Università degli Studi di Napoli “Federico II”

Scuola Politecnica e delle Scienze di Base
Area Didattica di Scienze Matematiche Fisiche e Naturali

Dipartimento di Fisica “Ettore Pancini”

Laurea magistrale in Fisica

Casimir Effect in Superconductor at high C.T.

Relatori:
Prof. Luigi Rosa

Candidato:
Sergio Balestrieri
Matricola N94000473

A.A. 2018/2019

Contents

Introduction

The following study aims to evaluate the Casimir energy in a generical number of cavity, when the distance from the plates is in the order of nanometers. The dielectric model [21] doesn't work in this case because the distance from the plates are too small for this reason using the Kempf work [20] we create a first approximation for the Casimir energy in this model.

The Archimedes experiment aims to calculate the Casimir energy in a plates of Ybco in the order of few centimeters that contains a large number of cavities ($\simeq 10^7$) in the order of nanometers. The following study allow to give us a first analysis of the experimental result of Archimedes.

In the first chapter we have a historical description about the problem of the black body and the Planck model that gave birth at the study of the zero point energy.

In the second chapter we explain how to quantize the electromagnetic field and obtain, in this case too, the zero-point energy.

After that we show the Casimir calculus, i.e., the macroscopic effect of the zero point energy for two plates and extend this for a plates in a dielectric material.

In the third chapter we analyze the Barton model [14] for a cavities (plasma sheet model) and we calculate the Casimir energy for one and more cavities (for instance 2,3). After that we consider the difference between a quantum system with zero temperature and a quantum system with finite temperature and evaluate the Casimir energy when have finite temperature.

Finally we generalize the Casimir energy formula for a number n of cavities and note the difference between a dielectric case and the Barton model.

In the fourth chapter we analyze the evolution of the generalization to n cavity of the Casimir energy in function of distance and number of cavities and we have evaluate that this evolution follows the same properties of the dielectric case (see Bordag [17]).

Chapter 1

The Planck model

The Casimir effect in its simplest form is the interaction of a pair of neutral, parallel conducting planes due to disturbance of the vacuum of the electromagnetic field. It is a pure quantum effect, there is no force between the plates in classical electrodynamics. In the ideal situation, at zero temperature for instance, there are no real photon in between the plates. So it's only the vacuum, i.e., the ground state of quantum electrodynamics which causes the plates to attract each other.

In this section we give a brief historical introduction about the discovery of zero-point energy. We analyze the blackbody problem that is of fundamental importance in the development of quantum field theory, and led to the concept of zero-point energy.

1.1 The blackbody problem

In 1860 Kirchhoff derived a general relation between the radiative and absorptive strengths of a body at fixed temperature T. According to Kirchhoff's law, the ratio of the radiative strength to the absorption coefficient of the radiation of wavelength λ is the same for all bodies at temperature T, and defines a universal function $F(\lambda, T)$. This led to the abstraction of an ideal blackbody for which the absorption coefficient is unity at every wavelength, corresponding to total absorption. Thus $F(\lambda, T)$ characterizes the radiative strength at wavelength λ of a blackbody at temperature T. The problem was to determine the universal function $F(\lambda, T)$.

An important step was taken in 1884 by Boltzmann, who invoked several aspect of Maxwell's electromagnetic theory. The most important of these for the present discussion is the result that isotropic radiation exerts on a perfectly reflecting surface a pressure $u/3$ where u is the energy density of the radiation. Boltzmann considered blackbody radiation confined in a cylinder of volume V, one end of which is a perfectly reflecting piston. The radiation pressure of the piston increases the volume by dV, and in order to maintain a constant temperature, according to the first law of thermodynamics, an amount of heat must be added :

$$dQ = V du + \frac{4}{3} u dV \tag{1.1.1}$$

Since $dS = \frac{dQ}{dt}$ is an exact differential we obtain Stefan-Boltzmann-law:

$$u = bT^4 \tag{1.1.2}$$

where b is a universal constant (the Stefan constant).

The Stefan-Boltzann law is in conflict with elementary classical models of equilibrium between radiation and matter. Indeed let us consider the classical oscillator model of an atom, where an electron is assumed to be bound by an elastic restoring force. In a state of equilibrium between radiation and matter, the energy absorption rate should equal the emission rate:

$$\frac{\pi e^2}{3m} \rho(\nu_0) = \frac{32\pi^4 e^2 \nu_0^4}{3c^3} x^2 \tag{1.1.3}$$

Where $\rho(\nu)d\nu$ denotes the energy for unit volume of radiation in the frequency interval $[\nu, \nu + d\nu]$, ν_0 is the natural oscillation frequency of the electron in the atom and x is the electron displacement from its equilibrium position in the classical oscillator model in the atom. Now according to the virial theorem of classical mechanics the average potential energy of the electron oscillator is equal to the average kinetic energy and their sum is the total oscillator energy $U = 4\pi^2 m\nu_0^2 x^2$ obtaining that

$$\rho(\nu) = \frac{8\pi\nu}{c^3} U \tag{1.1.4}$$

for a blackbody, which absorb all frequencies ν. Finally the equipartition theorem of a classical statistical mechanics demand that the average value of U in thermal equilibrium is kT, where k is Boltzmann's constant, so that the spectral energy density of thermal radiation must be:

$$\rho(\nu) = \left(\frac{8\pi\nu}{c^3}\right) kT \tag{1.1.5}$$

this is the Rayleigh-Jeans distribution. The total electromagnetic energy density

$$u = \int_0^\infty \rho(\nu) d\nu \tag{1.1.6}$$

violates the Stefan-Boltzmann law. Furthermore the Rayleigh-Jeans law suffer from the ultraviolet catastrophe: u diverges when (1.1.6) is used for $\rho(\nu)$.

The Rayleigh-Jeans distribution obeys to another classical result, due to Wien in 1893. Wien basically followed Boltzmann's model of radiation contained in a cylinder with a piston, but included the Doppler shift of radiation reflected by the moving piston. This allowed radiant energy to be exchanged among different frequencies. In this way Wien showed that the spectral energy density must follow the general form:

$$\rho(\nu) = \nu^3 \phi_1(\nu/T) \tag{1.1.7}$$

or, in term of the wavelength:

$$\rho(\lambda) = \lambda^{-5} \phi_2(\lambda/T) \tag{1.1.8}$$

where ϕ_1 and ϕ_2 are undetermined functions.

1.2 Planck's solution and zero point energy

In 1890, Planck produced a derivation of the Wien distribution from general thermodynamic considerations plus the assumption that the entropy of a collection of radiators depends only on their total energy. An important result was the following relation between the entropy S and the average energy U of an elementary radiator in thermal equilibrium with radiation at the temperature T:

$$\frac{\partial^2 S}{\partial U^2} = -\frac{A}{U} \tag{1.2.1}$$

where A is a constant dependent on the frequency of a given radiator. From this equation and the general relation $\partial S/\partial U = 1/T$ it follows that:

$$U = Be^{-1/AT} \tag{1.2.2}$$

where B is a constant that, like A, may depend on the frequency of a given radiator. This result together with eq.(1.1.4), yields the radiation spectral energy density:

$$\rho(\nu) = f(\nu)e^{-1/AT} \tag{1.2.3}$$

where $f(\nu)$ is some function of ν. Wien's displacement law implies that $f(\nu)$ and A are proportional to ν^3 and ν^{-1}, respectively, so that:

$$\rho(\nu) = C\nu^3 e^{-D\nu/T} \tag{1.2.4}$$

where C and D are constant or

$$\rho(\lambda) = \alpha\lambda^{-5}e^{-\beta/\lambda T} \tag{1.2.5}$$

where α and β are constant the equations (1.2.4) and (1.2.5) is know as Wien distribution.

The Wien distribution, however, was soon found to be incorrect in higher wavelength's experiment. The data indicated that $\rho(\nu)$ was proportional to temperature T for small ν and large T, indeed the Rayleigh-Jeans distribution fits with the experimental data in this frequency range.

For small ν and large T, the experimental result $\rho(\nu) \propto T$ and equation (1.1.4) imply $U \propto T$ and therefore, since $\partial S/\partial U = T^{-1}$, $\partial^2 S/\partial U^2 = U^{-2}$ and $S \propto \log U$. On the other hand (1.2.1) leads to Wien distribution, which has the correct form for large ν and small T, for this reason Planck proposed the interpolation:

$$\frac{\partial^2 S}{\partial U^2} = \frac{-A}{U(B+U)} \tag{1.2.6}$$

where A and B are constant. Using again the relation $\partial S/\partial U = T^{-1}$, equation (1.1.4) and the Wien displacement law, one obtain from (1.2.6) the spectral energy density:

$$\rho(\nu) = \frac{\alpha\lambda^{-5}}{e^{-\beta/\lambda T} - 1} \tag{1.2.7}$$

where α and β are constant. This formula was found to agree with all the existing data.

The Planck's reasoning may be summarized as follows([1], [2], [3], [4]). Consider N radiators of frequency ν and total energy $U_N = NU = P\epsilon$, where P is a large integer and ϵ is a finite element of energy. The entropy $S_N = NS = N \log W_N$, where W_N is the number of ways in which the P energy element can be distributed among N radiators. If the energy element are assumed to be indistinguishable we have:

$$W_N = \frac{(N-1+P)!}{P!(N-1)!} \tag{1.2.8}$$

use the Stirling's approximation, then gives, for $N, P \gg 1$

$$S \simeq k\left[(1+\frac{U}{\epsilon})\log(1+\frac{U}{\epsilon}) - \frac{U}{\epsilon}\log\frac{U}{\epsilon}\right] \tag{1.2.9}$$

thus:

$$\frac{\partial S}{\partial U} = \frac{1}{T} = \log(1+\frac{\epsilon}{U}) \to U = \frac{\epsilon}{e^{\epsilon/kT}-1} \tag{1.2.10}$$

for the average energy of the radiators. The excellent agreement between (1.2.7) and experiment, together with the eq.(1.1.4), suggests that ϵ is inversely proportional to the wavelength, or directly proportional to the frequency of the oscillator:

$$\epsilon = h\nu \tag{1.2.11}$$

then

$$U = \frac{h\nu}{e^{h\nu/KT}-1} \tag{1.2.12}$$

and (1.1.4) implies

$$\rho(\nu) = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT}-1} \tag{1.2.13}$$

for the spectral energy density of thermal radiation.

A revolutionary aspect of Planck's calculation, of course, is the physical significance it attaches to the energy element of size ϵ, and the relation (1.2.11) between ϵ and the frequency ν of a material oscillator. Boltzmann had also employed energy element in his counting of ways, but in his calculation ϵ had no particular significance and in fact could ultimately be taken to be zero once a formula of W_N had been obtained. If Planck had taken the limit $\epsilon \to 0$ in equation (1.2.10), however, then $\partial S/\partial U \to k/U$ and $\partial^2 S/\partial U^2 \to -k/U^2$ which leads to the Rayleigh-Jeans distribution. In Planck's derivation of his spectrum, therefore, the quantization of energy was absolutely essential.

Einstein and Stern noted a circumstance about the zero-point energy is worth mentioning. Consider the classical limit $kT \gg h\nu$ of the expression (1.2.12) for the average energy of an oscillator in thermal equilibrium with radiation:

$$U = \frac{h\nu}{e^{h\nu/KT} - 1} \simeq \frac{h\nu}{1 + \frac{h\nu}{kT} + \frac{1}{2}(\frac{k\nu}{kT})^2 - 1} = \frac{kT}{1 + \frac{1}{2}\frac{h\nu}{kT}} \simeq kT - \frac{1}{2}h\nu \tag{1.2.14}$$

thus U contains a first-order temperature-independent correction to kT, the energy predicted by the equipartition theorem in the classical limit but:

$$U + \frac{1}{2}h\nu = \frac{h\nu}{e^{h\nu/KT} - 1} + \frac{1}{2}h\nu \tag{1.2.15}$$

which includes the zero point energy $\frac{1}{2}h\nu$, does not have a first-order correction to kT in the classical limit. In the Planck's second theory U was in fact replaced by $U + \frac{1}{2}h\nu$.

In his second theory, Planck considers that the absorption of radiation was assumed to proceed according to classical theory, whereas emission of radiation occurred discontinuously in discrete quanta of energy. Assumes that an oscillator can radiate only after it has continuously absorbed an energy $h\nu$. Let P_n be the probability that it has energy between $(n-1)h\nu$ and $nh\nu$, there is a probability p that it will lose all its energy in the form of radiation, and a probability $1-p$ that it continues to absorb without emission of radiation. Thus $P_2 = P_1(1-p)$, $P_3 = P_2(1-p) = P_1(1-p)^2$.... and $P_n = P_1(1-p)^{n-1}$and

$$\sum_{n=1}^{\infty} P_n = 1 = \sum_{n=1}^{\infty} P_1(1-p)^{n-1} = P_1/p \tag{1.2.16}$$

or $P_1 = p$ is the probability that an oscillator in equilibrium with radiation has energy between 0 and $h\nu$, $P_2 = p(1-p)$ is the probability that it has the energy between $h\nu$ and $2h\nu$, and $P_n = p(1-p)^{n-1}$ is the probability that it has energy between $(n-1)h\nu$ and $nh\nu$. Following Boltzmann, Planck defines the oscillator entropy as:

$$S = -k\sum_{n=1}^{\infty} P_n \log P_n = -k\sum_{n=1}^{\infty} p(1-p)^{n-1}\log[p(1-p)^{n-1}] = -k\left[\frac{1}{p}\log p + (\frac{1}{p} - 1)\log(\frac{1}{p} - 1)\right] \tag{1.2.17}$$

Planck now assumes that all energies between $(n-1)h\nu$ and $nh\nu$ are equally likely, so that the average energy of the oscillators with energy between $(n-1)h\nu$ and $nh\nu$ is $\frac{1}{2}(n+n-1)h\nu = (\frac{1}{p} - \frac{1}{2})h\nu$; for (1.2.17), therefore,

$$S = k\left[(\frac{U}{h\nu} + \frac{1}{2})\log(\frac{U}{h\nu} + \frac{1}{2}) - (\frac{U}{h\nu} - \frac{1}{2})\log(\frac{U}{h\nu} - \frac{1}{2})\right] \tag{1.2.18}$$

using once again the relation $\partial S/\partial U = \frac{1}{T}$, Planck obtained:

$$U = \frac{h\nu}{e^{h\nu/kT} - 1} + \frac{1}{2}h\nu \tag{1.2.19}$$

this implies that when $T \to 0$ $U \to \frac{1}{2}h\nu$. Planck equation (1.2.19) marked the birth of the concept of zero-point energy.

