x;y) - G(0;x)G(x,-x;y) + G(0;y)G(x,-x;y) + G(1;x)G(x,-x;y)\right)$$
$$+ 32\left(5x^4y^2+x^2\left(5y^4-7\right)-3y^2\right)\left(G(0,0,-1;x)+G(0,0,1;x)\right) + 128\left(x^4y^2+2x^2\left(y^4-1\right)-y^2\right)\left(G(0,0,-i;x)+G(0,0,i;x)\right)$$
$$- 8\left(x^4\left(y^2+1\right)^2+2x^2\left(y^4+10y^2+1\right)+\left(y^2+1\right)^2\right)\left(G(0;x)G\left(0,0,\frac{1}{x};y\right)+G(0;x)G\left(0,0,-\frac{1}{x};y\right)\right)$$
$$+ 8\left(3x^4\left(y^2+1\right)^2+x^2\left(6y^4-4y^2+6\right)+3\left(y^2+1\right)^2\right)\left(G(0;x)G(0,0,-x;y)+G(0;x)G(0,0,x;y)\right)$$
$$- 32\left(x^4y^2-x^2\left(y^4+1\right)+y^2\right)\left(-\frac{1}{2}G(0;x)^2G\left(\frac{1}{x};y\right)-\frac{1}{2}G(0;x)^2G\left(-\frac{1}{x};y\right)+\frac{1}{2}G(-1;x)G(0;x)G\left(\frac{1}{x};y\right)\right.$$
$$+ \frac{1}{2}G(1;x)G(0;x)G\left(\frac{1}{x};y\right) + \frac{1}{2}G(-1;x)G(0;x)G\left(-\frac{1}{x};y\right) + \frac{1}{2}G(1;x)G(0;x)G\left(-\frac{1}{x};y\right)$$
$$- \frac{1}{2}G(-1;x)G(0;x)G(-x;y) - \frac{1}{2}G(1;x)G(0;x)G(-x;y) - \frac{1}{2}G(-1;x)G(0;x)G(x;y)$$
$$- \frac{1}{2}G(1;x)G(0;x)G(x;y) - \frac{1}{2}G(0;x)G\left(\frac{1}{x},-1;y\right) - \frac{1}{2}G(0;x)G\left(\frac{1}{x},1;y\right)$$
$$- \frac{1}{2}G(0;x)G\left(-\frac{1}{x},-1;y\right) - \frac{1}{2}G(0;x)G\left(-\frac{1}{x},1;y\right) + \frac{1}{2}G(0;x)G(-x,-1;y)$$
$$+ \frac{1}{2}G(0;x)G(-x,1;y) + \frac{1}{2}G(0;x)G(x,-1;y) + \frac{1}{2}G(0;x)G(x,1;y) - G(-1;x)G(0,-1;y) - G(1;x)G(0,-1;y)$$
$$- G(-1;x)G(0,1;y) - G(1;x)G(0,1;y) - G(0;y)G(0,-1;y) - G(0;y)G(0,1;y) + G(0,-1,-1;y)$$
$$\left.+G(0,-1,1;y) + 2G(0,0,-1;y) + 2G(0,0,1;y) + G(0,1,-1;y) + G(0,1,1;y)\right)$$
$$- 16\left(x^4y^2+x^2\left(y^4+1\right)-3y^2\right)\left(2G\left(0,0,\frac{1}{x};y\right)+2G\left(0,0,-\frac{1}{x};y\right)+G\left(0,\frac{1}{x},-1;y\right)\right.$$
$$\left.+G\left(0,\frac{1}{x},1;y\right)+G\left(0,-\frac{1}{x},-1;y\right)+G\left(0,-\frac{1}{x},1;y\right)\right)$$
$$- 16\left(-3x^4y^2+x^2\left(y^4+1\right)+y^2\right)\left(2G(0,0,-x;y)+2G(0,0,x;y)+G(0,-x,-1;y)+G(0,-x,1;y)+G(0,x,-1;y)\right.$$
$$\left.+G(0,x,1;y)\right) - 128\left(x^2-1\right)\left(x^2+1\right)y^2\left(G(-i,0,-1;x)+G(-i,0,1;x)+G(i,0,-1;x)+G(i,0,1;x)\right)$$
$$- 32\left(x^4y^2+x^2\left(y^4-1\right)-y^2\right)\left(-\frac{1}{12}\pi^2G\left(\frac{1}{x};y\right)-G(0,-1;x)G\left(\frac{1}{x};y\right)-G(0,1;x)G\left(\frac{1}{x};y\right)\right.$$
$$- \frac{1}{12}\pi^2G\left(-\frac{1}{x};y\right) - G(0,-1;x)G\left(-\frac{1}{x};y\right) - G(0,1;x)G\left(-\frac{1}{x};y\right)$$
