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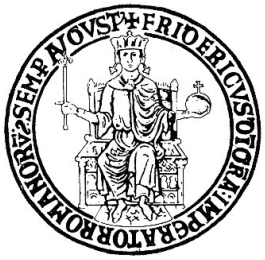

Scuola Politecnica e delle Scienze di Base

Area Didattica di Scienze Matematiche Fisiche e Naturali

Dipartimento di Fisica "Ettore Pancini"

Laurea Triennale in Fisica

Quantum mechanics of two level systems

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Introduction

The application of quantum mechanical theory to the analysis of physical systems with a two dimensional state space is truly fascinating. The topic's interest is not limited to its mathematical simplicity but is mainly based on the physical importance of its applications.

Unfortunately none of the considered physical systems can be fully analyzed with this model and it imposes to make an approximation in all the examined settings. Therefore, defining the experimental conditions in which the approximation can be done is of fundamental importance and we will try on each occasion to characterize the validity range of the theory.

Herein will be presented some of the many system which are suitable to be studied by the two level system model.

First we will analyze the most important example: the spin $\frac{1}{2}$ system. We will review the experimental evidence which led to the introduction of intrinsic degrees of freedom, justifying the two level system approximation in that scenario. Then, we will take into account the interaction with a magnetic field (first static then time-dependent); finally we will see how the temporal evolution has a simple geometrical interpretation in a new abstract space.

The complete knowledge of spin $\frac{1}{2}$ systems is crucial in the analysis of the most general of the two level systems. We shall see that the original system can be replaced introducing a fictitious spin $\frac{1}{2}$ interacting with a fictitious magnetic field, whose direction and modulus depends on the physical framework, so that every two level system has an immediate and simple interpretation.

The application of the two level system theory to concrete physical situation will allow a global understanding in an easy and elegant way.

First, we will study the ammonia molecule, introducing the maser function; then we will give a qualitative explanation to the neutrino oscillation phenomenon.

In particular the neutrino oscillation has to be studied with the quantum density matrix formalism. We will introduce it always looking for a geometrical approach as intuitive as the pure state case.

Chapter 1

Spin 1/2 systems

1.1 The Stern Gerlach experiment

The Stern Gerlach experiment is the first experimental evidence of the existence of spin degrees of freedom. The experiment consists of studying the deflection of a beam of neutral paramagnetic atoms in a highly inhomogeneous magnetic field. The interaction of an atomic system with a magnetic field is governed by the total angular momenta $\vec{J}$ arising both from the orbital and the spin angular momenta. When $J \neq 0$ the atom posseses a permanent magnetic moment

$$< \vec{\mu} >= -g\mu_B < \frac{\vec{J}}{\hbar} > \tag{1.1}$$

so that the paramagnetism is observed when the magnetic moments are partially aligned by the field. Silver atoms are contained in a furnace which is heated to high temperature, they leave through a small opening an are selected by a collimating slit which selects those atoms whose velocity is parallel to a particular direction (Oy in this case).

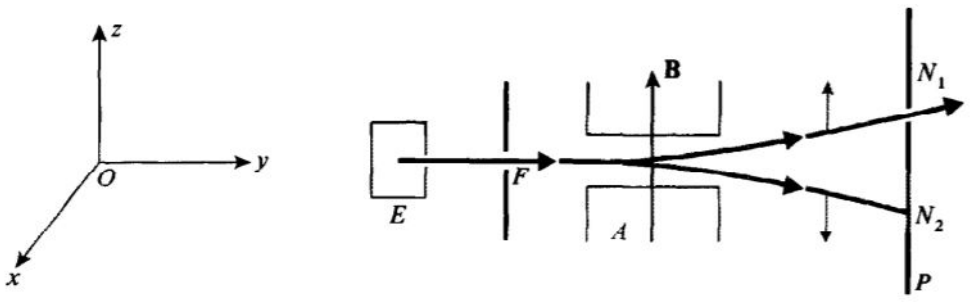

Figure 1.1: Schematic drawing of the Stern Gerlach experiment.

The atomic beam then enter the region with the electromagnetic field which has its largest component along Oz which is strongly variable. Since the silver atoms are neutral they are not subject to Lorentz force, so neglecting the terms proportional to μ_x and μ_y the force can be calculated by:

$$\vec{F} = \boldsymbol{\nabla}\mu_z B_z. \tag{1.2}$$

Since $\boldsymbol{\nabla}B_z$ has only z-component the force on the atom is parallel to Oz and proportional to μ.

The results of the experiment are outstanding since we do observe two spots symmetrical with respect to the center. This brought to the introduction of a new intrinsic angular momentum whose discrete spectrum includes only two eigenvalues.

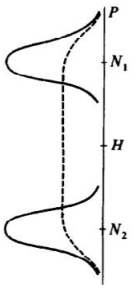

Figure 1.2: Spot observed in the Stern Gerlach screen. According to classical mechanics one should observe one spot (dashed line), in reality two symmetrical spots centered at N_1 and N_2 are observed.

We now know that silver atoms are in their ground state spin $\frac{1}{2}$ system so that such a result can be achieved only if the system is described totally by its two dimensional spin Hilbert space and the external degrees of freedom shall not be studied quantum mechanically. This is obviously an approximation but all along this work we want to study physical systems with a two dimensional Hilbert space. This process will be always an approximation which we will call "Two level system approximation".

We want to apply the approximation in the present case. We shall verify that it is possible, in order to describe the motion of silver atoms, to construct wave packets whose dispersion Δp and Δz are completely negligible. The dispersion must satisfy Heisenberg relation and we can see numerically that:

$$\Delta z \Delta v_z \geq 10^{-9} \quad \frac{m}{s^2} \tag{1.3}$$

Considering the lengths and velocities involved in the problem:

- width of the slit $F \sim 0.1$ mm
- separation N_1N_2 several millimeters
- $\frac{B}{\frac{\partial B}{\partial z}} \sim 1$ mm
- velocity of the silver atoms leaving a furnace at 1000 K$\sim 500\ \frac{m}{s}$

we can see that while satisfying (1.3) they are negligible on the scale concerned, enabling us to reason in terms of quasi-pointlike wave packet moving along classical trajectories.

1.2 Formal theory of spin

The formal theory of spin is a carbon-copy of the orbital angular momenta one, we will follow [1]. Beginning with the fundamental commutation relation:

$$[S_x, S_y] = i\hbar\epsilon_{xyz}S_z \tag{1.4}$$

It follows that the eigenvectors of S^2 and S_z satisfy

$$S^2 |s,m\rangle = s(s+1) |s,m\rangle \qquad S_z |s,m\rangle = \hbar |s,m\rangle \tag{1.5}$$

and

$$S_\pm |s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\, |s, m\pm 1\rangle \tag{1.6}$$

where $S_\pm = S_x \pm iS_y$.

However this time the eigenvector are not spherical harmonics(they are not functions of the polar angles at all). There is no a priori reason to exclude half integer values of s and m.

It so happens that every elementary particle has a specific and immutable value of s.

The silver beam used by Stern and Gerlach is as spin $\frac{1}{2}$ system which is by far the most important case.

As we expect from the experiment there are just two eigenstates

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle \equiv |\uparrow\rangle \qquad \left|\frac{1}{2}, \frac{-1}{2}\right\rangle \equiv |\downarrow\rangle \tag{1.7}$$

An element of the Hilbert space $\mathcal{H}_s$ spanned by $(|\uparrow\rangle, |\downarrow\rangle)$ is called a spinor and can be expressed as a two-element column vector. Meanwhile any operator becomes a 2x2 matrix, allowing us to work out any useful calculation in a simple way.

Using relations (1.5) (1.6) we can find in particular the expression of the spin operators:

$$S_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \tag{1.8}$$

$$S_y = \begin{pmatrix} 0 & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & 0 \end{pmatrix} \tag{1.9}$$

$$S_x = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} \tag{1.10}$$

They are obviously all hermitian and can be written in the more compact form:

$$S_i = \frac{\hbar}{2}\sigma_i \tag{1.11}$$

where the σ_i are the "Pauli matrices".

The observable corresponding to $\vec{S}\cdot\hat{u}$ where $\hat{u}$ is a direction identified by (θ,ϕ) in the $(|\uparrow\rangle, |\downarrow\rangle)$ base is directly obtained by matrix multiplication

$$S_u = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z\cos\phi \tag{1.12}$$

and its eigenvectors can be found to be:

$$|\uparrow\rangle_u = \cos\frac{\theta}{2}\exp\left\{-i\frac{\phi}{2}\right\}|\uparrow\rangle + \sin\frac{\theta}{2}\exp\left\{i\frac{\phi}{2}\right\}|\downarrow\rangle \tag{1.13}$$

$$|\downarrow\rangle_u = -\sin\frac{\theta}{2}\exp\left\{-i\frac{\phi}{2}\right\}|\uparrow\rangle + \cos\frac{\theta}{2}\exp\left\{i\frac{\phi}{2}\right\}|\downarrow\rangle \tag{1.14}$$

In a two dimensional complex Hilbert spaces there should be four real degrees of freedom, but one is removed by the normalization bound and the other from the overall phase invariance of quantum mechanics. We can identify any spinor by the angles (θ,ϕ) which preserve the characteristic of being polar angles in a new fictitious space.

We see that a spinor has a geometrical representation given by a vector directed along (θ,ϕ) called Bloch vector pointing the surface of an unitary sphere called Bloch sphere.

It is important to note that orthogonal states in $\mathcal{H}$ are represented by antipodal points on the Bloch sphere, but obviously in our Hilbert space orthogonal vectors have an angular separation of $\frac{\pi}{2}$. Hence the division by two in (1.13) and (1.14).

Preparation of spin state along any direction can always be made by clever manipulation of Stern Gerlach apparati; in fact if we take that the most general spin state $\psi \in \mathcal{H}_S$ we can write:

$$|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle \tag{1.15}$$

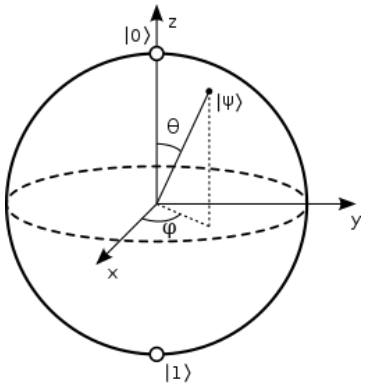

Figure 1.3: Geometrical representation on the Bloch sphere.