Chapter 2

The electromagnetic vacuum

The quantum field theory of the electromagnetic field in absence of any sources was formulated by Born, Heisenberg and Jordan (1926). The new quantum electrodynamics predicted a fluctuating zero-point of vacuum field existing even in the absence of any sources.

In this section , after having quantized the electromagnetic field, we study the Casimir effect. This is a macroscopic observable effect due to zero-point energy and we will extend the results to dielectric materials.

2.1 The harmonic oscillator

A monochromatic electromagnetic field is mathematically equivalent to a harmonic oscillator of the same frequency. Before showing this we will briefly review the harmonic oscillator in quantum mechanics. The Hamiltonian has the same form as in classical mechanics:

$$H = p^2/2m + \frac{1}{2}m\omega^2 q^2 \tag{2.1.1}$$

where now q and p are quantum mechanical operators in a Hilbert space. The Heisenberg equations of motion have the same form as classical Hamilton equations:

$$\begin{aligned} \dot{q} &= (i\hbar)^{-1}[q; H] = p/m \\ \dot{p} &= (i\hbar)^{-1}[p; H] = -m\omega^2 q \end{aligned} \tag{2.1.2}$$

these follow from the commutation rule $[q,p]=i\hbar$. We define the non Hermitian operator

$$a=\frac{1}{\sqrt{2m\hbar\omega}}(p-im\omega q) \tag{2.1.3}$$

or equivalently:

$$\begin{aligned} q &= i\sqrt{\frac{\hbar}{2m\omega}}(a-a^\dagger) \\ p &= \sqrt{\frac{m\hbar\omega}{2}}(a+a^\dagger) \end{aligned} \tag{2.1.4}$$

where $a^\dagger$is the adjoint of a. From $[q,p]=i\hbar$ it follows that

$$[a,a^\dagger]=1 \tag{2.1.5}$$

equations (2.1.4) allow us to write the Hamiltonian (2.1.1) in the form:

$$H=\frac{1}{2}\hbar\omega(aa^\dagger+a^\dagger a)=\hbar\omega(a^\dagger a+\frac{1}{2}). \tag{2.1.6}$$

The energy levels of the harmonic oscillator are thus determined by the eigenvalues of the operator $N=a^\dagger a$. We denote the eigenvalues and eigenket of N by n and $|n\rangle$, respectively

$$N|n\rangle=n|n\rangle \tag{2.1.7}$$

now $\langle n|N|n\rangle=\langle n|a^\dagger a|n\rangle$ is the scalar product of the vector $a|n\rangle$ with itself. It then follows from (2.1.7) that $n\langle n|n\rangle=n$ is real and positive.

Consider the effect on the vector $a|n\rangle$ of the operator N. Obviously $Na|n\rangle=aN|n\rangle+[N,a]|n\rangle$, but (2.1.5) imply $[N,a]=-a$, and therefore $Na|n\rangle=(n-1)a|n\rangle$. In other words, if $|n\rangle$ is an eigenstate of N with eigenvalue n, then $a|n\rangle$ is an eigenstate of N with eigenvalue $n-1$: $a|n\rangle=C|n-1\rangle$. Taking the norm of both sides of this equation we obtain $|C|^2=n$, and without any loss of generality we can choose the phase such that $C=\sqrt{n}$. Thus

$$a|n\rangle=\sqrt{n}|n-1\rangle \tag{2.1.8}$$

we find similarly that:

$$a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle \tag{2.1.9}$$

for obvious reasons a and $a^\dagger$are called lowering and raising operators.

We have already noted that the eigenvalues are positive ($n \geq 0$). But the equation (2.1.8) shows that we can generate eigenstates with lower and lower eigenvalues by successive applications of lowering operator a. Consistency then required that $a\,|n\rangle = 0$ for $n < 1$ and (2.1.8) indicates that this is satisfied only for $n = 0$. The eigenvalues n of N are therefore zero and all the positive integers. That is, the energy level of the harmonic oscillator are given by

$$E_n = (n + \frac{1}{2})\hbar\omega. \tag{2.1.10}$$

2.2 Quantization of Field mode

We will now take the most elementary route to the quantization of the electromagnetic field. The first step is to show that a field mode is equivalent to a harmonic oscillator.

The Maxwell equations for the "free" field i.e. the field in a region where there are no sources, are:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{2.2.1}$$

we introduce the vector potential $\vec{A}$ by writing $\vec{B} = \vec{\nabla} \times \vec{A}$. Since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ the second equation of (2.2.1) is automatically satisfied. the third equation of (2.2.1) implies $\vec{E} = -(1/c)\partial\vec{A}/\partial t + \vec{\nabla}\phi$, where ϕ is the scalar potential. From the fourth equation of (2.2.1) we have:

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \tag{2.2.2}$$

in the Coulomb gauge defined by $\vec{\nabla} \cdot \vec{A} = 0$ and, in absence of any sources, $\phi = 0$. Obviously the first equation of (2.2.1) is then also satisfied. Thus we can obtain a solution of free-space Maxwell equations by solving (2.2.2) for the Coulomb-gauge vector potential subject to appropriate boundary conditions. Separation of variables gives monochromatic solutions

$$\vec{A}(r;t) = \alpha(t)\vec{A}_0(r) + \alpha^*(t)\vec{A}_0^*(r) \tag{2.2.3}$$

$$= \alpha(0)e^{i\omega t}\vec{A}_0(r) + \alpha^*(0)e^{-i\omega t}\vec{A}_0^*(r) \tag{2.2.4}$$

where $\vec{A}_0(r)$ satisfies the Helmholtz equation:

$$\nabla^2 \vec{A}_0(r) + k^2 A_0(r) = 0 \qquad k = \frac{\omega}{c} \tag{2.2.5}$$

and $\alpha(t)$ satisfies $\ddot{\alpha}(t) = -\omega^2 \alpha(t)$. The electric and magnetic field vectors are given by:

$$\begin{aligned} \vec{E}(r,t) &= -\frac{1}{c}[\dot{\alpha}(t)\vec{A}_0(r) + \dot{\alpha}^*(t)\vec{A}_0^*(r)] \\ \vec{B}(r,t) &= \alpha(t)\vec{\nabla} \times \vec{A}_0(r) + \alpha^*(t)\vec{\nabla} \times \vec{A}_0^*(r). \end{aligned} \tag{2.2.6}$$

We show that the electromagnetic energy are:

$$H_F = \frac{1}{8\pi}\int d^3r(E^2 + B^2) = \frac{k^2}{2\pi}|\alpha(t)|^2 \tag{2.2.7}$$

where, without any loss of generality, we assume the "mode function" $\vec{A}_0(r)$ is normalized such that

$$\int d^3r \ |\vec{A}_0(r)|^2 = 1. \tag{2.2.8}$$

Define the real quantities

$$q(t) = \frac{i}{c\sqrt{4\pi}}[\alpha(t) - \alpha^*(t)] \tag{2.2.9}$$

$$p(t) = \frac{k}{\sqrt{4\pi}}[\alpha(t) + \alpha^*(t)] \tag{2.2.10}$$

in terms of which equation (2.2.7) is

$$H_F = \frac{1}{2}(p^2 + \omega^2 q^2). \tag{2.2.11}$$

The notation suggest that our field mode of frequency ω is mathematically equivalent to a harmonic oscillator of frequency ω. To prove this we must, of course, show that q and p are indeed canonically conjugate coordinate and momentum variables. But this is trivial: from the definitions (2.2.9) and (2.2.10) and $\dot{\alpha}(t) = -i\omega t$, we have $\dot{q} = p$ and $\dot{p} = -\omega^2 q$, which are the Hamilton equations that follows from the Hamiltonian H_F.

To describe a field mode quantum mechanically, we simply describe the equivalent harmonic oscillator quantum mechanically. Since the oscillator with Hamiltonian (2.2.11) has unitary mass, we introduce the lowering and raising operators

a and $a^\dagger$ using (2.1.4) . Comparing with eq.(2.2.9) and eq.(2.2.10), we see that this quantization procedure, except for a trivial constant, is equivalent to replace the classical variables $\alpha(t)$ and $\alpha(t)^*$ by the quantum mechanical operator $a(t)$ and $a^\dagger(t)$.

The classical vector potential (2.2.4) is thus replaced by the operator

$$\vec{A}(r;t) = \left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2}[a(t)\vec{A}_0(r) + a^\dagger(t)\vec{A}_0^*(r)] \tag{2.2.12}$$

and the operators corresponding to the electric and magnetic field are similar:

$$\vec{E}(r,t) = i(2\pi\hbar\omega)^{1/2}[a(t)\vec{A}_0(r) + a^\dagger(t)\vec{A}_0^*(r)] \tag{2.2.13}$$

$$\vec{B}(r,t) = \left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2}[a(t)\vec{\nabla}\times\vec{A}_0(r) + a^\dagger(t)\vec{\nabla}\times\vec{A}_0^*(r)]. \tag{2.2.14}$$

The Hamiltonian (2.2.11) for the quantized field mode is now obviously equivalent to:

$$H_F = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{2.2.15}$$

the energy eigenvalues of a field mode of frequency ω are given by equation (2.1.10). The integer n is the number of photons in the field mode described by the state $|n\rangle$. The vacuum state $|0\rangle$ has no photon, but it nevertheless has an energy $\frac{1}{2}\hbar\omega$. The quantum theory of radiation thus predict the existence of a zero-point electromagnetic field. In the vacuum state and in all stationary state $|n\rangle$, the expectation values of electric and magnetic field vanish:

$$\left\langle \vec{E}(r,t)\right\rangle = \left\langle \vec{B}(r,t)\right\rangle = 0 \tag{2.2.16}$$

since $\langle n|a|n\rangle = 0$. This means that the electric and magnetic field vectors fluctuate with zero mean in the state $|n\rangle$, although the field has a definite, nonfluctuating energy $(n+\frac{1}{2})\hbar\omega$.

Consider the expectation value of the square of the electric field, from (2.2.13) this is given by:

$$\left\langle E^2(r)\right\rangle = 4\pi\hbar\omega|A_0(r)|^2 n + \left\langle E^2(r)\right\rangle_0 \tag{2.2.17}$$

the first term is the probability distribution for finding the photon , which is the intensity in the classical wave theory, and n is the number of photons. The term $\left\langle E^2(r)\right\rangle_0$ is the vacuum contribution and we consider like a constant to add in equation as in (2.2.17). Physical measurement will therefore reveal only deviations from the vacuum state. Thus the field Hamiltonian (2.2.15), for example, can be replaced by:

$$H_F - \langle 0|H_F|0\rangle = \hbar\omega a^\dagger a \tag{2.2.18}$$

without affecting any physical predictions of the theory. The new Hamiltonian (2.2.18) is said to be normally ordered, the raising operator $a^\dagger$ appearing to the left of the lowering operator a. The normally ordered Hamiltonian is denoted $: H_F :$

$$: H_F := \hbar\omega a^\dagger a \tag{2.2.19}$$

in other words, within the normal ordering symbol we can commute a and $a^\dagger$. Since the zero-point energy is intimately connected to the noncommutativity of a and $a^\dagger$, the normal ordering procedure eliminates any contribution from zero-point field. This is especially reasonable in the case of the field Hamiltonian, since the zero-point term merely adds a constant energy which can be eliminate by a simple redefinition of the zero of the energy. Moreover, this constant energy in Hamiltonian obviously commutes with a and $a^\dagger$ and so cannot have any effect on the quantum dynamics described by the Heisenberg equations of the motion, furthermore we shall see that it possible to attribute measurable effect, such as the Casimir force, to change the zero-energy.

The generalization of the quantization procedure to a multimode field is straightforward. We consider the field in free space with no physical boundaries, in which case the number of allowed modes is infinite.

Obviously the field intensity for infinite free space should be independent of position so that, from (2.2.17), $|A_0(r)|^2$ should be independent of r for each mode of field. Of course $\vec{A}_0(r)$ must still satisfy the Helmholtz equation (2.2.5). A mode function satisfying these conditions is obviously $\vec{A}_0(r) = \vec{e}_k e^{i\vec{k}\cdot\vec{r}}$, where $\vec{k}\cdot\vec{e}_k = 0$ in order to have the transversality condition $\vec{\nabla}\cdot\vec{A} = 0$ satisfied for the Coulomb gauge in which we are working.

We also wish to normalize our mode function according to equation (2.2.8). To achieve the desired normalization we pretend that space is divided into cubes of volume $V = L^3$ and impose on the field the periodic boundary condition

$$A(x+L; y+L; z+L) = A(x; y; z) \tag{2.2.20}$$

or equivalently

$$(k_x; k_y; k_z) = \frac{2\pi}{L}(n_x; n_y; n_z) \tag{2.2.21}$$

where each n can assume any integer value. Of course this artificial periodic boundary condition will have no physical consequences if L is very large compared with any physical dimensions of interest. It allows us to consider the field in any one of the imaginary cubes, and to define a mode function $\vec{A_k}(r) = V^{-\frac{1}{2}}\vec{e_k}e^{i\vec{k}\cdot\vec{r}}$ satisfying the Helmholtz equation, transversality and the "box normalization"

$$\int_V d^3r|A_k(r)|^2 = 1 \tag{2.2.22}$$

where $\vec{e_k}$ is chosen to be a unit vector. The unit vector $\vec{e}_k$, which we take to be real, specifies the polarization of the field mode. The condition $\vec{k}\cdot\vec{e_k} = 0$ means that there are two independent choices for $\vec{e}_k$, which we call e_{k1} and e_{k2}, $e_{k1}\cdot e_{k2} = 0$ and $e_{k1}^2 = e_{k2}^2 = 1$, thus we can define the mode functions

$$\vec{A}_{k\lambda}(r) = V^{-\frac{1}{2}}\vec{e}_{k\lambda}e^{i\vec{k}\cdot\vec{r}} \quad (\lambda = 1, 2) \tag{2.2.23}$$

in term of which the vector potential (2.2.12) becomes:

$$\vec{A}_{k\lambda} = \left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}[a_{k\lambda}(t)e^{i\vec{k}\cdot\vec{r}} + a_{k\lambda}^\dagger(t)e^{-i\vec{k}\cdot\vec{r}}]\vec{e}_{k\lambda} \tag{2.2.24}$$

where $\omega_k = kc$ and $a_{k\lambda}$ and $a_{k\lambda}^\dagger$ are respectively the photon annihilation and creation operators for the mode with wave vector $\vec{k}$ and polarization λ. This gives the vector potential for a plane-wave mode of the field. The condition (2.2.21) shows that there is an infinite number of such modes. The linearity of Maxwell's equations allow us to write

$$\vec{A}(r,t) = \sum_{k\lambda}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}[a_{k\lambda}(t)e^{i\vec{k}\cdot\vec{r}} + a_{k\lambda}^\dagger(t)e^{-i\vec{k}\cdot\vec{r}}]\vec{e}_{k\lambda} \tag{2.2.25}$$