$$+ G(-1;x)G\left(\frac{1}{x},-\frac{1}{x};y\right) - G(0;x)G\left(\frac{1}{x},-\frac{1}{x};y\right) + G(0;y)G\left(\frac{1}{x},-\frac{1}{x};y\right)$$
$$+ G(1;x)G\left(\frac{1}{x},-\frac{1}{x};y\right) - G(-1;x)G\left(\frac{1}{x},-x;y\right)$$
$$+ G(0;x)G\left(\frac{1}{x},-x;y\right) - G(0;y)G\left(\frac{1}{x},-x;y\right) - G(1;x)G\left(\frac{1}{x},-x;y\right)$$
$$+ G(-1;x)G\left(-\frac{1}{x},\frac{1}{x};y\right) - G(0;x)G\left(-\frac{1}{x},\frac{1}{x};y\right) + G(0;y)G\left(-\frac{1}{x},\frac{1}{x};y\right)$$
$$+ G(1;x)G\left(-\frac{1}{x},\frac{1}{x};y\right) - G(-1;x)G\left(-\frac{1}{x},x;y\right) + G(0;x)G\left(-\frac{1}{x},x;y\right)$$
$$- G(0;y)G\left(-\frac{1}{x},x;y\right) - G(1;x)G\left(-\frac{1}{x},x;y\right) - G\left(0,\frac{1}{x},-\frac{1}{x};y\right)$$
$$+ G\left(0,\frac{1}{x},-x;y\right) - G\left(0,-\frac{1}{x},\frac{1}{x};y\right) + G\left(0,-\frac{1}{x},x;y\right) - G\left(\frac{1}{x},0,-\frac{1}{x};y\right) + G\left(\frac{1}{x},0,-x;y\right)$$
$$- G\left(\frac{1}{x},-\frac{1}{x},-1;y\right) - G\left(\frac{1}{x},-\frac{1}{x},1;y\right) + G\left(\frac{1}{x},-x,-1;y\right)$$
$$+ G\left(\frac{1}{x},-x,1;y\right) - G\left(-\frac{1}{x},0,\frac{1}{x};y\right) + G\left(-\frac{1}{x},0,x;y\right) - G\left(-\frac{1}{x},\frac{1}{x},-1;y\right)$$

$$-G\left(-\frac{1}{x},\frac{1}{x},1;y\right)+G\left(-\frac{1}{x},x,-1;y\right)+G\left(-\frac{1}{x},x,1;y\right)+4G(0;x)G(0,-i;x)+4G(0;x)G(0,i;x)\Big)$$
$$+32\left(x^4y^2-x^2\left(y^4-1\right)-y^2\right)\left(G\left(0,-x,\frac{1}{x};y\right)-G(0,-x,x;y)+G\left(0,x,-\frac{1}{x};y\right)\right.$$
$$-G(0,x,-x;y)+G\left(-x,0,\frac{1}{x};y\right)-G(-x,0,x;y)+G\left(-x,\frac{1}{x},-1;y\right)+G\left(-x,\frac{1}{x},1;y\right)$$
$$-G(-x,x,-1;y)-G(-x,x,1;y)+G\left(x,0,-\frac{1}{x};y\right)-G(x,0,-x;y)+G\left(x,-\frac{1}{x},-1;y\right)$$
$$\left.+G\left(x,-\frac{1}{x},1;y\right)-G(x,-x,-1;y)-G(x,-x,1;y)\right)$$
$$-64((1+y^2)^2+x^4(1+y^2)^2+2x^2(1-10y^2+y^4))(G[0,0,0,-1,x]+G[0,0,0,1,x])$$
$$+128x^2y^2\left(2G(0,-i;x)G(0;x)^2+2G(0,i;x)G(0;x)^2-4G(0;y)G(0,-i;x)G(0;x)-4G(0;y)G(0,i;x)G(0;x)\right.$$
$$-8G(0,0,-i;x)G(0;x)-8G(0,0,i;x)G(0;x)+G\left(0,\frac{1}{x},-\frac{1}{x};y\right)G(0;x)$$
$$-G\left(0,\frac{1}{x},-x;y\right)G(0;x)+G\left(0,-\frac{1}{x},\frac{1}{x};y\right)G(0;x)-G\left(0,-\frac{1}{x},x;y\right)G(0;x)$$
$$-G\left(0,-x,\frac{1}{x};y\right)G(0;x)+G(0,-x,x;y)G(0;x)-G\left(0,x,-\frac{1}{x};y\right)G(0;x)$$
$$+G(0,x,-x;y)G(0;x)+G(0,-1;x)G\left(0,\frac{1}{x};y\right)+G(0,1;x)G\left(0,\frac{1}{x};y\right)+G(0,-1;x)G\left(0,-\frac{1}{x};y\right)$$