Since the state vector is normalized, exists θ such that

$$|c_1| = \cos\frac{\theta}{2} \tag{1.16}$$

$$|c_2| = \sin\frac{\theta}{2} \tag{1.17}$$

Imposing

$$\tan\frac{\theta}{2} = \frac{c_1}{c_2} \tag{1.18}$$

$$\phi = Arg(c_2) - Arg(c_1) \tag{1.19}$$

$$\chi = Arg(c_2) + Arg(c_1) \tag{1.20}$$

$$\tag{1.21}$$

the state can be written as :

$$|\psi\rangle = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}|\downarrow\rangle = |\uparrow\rangle_u \tag{1.22}$$

with (θ, ϕ) defined above, so that to prepare a system in state $|\psi\rangle$ it suffices to place the SG apparatus so that its axis point along $\vec{u}$.

Once prepared the state we have to follow the postulate of quantum mechanics to predict the results of spin measurement. Often experiments uses two SG magnets one after the other in quite a similar way to polarizer and analyzer of optical polarization experiment.

For the intrinsic probabilistic nature we will argument only of probability that a certain outcome will occur repeating the same experiment N times, but if we calculate the mean values of spin operator we will find the components of a classical

angular momentum of modulus $\frac{\hbar}{2}$ oriented along the vector $\vec{u}$:

$$\begin{cases} \langle\uparrow_u|S_z|\uparrow_u\rangle &= \frac{\hbar}{2}\cos\theta \\ \langle\uparrow_u|S_y|\uparrow_u\rangle &= \frac{\hbar}{2}\sin\theta\sin\phi \\ \langle\uparrow_u|S_x|\uparrow_u\rangle &= \frac{\hbar}{2}\sin\theta\cos\phi \end{cases} \tag{1.23}$$

confirming that quantum mechanic is mapped onto classical mechanics for high quantum numbers.

1.3 Interaction with magnetic field

1.3.1 Static Magnetic field

Let the spin system be in an uniform magnetic field $\vec{B_0}$ along $\hat{k}$. The hamiltonian is given by:

$$H = -\vec{\mu}\vec{B_0} = g\mu_B B_0 \frac{\sigma_z}{2} = -\gamma B_0 S_z \tag{1.24}$$

where γ is the gyromagnetic ratio.

It's easy to solve the stationary Schroedinger equation. Setting $\omega_0 = -\gamma B_0$ we find the two energy levels

$$E_\pm = \pm\frac{1}{2}\hbar\omega_0 \tag{1.25}$$

defining a single Bohr frequency, while the eigenvectors are those of S_z.

If we assume that at $t = 0$ the state vector is $|\psi(0)\rangle = |\uparrow\rangle_u$ where $\hat{u} = \hat{u}(\theta, \phi)$, we can calculate

$$|\psi(t)\rangle = \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}e^{iE_+t}|\uparrow\rangle + \sin\frac{\theta}{2}e^{+i\frac{\phi}{2}}e^{-iE_-t}|\downarrow\rangle \tag{1.26}$$

We see on the Bloch sphere that the spin revolves around $0z$ with angular velocity ω_0 proportional to the magnetic field

$$\theta(t) = \theta \tag{1.27}$$

$$\phi(t) = \phi + \omega_0 t \tag{1.28}$$

which is the quantum mechanical Larmor precession.

Obviously the mean value will behave like the components of a classical angular momentum of modulus $\frac{\hbar}{2}$ undergoing classical Larmor precession.

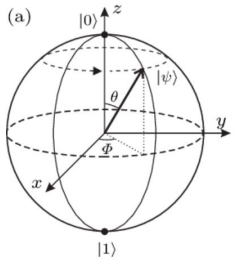

Figure 1.4: Quantum mechanical Larmor precession on the Bloch sphere.

1.3.2 Time dependent magnetic field

Suppose now to turn on a weak oscillating magnetic field along the $\hat{x}$ axis. The Hamiltonian is

$$H = -\vec{\mu} \cdot \vec{B} = \frac{g\mu_B}{\hbar}(S_x B_x \cos\omega t + S_z B_z) \tag{1.29}$$

The wave function is a two-component spinor

$$\psi(t) = \begin{pmatrix} c_\uparrow(t) \\ c_\downarrow(t) \end{pmatrix} \tag{1.30}$$

When the perturbation B_x is absent the unperturbed eigenfunctions are:

$$\psi_{\frac{1}{2}} = \begin{pmatrix} e^{-i\frac{\omega_0 t}{2}} \\ 0 \end{pmatrix} \qquad \psi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ e^{\frac{i\omega_0 t}{2}} \end{pmatrix} \tag{1.31}$$

H is now time dependent and under certain conditions the problem can be solved by first order perturbation theory but we shall be interested in large values of t, so that we proceed by using the explicit forms of the Pauli matrices to write Schroedinger equation as a pair of coupled differential equations for $c_{\uparrow\downarrow}(t)$ as presented in [2].

$$i\frac{\mathrm{d}c_\uparrow(t)}{\mathrm{d}t} = \frac{\omega_0}{2}c_\uparrow(t) + \frac{\tilde{\omega_0}}{2}\cos(\omega t)c_\downarrow(t) \tag{1.32}$$

$$i\frac{\mathrm{d}c_\downarrow(t)}{\mathrm{d}t} = \frac{\tilde{\omega_0}}{2}\cos(\omega t)c_\uparrow(t) - \frac{\omega_0 t}{2}c_\downarrow(t) \tag{1.33}$$

where:

$$\tilde{\omega_0} = g\mu_B\frac{B_x}{\hbar} = -\gamma B_x \tag{1.34}$$

It is convenient to define new functions by setting

$$C_+(t) = e^{\frac{i\omega_0 t}{2}} c_\uparrow(t) \tag{1.35}$$

$$C_-(t) = e^{\frac{-i\omega_0 t}{2}} c_\downarrow(t) \tag{1.36}$$

this removes the term in ω_0 giving

$$i\frac{dC_+(t)}{dt} = \frac{\tilde{\omega_0}}{2}\cos\omega t e^{i\omega_0 t} C_- \tag{1.37}$$

$$i\frac{dC_-(t)}{dt} = \frac{\tilde{\omega_0}}{2}\cos\omega t e^{-i\omega_0 - t} C_+ \tag{1.38}$$

By recognising in the product $\cos(\omega t)e^{\pm\omega_0 t}$ terms in $e^{\pm(\omega\pm\omega_0)t}$ we can neglect $e^{\pm i(\omega+\omega_0)t}$ because they oscillate extremely rapidly and on the average they make little contribution to the equation.

Dropping these terms the approximate Schroedinger equation is

$$i\frac{dC_+(t)}{dt} = \frac{\tilde{\omega_0}}{4} e^{i(\omega-\omega_0)t} C_- \tag{1.39}$$

$$i\frac{dC_-(t)}{dt} = \frac{\tilde{\omega_0}}{4} e^{-i(\omega_0-\omega)t} C_+ \tag{1.40}$$

It is possible now to solve exactly, the general solution is

$$C_+(t) = pe^{i\eta_+ t} + qe^{i\eta_- t} \tag{1.41}$$

$$C_-(t) = -\frac{4}{\tilde{\omega_0}}[p\eta_+ e^{i\eta_+ t} + q\eta_- e^{i\eta_- t}]e^{i(\omega-\omega_0)t} \tag{1.42}$$

where p and q are costant of integration and

$$\eta_\pm = \frac{1}{2}[(\omega_0 - \omega) \pm \sqrt{(\omega-\omega_0)^2 + \frac{\tilde{\omega_0}^2}{4}}] \tag{1.43}$$

Preparing the initial state such that $C_+(0) = 1$ and $C_-(0) = 0$ we can calculate the probability of transition from upper to lower level obtaining:

$$P(+\to+) = \cos^2\frac{\omega_R t}{2} + [\frac{(\omega-\omega_0)^2}{\omega_R^2}] sin^2 \frac{\omega_R t}{2} \tag{1.44}$$

$$P(+\to-) = \frac{\frac{\tilde{\omega_0}^2}{4}}{(\omega_R)^2}[\sin^2\frac{\omega_R t}{2}] \tag{1.45}$$

where ω_R is known as the "Rabi flopping frequency" and is

$$\omega_R = \eta_+ - \eta_- = \sqrt{(\omega-\omega_0)^2 + \frac{\tilde{\omega_0}^2}{4}} \tag{1.46}$$

This process is called spin resonance (SR).

Chapter 2

Two level systems

2.1 Introduction

In this section we now consider a general system whose state space is two dimensional, following [3]. We have discussed before that this model can be used whenever the TLS approximation is valid. As we will se this is often the case when we have two states whose energies are close together and very different from those of all other states of the system. We want to evaluate the effect of an external perturbation(or an internal interactions previously neglected) and if the approximation of two level system is good we can ignore all the other energy levels staying in our two dimensional subspace.

2.2 General treatment of Two level system

Suppose that $H = H_0 + W$ where H_0 is the unperturbed hamiltonian and W is the perturbation or coupling.

For a basis, we choose the orthonormal basis $(|1\rangle, |2\rangle)$ eigenkets of H_0 with eigenvalues (E_1, E_2).

We assume that W is time independent in this space, and in the $(|1\rangle, |2\rangle)$ base is represented by the hermitian matrix

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \tag{2.1}$$

Let the eigenstates of H be $(|+\rangle, |-\rangle)$ with eigenvalues (E_+, E_-).

The consequences of the coupling are clear:

- E_1 and E_2 are no longer possible energies of the system and a measurement of energy can yield $E_\pm$.

- $|1\rangle$ and $|2\rangle$ are no longer stationary states of the system .