for the total vector potential in the free space. Using the fact that:

$$\int_V d^3r\vec{A}_{k\lambda}(r)\cdot\vec{A}^*_{k'\lambda'}(r) = \delta^3_{kk'}\delta_{\lambda\lambda'} \tag{2.2.26}$$

we find that the Hamiltonian of the field can be written:

$$H_F = \sum_{k\lambda} \hbar\omega_k (a^\dagger_{k\lambda} a_{k\lambda} + \frac{1}{2}) \tag{2.2.27}$$

for the infinity modes in free space. This is Hamiltonian for an infinite number of uncoupled harmonic oscillator. Thus the different modes of the field are independent and satisfy the commutation relation:

$$[a_{k\lambda}(t), a^\dagger_{k'\lambda'}(t)] = \delta^3_{kk'}\delta_{\lambda\lambda'} \tag{2.2.28}$$

and $[a_{k\lambda}, a_{k'\lambda'}] = [a^\dagger_{k\lambda}, a^\dagger_{k'\lambda'}] = 0$. From the (2.2.25) it follows that:

$$\vec{E}(r,t) = i\sum_{k\lambda}\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2}[a_{k\lambda}(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger_{k\lambda}(t)e^{-i\vec{k}\cdot\vec{r}}]\vec{e}_{k\lambda} \tag{2.2.29}$$

$$\vec{B}(r,t) = i\sum_{k\lambda}\left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}[a_{k\lambda}(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger_{k\lambda}(t)e^{-i\vec{k}\cdot\vec{r}}]\vec{k}\times\vec{e}_{k\lambda} \tag{2.2.30}$$

it is worth nothing that the free space mode functions (2.2.23) form a complete set for transverse vector field satisfying our periodic boundary condition. That is, the plane-wave modes $\vec{A}_{k\lambda}(r)$ form a complete set in term of which any mode of the field may be expanded (this is essentially just a statement of Fourier's theorem about the completeness of sines and cosines). Of course $\vec{A}_{k\lambda}(r)$ are complete only for modes satisfying the periodic boundary condition, but in slightly more sophisticated approach we can work with a complete continuum of plane wave mode functions in which the $\vec{k}$ vectors are not restricted to the discrete spectrum. This has a formal consequences such as the replacement of $\delta^3_{kk'}$ in (2.2.26) and (2.2.28), but since it has no physical consequence here, we just stick to the periodic boundary condition.

The linear momentum of the field is given classically by $\vec{P} = (1/4\pi c)\int_V d^3r(\vec{E}\times\vec{B})$. In the case of quantized field we use (2.2.29) and (2.2.30) in this expression and obtain, after straightforward manipulations,

$$P = \sum_{k\lambda}\hbar\vec{k}(a^\dagger_{k\lambda}a_{k\lambda} + \frac{1}{2}) \tag{2.2.31}$$

obviously $[P, H_F] = 0$, so that the linear momentum of the field in the absence of any source is a constant of motion. It is also obvious that the eigenvalues of P are $\sum_{k\lambda}\hbar\vec{k}(n_{k\lambda} + \frac{1}{2})$ where each n is a positive integer or zero. A stationary state of the free field is thus characterized by the set of the photon numbers $\{n_{k\lambda}\}$. The state $|\{n_{k\lambda}\}\rangle$ has a total photon number $\sum_{k\lambda} n_{k\lambda}$, an energy

$$E = \sum_{k\lambda} \hbar\omega_k (n_{k\lambda} + \frac{1}{2}) \tag{2.2.32}$$

and linear momentum:

$$\vec{P} = \sum_{k\lambda} \hbar\vec{k}(n_{k\lambda} + \frac{1}{2}) \tag{2.2.33}$$

or:

$$E = \sum_{k\lambda} \hbar\omega_k n_{k\lambda} \qquad \vec{P} = \sum_{k\lambda} \hbar\vec{k} n_{k\lambda} \tag{2.2.34}$$

if the zero point energy and linear momentum associated with the vacuum state are discarded. Note that the zero-point momentum $\sum_{k\lambda} \frac{1}{2}\hbar\vec{k}$ in fact vanishes since for each $\vec{k}$ there is a equal contribution from $-\vec{k}$ in the summation. We have thus arrived at the quantum theory of the free electromagnetic field in which stationary states are described by photons of energy $\hbar\omega_k$ and linear momentum $\hbar\vec{k}$. Since $E^2 - P^2c^2 = \hbar^2(\omega_k^2 - k^2c^2) = 0$ for each photon, the photon have zero rest mass. The theory also implies that photons are bosons, i.e., that the stationary states are symmetric with respect to permutation of identical photons. To see this, note from equation (2.1.9) that the n-photon state $|n\rangle$ of a field mode may be written in the form:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{2.2.35}$$

which is obviously symmetric with respect to any permutation of the n photons. Of course the boson character of photons is just a consequence of the commutation rule (2.2.28), from which the (2.2.35) follows.

2.3 Casimir Force

In the previous section we assumed that the zero-point energy cannot have any effect in the physical measurement. Casimir showed in 1948 that one consequence of the zero-point field is an attractive force between two uncharged, perfectly conducting parallel plates (Figure 2.1).

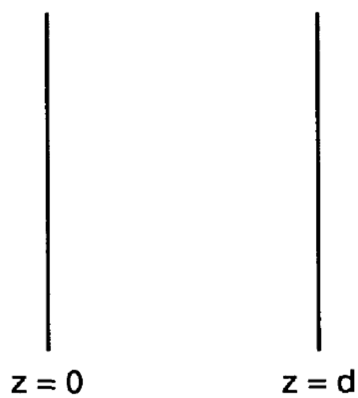

Figure 2.3.1: Two conducting parallel planes experience an attractive force attributable to the zero-point electromagnetic field

The physical situation shown in Figure 2.1 leads us to consider a different set of modes than the free-space plane-wave modes we have dealt with so far. Consider first the modes appropriate to the interior of a rectangular parallelepiped of sides $L_x = L_y = L$ and L_z. For conducting walls the mode functions such that the tangential component of the electric field vanishes on the walls are $\vec{A}(r) = A_x(r)\hat{\imath} + A_y(r)\hat{\jmath} + A_z(r)\hat{k}$ where:

$$A_x(r) = (8/V)^{1/2} a_x \cos(k_x x) \sin(k_y y) \sin(k_z z) \tag{2.3.1}$$
$$A_y(r) = (8/V)^{1/2} a_y \sin(k_x x) \cos(k_y y) \sin(k_z z) \tag{2.3.2}$$
$$A_z(r) = (8/V)^{1/2} a_z \sin(k_x x) \sin(k_y y) \cos(k_z z) \tag{2.3.3}$$

with $a_x^2 + a_y^2 + a_z^2 = 1$ $V = L^2 L_z$ and

$$k_x = \frac{l\pi}{L} \quad k_y = \frac{m\pi}{L} \quad k_z = \frac{n\pi}{L_z} \tag{2.3.4}$$

with l, m and n each taking on all positive integer values and zero. In order to satisfy the transversality condition $\nabla \cdot A = 0$ we also require

$$k_x A_x + k_y A_y + k_z A_z = \frac{\pi}{L}(lA_x + mA_y) + \frac{\pi}{L_z}(nA_z) = 0. \tag{2.3.5}$$

Thus there are two independent polarization, unless one of integers l, m or n is zero, in which case (2.3.5) indicates that there is only one polarization. It is easy to check that equations (2.3.1) – (2.3.3) define transverse mode function satisfying the Helmholtz equation (2.2.5) as well as the condition that the transverse components of $\vec{E}$ vanish on the cavity walls. Furthermore these mode functions are orthogonal and satisfy the normalization condition, i.e,

$$\int_0^L dx \int_0^L dy \int_0^{L_z} dz[A_x(r)^2 + A_y(r)^2 + A_z(r)^2] = 1 \tag{2.3.6}$$

actually all we really require for the calculation of the Casimir force are the allowed frequencies defined by (2.3.4):

$$\omega_{lmn} = k_{lmn}c = \pi c \left[\frac{l^2}{L^2} + \frac{m^2}{L^2} + \frac{n^2}{L_z^2}\right]^{1/2} \tag{2.3.7}$$

the zero-point energy of the field inside the cavity is therefore

$$\sum_{l,m,n}{}' (2)\frac{1}{2}\hbar\omega_{lmn} = \sum_{lmn}{}' \pi\hbar c \left[\frac{l^2}{L^2} + \frac{m^2}{L^2} + \frac{n^2}{L_z^2}\right]^{1/2} \tag{2.3.8}$$

the factor 2 arises from the two independent polarization of modes with $l, m, n \neq 0$, and the prime on the summation symbol implies that a factor 1/2 should be inserted if all this integers are zero, for then we have just one independent polarization as noted earlier.

In the physical situation of interest L is so large compared with $L_z = d$ that we may replace the sums over l and m in (2.3.8) by integrals: $\sum_{lmn} \to \sum_n' (L/\pi)^2 \int\int dk_x dk_y$ and

$$E(d) = \frac{L^2}{\pi^2}(\hbar c)\sum_n{}' \int_0^\infty dk_x \int_0^\infty dk_y (k_x^2 + k_y^2 + \frac{n^2\pi^2}{d^2})^{1/2} \tag{2.3.9}$$

this is infinite; the zero-point energy of the vacuum is infinite in any finite volume. If d were also made arbitrarily large, the sum over n could be replaced by an integral. Then the zero-point energy (2.3.9) would be:

$$E(\infty) = \frac{L^2}{\pi^2}(\hbar c)\frac{d}{\pi}\int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z (k_x^2 + k_y^2 + k_z^2)^{1/2} \tag{2.3.10}$$

which is also infinite.

The potential energy of the system when the plates are separated by a distance

d is $U(d) = E(d) - E(\infty)$, the energy required to bring the plates from a large separation to the separation d

$$U(d) = \frac{L^2\hbar c}{\pi^2}[\sum_n{}' \int_0^\infty dk_x \int_0^\infty dk_y (k_x^2 + k_y^2 + \frac{n^2\pi^2}{d^2})^{1/2} - \\ + \frac{d}{\pi}\int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z (k_x^2 + k_y^2 + k_z^2)^{1/2}] \quad (2.3.11)$$

this is the difference between two infinites quantities, but we shall now show that it is nonetheless possible to extract from it a physically meaningful, finite value. In polar coordinate u, θ in the k_x, k_y plane ($dk_x dk_y = u du d\theta$) we have:

$$U(d) = \frac{L^2\hbar c}{\pi^2}\left(\frac{\pi}{2}\right)[\sum_{n=0}^{\infty}{}' \int_0^\infty du u \left(u^2 + \frac{n^2\pi^2}{d^2}\right)^{1/2} + \quad (2.3.12) \\ - \left(\frac{d}{\pi}\right)\int_0^\infty dk_z \int_0^\infty du u (u^2 + k_z^2)^{1/2}]$$

since θ ranges from 0 to $\frac{\pi}{2}$ for $k_x, k_y > 0$. We now introduce a cutoff function $f(k) = f([u^2 + k_z^2]^{1/2})$ such that $f(k) = 1$ for $k \ll k_m$ and $f(k) = 0$ for $k \gg k_m$. Physically, it can be argued that $f(k)$ is necessary because the assumption of perfectly conducting walls breaks down at small wavelengths and especially for wavelengths small compared with an atomic dimension. We might then suppose that $k_m \simeq 1/a_0$ where a_0 is the Bohr radius. What we are assuming here is that the Casimir effect is primarily a low-frequencies non relativistic effect. We thus replace (2.3.12) by:

$$U(d) = \frac{L^2\hbar c}{\pi^2}\left(\frac{\pi}{2}\right)[\sum_{n=0}^{\infty}{}' \int_0^\infty du u \left(u^2 + \frac{n^2\pi^2}{d^2}\right)^{1/2} f\left([u^2 + \frac{n^2\pi^2}{d^2}]^{1/2}\right) + \\ - \left(\frac{d}{\pi}\right)\int_0^\infty dk_z \int_0^\infty du u (u^2 + k_z^2)^{1/2} f([u^2 + k_z^2]^{1/2})] \quad (2.3.13)$$

$$U(d) = \frac{L^2\hbar c}{4\pi}\left(\frac{\pi^3}{d^3}\right)[\sum_{n=0}^{\infty}{}' \int_0^\infty dx (x + n^2)^{1/2} f(\frac{\pi}{d}[x + n^2]^{1/2}) + \\ - \int_0^\infty d\kappa \int_0^\infty dx (x + \kappa^2)^{1/2} f(\frac{\pi}{d}[x + \kappa^2]^{1/2})] \quad (2.3.14)$$

where we have defined the new integration variables $x = u^2 d^2/\pi^2$ and $\kappa =$

$k_z d/\pi$. Now

$$U(d) = \left(\frac{\pi^2\hbar c}{4d^3}\right) L^2 \left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^{\infty} d\kappa F(\kappa)\right] \tag{2.3.15}$$

where

$$F(\kappa) = \int_0^{\infty} dx(x+\kappa^2)^{1/2} f([x+\kappa^2]^{1/2}) \tag{2.3.16}$$

according to Euler-MacLaurin ([1], [5], [6]) summation formula:

$$\sum_{n=1}^{\infty} F(n) - \int_0^{\infty} F(\kappa)d\kappa = -\frac{1}{2}F(0) - \frac{1}{12}F'(0) + \frac{1}{720}F'''(0) \tag{2.3.17}$$

for $F(\infty) \to 0$. To evaluate the n-th derivative $F^{(n)}(0)$ we note that

$$F(\kappa) = \int_{\kappa^2}^{\infty} du\sqrt{u} f(\frac{\pi}{d}\sqrt{u}) \qquad F^{'}(\kappa) = -2\kappa^2 f(\frac{\pi}{d}\kappa) \tag{2.3.18}$$

then $F'(0) = 0$ and $F^{'''}(0) = 0$ and all higher derivative $F^{(n)}(0)$ vanish if we assume that all derivatives of the cutoff function vanish at $\kappa = 0$. Thus $\sum_{n=1}^{\infty} F(n) - \int_0^{\infty} F(\kappa)d\kappa = -\frac{1}{2}F(0) - \frac{4}{720}$ and

$$U(d) = \left(\frac{\pi^2\hbar c}{4d^3}\right) L^2 \left(-\frac{4}{720}\right) = -\left(\frac{\pi^2\hbar c}{720d^3}\right) L^2 \tag{2.3.19}$$

which is finite and independent of the cutoff function. The attractive force per unit area between the plates is then $F(d) = -\pi^2\hbar c/240d^4$.