$$+G(0,1;x)G\left(0,-\frac{1}{x};y\right)-G(0,-1;x)G(0,-x;y)-G(0,1;x)G(0,-x;y)$$
$$-G(0,-1;x)G(0,x;y)-G(0,1;x)G(0,x;y)$$
$$+8G(0;y)G(0,0,-i;x)+8G(0;y)G(0,0,i;x)-G(-1;x)G\left(0,\frac{1}{x},-\frac{1}{x};y\right)-G(0;y)G\left(0,\frac{1}{x},-\frac{1}{x};y\right)$$
$$-G(1;x)G\left(0,\frac{1}{x},-\frac{1}{x};y\right)+G(-1;x)G\left(0,\frac{1}{x},-x;y\right)+G(0;y)G\left(0,\frac{1}{x},-x;y\right)$$
$$+G(1;x)G\left(0,\frac{1}{x},-x;y\right)-G(-1;x)G\left(0,-\frac{1}{x},\frac{1}{x};y\right)-G(0;y)G\left(0,-\frac{1}{x},\frac{1}{x};y\right)$$
$$-G(1;x)G\left(0,-\frac{1}{x},\frac{1}{x};y\right)+G(-1;x)G\left(0,-\frac{1}{x},x;y\right)+G(0;y)G\left(0,-\frac{1}{x},x;y\right)$$
$$+G(1;x)G\left(0,-\frac{1}{x},x;y\right)+G(-1;x)G\left(0,-x,\frac{1}{x};y\right)+G(0;y)G\left(0,-x,\frac{1}{x};y\right)+G(1;x)G\left(0,-x,\frac{1}{x};y\right)$$
$$-G(-1;x)G(0,-x,x;y)-G(0;y)G(0,-x,x;y)-G(1;x)G(0,-x,x;y)+G(-1;x)G\left(0,x,-\frac{1}{x};y\right)$$
$$+G(0;y)G\left(0,x,-\frac{1}{x};y\right)+G(1;x)G\left(0,x,-\frac{1}{x};y\right)$$
$$-G(-1;x)G(0,x,-x;y)-G(0;y)G(0,x,-x;y)-G(1;x)G(0,x,-x;y)+12G(0,0,0,-i;x)+12G(0,0,0,i;x)$$
$$+2G\left(0,0,\frac{1}{x},-\frac{1}{x};y\right)-2G\left(0,0,\frac{1}{x},-x;y\right)+2G\left(0,0,-\frac{1}{x},\frac{1}{x};y\right)$$
$$-2G\left(0,0,-\frac{1}{x},x;y\right)-2G\left(0,0,-x,\frac{1}{x};y\right)+2G(0,0,-x,x;y)-2G\left(0,0,x,-\frac{1}{x};y\right)+2G(0,0,x,-x;y)$$
$$-4G(0,-i,0,-1;x)-4G(0,-i,0,1;x)-4G(0,i,0,-1;x)$$
$$-4G(0,i,0,1;x)+G\left(0,\frac{1}{x},0,-\frac{1}{x};y\right)-G\left(0,\frac{1}{x},0,-x;y\right)+G\left(0,\frac{1}{x},-\frac{1}{x},-1;y\right)$$
$$+G\left(0,\frac{1}{x},-\frac{1}{x},1;y\right)-G\left(0,\frac{1}{x},-x,-1;y\right)-G\left(0,\frac{1}{x},-x,1;y\right)$$
$$+G\left(0,-\frac{1}{x},0,\frac{1}{x};y\right)-G\left(0,-\frac{1}{x},0,x;y\right)+G\left(0,-\frac{1}{x},\frac{1}{x},-1;y\right)$$
$$+G\left(0,-\frac{1}{x},\frac{1}{x},1;y\right)-G\left(0,-\frac{1}{x},x,-1;y\right)-G\left(0,-\frac{1}{x},x,1;y\right)$$
$$-G\left(0,-x,0,\frac{1}{x};y\right)+G(0,-x,0,x;y)-G\left(0,-x,\frac{1}{x},-1;y\right)-G\left(0,-x,\frac{1}{x},1;y\right)$$
$$+G(0,-x,x,-1;y)+G(0,-x,x,1;y)-G\left(0,x,0,-\frac{1}{x};y\right)+G(0,x,0,-x;y)-G\left(0,x,-\frac{1}{x},-1;y\right)$$
$$\left.-G\left(0,x,-\frac{1}{x},1;y\right)+G(0,x,-x,-1;y)+G(0,x,-x,1;y)\right)$$
$$-16\left(\left(y^2+1\right)^2x^4+2\left(y^4-6y^2+1\right)x^2+\left(y^2+1\right)^2\right)\left(-\frac{1}{2}G(-1;x)G(0;x)^3\right.$$
$$+\frac{1}{3}G(-1;y)G(0;x)^3-\frac{1}{2}G(1;x)G(0;x)^3+\frac{1}{3}G(1;y)G(0;x)^3+G(-1;x)G(0;y)G(0;x)^2$$