The diagonalization of the matrix presents no problems since is done in a 2-D space. We find eigenvalues:

$$E_{\pm} = \frac{E_1 + W_{11} + E_2 + W_{22}}{2} \pm \sqrt{(E_1 + W_{11} - E_2 - W_{22})^2 + 4|W_{12}|^2} \tag{2.2}$$

and eigenvectors:

$$|+\rangle = \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} |1\rangle + \sin\frac{\theta}{2} e^{\frac{i\phi}{2}} |2\rangle \tag{2.3}$$

$$|-\rangle = -\sin\frac{\theta}{2} e^{\frac{-i\phi}{2}} |1\rangle + \cos\frac{\theta}{2} e^{\frac{i\phi}{2}} |2\rangle \tag{2.4}$$

where (θ, ϕ) are defined by:

$$\tan\theta = \frac{2|W_{12}|}{E_1 + W_{11} - E_2 - W_{22}} \tag{2.5}$$

$$W_{21} = W_{12} e^{i\phi} \tag{2.6}$$

Obviously interesting effects arise from the fact that W is non diagonal so that we can set $W_{11} = W_{22} = 0$. In order to simplify the notation we introduce the parameters:

$$E_m = \frac{E_1 + E_2}{2} \qquad \Delta = \frac{E_1 - E_2}{2} \tag{2.7}$$

so that the eigenvalues reduces to:

$$E_{\pm} = E_m \pm \sqrt{\Delta^2 + |W_{12}|^2} \tag{2.8}$$

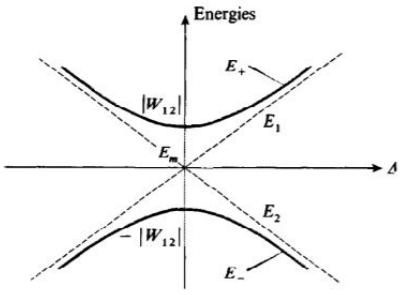

Figure 2.1: Anticrossing diagram.

From the diagram we can grasp fundamental physical information. In the absence of coupling the two levels "cross", but when the perturbation is present the two levels "repels each other" since for any Δ we have

$$|E_+ - E_-| > |E_1 - E_2| \tag{2.9}$$

Moreover the effects of coupling are much more important when the two unperturbed levels have the same energy as can be seen expanding in $|\frac{W_{12}}{\Delta}|$ near the asymptotes

$$E_\pm = E_m \pm \Delta(1 + \frac{|\frac{W_{12}}{\Delta}|}{2} + \dots) \tag{2.10}$$

which is of the second order while for small separation of Δ expanding in $|\frac{\Delta}{W_{12}}|$

$$E_\pm = E_m \pm |W_{12}|(1 + \frac{|\frac{\Delta}{W_{12}}|}{2}) \tag{2.11}$$

which is of the first order in the perturbation.

The effects of the coupling on the eigenstates can be calculated recalling that:

$$\tan\theta = |\frac{W_{12}}{\Delta}| \tag{2.12}$$

It follows that for strong coupling ($\Delta << W_{12}$):

$$|+\rangle = \frac{1}{\sqrt{2}}[e^{-i\frac{\phi}{2}}|1\rangle + e^{i\frac{\phi}{2}}|2\rangle] \tag{2.13}$$

$$|-\rangle = \frac{1}{\sqrt{2}}[-e^{-i\frac{\phi}{2}}|1\rangle + e^{i\frac{\phi}{2}}|2\rangle] \tag{2.14}$$

while for weak coupling ($\Delta >> W_{12}$) we have

$$|+\rangle = \exp\left\{-i\frac{\phi}{2}\right\}[|1\rangle + \exp\{-i\phi\}|\frac{W_{12}}{2\Delta}|\,|2\rangle + \dots] \tag{2.15}$$

$$|-\rangle = \exp\left\{i\frac{\phi}{2}\right\}[|2\rangle - \exp\{-i\phi\}\frac{W_{12}}{2\Delta}|1\rangle] \tag{2.16}$$

It is clear that the same pattern has emerged in fact for weak coupling apart from first order term the eigenstates remains unchanged while for strong coupling the new eigenstates are linear superposition with coefficients of the same modulus.

The time evolution of a state vector is then straight-forward, in fact the state vector will folllow:

$$i\hbar\frac{\mathrm{d}\,|\psi(t)\rangle}{\mathrm{d}t} = (H_0 + W)\,|\psi(t)\rangle \tag{2.17}$$

Let the state vector be:

$$|\psi(t)\rangle = c_1(t)\,|1\rangle + c_2(t)\,|2\rangle \tag{2.18}$$

If we project onto the $(|1\rangle, |2\rangle)$ basis we obtain:

$$i\hbar\frac{\mathrm{d}c_1(t)}{\mathrm{d}t} = E_1c_1(t) + W_{12}c_2(t) \tag{2.19}$$

$$i\hbar\frac{\mathrm{d}c_2(t)}{\mathrm{d}t} = W_{12}c_1(t) + E_2c_2(t) \tag{2.20}$$

This constitutes a system a linear system of homogeneous coupled differential equations, so we have to move into $(|+\rangle, |-\rangle)$ basis to analyze time evolution.

If the system at $t = 0$ is in the state

$$|\psi(0)\rangle = \lambda\,|+\rangle + \mu\,|-\rangle \tag{2.21}$$

we then have

$$|\psi(t)\rangle = \lambda e^{i\omega_+ t}\,|+\rangle + \mu e^{i\omega_- t}\,|-\rangle \tag{2.22}$$

This means that the ket $|\psi(t)\rangle$ oscillates between the two unperturbed states, to see this we can choose

$$|\psi(0)\rangle \equiv |1\rangle = e^{i\frac{\phi}{2}}[\cos\frac{\theta}{2}\,|+\rangle - \sin\frac{\theta}{2}\,|-\rangle] \tag{2.23}$$

from which we deduce:

$$|\psi(t)\rangle = e^{i\frac{\phi}{2}}[\cos\frac{\theta}{2}e^{i\omega_+ t}\,|+\rangle - \sin\frac{\theta}{2}e^{i\omega_- t}\,|-\rangle] \tag{2.24}$$

The probability of finding the state at time t in the state $|2\rangle$ is then written:

$$P_{12}(t) = |\,\langle 1|2\rangle\,|^2 = \sin^2\theta\sin^2[\frac{E_+ - E_-}{2\hbar}t] \tag{2.25}$$

which can be expressed as

$$P_{12}(t) = \frac{4|W_{12}^2|}{4|W_{12}^2| + (E_1 - E_2)^2)}\sin^2[\sqrt{4|W_{12}^2| + (E_1 - E_2)^2}\frac{t}{2\hbar}] \tag{2.26}$$

Which is sometimes called Rabi's formula.

We observe that $P_{12}(t)$ oscillates at the Bohr frequency of the system and varies between 0 and $\sin^2\theta$, in function of the parameters $|W_{12}|$ and $E_1 - E_2$.

In the strong coupling case where $\Delta << |W_{12}|$, we know that $\sin^2\theta \sim 1$ and the system oscillates completely from one state to the other with frequency proportional to the coupling.

While in the weak coupling case it's not surprising that $\sin^2\theta$ becomes very small and the systems evolves very little over time.

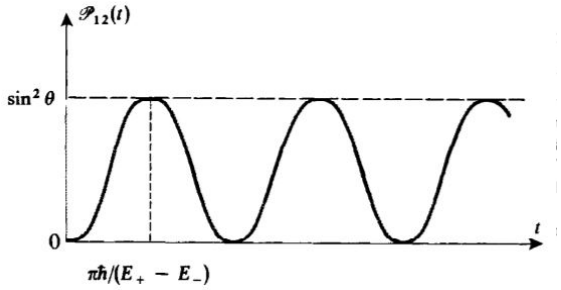

Figure 2.2: Rabi oscillation between unperturbed states.

2.3 Equivalence with spin 1/2 system

Consider a two-level system whose Hamiltonian is represented in $(|1\rangle, |2\rangle)$ basis as

$$H = \begin{pmatrix} \frac{H_{11}-H_{22}}{2} & H_{12} \\ H_{21} & -\frac{H_{11}-H_{22}}{2} \end{pmatrix} \tag{2.27}$$

where we choose $\frac{H_{11}+H_{22}}{2}$ as the new energy origin.

Although the two-level system under consideration is not spin $1/2$, we can always associate with it a fictitious spin $.1/2$ whose Hamiltonian is represented by the same matrix in the $(|\uparrow\rangle, |\downarrow\rangle)$ basis.

We shall see that H can be interpreted as describing the interaction of this fictious spin with a static fictitious magnetic field $\vec{B}$ whose direction and modulus are simply related to the parameters in H.

The Hamiltonian of the coupling between a spin $1/2$ and a magnetic field $\vec{B}$ can be written as:

$$\tilde{H} = -\gamma \vec{B} \cdot \vec{S} = -\gamma \frac{\hbar}{2}(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z) \tag{2.28}$$

This is equivalent to

$$\tilde{H} = -\gamma \frac{\hbar}{2} \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} \tag{2.29}$$

Therefore to make H equvalent to $\tilde{H}$ we must simply choose a fictitious field defined by

$$B_x = -\frac{2}{\gamma\hbar} ReH_{12} \tag{2.30}$$

$$B_y = \frac{2}{\gamma\hbar} ImH_{12} \tag{2.31}$$

$$B_z = \frac{1}{\gamma\hbar}(H_{22} - H_{11}) \tag{2.32}$$

According to formulas (2.5) and (2.6), the angles (θ, ϕ) associated to $H = \tilde{H}$ are

$$\tan\theta = \frac{|\gamma B_\perp|}{-\gamma B_z} \qquad 0 \leq \theta \leq \pi \tag{2.33}$$

$$-\gamma(B_x + iB_y) = |\gamma B_\perp| e^{i\phi} \qquad 0 \leq \phi \leq 2\pi \tag{2.34}$$

γ is a simple parameter in the model and can have arbitrary value. Choosing γ negative, relation shows that the angles (θ, ϕ) associated to the matrix H are simply the polar angles of the direction of the fictitious field $\vec{B}$ on the Bloch sphere (if we had chose γ positive they would be the opposite of the direction) where now the poles are the unperturbed states $|1\rangle$ and $|2\rangle$.

The geometrical formalism remains the same if we forget the two-level system from which we started and consider only the matrix H as representing the hamiltonian of a spin 1/2 in the $(|\uparrow\rangle, |\downarrow\rangle)$ basis, so that:

$$H = \omega S_u \tag{2.35}$$

where $S_u = \vec{S} \cdot \hat{u}$ is the operator associated to spin component along the direction $\hat{u}$ defined by (θ, ϕ) and ω is the Larmor angular velocity $\omega = |\gamma||\vec{B}|$. We can follow dynamical evolution on the Bloch sphere, easily interpreting Rabi oscillation where θ and ϕ introduced in (2.33) and (2.34) are the polar angles of the perturbed "Hamiltonian vector"(i.e fictitious field) about which the state will precess.

Chapter 3

Ammonia molecule

3.1 Introduction

In this chapter we shall see how the two level system model is suitable for studying particular configuration of the ammonia molecule which will led us to introduce the first electromagnetic amplifier: the ammonia maser.

In the ammonia molecule NH_3 the three hydrogen atoms form the base of a pyramid whose apex is the nitrogen atom. We study the molecule using a simplified model

- The nitrogen atom,much heavier, is motionless
- The potential energy of the system is a function of only one parameter, the distance x between the nitrogen and the plane defined by the hydrogen atoms.