2.4 Casimir effect in a stratified media

In the vacuum, we solved the eigenvalues problem for obtaining the solution of the Casimir effect. When we consider a stratified dielectric material characterized by a dielectric constant($\epsilon(\omega)$) the eigenvalue problem cannot be solved for this reason, in this section, we use an another approach to obtain the Casimir energy.

This is the configuration investigated by E.M. Lifshitz ([7], [8]) for which he obtained the general representation of the Van der Waals and Casimir force in terms of the frequency dependent dielectric permettivities for an arbitrary number of plane-parallel layers of different material.

The original Lifshitz derivation was based on the assumption that the dielectric materials can be considered as continuous media characterized by randomly fluctuating sources. The correlation function of these sources, situated at different point, is proportional to the δ- function of the radius-vector joining these point. The force per unit area acting upon one of the semispaces was calculated as the flux incoming momentum into this semispaces through the boundary plane. The flux is given by the appropriate component of the stress tensor ($zz-$component if xy is the boundary plane). Usual boundary condition on the boundary surfaces between different media were imposed on the Green's functions. To exclude the divergences, the value of all Green's function in vacuum were subtracted of their values in the dielectric media.

Here we present another derivation of the Lifshitz ([7], [9], [10]) result and their generalization starting directly from the zero-point energy of electromagnetic field. In doing so, the continuous media, characterized by the frequency dependent dielectric permettivities, and appropriate boundary conditions on the photon states, can be considered as some effective external field (which cannot be described, however, by a potential added to the left-hand side of wave equation).

In the experiment on the Casimir force measurement, symmetrical configuration are usually used, i.e., both interacting bodies are made of one and the same material which is covered by a thin layers of another material. In line with this let us consider the configuration presented in Figure 2.2. The layer is of thickness a if the permettivities is $\epsilon_2(\omega)$ while is of thickness d if the permettivities is $\epsilon_1(\omega)$.

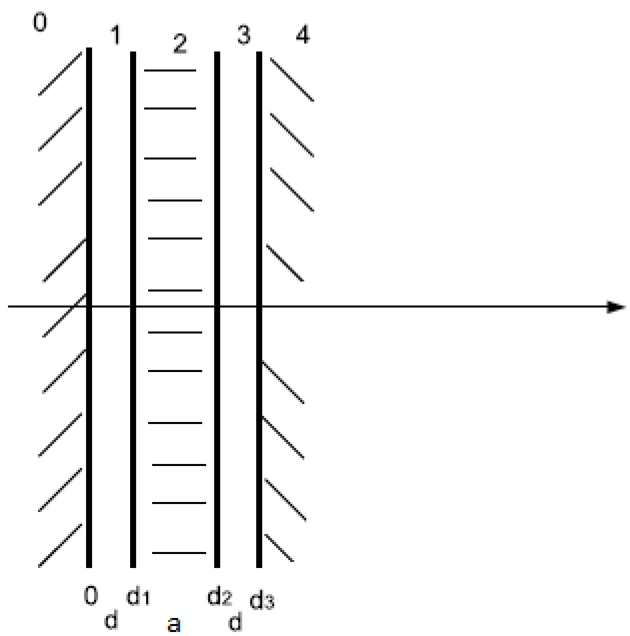

Figure 2.4.1: A two layer cavities, where the layer 1,3 have thickness d instead 2 have thickness a

In line with eq.(2.3.9) the non-renormalized vacuum energy density of electromagnetic field reads:

$$E_S(a,d) = \frac{E_0(a,d)}{S} = \frac{\hbar}{2}\int \frac{dk_1 dk_2}{(2\pi)^2}\sum_n(\omega^{(1)}_{k_\perp,n} + \omega^{(2)}_{k_\perp,n}) \tag{2.4.1}$$

where we have separated the proper frequencies of the modes with two different polarization of the electric field (parallel and perpendicular to the plane formed by $k_\perp$ and z-axis respectively). Here $k_\perp = (k_1, k_2)$ is the two dimensional propagation vector in $xy-$planes. For simplicity the $x-$axis is chosen to be parallel to $k_\perp$.

In order to solve this problem we use the formalism of surface modes ([7], [9], [10]) which are exponentially damped for $z > d_3 \to z > 2d + a$ and $z < 0$. These modes describe waves propagating parallel to the surface of the walls. To find these modes let us represent the renormalized set of negative-frequency solutions to Maxwell equations in the form

$$\begin{aligned} \vec{E}^{(i)}_{k_\perp,n}(t,\vec{r}) &= \vec{f}^{(i)}_\alpha(k_\perp, z)e^{i(k_x x + k_y y) - i\omega t} \\ \vec{B}^{(i)}_{k_\perp,n}(t,\vec{r}) &= \vec{g}^{(i)}_\alpha(k_\perp, z)e^{i(k_x x + k_y y) - i\omega t} \end{aligned} \tag{2.4.2}$$

where the index i numerates the same state of polarization as in eq.(2.4.1), index α numerates the regions show in Figure 2.2.

From Maxwell equations the wave equation for the $z-$dependent vector functions follows

$$\frac{d^2\vec{f}^{(i)}_\alpha}{dz^2} - R^2_\alpha \vec{f}^{(i)}_\alpha = 0 \qquad \frac{d^2\vec{g}^{(i)}_\alpha}{dz^2} - R^2_\alpha \vec{g}^{(i)}_\alpha = 0 \tag{2.4.3}$$

where the notation is introduced

$$R^2_\alpha = k^2_\perp - \epsilon_\alpha(\omega)\frac{\omega^2}{c^2} \quad k^2_\perp = k^2_1 + k^2_2 \quad \alpha = 0, 1, 2, 3, 4 \tag{2.4.4}$$

in obtaining eq.(2.4.3) we have assumed that the media are isotropic so that the electric displacement is $\vec{D}_\alpha = \epsilon_\alpha \vec{E}_\alpha$.

According to boundary conditions at the interface between two dielectrics the normal component of $\vec{D}$ and the tangential component of $\vec{E}$ should be continuous. Also $\vec{B_n}$ and $\vec{H_t} = \vec{B_t}$ (in our case of non-magnetic media) are continuous. It is easy to verify that all these conditions are satisfied automatically if the quantities $\epsilon_\alpha \vec{f}^{(1)}_\alpha$ and $df^{(1)}/dz$ or $f^{(2)}_\alpha$ and $df^{(2)}_\alpha/dz$ are continuous. Let us consider in detail the first of these conditions.

According to eq.(2.4.3) the surface modes $f^{(1)}_z$ in different regions of Figure 2.2

can be represented as the following combinations of exponents:

$$
\begin{aligned}
f_z^{(1)} &= Ae^{R_0 z} \quad z<0\\
f_z^{(1)} &= Be^{R_1 z} + Ce^{-R_1 z} \quad 0<z<d_1\\
f_z^{(1)} &= De^{R_2 z} + Ee^{-R_2 z} \quad d_1<z<d_2\\
f_z^{(1)} &= Fe^{R_3 z} + Ge^{-R_3 z} \quad d_2<z<d_3\\
f_z^{(1)} &= He^{-R_4 z} \quad z>d_3
\end{aligned}
\tag{2.4.5}
$$

where $R_0 = R_2 = R_4$ and $R_1 = R_3$, imposing the continuity conditions on $\epsilon_\alpha f_{z,\alpha}^{(1)}$ and $df_{z,\alpha}^{(1)}/dz$ at the points $z = 0, d_1, d_2, d_3$. If we consider the thickness of the layers assume that $d_1 = d,\ d_2 = a + d,\ d_3 = 2d + a$, taking to account eq.(2.4.5) we arrive at following system of equations:

$$
\begin{aligned}
A\epsilon_2 &= B\epsilon_1 + C\epsilon_1\\
AR_2 &= BR_1 - CR_1\\
\epsilon_1(Be^{R_1 d} + Ce^{-R_1 d}) &= \epsilon_2(De^{R_2 d} + Ee^{-R_2 d})\\
BR_1 e^{R_1 d} - CR_1 e^{-R_1 d} &= DR_2 e^{R_2 d} - ER_2 e^{-R_2 d}\\
\epsilon_2(De^{R_2(a+d)} + Ee^{-R_2(a+d)}) &= \epsilon_1(Fe^{R_1(a+d)} + Ge^{-R_1(a+d)})\\
DR_2 e^{R_2(a+d)} - ER_2 e^{-R_2(a+d)} &= FR_1 e^{R_1(a+d)} - GR_1 e^{-R_1(a+d)}\\
\epsilon_1(Fe^{R_1(2d+a)} + Ge^{-R_1(2d+a)}) &= \epsilon_2(He^{-R_2(2d+a)})\\
FR_1 e^{R_1(2d+a)} - GR_1 e^{-R_1(2d+a)} &= -R_2(He^{-R_2(2d+a)})
\end{aligned}
\tag{2.4.6}
$$

this is a linear homogeneous system of algebraic equations relating the unknown coefficient $A, B, ...H$. It has non trivial solutions under the condition that the determinant of its coefficient is equal to zero. This condition is, accordingly, the equation for the determination of the proper frequencies $\omega_{k_\perp,n}^{(1)}$ of the modes with a parallel polarization([7], [12]):

$$
\begin{aligned}
\Delta^{(1)}(\omega_{k_\perp,n}^{(1)}) = e^{(a+d)(R_1-R_2)}\{&(r_{12}^+)^4 + (r_{12}^-)^4 e^{-2(a+d)R_1} + (r_{12}^+)^2 (r_{12}^-)^2 [\\
&e^{-2a(R_2+R_1)} - e^{-2aR_2} - e^{-2a(R_2+R_1)-2dR_1} + e^{-2aR_2-2dR_1} - e^{-2dR_1} - e^{-2aR_1}]\}
\end{aligned}
\tag{2.4.7}
$$

here the following notations are introduced

$$
r_{ij}^\pm = \epsilon_i R_j \pm \epsilon_i R_j \qquad q_{ij}^\pm = R_j \pm R_i \quad i,j = 1,2 \tag{2.4.8}
$$

similarly the requirement that the quantities $f^{(2)}_{y,\alpha}$ and $df^{(2)}_{y,\alpha}/dz$ are continuous at the boundaries results in the equations for determination of the frequencies $\omega^{(2)}_{k_\perp,n}$ of the perpendicular polarized modes([7], [12]):

$$\Delta^{(2)}(\omega^{(2)}_{k_\perp,n}) = e^{(a+d)(R_1-R_2)}\{(q^+_{12})^4 + (q^-_{12})^4 e^{-2(a+d)R_1} + (q^+_{12})^2(q^-_{12})^2[\\ e^{-2a(R_2+R_1)} - e^{-2aR_2} - e^{-2a(R_2+R_1)-2dR_1} + e^{-2aR_2-2dR_1} - e^{-2dR_1} - e^{-2aR_1}]\}. \tag{2.4.9}$$

Given that we have a dielectric material, we not solve the eigenvalue problem (instead in the vacuum we obtain that $\omega_{k_\perp,n} = \sqrt{k_1^2 + k_2^2 + \frac{\pi^2 n^2}{d^2}}$). For this reason, summation in eq.(2.4.1) over the solutions (2.4.7) and (2.4.9) can be performed by applying the argument theorem([7], [9], [10]):

$$\sum_n \omega^{1,2}_{k_\perp,n} = \frac{1}{2\pi i}\left[\int_{i\infty}^{-i\infty} \omega d\log\Delta^{1,2}(\omega) + \int_{C_+} \omega d\log\Delta^{1,2}(\omega)\right] \tag{2.4.10}$$

where C^+ is a semicircle of infinite radius in the right one-half of the complex $\omega-$plane with a center at the origin. Notice that the functions $\Delta^{1,2}(\omega)$ defined in eq.(2.4.7) and eq.(2.4.9) have no poles. For this reason the sum over their poles is absent from (2.4.10).

The second integral in the right-side of (2.4.10) is simply calculated with the natural supposition that:

$$\lim_{\omega\to\infty} \epsilon_\alpha(\omega) = 1 \qquad \lim_{\omega\to\infty} \frac{d\epsilon_\alpha(\omega)}{dz} = 0 \tag{2.4.11}$$

along any radial direction in complex $\omega - plane$. The result is infinite, and does not depend on a:

$$\int_{C_+} \omega d\log\Delta^{1,2}(\omega) = 4\int_{C_+} d\omega. \tag{2.4.12}$$

Now we introduce a new variable $\xi = -i\omega$ in eq.(2.4.10), eq.(2.4.12). The result is

$$\sum_n \omega^{1,2}_{k_\perp,n} = \frac{1}{2\pi}\int_{\infty}^{-\infty} \xi d\log\Delta^{1,2}(i\xi) + \frac{2}{\pi}\int_{C_+} d\xi \tag{2.4.13}$$

where both contributions in the right-hand side diverge. To remove the divergences we use the renormalization procedure ([7], [13]) that the renormalized physical vacuum energy density vanishes for the infinitely separated interacting bodies. From eq.(2.4.7), eq.(2.4.9) and eq.(2.4.13) it follows:

$$\lim_{a,d\to\infty}\sum_n \omega^{1,2}_{k_\perp,n} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\xi d\log\Delta^{(1,2)}_\infty(i\xi) + \frac{2}{\pi}\int_{C_+} d\xi \tag{2.4.14}$$

where the asymptotic behavior of $\Delta^{1,2}$ at $d, a \to \infty$ is given by

$$\begin{aligned}\Delta^1_\infty &= e^{(a+d)(R_1-R_2)}(r^+_{12})^4\\ \Delta^2_\infty &= e^{(a+d)(R_1-R_2)}(q^+_{12})^4\end{aligned} \tag{2.4.15}$$

now the renormalized physical quantities are found with the help eq.(2.4.13)–eq.(2.4.15):

$$\begin{aligned}\left(\sum_n \omega^{(1,2)}_{k_\perp,n}\right)_{ren} &= \sum_n \omega^{1,2}_{k_\perp,n} - \lim_{a,d\to\infty}\sum_n \omega^{1,2}_{k_\perp,n}\\ &= \int_\infty^{-\infty}\xi d\ln\frac{\Delta^{(1,2)}(i\xi)}{\Delta^{(1,2)}_\infty(i\xi)}\end{aligned} \tag{2.4.16}$$

they can be transformed to a more convenient form with the help of integration by parts:

$$\left(\sum_n \omega^{(1,2)}_{k_\perp,n}\right)_{ren} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \ln\frac{\Delta^{(1,2)}(i\xi)}{\Delta^{(1,2)}_\infty(i\xi)} \tag{2.4.17}$$

where the term outside the integral vanishes. To obtain the physical renormalized Casimir energy density one should substitute the renormalized quantities eq.(2.4.17) instead of eq.(2.4.13) into eq.(2.4.1) with the result:

$$E^{ren}_S(a,d) = \frac{\hbar}{4\pi}\int_0^\infty k_\perp dk_\perp \int_0^\infty d\xi[\ln Q_1(i\xi) + \ln Q_2(i\xi)] \tag{2.4.18}$$

where we introduced polar coordinates in k_1, k_2 plane, and:

$$\begin{aligned}Q_1(i\xi) = \frac{\Delta^{(1)}(i\xi)}{\Delta^{(1)}_\infty(i\xi)} &= 1 + \frac{(r^-_{12})^4}{(r^+_{12})^4}e^{-2(a+d)R_1} + \frac{(r^-_{12})^2}{(r^+_{12})^2}[e^{-2a(R_2+R_1)} - e^{-2aR_2}+\\ &- e^{-2a(R_2+R_1)-2dR_1} + e^{-2aR_2-2dR_1} - e^{-2dR_1} - e^{-2aR_1}]\end{aligned} \tag{2.4.19}$$

$$\begin{aligned}Q_2(i\xi) = \frac{\Delta^{(2)}(i\xi)}{\Delta^{(2)}_\infty(i\xi)} &= 1 + \frac{(q^-_{12})^4}{(q^+_{12})^4}e^{-2(a+d)R_1} + \frac{(q^-_{12})^2}{(q^+_{12})^2}[e^{-2a(R_2+R_1)} - e^{-2aR_2}+\\ &- e^{-2a(R_2+R_1)-2dR_1} + e^{-2aR_2-2dR_1} - e^{-2dR_1} - e^{-2aR_1}]\end{aligned} \tag{2.4.20}$$

may be possible return to one cavity case if take the limit $a \to \infty$ while d is fixed:

$$\Delta^{'(1)} = \lim_{a\to\infty} \Delta^{(1)} = e^{d(R_1-R_2)}[(r_{12}^+)^4 - (r_{12}^+)^2(r_{12}^-)^2 e^{-2dR_1}] \tag{2.4.21}$$

$$\Delta^{'(2)} = \lim_{a\to\infty} \Delta^{(2)} = e^{d(R_1-R_2)}[(q_{12}^+)^4 - (q_{12}^+)^2(q_{12}^-)^2 e^{-2dR_1}] \tag{2.4.22}$$

note that this terms are not renormalized.