$$-\frac{1}{2}G(-1;y)G(0;y)G(0;x)^2+G(0;y)G(1;x)G(0;x)^2-\frac{1}{2}G(0;y)G(1;y)G(0;x)^2$$
$$+\frac{1}{2}G(-1;x)G\left(-1,\frac{1}{x};y\right)G(0;x)+\frac{1}{2}G(0;y)G\left(-1,\frac{1}{x};y\right)G(0;x)+\frac{1}{2}G(1;x)G\left(-1,\frac{1}{x};y\right)G(0;x)$$
$$+\frac{1}{2}G(-1;x)G\left(-1,-\frac{1}{x};y\right)G(0;x)+\frac{1}{2}G(0;y)G\left(-1,-\frac{1}{x};y\right)G(0;x)+\frac{1}{2}G(1;x)G\left(-1,-\frac{1}{x};y\right)G(0;x)$$
$$-\frac{1}{2}G(-1;x)G(-1,-x;y)G(0;x)-\frac{1}{2}G(0;y)G(-1,-x;y)G(0;x)-\frac{1}{2}G(1;x)G(-1,-x;y)G(0;x)$$
$$-\frac{1}{2}G(-1;x)G(-1,x;y)G(0;x)-\frac{1}{2}G(0;y)G(-1,x;y)G(0;x)-\frac{1}{2}G(1;x)G(-1,x;y)G(0;x)$$

$$
\begin{aligned}
&- G(-1;y)G(0,-1;x)G(0;x) - 2G(0;y)G(0,-1;x)G(0;x) - G(1;y)G(0,-1;x)G(0;x) - G(-1;y)G(0,1;x)G(0;x)\\
&- 2G(0;y)G(0,1;x)G(0;x) - G(1;y)G(0,1;x)G(0;x) - \frac{1}{2}G(-1;x)G\left(0,\frac{1}{x};y\right)G(0;x) - \frac{1}{2}G(1;x)G\left(0,\frac{1}{x};y\right)G(0;x)\\
&- \frac{1}{2}G(-1;x)G\left(0,-\frac{1}{x};y\right)G(0;x) - \frac{1}{2}G(1;x)G\left(0,-\frac{1}{x};y\right)G(0;x) + \frac{1}{2}G(-1;x)G(0,-x;y)G(0;x)\\
&+ \frac{1}{2}G(1;x)G(0,-x;y)G(0;x) + \frac{1}{2}G(-1;x)G(0,x;y)G(0;x)\\
&+ \frac{1}{2}G(1;x)G(0,x;y)G(0;x) + \frac{1}{2}G(-1;x)G\left(1,\frac{1}{x};y\right)G(0;x) + \frac{1}{2}G(0;y)G\left(1,\frac{1}{x};y\right)G(0;x)\\
&+ \frac{1}{2}G(1;x)G\left(1,\frac{1}{x};y\right)G(0;x) + \frac{1}{2}G(-1;x)G\left(1,-\frac{1}{x};y\right)G(0;x) + \frac{1}{2}G(0;y)G\left(1,-\frac{1}{x};y\right)G(0;x)\\
&+ \frac{1}{2}G(1;x)G\left(1,-\frac{1}{x};y\right)G(0;x) - \frac{1}{2}G(-1;x)G(1,-x;y)G(0;x)\\
&- \frac{1}{2}G(0;y)G(1,-x;y)G(0;x) - \frac{1}{2}G(1;x)G(1,-x;y)G(0;x) - \frac{1}{2}G(-1;x)G(1,x;y)G(0;x)\\
&- \frac{1}{2}G(0;y)G(1,x;y)G(0;x) - \frac{1}{2}G(1;x)G(1,x;y)G(0;x) + G(-1,0,-1;x)G(0;x) - G(-1,0,-1;y)G(0;x)\\
&+ G(-1,0,1;x)G(0;x)\\
&- G(-1,0,1;y)G(0;x) + G(-1,0,-x;y)G(0;x) + G(-1,0,x;y)G(0;x) - \frac{1}{2}G\left(-1,\frac{1}{x},-1;y\right)G(0;x)\\
&- \frac{1}{2}G\left(-1,\frac{1}{x},1;y\right)G(0;x) - \frac{1}{2}G\left(-1,-\frac{1}{x},-1;y\right)G(0;x)\\
&- \frac{1}{2}G\left(-1,-\frac{1}{x},1;y\right)G(0;x) + \frac{1}{2}G(-1,-x,-1;y)G(0;x) + \frac{1}{2}G(-1,-x,1;y)G(0;x)\\
&+ \frac{1}{2}G(-1,x,-1;y)G(0;x) + \frac{1}{2}G(-1,x,1;y)G(0;x) - \frac{1}{2}G\left(0,-1,\frac{1}{x};y\right)G(0;x)\\
&- \frac{1}{2}G\left(0,-1,-\frac{1}{x};y\right)G(0;x) + \frac{1}{2}G(0,-1,-x;y)G(0;x) + \frac{1}{2}G(0,-1,x;y)G(0;x)\\
&- 4G(0,0,-1;x)G(0;x) + G(0,0,-1;y)G(0;x) - 4G(0,0,1;x)G(0;x) + G(0,0,1;y)G(0;x)\\