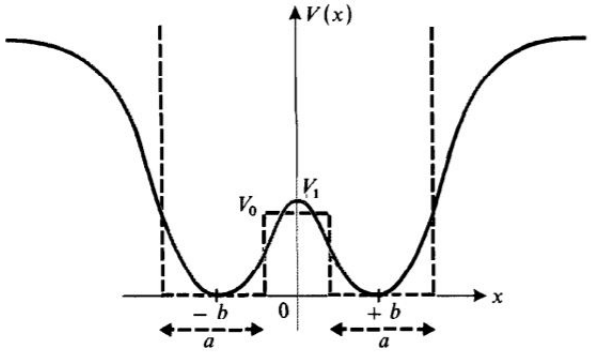

Figure 3.1: Potential energy for the molecule. Square well potential to approximate V(x) is shown in dashed lines.

The symmetry of the problem requires $V(x)$ to be even, with two minima corresponding to two symmetrical configurations in which classically is stable where we choose energy origin such that these energy are zero.

This model reduces the problem to a motion of a single fictitious particle of reduced mass$\mu = \frac{3m_H m_N}{3m_H+m_N}$ under the potential $V(x)$.

3.2 Infinite barrier approximation

Let's oversimplify the model assuming that the potential barrier on the hydrogen plane is infinite.

No tunnel effect is possible and we have in each region a simple infinite barrier problem. The possible energies are

$$E_n = \frac{\hbar^2 k_n^2}{2m} \tag{3.1}$$

with

$$k_n = \frac{n\pi}{a} \tag{3.2}$$

Each of the energy value is twofold degenerate since two wave functions correspond to it:

$$\psi_1^n(x) = \begin{cases} \frac{\sqrt{2}}{a}[\sin k_n(b+\frac{a}{2}-x)] & \text{if} \quad b-\frac{a}{2} \le x \le b+\frac{a}{2} \\ 0 \quad \text{everywhere else} \end{cases} \tag{3.3}$$

$$\psi_2^n(x) = \begin{cases} \frac{\sqrt{2}}{a}[\sin k_n(b+\frac{a}{2}+x)] & \text{if} \quad b-\frac{a}{2} \le -x \le b+\frac{a}{2} \\ 0 \quad \text{everywhere else} \end{cases} \tag{3.4}$$

The Bohr frequency associated with the first two levels correspond the to-and-fro motion of the particle between the two sides of a definite well and physically such an oscillation represents molecular vibration of the plane of the three hydrogen about its stable equilibrium position ($x = \pm b$) whose frequency of oscillation falls in the infrared.

Since $V(x)$ is even the Hamiltonian commutes with the parity operator Π.

The common basis of eigenvectors is symmetrical and antisymmetrical combinations of the ψ_i^n having:

$$\psi_s^n(x) = \frac{1}{\sqrt{2}}[\psi_1^n(x) + \psi_2^n(x)] \tag{3.5}$$

$$\psi_a^n(x) = \frac{1}{\sqrt{2}}[\psi_1^n(x) - \psi_2^n(x)] \tag{3.6}$$

where the particle can be found in both wells.

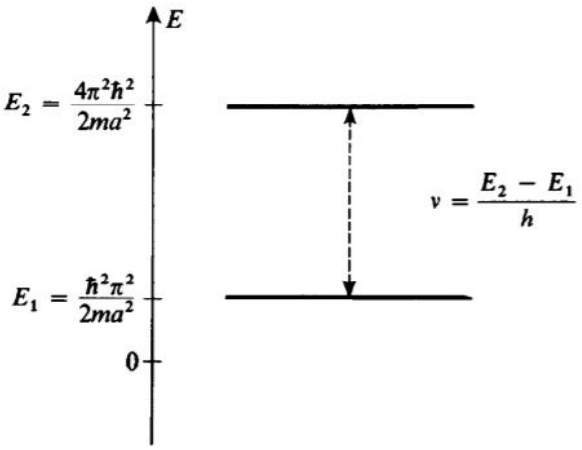

Figure 3.2: Energy spectrum in the infinite barrier approximation.

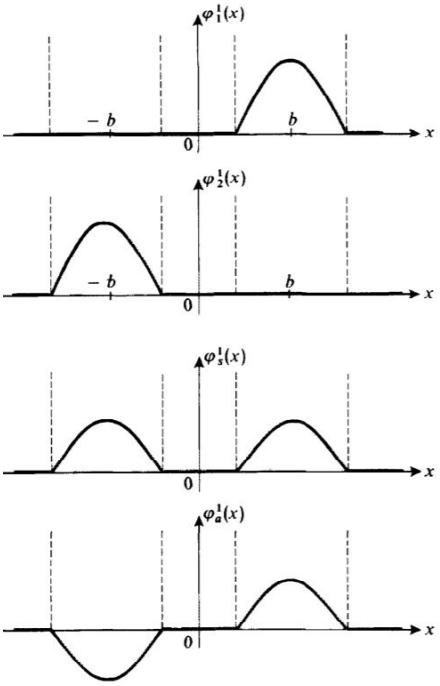

Figure 3.3: Eigenfunctions of the infinite barrier problem. The first two are localized on a definite side of the barrier while the last two the symmetry of the problem is used to choose as stationary states symmetrical and antisymmetrical states.

3.3 Finite barrier

Now we can come back to the original problem where V_0 has a finite value.

$V(x)$ is always even and we can look for eigenfunctions that are common to H and Π.

In the interval $-(b-\frac{a}{2}) \leq x \leq (b-\frac{a}{2})$ the wave function is no longer zero and we have to solve again the stationary Schroedinger equation obtaining the energy quantization condition:

$$\tan k_s a - \frac{k_s}{\sqrt{\alpha^2 - k_s^2}} \coth\left[\sqrt{\alpha^2 - k_s^2}(b - \frac{a}{2})\right] \tag{3.7}$$

$$\tan k_a a - \frac{k_a}{\sqrt{\alpha^2 - k_a^2}} \tanh\left[\sqrt{\alpha^2 - k_a^2}(b - \frac{a}{2})\right] \tag{3.8}$$

The trascendental equation can be solved graphically finding a certain number of roots $(k_s^1, k_s^2, \ldots, k_a^1 k_a^2, \ldots)$.

It is important to note that the energies E_s^n and E_a^n are now different.

Sketching the shape of the energy spectrum we see that the removal of the degeneracy of E_1 and E_2 gives rise to doublets.

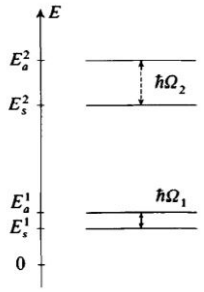

Figure 3.4: Energy spectrum in the finite barrier approximation.

The distance between the doublet is much greater than the spacing within each doublet (10^3 times). We can define new Bohr frequency

$$\Omega_1 = \frac{E_a^1 - E_s^1}{h}, \qquad \Omega_2 = \frac{E_a^2 - E_s^2}{h}, \quad \ldots \tag{3.9}$$

3.4 Two level system approximation

We see that the condition of two-level system is respected, we can work with (E_s^i, E_a^i) neglecting all the other energies and the removal of degeneracy can be easily interpreted, we will follow [4].

Suppose we are interested in the ground state (i.e $n = 1$). The molecule is in a definite state of rotation or translation, a physical model for the state space can be visualized as follows. Suppose the ammonia is rotating about an axis passing through nitrogen atom and perpendicular to the plane of the hydrogen atoms, there are two possible configurations:

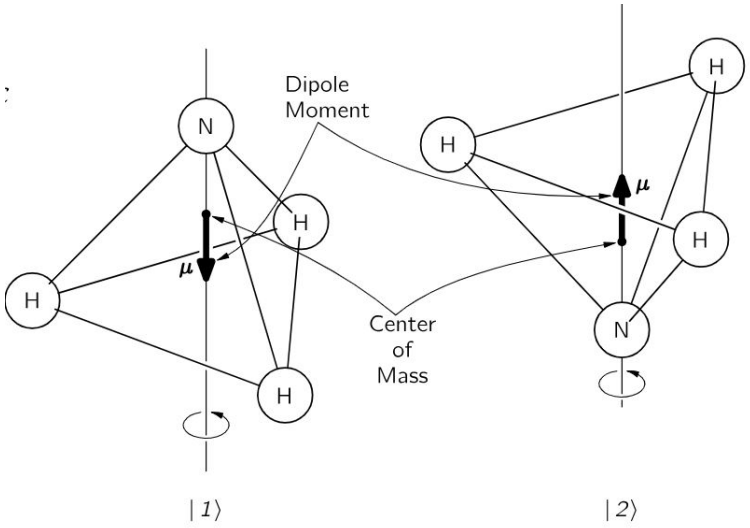

Figure 3.5: Two level system model for the Ammonia molecule in its ground state.

We take these states $(|1\rangle, |2\rangle)$[i.e $(|\psi_1^1, |\psi_2^1\rangle\rangle)$] as the basis of our space. Our mind thinks classically and we can be brought to think that these are stationary states, but the quantum possibility of tunnel effect denies that assumption. When the height of the barrier is infinite $(|1\rangle, |2\rangle)$ have the same energy and are indeed stationary states and the Hamiltonian is:

$$H_0 = E_0 I \tag{3.10}$$

To take into account phenomenologically the fact that the barrier is not infinite we introduce a coupling term W anti diagonal in $(|1\rangle, |2\rangle)$ base:

$$W = -A\sigma_x \tag{3.11}$$

The problem is nothing more than a specific case of the general one treated before.

If we want to find the stationary states we can now look at the general formula (2.2), noting that we are in the maximal splitting condition and $A < 0$(i.e $\phi = \pi$) we obtain:

$$E_\pm = E_0 \mp A \tag{3.12}$$

and the stationary states:

$$|+\rangle = \frac{1}{\sqrt{2}}[|1\rangle + |2\rangle] \tag{3.13}$$

$$|-\rangle = \frac{1}{\sqrt{2}}[|1\rangle - |2\rangle] \tag{3.14}$$

[i.e $(|\psi_s^1\rangle, |\psi_a^1\rangle)$].

It is important to highlight that now we have changed notation since $|+\rangle$ would have represented the antisymmetrical combination, so to avoid any confusion the sign will represent the eigenvalue of the parity operator.

We expect to see Rabi oscillation if the system starts in one of the unperturbed state.