These formulae eq.(2.4.19), eq.(2.4.20) can be very much simplified when $a = d$:

$$Q_1^{'}(i\xi) = \left(1 + \frac{(r_{12}^-)^2 e^{-2R_1 d}}{(r_{12}^+)^2}\right)^2 + e^{-2dR_2}\frac{(r_{12}^-)^2}{(r_{12}^+)^2}\left(-e^{-4R_1 d} + 2e^{-2dR_1} - 1\right) \tag{2.4.23}$$

$$Q_2^{'}(i\xi) = \left(1 + \frac{(q_{12}^-)^2 e^{-2R_1 d}}{(q_{12}^+)^2}\right)^2 + e^{-2dR_2}\frac{(q_{12}^-)^2}{(q_{12}^+)^2}\left(-e^{-4R_1 d} + 2e^{-2dR_1} - 1\right) \tag{2.4.24}$$

Chapter 3

Casimir Effect in plasma sheet model

In the previous section we have analyzed the Casimir energy for the dielectric model. Now we consider a new model for the description of the cavity, that so called "plasma sheet" model.

In this section after a brief description , we evaluate the Casimir energy for one cavity within two plasma sheet and extend the formula for a quantum system at non zero temperature. Finally we analyze what happen for different number of cavities.

3.1 Model Description

We start a brief outline of the Plasma sheet model ([14],[16]). Let us consider an infinitesimally thin and indefinitely extended flat sheet occupying the $xy-$plane, carrying a continuous fluid with mass and charge densities nm, ne per unit area, plus an immobile, uniformly distributed, overall-neutralizing background charge. The subscript $\|$ indicates vector component parallel to the sheet, and $\vec{r} = (\vec{s}, z)$, i.e., $\vec{s} = \vec{r}_{\|} = (x, y)$. The fluid displacement $\vec{\xi}$ is purely tangential, with surface charge and current densities given by:

$$\sigma = -ne\vec{\nabla}_{\|} \cdot \vec{\xi} \qquad \vec{J} = ne\dot{\vec{\xi}} \tag{3.1.1}$$

the motion is assumed to be nonrelativistic ($\dot{\xi} \ll c$), so that the Lorentz force is negligible and Newton's second law reads

$$\partial^2 \vec{\xi}(\vec{s}, t)/\partial t^2 = (e/m)\vec{E}_{\parallel}(\vec{s}, z = 0, t) \tag{3.1.2}$$

evidently the model mimic n delocalized particles per unit area, call them electrons, with charge and mass e, m. The surface density n is related to some mean inter-electron distance a by:

$$n = 1/a^2 \tag{3.1.3}$$

merely for orientation, we shall form rough estimates with a of the order of a few Bohr radii far longer than the classical electron radius r_0.

$$a \simeq a_B = \hbar^2/me^2 \quad r_0 = e^2/mc^2 \quad r_0/a \simeq (e^2/\hbar c)^2 = \alpha \simeq (1/137) \tag{3.1.4}$$

moreover, we impose a Debye-type cutoff K on the surface-parallel wave numbers of waves that the fluid can support, and by the same token also on Maxwell waves that can interact effectively with the fluid as such (as distinct from the individual charge carriers out of which in the last analysis the fluid is formed). In detail ([14], [16]). Thus:

$$\pi K^2/(2\pi)^2 = n \Rightarrow K = \sqrt{4\pi}/a \tag{3.1.5}$$

we define another characteristic wave number $q = (2\pi/c^2)(ne)^2/(nm)$ and observe that the input parameters of the model (ne, nm, K and through Maxwell's equations also c) admit one dimensionless combination X:

$$q = 2\pi ne^2/mc^2 = 2\pi n r_0 \qquad X = K/q = a/r_0\sqrt{\pi} \simeq 1/\alpha^2 \gg 1 \tag{3.1.6}$$

we list three widely considered limits that can highlight important features of the more complicated exact result.

- The nonretarded (NR) limit, $c \to \infty$ at fixed a and K, entailing $q \to 0$, with $c\sqrt{q}$ fixed and finite, and $X \to \infty$. In the nonretarded model there are no photons, and the only excitation of the sheet are surface plasmons.

- The perfect-reflector (PR) limit, designed to make the sheet reflect perfectly at all frequencies, which will be seen presently to require $q \to \infty$.

- The no-cutoff (NC) limit is artificial in that it abandons the Debye connection between a and K and contemplates $K \to \infty$ at fixed a and fixed q.

For the normal modes, with all time variation described by a common factor $exp(-i\omega t)$, equation (3.1.2) and Maxwell's equations plus (3.1.1), read:

$$\vec{\xi} = -(e/m\omega^2)\vec{E}_{\parallel} \quad \vec{J} = -i\omega n e\vec{\xi} \quad \sigma = -ne\vec{\nabla}_{\parallel} \cdot \vec{\xi} \tag{3.1.7}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} - i\omega\vec{B}/c = 0 \tag{3.1.8}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\delta(z)\sigma \qquad \vec{\nabla} \times \vec{B} + i\omega\vec{E}/c = 4\pi\delta(z)\vec{J}/c \tag{3.1.9}$$

To obtain the matching conditions on the field, we integrate equations (3.1.9) across the sheet, which amounts to applying Gauss' law and Ampère's law. They yield

$$[\vec{E}_{\parallel}]_S = 0 \quad [E_z] = 2q(c/\omega)^2\vec{\nabla}_{\parallel} \cdot \vec{E}_{\parallel} \tag{3.1.10}$$

$$[B_z] = 0 \quad [\vec{B}_{\parallel}]_S = -i2q(c/\omega)\hat{z} \times \vec{E}_{\parallel} \tag{3.1.11}$$

3.2 Vacuum energy

In the following section, after having described the plasma sheet model, we study the Casimir effect for thin plasma sheet using the same reasoning as the dielectric case.

Let us start with eq.(3.1.8), eq.(3.1.9), the solutions can be written:

$$\vec{E}(\vec{x},t) = \vec{f}_{\alpha}(\vec{k_{\parallel}}, z)e^{i(\vec{k_{\parallel}}\cdot\vec{x_{\parallel}} - \omega t)} \qquad \vec{B}(\vec{x},t) = \vec{g}_{\alpha}(\vec{k_{\parallel}}, z)e^{i(\vec{k_{\parallel}}\cdot\vec{x_{\parallel}} - \omega t)} \tag{3.2.1}$$

where $\vec{f} = (f_1, f_2, f_3)$, $\vec{g} = (g_1, g_2, g_3)$, $\vec{k_{\parallel}} = (k_1, k_2)$ and $\vec{x_{\parallel}} = (x_1, x_2)$ and $\alpha = 0, 1, ...n$ numerates the regions (for instance see figure 2.2).

We can show that (see Appendix A) $f_1, f_2, g_1 g_2$ can be written as functions of f_3, g_3 and their derivative, where f_3 and g_3 are obtained by:

$$\frac{\partial^2 f_3^{\alpha}}{\partial z^2} + \left(\frac{\epsilon_{\alpha}\omega^2}{c^2} - |k_{\parallel}^2|\right) f_3^{\alpha} = 0 \qquad \frac{\partial^2 g_3^{\alpha}}{\partial z^2} + \left(\frac{\epsilon_{\alpha}\omega^2}{c^2} - |k_{\parallel}^2|\right) g_3^{\alpha} = 0 \tag{3.2.2}$$

the solutions of this equation are:

$$f_3, g_3 \simeq A e^{iK_\alpha z} + B e^{-iK_\alpha z} \tag{3.2.3}$$

where the asymptotic behavior of f_3, g_3 is:

$$z \to -\infty \Rightarrow e^{iR_\alpha z} + r e^{-iR_\alpha z} \qquad z \to \infty \Rightarrow t e^{iR_\alpha z} \tag{3.2.4}$$

where r and t are the reflection and transmission coefficient and K_α is the wave number in the direction perpendicular to the planes. The frequency of these solutions follows from the dispersion relation $\epsilon_\alpha \omega^2/c^2 = k_\parallel^2 + K_\alpha^2$.

The matching conditions in the plasma sheet model demand $E_\parallel$ to be continuous across the surface and for the jump of the normal component of $\vec{D}$, $[D_z] = \frac{2\Omega}{\omega^2}\vec{\nabla}_\parallel \cdot \vec{E}_\parallel$ to hold. The parameter Ω is proportional to the density of the carriers in the plasma sheet and coincides with the parameter q in eq.(3.1.6). (For the details refer to [14]).

In this model, the polarization can be separated into TE and TM too and the matching conditions then read (see Appendix A):

$$g_3^+ - g_3^- = 0 \quad \partial_z g_3^+ - \partial_z g_3^- = 2\Omega g_3 \quad \text{(TE)} \tag{3.2.5}$$

$$\partial_z f_3^+ - \partial_z f_3^- = 0 \quad \epsilon^+ f_3^+ - \epsilon^- f_3^- = -2\frac{\Omega}{\omega^2}\partial_z f_3 \quad \text{(TM)} \tag{3.2.6}$$

The matching conditions for TE polarizations is the same of the dielectric case except for a delta function potential on a plane with streght Ω, instead the matching conditions for TM polarization depend by a potential δ' with streght $-2\frac{\Omega}{\omega^2}$.

Let us consider two parallel plasma sheet at distance L. We have two continuous media with $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ dielectric permettivities where we find in $z < 0$ and $z > L$ $\epsilon_1(\omega)$ otherwise we find $\epsilon_2(\omega)$. According to eq.(3.2.2) f_3, g_3 have the following combination of exponent:

$$\begin{aligned} f_3^{(0)} &= e^{iK_0 z} + r e^{-iK_0 z} \\ f_3^{(1)} &= r_1 e^{-iK_1 z} + t_1 e^{iK_1 z} \\ f_3^{(2)} &= t e^{iK_0 z} \end{aligned} \tag{3.2.7}$$

where in this case $K_2 = K_0$. Using the matching conditions eq.(3.2.5) in $z = 0$

and $z = L$ we obtain the following system of equation:

$$\begin{aligned} r_1 + t_1 - 1 - r &= 0 \\ iK_1(r_1 - t_1) - iK_0(1 - r) &= 2\Omega(r_1 + t_1) \\ te^{iK_0L} - r_1e^{-iK_1L} - t_1e^{iK_1L} &= 0 \\ iK_0te^{iK_0L} - iK_1(t_1e^{iK_1L} - r_1e^{-iK_1L}) &= 2\Omega te^{iK_0L} \end{aligned} \quad (3.2.8)$$

Now, solving the system eq.(3.2.8) we obtain:

$$\Delta_1^{(TE)} = e^{iL(K_0-K_1)}[(K_0 + K_1 + 2i\Omega)(-K_0 - K_1 - 2i\Omega) + e^{2iLK_1}(K_0 - K_1 + 2i\Omega)^2] \quad (3.2.9)$$

$$t^{(TE)}(k) = \frac{-4e^{-iL(K_0-K_1)}K_0K_1}{\Delta_1^{TE}}. \quad (3.2.10)$$

We can use the same reasoning with the boundary condition of TM case and obtain:

$$\Delta_1^{(TM)} = e^{iL(K_0-K_1)}[(iK_0\epsilon_1 + iK_1\epsilon_0 - 2\frac{\Omega}{\omega^2}K_0K_1)^2 - e^{2iLK_1}(iK_0\epsilon_1 - iK_1\epsilon_0 + 2\frac{\Omega}{\omega^2}K_0K_1)^2] \quad (3.2.11)$$

$$t^{(TM)} = \frac{-4e^{-iL(K_0-K_1)}K_0K_1\epsilon_0\epsilon_1}{\Delta_1^{(TM)}} \quad (3.2.12)$$

the non renormalized vacuum energy, in the plasma sheet model, can be written:

$$E_0 = \frac{1}{2}\int \frac{dk_\parallel}{(2\pi)^2}\left\{\int_0^\infty \frac{dk}{2\pi i}\omega(k_\parallel, k)\frac{\partial}{\partial k}\frac{t(k)}{t(-k)}\right\} \quad (3.2.13)$$

where $t(k)$ is the transmission coefficient eq.(3.2.10), eq.(3.2.12) for the TE and TM polarization respectively.