&- \frac{1}{2}G\left(0,1,\frac{1}{x};y\right)G(0;x) - \frac{1}{2}G\left(0,1,-\frac{1}{x};y\right)G(0;x) + \frac{1}{2}G(0,1,-x;y)G(0;x) + \frac{1}{2}G(0,1,x;y)G(0;x)\\
&+ \frac{1}{2}G\left(0,\frac{1}{x},-1;y\right)G(0;x) + \frac{1}{2}G\left(0,\frac{1}{x},1;y\right)G(0;x) + \frac{1}{2}G\left(0,-\frac{1}{x},-1;y\right)G(0;x) + \frac{1}{2}G\left(0,-\frac{1}{x},1;y\right)G(0;x)\\
&- \frac{1}{2}G(0,-x,-1;y)G(0;x) - \frac{1}{2}G(0,-x,1;y)G(0;x) - \frac{1}{2}G(0,x,-1;y)G(0;x)\\
&- \frac{1}{2}G(0,x,1;y)G(0;x) + G(1,0,-1;x)G(0;x) - G(1,0,-1;y)G(0;x) + G(1,0,1;x)G(0;x)\\
&- G(1,0,1;y)G(0;x) + G(1,0,-x;y)G(0;x) + G(1,0,x;y)G(0;x) - \frac{1}{2}G\left(1,\frac{1}{x},-1;y\right)G(0;x)\\
&- \frac{1}{2}G\left(1,\frac{1}{x},1;y\right)G(0;x) - \frac{1}{2}G\left(1,-\frac{1}{x},-1;y\right)G(0;x)\\
&- \frac{1}{2}G\left(1,-\frac{1}{x},1;y\right)G(0;x) + \frac{1}{2}G(1,-x,-1;y)G(0;x) + \frac{1}{2}G(1,-x,1;y)G(0;x) + \frac{1}{2}G(1,x,-1;y)G(0;x)\\
&+ \frac{1}{2}G(1,x,1;y)G(0;x) + G(-1;x)G(-1,0,-1;y) + G(0;y)G(-1,0,-1;y) + G(1;x)G(-1,0,-1;y)\\
&+ G(-1;x)G(-1,0,1;y) + G(0;y)G(-1,0,1;y) + G(1;x)G(-1,0,1;y) - \frac{1}{2}G(-1;x)G\left(-1,0,\frac{1}{x};y\right)\\
&- \frac{1}{2}G(0;y)G\left(-1,0,\frac{1}{x};y\right) - \frac{1}{2}G(1;x)G\left(-1,0,\frac{1}{x};y\right)\\
&- \frac{1}{2}G(-1;x)G\left(-1,0,-\frac{1}{x};y\right) - \frac{1}{2}G(0;y)G\left(-1,0,-\frac{1}{x};y\right) - \frac{1}{2}G(1;x)G\left(-1,0,-\frac{1}{x};y\right)\\
&- \frac{1}{2}G(-1;x)G(-1,0,-x;y) - \frac{1}{2}G(0;y)G(-1,0,-x;y) - \frac{1}{2}G(1;x)G(-1,0,-x;y) - \frac{1}{2}G(-1;x)G(-1,0,x;y)\\
&- \frac{1}{2}G(0;y)G(-1,0,x;y)\\
&- \frac{1}{2}G(1;x)G(-1,0,x;y) + G(-1;y)G(0,0,-1;x) + 2G(0;y)G(0,0,-1;x) + G(1;y)G(0,0,-1;x)\\
&- G(-1;x)G(0,0,-1;y) - G(0;y)G(0,0,-1;y) - G(1;x)G(0,0,-1;y) + G(-1;y)G(0,0,1;x)\\
&+ 2G(0;y)G(0,0,1;x) + G(1;y)G(0,0,1;x) - G(-1;x)G(0,0,1;y) - G(0;y)G(0,0,1;y)\\
&- G(1;x)G(0,0,1;y) + \frac{1}{2}G(-1;x)G\left(0,0,\frac{1}{x};y\right) + \frac{1}{2}G(0;y)G\left(0,0,\frac{1}{x};y\right) + \frac{1}{2}G(1;x)G\left(0,0,\frac{1}{x};y\right)\\
&+ \frac{1}{2}G(-1;x)G\left(0,0,-\frac{1}{x};y\right) + \frac{1}{2}G(0;y)G\left(0,0,-\frac{1}{x};y\right) + \frac{1}{2}G(1;x)G\left(0,0,-\frac{1}{x};y\right)\\
&+ \frac{1}{2}G(-1;x)G(0,0,-x;y) + \frac{1}{2}G(0;y)G(0,0,-x;y) + \frac{1}{2}G(1;x)G(0,0,-x;y) + \frac{1}{2}G(-1;x)G(0,0,x;y)\\
&+ \frac{1}{2}G(0;y)G(0,0,x;y) + \frac{1}{2}G(1;x)G(0,0,x;y) + G(-1;x)G(1,0,-1;y) + G(0;y)G(1,0,-1;y)\\
&+ G(1;x)G(1,0,-1;y) + G(-1;x)G(1,0,1;y)\\