Suppose that $|\psi(0)\rangle = |1\rangle$. By simple calculation the state at time t will be:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{iE_0 t}{\hbar}} [e^{i\frac{At}{\hbar}} |+\rangle + e^{-i\frac{At}{\hbar}} |-\rangle] \tag{3.15}$$

We again find the Rabi problem with $\mathcal{P}_{12}$ and $\mathcal{P}_{11}$ functions that oscillates at the Bohr frequency $\upsilon = \frac{2A}{h}$ [i.e $\frac{\Omega}{2}$], experimentally found to be 23800MHz which is in the microwave range called inversion frequency.

3.4.1 Interaction with electric field

We see in fig (3.5) that when the nitrogen atom flips about the plane of hydrogen also the dipole moment along x flips.

If we call D the observable associated to it, we assume that in $(|1\rangle, |2\rangle)$ basis is represented by

$$D = \begin{bmatrix} -\eta & 0 \\ 0 & \eta \end{bmatrix} \tag{3.16}$$

When the molecule is placed in a static electric field ξ pointing in the positive x direction the interaction energy of the molecule with the field is then

$$W'(\xi) = -\xi D = -\eta\xi\sigma_z \tag{3.17}$$

The total Hamiltonian is then:

$$H = H_0 + W + W'(\xi) = \begin{bmatrix} E_0 - \eta\xi & -A \\ -A & E_0 + \eta\xi \end{bmatrix} \tag{3.18}$$

Following the general formulas (2.2) (2.3) (2.4) we arrive to:

$$E'_{\pm} = E_0 \pm \sqrt{A^2 + \eta^2\xi^2} \tag{3.19}$$

$$|+'\rangle = \cos\frac{\theta}{2}|1\rangle - \sin\frac{\theta}{2}|2\rangle \tag{3.20}$$

$$|-'\rangle = \sin\frac{\theta}{2}|1\rangle + \cos\frac{\theta}{2}|2\rangle \tag{3.21}$$

Where

$$\tan\theta = -\frac{A}{\eta\xi} \qquad 0 \leq \theta < 0 \tag{3.22}$$

We can sketch the anti-crossing diagram in function of ξ (Which is responsible for splitting in unperturbed energy).

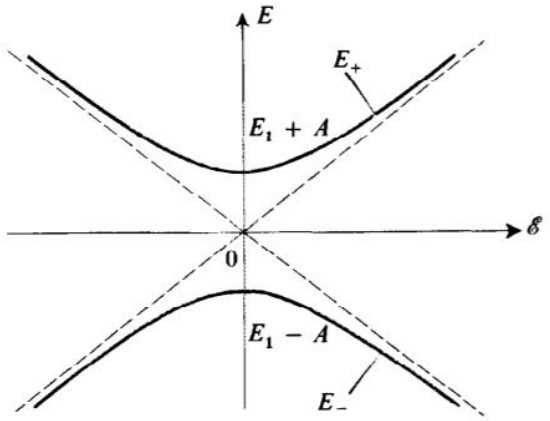

Figure 3.6: Anticrossing diagram for the Ammonia molecule in a static electric field pointing towards the axis of simmetry of the molecule.

So we see that this result from a compromise between the action of ξ and W. If we calculate the mean value of the induced electric dipole moment we find:

$$\langle\pm|D|\pm\rangle = \mp\frac{\eta^2\xi}{\sqrt{A^2+\eta^2\xi^2}} \tag{3.23}$$

Considering experimental conditions we can always use the approximation:

$$(\frac{\eta\xi}{A}) << 1 \tag{3.24}$$

Expanding (3.19) in weak field limit we see that:

$$\langle\pm|D|\pm\rangle = \mp\frac{\eta^2}{A}\xi \tag{3.25}$$

and we can simply calculate also the electrical susceptibility:

$$\varepsilon_\pm = \mp\frac{\eta^2}{A} \tag{3.26}$$

Moreover in this limit the eigenvalues become:

$$E'_{\pm} = E_0 \pm (A + \frac{\eta^2 \xi^2}{2A}) \tag{3.27}$$

3.4.2 Ammonia maser

It is very interesting now to see how all the phenomena discussed in this section were used to construct the Ammonia maser [5].

The idea is the following. Suppose we have a beam of ammonia molecules, because the energy separation of the levels is very small a normal population in thermal equilibrium contains very nearly equal numbers of molecules in each state. However, by passing the beam through an inhomogeneous electric field a separation can be achieved.

To see this we suppose to have a region where ξ is weak but where ξ^2 has a strong gradient in the $(0x)$ direction

$$\frac{\mathrm{d}\xi^2}{\mathrm{d}x} = \lambda \tag{3.28}$$

According to (3.27) we can find the force on the molecules in the x direction

$$(F_x)_{\pm} = -\frac{\mathrm{d}E'_{\pm}}{\mathrm{d}x} = \mp\frac{\lambda\eta^2}{2A} \tag{3.29}$$

which splits the two population.

Having obtained a population entirely in the state $|+\rangle$ the maser function can be obtained by stimulated emission of the transition $(+ \to -)$ which is reinforced by passing the beam through a cavity tuned to the required frequency.

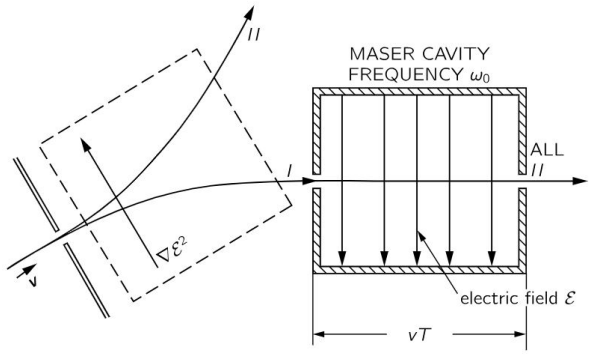

Figure 3.7: Schematic drawing of the Ammonia maser.

Let's assume the electric field in the cavity is

$$\xi(t) = \xi_0 \cos \omega t \tag{3.30}$$

The interaction energy can be represented in $(|+\rangle, |-\rangle)$ basis by a off diagonal matrix

$$W'' = \eta\xi(t)\sigma_x = \begin{bmatrix} 0 & \eta\xi(t) \\ \eta\xi(t) & 0 \end{bmatrix} \tag{3.31}$$

This is totally analogous to the SR experiment so we use the same non perturbative approach. The wave function at time t will be:

$$|\psi(t)\rangle = c_+(t)\,|+\rangle + c_-(t)\,|-\rangle \tag{3.32}$$

Solving the time dependent Schroedinger equation in the RWA approximation we arrive at the (1.39) (1.40):

$$i\frac{dC_+(t)}{dt} = \frac{\tilde{\omega_0}}{4} e^{i(\omega-\omega_0)t} C_- \tag{3.33}$$

$$i\frac{dC_-(t)}{dt} = \frac{\tilde{\omega_0}}{4} e^{-i(\omega_0-\omega)t} C_+ \tag{3.34}$$

where now:

$$\hbar\omega_0 = 2A \tag{3.35}$$

$$\hbar\tilde{\omega_0} = 2\eta\xi_0 \tag{3.36}$$

Following general solution (1.44) (1.45) (where we have set the initial conditions just to fit this particular situation) which in present case (i.e resonance) reads:

$$\mathcal{P}_{\pm} = |c_-(t)|^2 = \sin^2 \frac{\tilde{\omega_0} t}{4} \tag{3.37}$$

$$\mathcal{P}_{++} = |c_+(t)|^2 \cos^2 \frac{\tilde{\omega_0} t}{4} \tag{3.38}$$

We see that every ammonia molecule will undergo to the transition if the time T of transit is given by:

$$\frac{\tilde{\omega_0} T}{4} = \frac{\pi}{2} \tag{3.39}$$

Of course not all the molecules will have the same velocity, but if the most probable velocity is v then the length of the cavity L should be adjusted so that:

$$L = vT = \frac{2\pi v}{\tilde{\omega_0}} \tag{3.40}$$

Power outputs of $10^{-10}W$ can be obtained with line width as small as $10^{-2}Hz$ which makes the ammonia maser an excellent frequency standard.

Chapter 4

Density matrix formalism

4.1 Introduction

In many situations the state of the system is not perfectly determined, often system properties are known only statistically. We have to incorporate into quantum mechanical formalism this incomplete information following [3].

We formalyze saying that we deal with statistical mixture of states $(|\psi_1\rangle, |\psi_2\rangle, \dots)$ with probabilities of be in that state $(p_1, p_2, \dots)$

It's fundamental to note that probabilities intervene on two different levels:

- first,the partial information about the system which are necessarily statistical
- then, when the postulates concerning the measurement are applied

Moreover the system is not equivalent to a system having the probability $|c_k^2|$ of being in the state $|\psi_k\rangle$, due to interference effects between these states. It is impossible, in general, to describe the statistical mixture with an "average state vector".

4.2 Pure state

Before studying the general case we begin by analyzing the case where the state system is perfectly known, i.e pure state case.

Let $(|u_n\rangle)$ be a basis and $|\psi(t)\rangle = \sum_n c_n(t)\,|u_n\rangle$ the state vector.

If A is an observable, the mean value of A at the instant t is:

$$\langle\psi(t)|A|\psi(t)\rangle =< A > (t) = \sum_{n,p} c_n^*(t) c_p(t) A_{np} \tag{4.1}$$

Now we introduce the density operator $\rho(t)$ defined by

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \tag{4.2}$$

The density operator is represented in $(|u_n\rangle)$ basis by

$$\rho_{pn}(t) = c_n^*(t)c_p(t) \tag{4.3}$$

The introduction of this operator allow us to obtain all the physical preditions that can be calculated with the state vector formalism.

- The normalization of the state vector suggests:

$$\sum_n \rho_{nn}(t) = \mathrm{Tr}\{\rho(t)\} = 1 \tag{4.4}$$

  which express the conservation of probability.

- Rewriting the mean value of A:

$$< A > (t) = \sum_{np} \langle u_p|\rho|u_n\rangle \langle u_n|A|u_p\rangle = \sum_p \langle u_p|\rho A|u_p\rangle = \mathrm{Tr}\{\rho(t)A\} \tag{4.5}$$

- The time evolution of the density operator derive directly from Schroedinger equation

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = (\frac{\mathrm{d}\,|\psi\rangle}{\mathrm{d}t})\,\langle\psi(t)| + |\psi(t)\rangle\,(\frac{\mathrm{d}\,\langle\psi(t)|}{\mathrm{d}t}) = \frac{1}{i\hbar}[H(t), \rho(t)] \tag{4.6}$$

  known as "Von Neumann equation".