The eq.(3.2.13) is highly oscillating and it is difficult to calculate numerically. For this reason, in general we rotate k along imaginary axis, using argument theorem (2.4.13) and obtain:

$$E_0 = \frac{1}{2}\int \frac{dk_\parallel}{(2\pi)^2}\left\{\int_0^\infty \frac{dk}{\pi}\omega(k_\parallel, ik)\frac{\partial}{\partial k}\log t(ik)\right\} \quad (3.2.14)$$

where:

$$\Delta_1^{(TE)}(ik) = e^{-L(K_0-K_1)}[(K_0+K_1+2\Omega)^2 - e^{-2LK_1}(K_0-K_1+2\Omega)^2] \tag{3.2.15}$$

$$\Delta_1^{(TM)}(ik) = e^{-L(K_0-K_1)}[(K_0\epsilon_1+K_1\epsilon_0-2\frac{\Omega}{\omega^2}K_0K_1)^2 - e^{-2LK_1}(-K_0\epsilon_1+K_1\epsilon_0-2\frac{\Omega}{\omega^2}K_0K_1)^2] \tag{3.2.16}$$

$$t^{(TE)}(ik) = \frac{4e^{L(K_0-K_1)}K_0K_1}{\Delta_1^{(TE)}(ik)}. \tag{3.2.17}$$

$$t^{(TM)}(ik) = \frac{4e^{L(K_0-K_1)}K_0K_1\epsilon_0\epsilon_1}{\Delta_1^{(TM)}(ik)} \tag{3.2.18}$$

in this formula, the ultraviolet divergences are still present. We remove them by subtracting the contribution from one single surface. This is equivalent to subtracting the limit for large separation L. We define:

$$\Delta_1^{\infty(TE)}(ik) = \lim_{L\to\infty}\Delta_1^{(TE)}(ik) = e^{-L(K_0-K_1)}(K_0+K_1+2\Omega)^2 \tag{3.2.19}$$

$$\Delta_1^{\infty(TM)}(ik) = \lim_{L\to\infty}\Delta_1^{(TM)}(ik) = e^{-L(K_0-K_1)}(K_0\epsilon_1+K_1\epsilon_0-2\frac{\Omega}{\omega^2}K_0K_1)^2 \tag{3.2.20}$$

$$t^{\infty(TE)}(ik) = \lim_{L\to\infty}t^{(TE)}(ik) = \frac{4e^{L(K_0-K_1)}K_0K_1}{\Delta_1^{\infty(TE)}(ik)}. \tag{3.2.21}$$

$$t^{\infty(TM)}(ik) = \lim_{L\to\infty}t^{(TM)}(ik) = \frac{4e^{L(K_0-K_1)}K_0K_1\epsilon_0\epsilon_1}{\Delta_1^{\infty(TM)}(ik)} \tag{3.2.22}$$

the renormalized energy can be read:

$$E_0 = \frac{1}{2}\int\frac{dk_\parallel}{(2\pi)^2}\left\{\int_0^\infty\frac{dk}{\pi}\omega(k_\parallel,ik)\frac{\partial}{\partial k}\log t^{ren}(ik)\right\} \tag{3.2.23}$$

where:

$$\Delta_1^{(TE)ren}(ik) = 1 + e^{-2LK_1} \frac{(K_0 - K_1 + 2\Omega)(K_1 - K_0 - 2\Omega)}{(K_0 + K_1 + 2\Omega)^2} \tag{3.2.24}$$

$$\Delta_1^{(TM)ren}(ik) = 1 + e^{-2LK_1} \frac{(-K_0\epsilon_1 + K_1\epsilon_0 - 2\frac{\Omega}{\omega^2}K_0K_1)(+K_0\epsilon_1 - K_1\epsilon_0 + 2\frac{\Omega}{\omega^2}K_0K_1)}{(K_0\epsilon_1 + K_1\epsilon_0 - 2\frac{\Omega}{\omega^2}K_0K_1)^2} \tag{3.2.25}$$

$$t^{(TE)ren}(ik) = \frac{1}{\Delta_1^{(TE)ren}(ik)} \qquad t^{(TM)ren}(ik) = \frac{1}{\Delta_1^{(TM)ren}(ik)} \tag{3.2.26}$$

Thanks to eq.(3.2.26) we can write the eq.(3.2.23) in term of $\Delta_1^{(TE)(TM)}$:

$$E_0 = \frac{1}{2} \int \frac{dk_\parallel}{(2\pi)^2} \left\{ \int_0^\infty \frac{dk}{\pi} \omega(k_\parallel, ik) \frac{\partial}{\partial k} \log \frac{1}{\Delta_1^{ren}(ik)} \right\}. \tag{3.2.27}$$

For purpose of numerical evaluation it is meaningful to integrate by parts to get rid of the derivative, and obtain:

$$E_0 = \frac{1}{2} \int \frac{dk_\parallel}{(2\pi)^2} \left\{ \int_0^\infty \frac{dk}{\pi} \omega(k_\parallel, ik) [\log \Delta_1^{(TE)ren}(ik) + \log \Delta_1^{(TM)ren}(ik)] \right\}. \tag{3.2.28}$$

where we have summed over the polarization. This is the formula for the energy of one cavity described by “plasma sheet”, note that this formula is the same of the dielectric case eq.(2.4.18) but with different value of $\Delta^{(TE)(TM)}$.

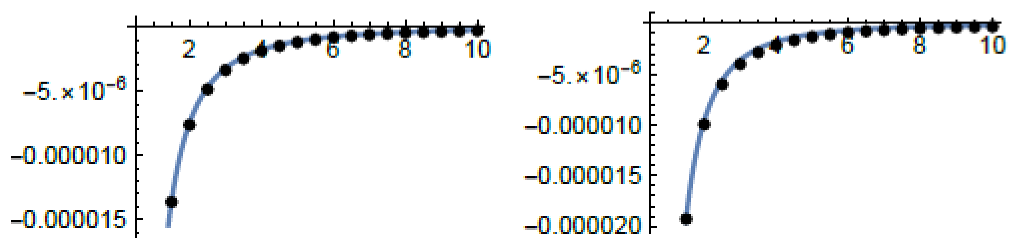

Figure 3.2.1: Evolution of the one cavity energy as function of d

3.3 Casimir energy at finite temperature

In all the previous results, we have assumed that the temperature is zero. In this section, we consider the behavior of a generic quantum system at temperature different from zero, especially we evaluate the case of the scalar field. Finally we obtain the equation of the Casimir energy at finite temperature.

Let's start with the generical quantum system to finite temperature T, the most fundamental quantity of interest is the partition function $\mathcal{Z}$. We employ the canonical ensemble whereby $\mathcal{Z}$ is function of T ; the partition function is defined by:

$$\mathcal{Z} = Tr[e^{-\beta\hat{H}}] \tag{3.3.1}$$

where the trace is taken over the full Hilbert space and $\hat{H}$ is the Hamiltonian operator and $\beta = \frac{1}{k_B T}$.

The eigenvalues of $\hat{H}$ often can be difficult to compute; for this reason it's important to have, for example, a useful representation of $\mathcal{Z}$, the path integral represent:

$$\mathcal{Z} = \lim_{N\to\infty} \int [\prod_{i=1}^{N} \frac{dx_i dp_i}{2\pi\hbar}] \exp\left\{ -\frac{1}{\hbar} \sum_{j=1}^{N} \epsilon \left[\frac{p_j^2}{2m} + ip_j \frac{x_{j+1} - x_j}{\epsilon} + V(x_j) \right] \right\} |_{x_{n+1}=x_1\ \epsilon=\frac{\hbar}{kTN}} \tag{3.3.2}$$

which is often symbolically expressed as a "continuum" path integral:

$$\mathcal{Z} = \int_{x(\beta\hbar)=x(0)} \frac{\mathcal{D}x\mathcal{D}p}{2\pi\hbar} \exp\left\{ -\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \left[\frac{p(\tau)^2}{2m} - ip(\tau)\dot{x(\tau)} + V(x(\tau)) \right] \right\} \tag{3.3.3}$$

The integration measure here is understood as the limit indicated by (3.3.2); the discrete x_1's have been collected into function $x(\tau)$; the maximal value of $\tau-$coordinate has been obtained from $\epsilon N = \beta\hbar$.

We now explicitly evaluate the path integral in the case of a harmonic oscillator eq.(2.1.1).

Let us start by representing an arbitrary function $x(\tau)$, $0 < \tau < \beta\hbar$ as a Fourier sum:

$$x(\tau) = T \sum_{n=-\infty}^{\infty} x_n e^{i\omega_n \tau} \tag{3.3.4}$$

where the factor T is a convention. In general, imposing the periodicity condition, $x(\beta\hbar^-) = x(0^+)$ we obtain a set of frequencies called Matsubara frequencies:

$$e^{i\omega_n\beta\hbar} = 1 \text{ i.e.} \omega_n\beta\hbar = 2\pi n \quad n \in Z \tag{3.3.5}$$

where the values $\omega_n = 2\pi n/\beta\hbar$ are Matsubara frequencies. The corresponding amplitudes x_n are called Matsubara modes. Apart from periodicity, we also impose reality on $x(\tau)$

$$x(\tau) \in \mathcal{R} \Rightarrow x^*(\tau) = x(\tau) \Rightarrow x_n^* = x_{-n} \tag{3.3.6}$$

if we write $x_n = a_n + ib_n$, it then follows that:

$$x_n^* = a_n - ib_n = x_{-n} = a_{-n} + ib_{-n} \Rightarrow \begin{cases} a_n = a_{-n} \\ b_n = -b_{-n} \end{cases} \tag{3.3.7}$$

and moreover that $b_0 = 0$ and $x_n x_{-n} = a_n^2 + b_n^2$. Thereby we now have the representation:

$$x(\tau) = T\left\{a_0 + \sum_{n=1}^{\infty}\left[(a_n + ib_n)e^{i\omega_n\tau} + (a_n - ib_n)e^{-i\omega_n\tau}\right]\right\} \tag{3.3.8}$$

where a_0 is called the Matsubara zero mode.

In according to previous result, we can show that [19] the partition function for the case of harmonic oscillator can read:

$$\mathcal{Z} = C^{'}\int_{-\infty}^{\infty} da_0 \int_{-\infty}^{\infty}[\prod_{n\geq 1} da_n db_n]\exp[-\frac{1}{2}mkT\omega^2 a_0^2 - mkT\sum_{n\geq 1}(\omega_n^2+\omega^2)(a_n^2+b_n^2)] \tag{3.3.9}$$

where $C^{'}$is a coefficient dependent by the integration measurement.

Making use of the Gaussian integral $\int_{-\infty}^{\infty} dxexp(cx^2) = \sqrt{\pi/c}$ with $c > 0$, as well as the above integration measure, the expression (3.3.9) becomes:

$$\mathcal{Z} = C^{'}\sqrt{\frac{2\pi}{mkT\omega^2}}\prod_{n=1}^{\infty}\frac{\pi}{mkT(\omega_n^2+\omega^2)}. \tag{3.3.10}$$

In quantum field theory, the form of the theory is most economically defined in terms of the corresponding classical (Minkowskian) Lagrangian $\mathcal{L}_M$, rather than the Hamiltonian $\hat{H}$; for instance the Lorentz symmetry is explicit only in $\mathcal{L}_M$. Let us therefore start

$$\mathcal{L}_M = \frac{1}{2}m\dot{x}^2 + V(x) \tag{3.3.11}$$

that is the Lagrangian of the quantum harmonic oscillator. We re-interpret x as an "internal" degree of freedom ϕ situated at the origin $\vec{0}$ of d-dimensional space and obtain:

$$S_M^{HO} = \int dt\mathcal{L}_M^{HO} \tag{3.3.12}$$

$$\mathcal{L}_M^{HO} = \frac{m}{2}\left(\frac{\partial\phi(t,\vec{0})}{\partial t}\right)^2 - V(\phi(t,\vec{0})) \tag{3.3.13}$$

We may compare this with the usual action of a scalar field theory in d-dimensional space:

$$S_M^{SFT} = \int dt \int_x \mathcal{L}_M^{SFT} \tag{3.3.14}$$

$$\mathcal{L}_M^{SFT} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - V(\phi) = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_i \phi)(\partial_i \phi) - V(\phi) \tag{3.3.15}$$

comparing eq.(3.3.13) with eq.(3.3.15), we see that scalar field theory is formally nothing but a collection of almost independent harmonic oscillator with $m = 1$ at every $\vec{x}$. These oscillators interact via the derivative term $(\partial_i \phi)(\partial_i \phi)$, which, in the language of statistical physics, couples nearest neighbors through:

$$\partial_i \phi \simeq \frac{\phi(t, \vec{x} + \vec{e_i}\epsilon) - \phi(t, \vec{x})}{\epsilon} \tag{3.3.16}$$

where $\vec{e_i}$ is a unit vector in the direction i.

Next, we note that a coupling of the above type does not change the derivation of the path integral in any essential way: it was only important that the Hamiltonian was quadratic in the canonical momenta $p = m\dot{x} \leftrightarrow \partial_t \phi$. In other words, the derivation of the path integral is only concerned with object having to do with time dependence, and this appear in eq.(3.3.13) and eq.(3.3.15) in identical manners. Therefore, we can directly take over the result for $\mathcal{Z}$ in scalar field theory:

$$\mathcal{Z}^{SFT}(T) = \int_{\phi(\beta\hbar, \vec{x}) = \phi(0, \vec{x})} \prod_x [C\mathcal{D}\phi(t, \vec{x})] \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int_x \mathcal{L}_E^{SFT}\right] \tag{3.3.17}$$

$$L_E = -\mathcal{L}_M^{SFT}(t \to i\tau) = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \sum_{i=1}^{d}\left(\frac{\partial\phi}{dx^i}\right)^2 + V(\phi) \tag{3.3.18}$$

where the periodicity condition $\phi(\beta\hbar, \vec{x}) = \phi(0, \vec{x})$ give the Matsubara modes like the example of harmonic oscillator (3.3.5).

Given that the component of the electromagnetic field are a scalar field, the equation of Casimir energy (3.2.28) is substituting by:

$$E = k_B T \sum_{l=0}^{\infty\,'} \int \frac{dk_\perp}{(2\pi)^2} (\log \Delta^{(TE)}(\xi_l) + \log \Delta^{(TM)}(\xi_l)) \tag{3.3.19}$$

where ξ_l are the Matsubara frequencies $l = 0, 1, 2...$; the superscript $'$ on the sum means that the zero mode must be multiplied by a factor $\frac{1}{2}$.

3.4 Generalization to n-cavities

The equation (3.3.19) is the Casimir energy of one cavity described as "plasma sheet". Formally the energy of n cavities can be given in the form (3.3.19) simply redefining the function $\Delta^{(TE)}, \quad \Delta^{(TM)}$.

In this section we show the result for two and three cavities (in Appendix B we show the result for four and five cavities).