&+ G(0;y)G(1,0,1;y) + G(1;x)G(1,0,1;y) - \frac{1}{2}G(-1;x)G\left(1,0,\frac{1}{x};y\right) - \frac{1}{2}G(0;y)G\left(1,0,\frac{1}{x};y\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}G(1;x)G\left(1,0,\frac{1}{x};y\right)-\frac{1}{2}G(-1;x)G\left(1,0,-\frac{1}{x};y\right)\\
&-\frac{1}{2}G(0;y)G\left(1,0,-\frac{1}{x};y\right)-\frac{1}{2}G(1;x)G\left(1,0,-\frac{1}{x};y\right)-\frac{1}{2}G(-1;x)G(1,0,-x;y)-\frac{1}{2}G(0;y)G(1,0,-x;y)\\
&-\frac{1}{2}G(1;x)G(1,0,-x;y)-\frac{1}{2}G(-1;x)G(1,0,x;y)-\frac{1}{2}G(0;y)G(1,0,x;y)-\frac{1}{2}G(1;x)G(1,0,x;y)\\
&-G(-1,0,-1,-1;y)-G(-1,0,-1,1;y)-2G(-1,0,0,-1;y)-2G(-1,0,0,1;y)+G\left(-1,0,0,\frac{1}{x};y\right)\\
&+G\left(-1,0,0,-\frac{1}{x};y\right)+G(-1,0,0,-x;y)+G(-1,0,0,x;y)-G(-1,0,1,-1;y)-G(-1,0,1,1;y)+\frac{1}{2}G\left(-1,0,\frac{1}{x},-1;y\right)\\
&+\frac{1}{2}G\left(-1,0,\frac{1}{x},1;y\right)+\frac{1}{2}G\left(-1,0,-\frac{1}{x},-1;y\right)+\frac{1}{2}G\left(-1,0,-\frac{1}{x},1;y\right)+\frac{1}{2}G(-1,0,-x,-1;y)\\
&+\frac{1}{2}G(-1,0,-x,1;y)+\frac{1}{2}G(-1,0,x,-1;y)+\frac{1}{2}G(-1,0,x,1;y)-G(0,-1,0,-1;x)\\
&-G(0,-1,0,-1;y)-G(0,-1,0,1;x)-G(0,-1,0,1;y)+\frac{1}{2}G\left(0,-1,0,\frac{1}{x};y\right)+\frac{1}{2}G\left(0,-1,0,-\frac{1}{x};y\right)\\
&+\frac{1}{2}G(0,-1,0,-x;y)+\frac{1}{2}G(0,-1,0,x;y)+G(0,0,-1,-1;y)+G(0,0,-1,1;y)+3G(0,0,0,-1;y)+3G(0,0,0,1;y)\\
&-\frac{3}{2}G\left(0,0,0,\frac{1}{x};y\right)-\frac{3}{2}G\left(0,0,0,-\frac{1}{x};y\right)-\frac{3}{2}G(0,0,0,-x;y)-\frac{3}{2}G(0,0,0,x;y)+G(0,0,1,-1;y)\\
&+G(0,0,1,1;y)-\frac{1}{2}G\left(0,0,\frac{1}{x},-1;y\right)-\frac{1}{2}G\left(0,0,\frac{1}{x},1;y\right)\\
&-\frac{1}{2}G\left(0,0,-\frac{1}{x},-1;y\right)-\frac{1}{2}G\left(0,0,-\frac{1}{x},1;y\right)-\frac{1}{2}G(0,0,-x,-1;y)-\frac{1}{2}G(0,0,-x,1;y)\\
&-\frac{1}{2}G(0,0,x,-1;y)-\frac{1}{2}G(0,0,x,1;y)-G(0,1,0,-1;x)-G(0,1,0,-1;y)-G(0,1,0,1;x)-G(0,1,0,1;y)\\
&+\frac{1}{2}G\left(0,1,0,\frac{1}{x};y\right)\\
&+\frac{1}{2}G\left(0,1,0,-\frac{1}{x};y\right)+\frac{1}{2}G(0,1,0,-x;y)+\frac{1}{2}G(0,1,0,x;y)-G(1,0,-1,-1;y)-G(1,0,-1,1;y)-2G(1,0,0,-1;y)\\
&-2G(1,0,0,1;y)+G\left(1,0,0,\frac{1}{x};y\right)+G\left(1,0,0,-\frac{1}{x};y\right)\\
&+G(1,0,0,-x;y)+G(1,0,0,x;y)-G(1,0,1,-1;y)-G(1,0,1,1;y)+\frac{1}{2}G\left(1,0,\frac{1}{x},-1;y\right)+\frac{1}{2}G\left(1,0,\frac{1}{x},1;y\right)\\
&+\frac{1}{2}G\left(1,0,-\frac{1}{x},-1;y\right)+\frac{1}{2}G\left(1,0,-\frac{1}{x},1;y\right)+\frac{1}{2}G(1,0,-x,-1;y)+\frac{1}{2}G(1,0,-x,1;y)\\
&+\frac{1}{2}G(1,0,x,-1;y)+\frac{1}{2}G(1,0,x,1;y)\Big)\\
&+\frac{1}{90}\left(-720\zeta(3)\left(-7x^4y^2+x^2\left(y^4+1\right)+5y^2\right)+120\pi^2\left(x^4\left(y^4+6y^2+1\right)-2x^2\left(y^2+1\right)^2+\left(y^2-1\right)^2\right)\right.\\