Nevertheless this formalism seems to be unnecessary in the pure state case there are some advantages that will be fundamental when we dell with statistical mixture.

First the global phase invariance is already contained in the definition of the density operator but more important is the fact that the formulas involves linear operation.

Another important properties are that:

- The density operator is hermitian

$$\rho(t) = \rho\,\dagger\,(t) \tag{4.7}$$

-

$$\rho^2(t) = \rho(t), \qquad \mathrm{Tr}\big\{\rho^2(t)\big\} = 1 \tag{4.8}$$

4.3 Statistical mixture

Let's now consider the general case. The various probabilities are arbitrary but must respect

$$0 \leq p_1, p_2, \cdots \leq 1, \qquad \sum_k p_k = 1 \tag{4.9}$$

We can calculate the probability that a measurement of A yield a_n with:

$$\mathcal{P}(a_n) = \sum_k p_k \mathcal{P}_k(a_n) \tag{4.10}$$

where $P_k(a_n)$ is defined by

$$\mathcal{P}_k(a_n) = \langle \psi_k | P_n | \psi_k \rangle \tag{4.11}$$

From (4.5) we have

$$\mathcal{P}_k(a_n) = \mathrm{Tr}\{\rho_k P_n\} \tag{4.12}$$

where:

$$\rho_k = |\psi_k\rangle\langle\psi_k| \tag{4.13}$$

is the density operator corresponding to the pure state ψ_k

Then exploiting linearity of the trace we have:

$$\mathcal{P}(a_n) = \mathrm{Tr}\{\rho P_n\} \tag{4.14}$$

where we have set:

$$\rho = \sum_k p_k \rho_k \tag{4.15}$$

Since the coefficients p_k are all real obviously we have:

$$\rho = \rho\dagger \tag{4.16}$$

Moreover:

- $$\mathrm{Tr}\{\rho\} = \mathrm{Tr}\left\{\sum_k p_k \rho_k\right\} = \sum_k p_k \,\mathrm{Tr}\{\rho_k\} = 1 \tag{4.17}$$

- It can be easily shown that:

$$< A >= \sum_n a_n \mathcal{P}(a_n) = \mathrm{Tr}\left\{\rho \sum_n a_n P_n\right\} = \mathrm{Tr}\{\rho A\} \tag{4.18}$$

- Using Schroedinger equation linearity it can be found the law for temporal evolution

$$i\hbar\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = [H(t), \rho(t)] \tag{4.19}$$

  Where we suppose that, despite the lack of information about the state vector, the hamiltonian is perfectly known.

- ρ is a positive operator

$$\langle u|\rho|u\rangle = \sum_k p_k \langle u|\rho_k|u\rangle = \sum_k p_k |\langle u|\psi_k\rangle|^2 \geq 0 \tag{4.20}$$

We have so generalized all the equation for the pure state density matrix to the general case with the exception of (4.8) since ρ is no longer a projector.

Each one of the matrix elements of the density operator has a precise meaning. If we analyze the diagonal one, we can write in the $(|u_n\rangle)$ basis:

$$\rho_{nn} = \sum_k p_k(\rho_k)_{nn} \tag{4.21}$$

introducing:

$$c_n^{(k)} = \langle u_n|\psi_k\rangle \tag{4.22}$$

(4.21) reduces to

$$\rho_{nn} = \sum_k p_k|c_n^{(k)}|^2 \tag{4.23}$$

so that ρ_{nn} represents the average probability of finding the system in the state $|u_n\rangle$.

For this reason ρ_{nn} is called population of the state $|u_n\rangle$.

For non-diagonal term with analogous calculation:

$$\rho_{np} = \sum_k p_k c_n^{(k)} c_p^{(k)*} \tag{4.24}$$

where $c_n^{(k)}$ and $c_p^{(k)*}$ represents interference terms between the states $|u_n\rangle$ and $|u_p\rangle$ which appear once the state $|\psi_k\rangle$ is a coherent linear superposition of these states. If $\rho_{np} \neq 0$ a certain coherence between these states exists, that is the reason why the non diagonal elements are called coherences.

This nomenclature obviously depends on the basis chosen. From (4.6) we know that ρ is hermitian, so it is always possible to find an orthonormal basis $(|\chi_i\rangle)$ where ρ is diagonal having:

$$\rho = \sum_i \pi_i |\chi_i\rangle\langle\chi_i| \tag{4.25}$$

Since ρ is positive and $\mathrm{Tr}\{\rho\} = 1$ the following is always true:

$$0 \leq \pi_i \leq 1 \tag{4.26}$$

$$\sum_i \pi_i = 1 \tag{4.27}$$

so that ρ describes a statistical mixture of χ_i with probabilities π_i and no coherences.

If in the chosen base $(|u_n\rangle)$ the Hamiltonian is diagonal and time-independent, Von Neumann equation reduces to:

$$i\hbar \frac{\mathrm{d}\rho_{nn}(t)}{\mathrm{d}t} = 0 \tag{4.28}$$

$$i\hbar \frac{\mathrm{d}\rho_{np}(t)}{\mathrm{d}t} = (E_n - E_p)\rho_{np} \tag{4.29}$$

equivalent to:

$$\rho_{nn}(t) = const \tag{4.30}$$

$$\rho_{np}(t) = e^{\frac{i(E_p - E_n)t}{\hbar}} \rho_{np}(0) \tag{4.31}$$

4.4 Partial trace and study of entangled systems

The formalism of density operator is fundamental when we want to consider two different systems (1) and (2) and the global system (1)+(2). The state space is $\varepsilon = \varepsilon(1) \otimes \varepsilon(2)$ with basis $(|u_n(1)\rangle\,|v_p(2)\rangle)$.

The density operator ρ acts in ε, but suppose that we are interested on physical predictions on system (1) or (2). We introduce a new operator $\rho(1)$ such that

$$\langle u_n(1)|\rho(1)|u_n(1)\rangle = \sum_p [\langle u_n(1)|\,\langle v_p(2)|\,\rho(|u_n(1)\rangle\,|v_p(2)\rangle)] \tag{4.32}$$

That is equivalent to say:

$$\rho(1) = \mathrm{Tr}_2\,\rho \tag{4.33}$$

and similarly for $\rho(2)$.

The operation Tr_i is called partial trace, and it is clear why noting that:

$$\mathrm{Tr}\{\rho\} = \mathrm{Tr}_1\,\mathrm{Tr}_2\,\rho = \mathrm{Tr}_2\ Tr_1\rho \tag{4.34}$$

It can be demonstrated that these new operators satisfy all properties of density operators, allowing us to make physical prediction about a measurement of $A(1)$ acting only on space $\varepsilon(1)$.

We know that:

$$< A(1) >= \mathrm{Tr}\{\rho A(1)\} \tag{4.35}$$

This expression can be simplified by using definition of trace and closure relation obtaining:

$$< A(1) >= \mathrm{Tr}\{\rho(1)A(1)\} \tag{4.36}$$

Enabling us to calculate mean values of observables acting on system (1) as if the system was isolated and has $\rho(1)$ as density operator.

This is a very useful tool when the state of the system (1) + (2) is not in a product state so that we cannot assign a state vector to system (1). Nonetheless this have some consequences in fact even if ρ describes a pure state this is not in general true for $\rho(1)$ and $\rho(2)$, and it is in general impossible to find an Hamiltonian operator relating to system (1) alone which would enable us to write a Liouville equation for density operator $\rho(1)$ which instead is much more difficult.

4.5 Density matrix in two-level system

4.5.1 Pure state

Consider a spin 1/2 system coming out from a polaryizer in the eigenstate $|\uparrow\rangle_u$ of the $\vec{S}\cdot\hat{u}$ component of the spin. This is a pure state and the for each spin $< \vec{S} >= \frac{\hbar}{2}\hat{u}$.

In the $(|\uparrow\rangle, |\downarrow\rangle)$ basis the density matrix take on the simple form

$$\rho(\theta,\phi) = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix} \tag{4.37}$$

The matrix is in general non diagonal, and we note that

- The population are related to the longitudinal polarization $< S_z >$

$$\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow} = \cos\theta = \frac{2}{\hbar} < S_z > \tag{4.38}$$

- The coherences are related to the transverse polarization $< S_\perp >$

$$|\rho_{\uparrow\downarrow}| = |\rho_{\downarrow\uparrow}| = \frac{1}{\hbar} < \vec{S_\perp} > \tag{4.39}$$

4.5.2 Statistical mixture

Let us consider now spin of silver atom leaving a furnace. We have just one but important information: the spin is equiprobable to point in any direction of space.

To define the density matrix in this case we use the definition (4.15) where the sum is replaced by integral over all the possible directions:

$$\rho = \frac{1}{4\pi}\int d\Omega \rho(\theta,\phi) \tag{4.40}$$

giving:

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{4.41}$$

The ensemble is maximally unpolarized in fact we note that $\rho^2 = \frac{\rho}{2}$ and if we calculate the mean value of the spin components:

$$< S_i >= \mathrm{Tr}\{\rho S_i\} = 0 \tag{4.42}$$

It is clear that this statistical mixture cannot be described by an average state vector since we demonstrated that if

$$|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle \tag{4.43}$$

then we can associate for alll c_1 and c_2 two polar angles (θ,ϕ) fixing the direction of polarization of $|\psi\rangle$.

4.5.3 Geometrical interpretation

We have seen that with pure state system Bloch sphere formalism offered us an intutive point of view of how the two level system would have evolved in time and what was the physical situation, in this section we see that a similar formalism can be applied also with statistical mixture as presented in [6].

We exploit the fundamental property of the σ_i to be basis of $\mathcal{M}_2$.

We can expand $M \in \mathcal{M}_2$ (Space of 2x2 Hermitian matrices) in terms of the σ_i:

$$M = a_0 I + \vec{a}\cdot\vec{\sigma} \tag{4.44}$$

where the coefficients are:

$$a_x = \frac{Tr\rho\sigma_x}{2} \quad a_y = \frac{\mathrm{Tr}\{\rho\sigma_y\}}{2} \quad a_z = \frac{\mathrm{Tr}\{\rho\sigma_z\}}{2} \tag{4.45}$$

and:

$$a_0 = \frac{\mathrm{Tr}\{\rho\}}{2} \tag{4.46}$$

Formula (4.44) can be simplified if the matrix M is a density matrix of some statistical mixture of spin $\frac{1}{2}$ obtaining:

$$M = \rho = \frac{1}{2}I + \frac{1}{\hbar} < \vec{S} > \cdot\vec{\sigma} \tag{4.47}$$

First we understand that the system is in a pure state if

$$\rho^2 = \rho \tag{4.48}$$

that is true if and only if

$$< S^2 >= \frac{\hbar^2}{4} \tag{4.49}$$

i.e. the system is polarized along some direction $\hat{u}$.