Let us start with the equations of $\Delta_1^{ren(TE)}$ and $\Delta_1^{ren(TM)}$:

$$\Delta_1^{(TE)ren} = 1 + e^{-2LK_1} \frac{(K_0 - K_1 + 2\Omega)(K_1 - K_0 - 2\Omega)}{(K_0 + K_1 + 2\Omega)^2} \tag{3.4.1}$$

$$\Delta_1^{(TM)ren} = 1 + e^{-2LK_1} \frac{(-K_0\epsilon_1 + K_1\epsilon_0 - 2\frac{\Omega}{\omega^2}K_0K_1)(+K_0\epsilon_1 - K_1\epsilon_0 + 2\frac{\Omega}{\omega^2}K_0K_1)}{(K_0\epsilon_1 + K_1\epsilon_0 - 2\frac{\Omega}{\omega^2}K_0K_1)^2} \tag{3.4.2}$$

We define for TE polarization:

$$R_{TE}^{ij} = \frac{K_i - K_j + 2\Omega}{K_i + K_j + 2\Omega} \quad S_{TE}^{ij} = \frac{K_i - K_j - 2\Omega}{K_i + K_j + 2\Omega} \quad T_{TE}^{ij} = \frac{K_i + K_j - 2\Omega}{K_i + K_j + 2\Omega} \tag{3.4.3}$$

instead for TM polarization:

$$R_{TM}^{ij} = \frac{\epsilon_j K_i - \epsilon_i K_j + 2\frac{\Omega}{\omega^2}K_iK_j}{\epsilon_j K_i + \epsilon_i K_j - 2\frac{\Omega}{\omega^2}K_iK_j} \quad S_{TM}^{ij} = \frac{\epsilon_j K_i - \epsilon_i K_j - \frac{2\Omega}{\omega^2}K_iK_J}{\epsilon_j K_i + \epsilon_i K_j - 2\frac{\Omega}{\omega^2}K_iK_j} \quad T_{TM}^{ij} = \frac{\epsilon_j K_i + \epsilon_i K_j + 2\frac{\Omega}{\omega^2}K_iK_j}{\epsilon_j K_i + \epsilon_i K_j - 2\frac{\Omega}{\omega^2}K_iK_j} \tag{3.4.4}$$

For simplicity, we define an index $\alpha = 1, 2$ where $\alpha = 1$ indicate the TE polarization while $\alpha = 2$ indicate the TM polarization. We can substitute the eq.(3.4.3) and eq.(3.4.4) in eqs.(3.4.1) and (3.4.2) and obtain:

$$\Delta_1^{\alpha} = 1 + e^{-2LK_1} R_{\alpha}^{01} S_{\alpha}^{12} \tag{3.4.5}$$

remember that $R_0 = R_2$ and $\epsilon_2 = \epsilon_0$.We define:

$$E_{\alpha}^{ijk} = 1 + e^{-2LK_j} R_{\alpha}^{ij} S_{\alpha}^{jk} \tag{3.4.6}$$

substituting the eq.(3.4.6) into eq.(3.4.5) obtain:

$$\Delta_1^{\alpha} = E_{\alpha}^{012} \tag{3.4.7}$$

consider:

$$I_1^\alpha = E_\alpha^{012} \tag{3.4.8}$$

using eq.(3.4.8) we obtain:

$$\Delta_1^\alpha = I_1^\alpha \tag{3.4.9}$$

Let us consider three parallel plasma sheet that are put at a distance L from each others (see figure 3.4.1) . We have two continuous media with $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ dielectric permettivities where:

$$\begin{array}{ll} \epsilon_1(\omega) & z < 0, \quad L < z < 2L \\ \epsilon_2(\omega) & \text{otherwise} \end{array} \tag{3.4.10}$$

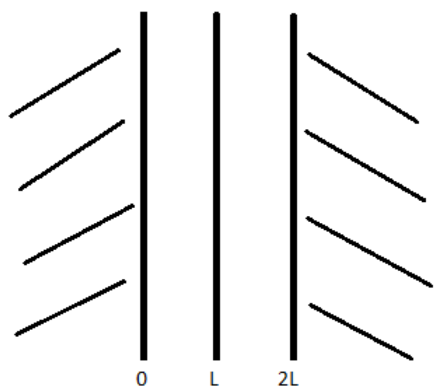

Figure 3.4.1: three parallel "plasma sheet"

Using the same method of section 3.2 we obtain the following combination of exponential:

$$\begin{aligned} f^{(0)} &= re^{-iK_0 z} + e^{iK_0 z} \\ f^{(1)} &= r_1 e^{-iK_1 z} + t_1 e^{iK_1 z} \\ f^{(2)} &= r_2 e^{-iK_0 z} + t_2 e^{iK_0 z} \\ f^{(3)} &= te^{iK_1 z} \end{aligned} \tag{3.4.11}$$

using the boundary condition eq.(3.2.5) in $z = 0$, $z = L$ and $z = 2L$ and solving the system we obtain: (in this section we study the TE polarization only because the TM polarization can be obtained by the $R_{TE} \to R_{TM}$ substitution)

$$\Delta_2 = e^{-iL(K_0-K_1)}[A + Be^{2iLK_1} + Ce^{2iLK_0} + De^{2iL(K_0+K_1)}] \qquad (3.4.12)$$

where:

$$A = i(K_0 + K_1 + 2i\Omega)^3 \quad B = -i(K_0 - K_1 + 2i\Omega)^2(K_0 + K_1 + 2i\Omega) \quad (3.4.13)$$

$$C = -i(-K_0 + K_1 + 2i\Omega)^2(K_0 + K_1 + 2i\Omega) \qquad (3.4.14)$$

$$D = i(K_0 - K_1 - 2i\Omega)(K_0 + K_1 - 2i\Omega)(K_0 - K_1 + 2i\Omega) \qquad (3.4.15)$$

and:

$$K_\alpha = k_\perp^2 + \frac{\epsilon_\alpha(\omega)\omega^2}{c^2} \qquad \alpha = 0,1,2,3 \quad R_2 = R_0 \quad R_1 = R_3 \qquad (3.4.16)$$

in according to the rotation along the imaginary axis, we substitute $K_0 \to iK_0$ and $K_1 \to iK_1$ and obtain the eqs. (3.4.12)-(3.4.15) in terms of real coefficient. Let us now consider the renormalization coefficient $\Delta_2^{ren} = \Delta_2/\Delta_2^\infty$, to obtain this we calculate $\Delta_2^\infty = \lim_{L\to\infty} \Delta_2$ and obtain:

$$\Delta_2^\infty = e^{iL(K_0-K_1)}A \qquad \Delta_2^{ren} = [1 + B'e^{-2LK_1} + C'e^{-2LK_0} + D'e^{-2L(K_0+K_1)}] \qquad (3.4.17)$$

where $B' = B/A$, $C' = C/A$ and $D' = D/A$, substituting eq.(3.4.3) into eq.(3.4.17) we obtain:

$$B' = R_{TE}^{01}S_{TE}^{12} \quad C' = R_{TE}^{12}S_{TE}^{01} \quad D' = R_{TE}^{01}S_{TE}^{01}T_{TE}^{12} \qquad (3.4.18)$$

and substituting eq.(3.4.18) into eq.(3.4.17) :

$$\Delta_2^{ren} = [1 + R_{TE}^{01}S_{TE}^{12}e^{-2LK_1} + R_{TE}^{12}S_{TE}^{01}e^{-2LK_0} + R_{TE}^{01}S_{TE}^{01}T_{TE}^{12}e^{-2L(K_0+K_1)}] \qquad (3.4.19)$$

and gathering this terms:

$$\Delta_2^{ren} = (1 + R_{TE}^{01}S_{TE}^{12}e^{-2LK_1}) + S_{TE}^{01}(R_{TE}^{12} + R_{TE}^{01}T_{TE}^{12}e^{-2LK_1})e^{-2LK_0}. \qquad (3.4.20)$$

Let us define:

$$F_{TE}^{ijk} = e^{-2d_j K_j} R_{TE}^{ij} T_{TE}^{jk} + R_{TE}^{jk} \quad G'^{ij}_{TE} = S_{TE}^{ij} \tag{3.4.21}$$

substituting eq.(3.4.21), eq.(3.4.6) in eq.(3.4.20) we finally obtain:

$$\Delta_2^{ren} = E_{TE}^{012} + F_{TE}^{012} {G'_{TE}}^{01} e^{-2K_0 L} \tag{3.4.22}$$

and defining

$$I_2^{'} = F_{TE}^{012} {G'_{TE}}^{01} e^{-2K_0 L} \tag{3.4.23}$$

using eq.(3.4.8) and eq.(3.4.23) in eq.(3.4.22) obtain that:

$$\Delta_2^{ren} = I_1 + I_2'. \tag{3.4.24}$$

Let us now consider four parallel plasma sheet that are put at distance L from each other (see figure 3.4.2). We have two continuous media with $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ dielectric permettivities where:

$$\begin{aligned} &\epsilon_1(\omega) \qquad z < 0, \quad L < z < 2L \quad z > 3L \\ &\epsilon_2(\omega) \qquad \text{otherwise} \end{aligned} \tag{3.4.25}$$

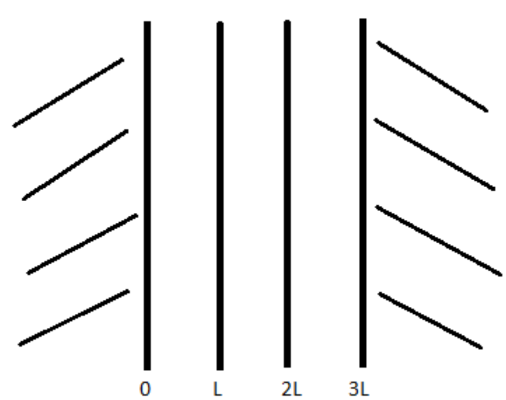

Figure 3.4.2: four parallel “plasma sheet”

Using the same method of section 3.2 we obtain the following combination of exponential:

$$\begin{aligned}
f^{(0)} &= e^{iK_0 z} + re^{-iK_0 z} \\
f^{(1)} &= t_1 e^{iK_1 z} + r_1 e^{-iK_1 z} \\
f^{(2)} &= t_2 e^{iK_0 z} + r_2 e^{-iK_0 z} \\
f^{(3)} &= t_3 e^{iK_1 z} + r_3 e^{-iK_1 z} \\
f^{(4)} &= te^{iK_0 z}
\end{aligned} \tag{3.4.26}$$

using the boundary condition eq.(3.2.5) in $z = 0$, $z = L$, $z = 2L$ and $Z = 3L$ and solving the system obtain:

$$\Delta_3 = e^{iL(K_0-K_1)}[A+Be^{2iK_1L}+Ce^{4iK_1L}+De^{2iK_0L}+Ee^{2iL(K_0+K_1)}+Fe^{2iL(K_0+2K_1)}] \tag{3.4.27}$$

where:

$$\begin{aligned}
A &= (K_0 + K_1 + 2i\Omega)^4 \\
B &= -2(K_0 - K_1 + 2i\Omega)^2(K_0 + K_1 + 2i\Omega)^2 \\
C &= (K_0 - K_1 + 2i\Omega)^4 \\
D &= -(K_0 - K_1 - 2i\Omega)^2(K_0 + K_1 + 2i\Omega)^2 \\
E &= 2(K_0^4 - 2K_0^2(K_1^2 - 4\Omega^2) + (K_1^2 + 4\Omega^2)^2 \\
F &= -(K_0 + K_1 - 2i\Omega)^2(K_0 - K_1 + 2i\Omega)^2
\end{aligned} \tag{3.4.28}$$

and:

$$K_\alpha = k_\perp^2 + \frac{\epsilon_\alpha(\omega)\omega^2}{c^2} \qquad \alpha = 0, 1, 2, 3, 4 \quad R_2 = R_0 = R_4 \quad R_1 = R_3 \tag{3.4.29}$$

in according to the rotation along the imaginary axis, we substitute $K_0 \to iK_0$ and $K_1 \to iK_1$ and obtain the eqs. (3.4.27),(3.4.28) in terms of real coefficient. Let us now consider the renormalization coefficient $\Delta_3^{ren} = \Delta_3/\Delta_3^\infty$, for obtain this we calculate $\Delta_3^\infty = \lim_{L\to\infty} \Delta_3$ and obtain:

$$\Delta_3^\infty = e^{-L(K_0-K_1)}A \tag{3.4.30}$$

$$\Delta_3^{ren} = [1 + B'e^{-2K_1L} + C'e^{-4K_1L} + D'e^{-2K_0L} + E'e^{-2L(K_0+K_1)} + F'e^{-2L(K_0+2K_1)}] \tag{3.4.31}$$

where $B' = B/A$, $C' = C/A$ and $D' = D/A$,$E' = E/A$,$F' = F/A$ substituting $eq.$(3.4.3) into $eq.$(3.4.31) obtain that:

$$B' = 2R^{01}_{TE}S^{12}_{TE} \quad C' = (R^{01}_{TE})^2(S^{12}_{TE})^2 \quad D' = S^{01}_{TE}R^{12}_{TE} \tag{3.4.32}$$

$$E' = R^{01}_{TE}S^{01}_{TE}T^{12}_{TE} + S^{12}_{TE}R^{12}_{TE}T^{01}_{TE} \quad F' = T^{12}_{TE}T^{01}_{TE}R^{01}_{TE}S^{12}_{TE} \tag{3.4.33}$$

let us define:

$$G^{ijk}_{TE} = S^{jk}_{TE}T^{ij}_{TE}e^{-2d_jK_j} + S^{ij}_{TE} \tag{3.4.34}$$

substituting $eq.$(3.4.34) , $eq.$3.4.21) and $eq.$(3.4.6) into $eq.$(3.4.31), using the same process of two cavities case, we find that:

$$\Delta^{ren}_3 = (E^{012}_{TE})^2 + F^{012}_{TE}e^{-2K_0L}G^{012}_{TE} \tag{3.4.35}$$

consider:

$$I_2 = F^{012}_{TE}G^{012}_{TE}e^{-2K_0L} \tag{3.4.36}$$

using $eq.$(3.4.36) and $eq.$(3.4.23) into $eq.$(3.4.35):

$$\Delta^{ren}_3 = I_1^2 + I_2 \tag{3.4.37}$$

We can generalize the eq.(3.4.9), (3.4.24), (3.4.37) to N cavities. Let us consider for instance $N = 3$ cavities, Δ_3 can be written as:

$$\Delta_3 = I_1^3 + 2I_1I_2 + I_3 \tag{3.4.38}$$

where the first term consider the three cavities that don't interact from each other, the second one is the interaction from two cavities and the third that doesn't interact (we have a factor 2 because we have two possibilities for this interaction or the first two cavities interact and the third doesn't interact or the last two cavities interact and the first doesn't interact), the third term is the interaction of three cavities. We can generalize this formula for N cavities and obtain:

$$\Delta_N = \sum_J Q_J(I_{k_1}.....I_{k_j}) \tag{3.4.39}$$

where $\{k_1, k_2..........k_j\}$ are the j-th partition of N and Q_j is his molteplicity (the number of combination that only differ among each other with respect the order in which the various I_k are distributed)

Now, let us consider n plasma sheet cavities, in this case we find two generalization depending if the value of n is odd or even:

- Let us consider n an odd number remember that if the number of cavities is five, for instance, we have six interface, in the dielectric case we can see that six interface represent three cavities (see Appendix B):

$$\Delta_3^d = \Delta_5^p = I_1^3 + 2I_1I_2 + I_3 \tag{3.4.40}$$

This occur because the plates in "plasma sheet" are infinitesimal with respect to the dielectric case. For this reason when we add two interface in the dielectric case obtain another cavity instead in the "plasma sheet" case we obtain another two cavities. Generalizing this formula for N cavities:

$$\Delta_N = \Delta_{2n-1} \tag{3.4.41}$$

where N is the number of dielectric cavities and n is the number of plasma sheet cavities.