&\left.-\pi^4\left(x^4\left(y^2+1\right)^2+2x^2\left(y^4-22y^2+1\right)+\left(y^2+1\right)^2\right)\right)+\frac{1}{y^2\left(y^2-1\right)}\\
&\Big(-32\left(x^2-1\right)\left(x^2+1\right)y^2\left(y^2-1\right)\left(y^4-1\right)(G(0;x)G(-i;x)+G(0;x)G(i;x))\\
&+64\left(x^2-1\right)\left(x^2+1\right)y^4\left(y^2-1\right)\Big(-2G(-i;x)G(0;x)G(0;y)-2G(i;x)G(0;x)G(0;y)+G(-i;x)G(0;x)^2\\
&+G(i;x)G(0;x)^2-\frac{1}{6}\pi^2G(-i;x)-\frac{1}{6}\pi^2G(i;x)\Big)\\
&-\frac{4}{3}y^2\left(y^2-1\right)\left(6\left(x^4-1\right)\left(y^4-1\right)+\pi^2\left(x^4\left(y^2+1\right)^2+2x^2\left(y^4-6y^2+1\right)+\left(y^2+1\right)^2\right)\right)(G(-1;x)G(0;x)+G(1;x)G(0;x))\\
&+64x^2y^2(-1+y^2)(-1+y^4)(G[-1,x]G[0,x]^2+G[0,x]^2G[1,x])\\
&+8x^2\left(y^2-1\right)\left(y^4-1\right)\left(-2x^2y^2+y^4+1\right)(G(0;x)G(-1;y)+G(0;x)G(1;y))\\
&+8\left(y^2-1\right)\left(y^4-1\right)\left(x^4y^2-x^2\left(y^4+1\right)+y^2\right)(G(-1;x)G(-1;y)+G(1;x)G(-1;y)+G(-1;x)G(1;y)\\
&+G(1;x)G(1;y)+G(0;y)G(-1;y)+G(0;y)G(1;y))\\
&-4\left(y^2-1\right)\left(x^4y^2\left(y^4+8y^2-1\right)+8x^3\left(y^5+y^3\right)-x^2\left(y^8-1\right)-8x\left(y^5+y^3\right)+y^2\left(y^4-8y^2-1\right)\right)\left(G(-1;x)G\left(-\frac{1}{x};y\right)\right.\\
&\left.-G(0;x)\ G\left(-\frac{1}{x};y\right)+G(0;y)G\left(-\frac{1}{x};y\right)+G(1;x)G\left(-\frac{1}{x};y\right)\right)\\
&+16y^2\left(y^2-1\right)\left(x^4y^2+x^2\left(3y^4-1\right)-3y^2\right)\left(G(0;x)G(0;y)G\left(\frac{1}{x};y\right)+G(0;x)G(0;y)G\left(-\frac{1}{x};y\right)\right)\\
&-4\left(y^2-1\right)\Big(x^4y^2\left(y^4-8y^2-1\right)-8x^3\left(y^5+y^3\right)-x^2\left(y^8-1\right)+8x\left(y^5+y^3\right)\\
&+y^2\left(y^4+8y^2-1\right)\Big)(G(-1;x)G(-x;y)-G(0;x)G(-x;y)+G(0;y)G(-x;y)+G(1;x)G(-x;y))\\
&+16y^2\left(y^2-1\right)\left(-3x^4y^2+x^2\left(3y^4-1\right)+y^2\right)\Big(G(0;x)^2G(-x;y)+G(0;x)^2G(x;y)\\
&-G(0;x)G(0;y)G(-x;y)-G(0;x)G(0;y)G(x;y)\Big)\\
&-4\left(y^2-1\right)\Big(x^4y^2\left(y^4-8y^2-1\right)+8x^3\left(y^5+y^3\right)-x^2\left(y^8-1\right)-8x\left(y^5+y^3\right)\\
&+y^2\left(y^4+8y^2-1\right)\Big)(G(-1;x)G(x;y)-G(0;x)G(x;y)+G(0;y)G(x;y)+G(1;x)G(x;y))\\
&-4y^2\left(y^2-1\right)\Big(x^4\left(y^2+1\right)\left(y^2(\zeta(3)-4)+\zeta(3)+4\right)+2x^2\left(y^4(\zeta(3)+4)-6y^2\zeta(3)+\zeta(3)-4\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(y^2+1\right)\left(y^2(\zeta(3)-4)+\zeta(3)+4\right)\right)(G(-1;y)+G(1;y))\\
&\frac{1}{3}(-2)y^2\left(y^2-1\right)\left(24\left(x^2-1\right)^2\left(y^4-1\right)G(-1;x)-16y^2\left(-5x^4+4x^2y^2+1\right)G(0;x)^3\right.\\