Let's consider now a statistical mixture of general two level systems. From (4.44) we know that to any projector onto $|a_i\rangle$ it can be associated a Bloch vector $\vec{a}$ such that

$$|a_i\rangle\langle a_i| = \frac{1}{2}[I + \vec{a_i} \cdot \vec{\sigma}] \tag{4.50}$$

We have learned that ρ is nothing else that a weighted projector, so if we suppose that our statistical mixture is composed of $(|a_i\rangle)$ with probabilities $(p_1, p_2, \dots)$ we are in the position to substitute (4.53) in the definition of ρ to have

$$\rho = \frac{1}{2}[(p_1 + p_2 + \dots]I + (\sum_i p_i\vec{a_i} \cdot \vec{\sigma}) = \frac{1}{2}[I + \vec{r} \cdot \vec{\sigma}] \tag{4.51}$$

where

$$\vec{r} = \sum_i p_i a_i = r\hat{r} \tag{4.52}$$

The statistical mixture is totally described by $\vec{r}$ which is analogous to a "center of mass" of the system. By comparing formulas (4.50) and (4.54) we see that:

$$\vec{r_i} = \mathrm{Tr}\{\rho\sigma_i\} = \frac{2}{\hbar} < \vec{S_i} > \tag{4.53}$$

We see that the center of mass $\vec{r}$ plays the role of the fictitious spin Bloch vector but with the fundamental difference that now the tip of the vector points inside the Bloch sphere going to the limit to zero when the ensemble is totally unpolarized.

The geometrical formalism remains unaltered and it is clear that two statistical mixture with the same "center of mass" will describe physically indistiguishable equivalent mixtures.

The density matrix is hermitian and the spectral decomposition is:

$$\rho = r_1 |r_1\rangle\langle r_1| + r_2 |r_2\rangle\langle r_2| \tag{4.54}$$

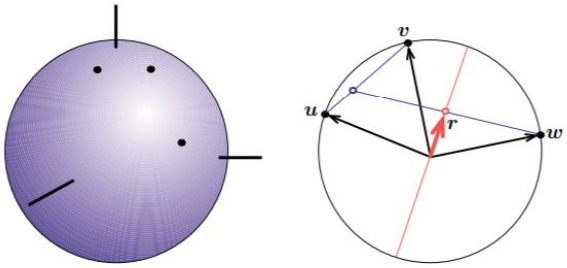

Figure 4.1: At left three weighted points on the unit 3-ball represent a mixture of three quantum states. On the right one dimension is discarded, construction indicates how one might compute the location of **r** which enters the "eigenrepresentation" of the mixture.

where r_1 and r_2 are defined by:

$$r_1 + r_2 = 1 \qquad r_1 - r_2 = r \tag{4.55}$$

It is natural then to introduce

$$Q = 1 - r = \begin{cases} 0 & for \quad pure \quad states \\ 1 & for \quad maximally \quad mixed \quad states \end{cases} \tag{4.56}$$

which is analogous to the degree of polarization of classical light theory.

In this framework an useful quantity is also

$$S = -\operatorname{Tr}\{\rho \log \rho\} = -r_1 \log r_1 - r_2 \log r_2 \tag{4.57}$$

and we have:

$$S = \begin{cases} 0 & for \quad pure \quad states \\ \log 2 & for \quad maximally \quad mixed \quad states \end{cases} \tag{4.58}$$

defined as the entropy of the two level system.

So far ρ can be represented on the Bloch sphere by the center of mass $\vec{r}$ that gives us information of how much is "polarized" the mixture. We are interested now in treating dynamical evolution.

As always we are analyzing any two-level system in total generality described by ρ under the Hamiltonian H:

$$H = \omega S_u \tag{4.59}$$

where $\hat{u}$ is the direction of the fictitious field $\vec{B}$

We expand easily in terms of the Pauli matrices obtaining:

$$H = \frac{\hbar\omega_0}{2}\hat{u}\cdot\vec{\sigma} \tag{4.60}$$

.
We now decompose $\vec{r}$ along the "Hamiltonian vector" $\hat{u}$

$$\vec{r} = \vec{r_\parallel} + \vec{r_\perp} \tag{4.61}$$

and the corresponding density matrices results

$$\rho_\parallel = \frac{1}{2}(I + \vec{r_\parallel}\cdot\vec{\sigma}) \tag{4.62}$$

$$\rho_\perp = \frac{1}{2}\vec{r_\perp}\cdot\vec{\sigma} \tag{4.63}$$

with $\rho = \rho_\parallel + \rho_\perp$
Using Von Neumann equation we are led to:

$$\frac{\mathrm{d}\vec{r_\parallel}}{\mathrm{d}t} = \vec{0} \tag{4.64}$$

$$\frac{\mathrm{d}\vec{r_\perp}}{\mathrm{d}t} = \omega_0\hat{u}\times\vec{r_\perp} \tag{4.65}$$

We see that upon substituting the Bloch vector associated to pure state $|\psi\rangle$ with $\vec{r}$, we again find precession about an "Hamiltonian" axis with angular velocity $\omega_0 = \frac{E_+ - E_-}{\hbar}$ dictated by energy difference .

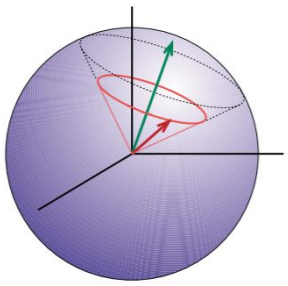

Figure 4.2: The long green arrow is set by the hamiltonian. The shorter red arrow twirls around the hamiltonian axis.

Moreover we note that the length of $\vec{r}$ as the precession goes does not change and it can be read as an illustration of the isoentropicity of quantum dynamical motion.

Chapter 5

Neutrino oscillation

5.1 Introduction

In this section we want to give a qualitative overwiev of neutrino oscillations, using the tools developed before, following [7]. All along this chapter we will work in natural units to lighten up the notation. Chadwick in 1914 demonstrated that the β^- decay spectrum was continuous, incompatible with the accepted theory, which modelled the phenomena as a two body decay. This surprising result led Bohr and many other scientists to think that the conservation of energy held only in a statistical sense. To remedy this serious problem in 1930 Pauli proposed that the existence of neutral weakly interacting fermion could solve the problem which was called by Fermi neutrino. Fermi theory will be a success but it is based on the fundamental fact that neutrino was massless.

Today the neutrino,in agreement with the standard model, belongs to the lepton family in which we find three doublets of particles

$$\begin{bmatrix} e^- \\ \upsilon_e \end{bmatrix} \tag{5.1}$$

$$\begin{bmatrix} \mu^- \\ \upsilon_\mu \end{bmatrix} \tag{5.2}$$

$$\begin{bmatrix} \tau^- \\ \upsilon_\tau \end{bmatrix} \tag{5.3}$$

formed by a charged lepton and its neutrino. Obviously there are the correspondent doublets of antiparticle.

What distinguish the various neutrino is the reaction to which they take part. For example if we have a neutrino produced in the decay:

$$\pi^+ \to \mu^+ + \upsilon_\mu \tag{5.4}$$

it can interacts with the neutron

$$\upsilon_\mu + n \rightarrow \mu^- + p \tag{5.5}$$

at it is what it identifies the muonic neutrino.

It is important to highlight that the standard model gives zero mass to neutrinos but as we will see is the mass difference between them that causes the oscillation phenomena.

5.2 Pure state oscillation

Each of the neutrino presented above is associated with an eigenstate of weak flavor $|\alpha\rangle$.

In the standard theory of neutrino oscillation a neutrino with flavor α and momentum $\vec{p}$ is described by the flavor state

$$|\upsilon_\alpha\rangle = \sum_k U^*_{\alpha k} |\upsilon_k\rangle \tag{5.6}$$

The $|\upsilon_k\rangle$ are the massive neutrino states and are eigenstates of the Hamiltonian

$$H |\upsilon_k\rangle = E_k |\upsilon_k\rangle \tag{5.7}$$

with energy eigenvalues

$$E_k = \sqrt{\vec{p}^2 + m_k^2} \tag{5.8}$$

so that the massive neutrino states evolve in time as plane waves.

For ultrarelativistic neutrinos (5.8) can be approximated by

$$E_k \simeq p + \frac{m_k^2}{2p} \tag{5.9}$$

Then it is straightforward to calculate

$$E_k - E_j \simeq \frac{\Delta m_{kj}^2}{2p} \tag{5.10}$$

It is clear that this difference of energy will become a phase difference in time evolution and the presence of the weight $U^*_{\alpha k}$ will cause the oscillation phenomena.

5.3 Two level system approximation

We want to focus on two neutrino mixing to see how the tools developed before can help us understanding the problem.

First of all we need to understand what means the TLS approximation in this case. We are going to negletct the coupling of the flavor neutrinos with the third massive neutrino that exists in nature. The two flavor neutrinos under analysis can be pure flavor neutrinos or also linear combination of them. This approximation is very useful in practice for two reasons:

- Oscillation formula depend on fewer parameters
- Many experiments are not sensitive to the influence of three neutrino mixing, so that we can analyze the data with the TLS model.

The two flavor neutrinos are linear superposition of the two massive neutrinos. The coefficients are given by the unitary matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5.11}$$

where θ is called the mkixing angle with an interval of $0 \leq \theta \leq \frac{\pi}{2}$.

We have now everything to use the tools developed before. The Hamiltonian in the $(|\upsilon_1\rangle, |\upsilon_2\rangle)$ basis is of course diagonal and given by

$$H = \begin{pmatrix} p + \frac{m_1^2}{2p} & 0 \\ 0 & p + \frac{m_2^2}{2p} \end{pmatrix} \tag{5.12}$$

We can easily find the expression in the $(|\upsilon_\alpha\rangle, |\upsilon_\beta\rangle)$ basis:

$$H_F = U \dagger HU = pI + \frac{1}{2p} \begin{pmatrix} m_1^2 \cos^2\theta + m_2^2 \sin^2\theta & \frac{\Delta m^2}{2} \sin 2\theta \\ \frac{\Delta m^2}{2} \sin 2\theta & m_2^2 \cos^2\theta + m_1^2 \sin^2\theta \end{pmatrix} \tag{5.13}$$

We know that the diagonal terms does not produce any phase difference between the states so that we are free to add

$$-\frac{m_1^2 + m_2^2}{4p} I \tag{5.14}$$

obtaining:

$$H_F = H_0 + H' = (p - \frac{m_1^2 + m_2^2}{4p})I + \frac{\Delta m^2}{4p} \begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos(2\theta) \end{bmatrix} \tag{5.15}$$

We are now again in the condition to apply our general formulas in fact we can identify $(|\alpha\rangle, |\beta\rangle)$ with $(|1\rangle, |2\rangle)$ as our unperturbed states.