&-4G(0;x)\left(\left(3x^4\left(y^4+8y^2-1\right)-8\pi^2x^2y^2-3\left(y^4+8y^2-1\right)\right)G(0;y)+2\left(3x^4\left(2y^4+\pi^2y^2-2\right)\right.\right.\\
&\left.\left.+x^2\left(-2\left(3+\pi^2\right)y^4+36y^2\zeta(3)+6\right)-\pi^2y^2\right)\right)\\
&+2\left(12\left(x^2-1\right)^2\left(y^4-1\right)G(1;x)-3\left(x^2-1\right)^2\left(y^2+1\right)^2G(0;y)^2+2\left(x^4\left(3\left(y^2+1\right)\left(y^2(\zeta(3)+2)+\zeta(3)-2\right)+2\pi^2y^2\right)\right.\right.\\
&\left.\left.+6x^2\left(y^4(\zeta(3)-2)+6y^2\zeta(3)+\zeta(3)+2\right)+3\left(y^2+1\right)\left(y^2(\zeta(3)+2)+\zeta(3)-2\right)-2\pi^2y^2\right)G(0;y)\right)\\
&+G(0;x)^2\left(24\left(-7x^4y^2+x^2\left(5y^4+1\right)+y^2\right)G(0;y)+\pi^2\left(x^4\left(y^2+1\right)^2+2x^2\left(y^4-6y^2+1\right)+\left(y^2+1\right)^2\right)\right.\\
&\left.\left.\left.-6\left(x^4\left(9y^4-8y^2-9\right)+2x^2\left(y^4-1\right)+y^4+8y^2-1\right)\right)\right)\right) \qquad \text{(C.1)}
\end{aligned}
$$

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Ringraziamenti

Ringrazio il mio relatore, il prof Francesco Tramontano per la disponibilità dimostrata nei miei confronti e la fiducia accordatami. Ringrazio Amedeo Primo per avermi aiutato e guidato nell'applicazione di questi nuovi metodi. Ringrazio il dottore Gabor Somogyi per il supporto ricevuto. Infine ringrazio la mia correlatrice la prof Giulia Ricciardi.

Ringrazio i miei genitori per avermi sempre sostenuto, sia nelle scelte che nei momenti di maggiore dubbio. Ringrazio Fabio per tutte le mattine insieme in macchina e per aver tollerato i miei ritardi, scusa.

Il mio percorso di studi nulla sarebbe stato senza gli amici che lo hanno accompagnato: ringrazio i miei compagni di corso e gli amici della π, con voi anche i momenti di maggiore stress sono stati corredati da leggerezza e spensieratezza. Ringrazio Antonio Forente per aver scalfito la mia barriera ed essersi sempre mostrato un amico. Ringrazio i membri del gruppo Dumb-Mines, Raffaele Campanile e Federico Luongo, amici e soci di progetti di ampie vedute.

Un ringraziamento speciale va a te Mery di Luca. Grazie di esserci, grazie per avermi supportato e sopportato sia nei momenti di gioia che ti tensione, grazie per essere stata il mio Editor in Chief, per aver letto e corretto la mie sintesi eccessive ed essere stata un riferimento per ogni scelta. Grazie di tutto e per tutto. Questo lavoro non sarebbe stato lo stesso senza di te e forse senza di te non sarebbe mai nato.