We see that the presence of a mixing angle apart from introducing the coupling term affects also the difference in the unperturbed energy Δ, we have in particular:

$$\Delta = \cos 2\theta \tag{5.16}$$

So it determines if we are in a strong or weak coupling environment.

We can now easily solve the Rabi problem and find the probability to go from one flavor to another. Using (2.26) we obtain:

$$\mathcal{P}_{\alpha\to\beta}(t,\theta) = \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E_0} \tag{5.17}$$

where L is the length of oscillation and is approximately equal to t.

5.3.1 Interaction with matter

In this analysis no term that involves interaction with matter is taken into account. In fact what is presented above results only supposing that oscillation take place in vacuum.

Possible interaction with matter are neutral current interaction (NC) and Charged current interaction (CC). The first are independent from the flavor and so insignificant to our scope since give only a common phase factor. The second are direct consequence of presence of electron which can interact with electronic neutrino in scattering phenomena:

$$\upsilon_e + e^- \to \upsilon_e + e^- \tag{5.18}$$

affecting the oscillation phenomena.

The Hamiltonian is given, at first order in Fermi constant G_F by:

$$\hat{H}'' = \sqrt{2} G_F N_e \, |\upsilon_e\rangle\langle\upsilon_e| \tag{5.19}$$

where N_e,constant in our treatment,is the difference of the density of positron and electron in the medium. We have to suppose now that one flavour state is the electronic one, leaving arbitrariness on the other.

Calling $V_{CC} = \sqrt{2} G_F N_e$ we see that H_F becomes

$$H_F^m = (p - \frac{m_1^2 + m_2^2}{4p})I + \frac{\Delta m^2}{4p} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos(2\theta) \end{pmatrix} + \begin{pmatrix} V_{CC} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.20}$$

where the apex m stays for medium. No problem arise from this interaction since affects only the difference on the "unperturbed energy". We are now interested to put in a nicer form to calculate transition probabilities. Adding the

diagonal terms $-\frac{V_{CC}}{2}I$ and neglecting the term proportional to I redefining the energy origin we find :

$$W^m = \frac{1}{4p}\begin{bmatrix} -\Delta m^2\cos 2\theta + 2pV_C C & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta - 2pV_{CC} \end{bmatrix} \tag{5.21}$$

We can define new parameters θ_m $(\Delta m^2)_m$ so that the Hamiltonian is equal to the vacuum one:

$$\Delta m^2 \cos 2\theta - 2pV_{CC} = (\Delta^2 m)_m \cos 2\theta_m \tag{5.22}$$

$$\Delta m^2 \sin 2\theta = (\Delta m^2)_m \sin 2\theta_m \tag{5.23}$$

and it is possible by setting:

$$\tan 2\theta_m = \frac{\sin 2\theta}{\cos 2\theta - \frac{2pV_{CC}}{\Delta m^2}} \tag{5.24}$$

$$(\Delta m^2)_m = \Delta m^2 \sqrt{\sin^2(2\theta) + [\cos^2 2\theta - \frac{2pV_{CC}}{\Delta m^2}]^2} \tag{5.25}$$

enabling us to write in the compact form:

$$W^m = \frac{(\Delta m^2)_m}{4p}\begin{bmatrix} -\cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & \cos 2\theta_m \end{bmatrix} \tag{5.26}$$

We arrive to the conclusion that for the Hamiltonian for the oscillation phenomena remains unchanged as long as we make the substitution with the parameters defined above.

We know that the maximal splitting condition looking at the anticrossing diagram is achieved when $E_1 - E_2 = E_{\upsilon_e} - E_{\upsilon_\mu} = 0$ that is when:

$$\cos 2\theta - \frac{2pV_{CC}}{\Delta m^2} = 0 \tag{5.27}$$

obtaining the "resonance condition":

$$\sqrt{2} G_F N_e = \frac{\Delta m^2}{2p}\cos 2\theta \tag{5.28}$$

The Hamiltonian matrix becomes:

$$W_R = \frac{(\Delta m^2)_m \sin 2\theta_m}{4p}\sigma_x \tag{5.29}$$

and its eigenstates are:

$$\left|\upsilon_+^m\right\rangle = \frac{[(|\upsilon_e\rangle + |\upsilon_\mu\rangle)]}{\sqrt{2}} \tag{5.30}$$

$$\left|\upsilon_-^m\right\rangle = \frac{[(|\upsilon_e\rangle - |\upsilon_\mu\rangle)]}{\sqrt{2}} \tag{5.31}$$

Moreover we note that in resonance condition $(\Delta m^2)_m$ reach its minimum.

5.4 Statistical mixture

However in real physics application neutrinos has to be studied with the density matrix formalism developed before.

We are interested in applying the geometrical approach exposed in chapter four.

Since the TLS approximation is valid we can use the property of Pauli matrices and write:

$$H = -\frac{1}{2}\vec{\sigma}\cdot\vec{B} \tag{5.32}$$

$$\rho = \frac{1}{2}I + \frac{1}{2}\vec{\sigma}\cdot\vec{r} \tag{5.33}$$

We remember that:

$$r_1 = 2\,\mathrm{Re}\{\rho_{e\mu}\} \qquad r_2 = -2\,\mathrm{Im}\{\rho_{e\mu}\} \qquad r_3 = \rho_{ee} - \rho_{\mu\mu} \tag{5.34}$$

$$B_1 = -2\,\mathrm{Re}\{H_{e\mu}\} \qquad B_2 = 2\,\mathrm{Im}\{H_{e\mu}\} \qquad B_3 = H_{ee} - H_{\mu\mu} \tag{5.35}$$

So that now our problem is reduced to the well known precession of a magnetic moment $\vec{r}$ in a magnetic field $\vec{B}$ in our fictitious geometrical space.

Supposing N_e constant all the proposition will be valid in the vacuum case by applying the substitution (5.24) and (5.25).

Considering the Hamiltonian (5.21) the vector $\vec{B}$ is given by:

$$B_1 = -\frac{(\Delta m^2)_m}{2p}\sin 2\theta_m \tag{5.36}$$

$$B_2 = 0 \tag{5.37}$$

$$B_3 = \frac{(\Delta m^2)_m}{2p}\sin 2\theta_m \tag{5.38}$$

Let's call $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ the fictitious geometrical basis. We see that B lies on the Π_{13} plane inclined of $2\theta_m$ with respect of $\vec{e_3}$.

Suppose that at $t = 0$ the statistical mixture is in a incoherent superposition of flavor eigenstates. We can set in total generality the initial condition to be:

$$r_1(0) = 0 \tag{5.39}$$

$$r_2(0) = 0 \tag{5.40}$$

$$r_3(0) = W_e - W_\mu \tag{5.41}$$

Since $\gamma = 1$ in our units the "Larmor angular frequency" is:

$$\omega = -|\vec{B}| = -\frac{(\Delta m^2)_m}{2p} \tag{5.42}$$

Following formula (4.68) the fictitious spin vector at time t will be:

$$r_1(t) = \frac{1}{2}\sin 4\theta_m(W_e - W_\mu)[\cos\frac{(\Delta m^2)_m}{2p}t - 1] \tag{5.43}$$

$$r_2(t) = \sin 2\theta_m(W_e - W_\mu)\sin\frac{(\Delta m^2)_m t}{2p} \tag{5.44}$$

$$r_3(t) = (W_e - W_\mu)[1 - 2\sin^2 2\theta_m \sin^2\frac{(\Delta m^2)_m t}{4p}] \tag{5.45}$$

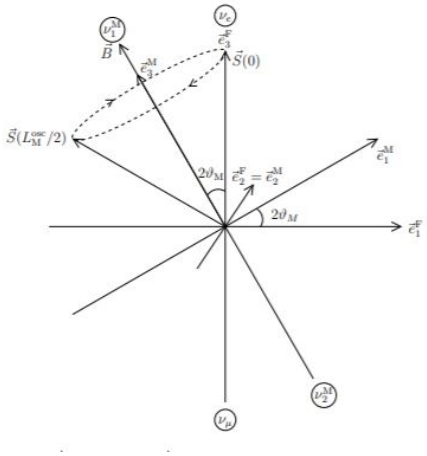

Figure 5.1: Neutrino oscillation on the Bloch sphere.

So that from (4.54) we relate the matrix element to the fictitious spin obtaining:

$$P_e(t) = \rho_{ee}(t) = \frac{1}{2} + (W_e - W_\mu)[\frac{1}{2} - \sin^2(2\theta_m)\sin^2\frac{(\Delta m^2)_m t}{4p}] \tag{5.46}$$

$$P_\mu(t) = \rho_{\mu\mu}(t) = \frac{1}{2} - (W_e - W_\mu)[\frac{1}{2} - \sin^2(2\theta_m)\sin^2\frac{(\Delta m^2)_m t}{4p}] \tag{5.47}$$

that for the pure state case (i.e $W_{e/\mu} = 1$) reduces to (2.26).

Chapter 6

Conclusion

The aim of this work was to underline with simple but efficient argument the net difference between classical and quantum mechanical modelization of nature. This was done with the application of quantum mechanical theory in its (computing-wise) easiest case. The two level system theory is used to model many other physical settings and plays a fundamental role in different interesting applications. The spin resonance experiments are the theoretical basis for developing the Magnetic Resonance Imaging technique, used in radiology to form pictures of the anatomy and physiological processes of the body, without involving X-Rays or ionizing radiations. Quantum computing employes the two level systems control (i.e. Quibit) to build a new theory of information with outstanding results in which spin resonance-type experiments play a crucial role [8]. As described for the Ammonia molecule and neutrino flavor eigenstates, the presence of a quantum mechanical coupling between "stationary states" can be strongly related to the cognition process which is currently modeled as a quantum process. In this regard, simulation of the dynamic perception of the Necker cube in terms of a two level quantum systems has led to the introduction of a neuro-physiological action similar to the Planck constant [9] .

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