- Let us consider n an even number, if the number of cavities is four, for instance, we have five interface that don't link with any dielectric case. In order to solve this problem we consider the solution of five cavities (that have six interface) and we carry out the last plates to the infinite that means $d_5 \to \infty$ and obtain (see Appendix B):

$$\Delta_4 = \Delta_5|_{d_5\to\infty} \tag{3.4.42}$$

generalizing to n cavity we obtain:

$$\Delta_n = \Delta_{n+1}|_{d_{n+1}\to\infty} \tag{3.4.43}$$

In form of partition we can consider two interface as a "integer cavity" and one interface as a " half cavity". If we interact two integer cavities obtain I_k terms instead if we interact an half cavity and a integer cavity obtain $I_k^{'}$ terms that are:

$$I_1^{'} = 1 \quad I_k^{'} = \lim_{G\to G'} I_k \quad k \neq 1 \tag{3.4.44}$$

where G' given by eq.(3.4.21).In four cavities case we have two integer cavities and one half cavity obtaining:

$$\Delta_4 = I_1^2 + I_1 I_2^{'} + I_2 + I_3^{'} \tag{3.4.45}$$

where the first term consider the three cavities that don't interact from each other (obtaining $I_1 \cdot I_1 \cdot I_1^{'} = I_1^2$ because we have an half cavity), the second one from two cavities and the third that doesn't interact (in this case when we consider the integer cavities that interact and the half cavity that don't interact we obtain $I_2 I_1^{'} = I_2$ otherwise we obtain $I_2^{'} I_1$), the third term is the interaction from all the cavities (since we have a half cavity we obtain $I_3^{'}$).

Generalizing to n cavities obtain:

$$\Delta_N = Q_j \sum_N (I_{k_1.....k_{n-1}} I_k^{'}) \tag{3.4.46}$$

where $Q_j$is obtained by the same method of the previous case.

Chapter 4

Numerical simulation

In the previous chapter we consider the "plasma sheet" model and we suppose a method for the calculation of the Casimir energy. In the dielectric case the Casimir energy have two fundamental properties:

- when the distance grows, the energy tend to zero
- when the number of cavities grow, the energy tend to a constant

For this reason, we want to obtain, using numerical simulation, that the Casimir energy formula for a "plasma sheet" has the same properties of the dielectric case.

Let us consider odd cavity of "plasma sheet" model, because the eq.(3.4.39) allow to link at the dielectric case. We plot the Casimir energy in function of distance for odd cavities (for instance we show only the one cavity and three cavities case), and obtain:

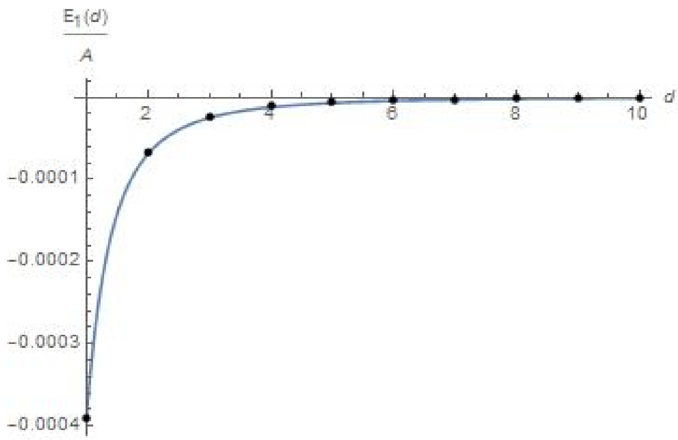

Figure 4.0.1: Casimir energy in function of distance (nm) for one cavity case

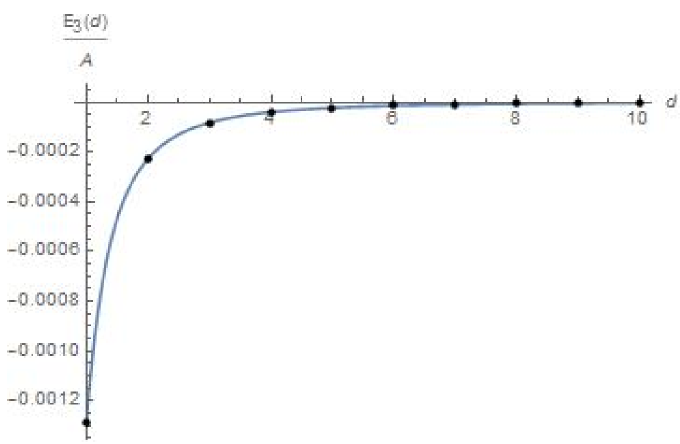

Figure 4.0.2: Casimir energy in function of distance (nm) for three cavities case

If we can fit this behavior, with a model function:

$$y = \frac{a}{x^b} \tag{4.0.1}$$

we obtain for the parameters a, b:

$$\begin{aligned} a &= -0.000390947 \qquad b = 2.54978 \quad \text{for one cavity} \\ a &= -0.00128623 \qquad b = 2.49953 \quad \text{for three cavities} \end{aligned} \tag{4.0.2}$$

This result allows to consider the evolution of the Casimir energy as a function of the distance, the same for all odd cavities because the b parameters are equal in the error limit. We can see that this evolution is the same of the dielectric case (see Bordag [17]).

It's possible to plot E_n/n where E_n is the Casimir energy and n is the number of cavities in function of the number of cavities. We can do this because allow to know in which number of cavities we can neglect the interaction terms:

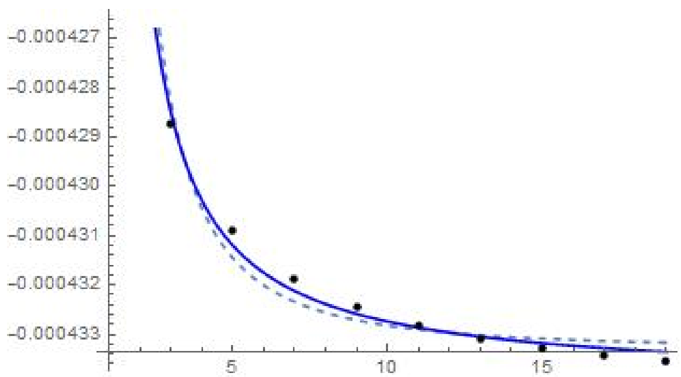

Figure 4.0.3: E_n/n as a function of number of cavities, where the filled line is the exponential fit and the dashed line is the fit of the type ax^{-b}

we can fit this behavior for example, with two possible model functions:

$$y = \frac{a}{x^b} + c \quad y = d - ge^{-\frac{f}{x}} \tag{4.0.3}$$

and obtain the following values for the parameters:

$$\begin{aligned} &a = 0.00004235 \quad b = 1.94439 \quad c = -0.000433 \\ &d = -0.0004386 \quad f = -2.32466 \quad g = -4.6644 \times 10^{-6} \end{aligned} \tag{4.0.4}$$

from the figure 4.0.3, we obtain that the exponential model is the best fit for this point. We don't have any physics explanation for this behavior.

Let us consider the Casimir energy for an even number of cavities, the formulation of the eq.(3.4.42) is different from the previous case for this reason we want to verify the consistency of this formula. To obtain this we compare the result of the eq.(3.4.42) and the equivalent point in the fit (for instance we can see two and four cavities case), using this formula:

$$\frac{E[2]}{2} = \alpha(2) \qquad \frac{E[4]}{4} = \alpha(4) \tag{4.0.5}$$

where $\alpha(n)$ is the equivalent point in the fit. we obtain:

$$\begin{aligned} E_2^a &= -0.0008474 \quad E_2^b = -0.0008527 \\ E_4^a &= -0.001721 \quad E_4^b = -0.001720 \end{aligned} \tag{4.0.6}$$

where E_2^a is the value of the eq.(4.0.5) and E_2^b is the value obtaining by eq.(3.4.42). We can plot the Casimir Energy as a function of the distance for even cavities (for instance we can see two cavities):

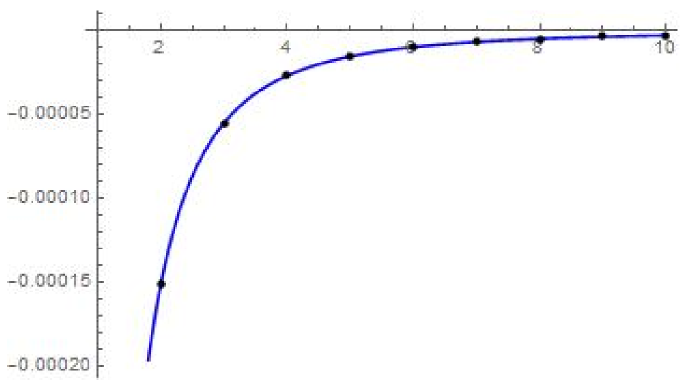

Figure 4.0.4: Casimir energy in function of the distance for two cavities

If we can fit this behavior, with a model function:

$$y = \frac{a}{x^b} \tag{4.0.7}$$

and obtain the following parameters:

$$a = -0.00085279 \quad b = 2.49956 \tag{4.0.8}$$

The result of the even case have the same evolution of the odd case in accordance with the Bordag result [17].

Conclusion

In this thesis's work we have studied the Casimir effect at finite temperature to the n coupled cavities case. Starting from the standard formula for two and three dielectric cavities we extended the result to n cavities.

We have analyzed the Barton model ("plasma sheet" model) [14] using the Kemp's hypothesis [20]. We have find a generalization for n cavities of this model because this represents a first approximation to have a macroscopic description of Casimir effect for the Archimedes experiment.

We have analyzed the general formula of Casimir energy of "plasma sheet" as a function of distance among cavities and number of cavities using numerical simulation so to compare the difference and the similarity with the dielectric case. We found a gain in energy of the order 1% respect a situation in which the plates don't interact each other furthermore we find an important gain of energy respect to the dielectric case and this is the reason because we use this model for the Archimedes experiment.

However this model doesn't represent completely the Ybco plates because we don't consider the microscopic properties of the material. Thus we obtain only a first approximation to what expected for the Archimedes experiment.

An important improvement will be to evaluate the Casimir effect starting with a microscopic model of the YBCO.

Appendix A

Boundaries condition in Plasma sheet model

We start with the Maxwell equations in vacuum:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{A.0.1}$$

we consider a typical condition of Casimir effect that is two parallel conducting plates in the vacuum, for this reason we use the boundary condition across the surface:

$$disc(\vec{E}_{\parallel}) = 0 \quad disc(B_z) = 0 \tag{A.0.2}$$
$$disc(E_z) = 0 \quad disc(\vec{B}_{\parallel}) = 0 \tag{A.0.3}$$

for this case, i need to consider solutions of this type:

$$\vec{E}(\vec{x}, t) = \vec{f}(\vec{k_{\parallel}}, z)e^{i(\vec{k_{\parallel}} \cdot \vec{x_{\parallel}} - \omega t)} \qquad \vec{B}(\vec{x}, t) = \vec{g}(\vec{k_{\parallel}}, z)e^{i(\vec{k_{\parallel}} \cdot \vec{x_{\parallel}} - \omega t)} \tag{A.0.4}$$

where $\vec{f} = (f_1, f_2, f_3)$, $\vec{g} = (g_1, g_2, g_3)$, $\vec{k_{\parallel}} = (k_1, k_2)$ and $\vec{x_{\parallel}} = (x_1, x_2)$.

Substituting eq.$(A.0.4)$ in the expansion of the rotor in eq.$(A.0.1)$ one obtain:

$$ik_2 f_3 - \partial_z f_2 = \frac{i\omega}{c} g_1 \quad ik_2 g_3 - \partial_z g_2 = -\frac{i\omega}{c} f_1 \tag{A.0.5}$$

$$-ik_1 f_3 + \partial_z f_1 = \frac{i\omega}{c} g_2 \quad -ik_1 g_3 + \partial_z g_2 = -\frac{i\omega}{c} f_2 \tag{A.0.6}$$

$$k_1 f_2 - k_2 f_1 = \frac{\omega}{c} g_3 \quad k_1 g_2 - k_2 g_1 = -\frac{i\omega}{c} f_3 \tag{A.0.7}$$

deriving the each members in $eq.(A.0.7)$ and substituting eq.$(A.0.5)$, eq.$(A.0.6)$ in eq.$(A.0.7)$ we obtain the following relations for g_1, g_2, f_1, f_2 :

$$g_2 = \frac{ick_2\partial_z g_3 - \omega k_1 f_3}{c|k_\parallel|^2} \qquad g_1 = \frac{ick_1\partial_z g_3 + \omega k_2 f_3}{c|k_\parallel|^2} \tag{A.0.8}$$

$$f_2 = \frac{ick_2\partial_z f_3 + k_1 \omega g_3}{c|k_\parallel|^2} \qquad f_1 = \frac{ick_1\partial_z f_3 - \omega k_2 g_3}{c|k_\parallel|^2} \tag{A.0.9}$$

where finally f_3 and g_3 are obtained by:

$$\frac{\partial^2 f_3}{\partial z^2} + \left(\frac{\omega^2}{c^2} - |k_\parallel^2|\right) f_3 = 0 \qquad \frac{\partial^2 g_3}{\partial z^2} + \left(\frac{\omega^2}{c^2} - |k_\parallel^2|\right) g_3 = 0 \tag{A.0.10}$$

for TE modes, we have $f_3 = 0$ so that:

$$f_1 = -\frac{k_2 \omega g_3}{c|k_\parallel|^2} \quad g_1 = \frac{ik_1 \partial_z g_3}{|k_\parallel|^2} \quad f_2 = \frac{k_1 \omega g_3}{c|k_\parallel|^2} \quad g_2 = \frac{ik_2 \partial_z g_3}{|k_\parallel|^2} \tag{A.0.11}$$

in plasma sheet model we can show (see Barton [14]) that from the eq.$(A.0.3)$ and eq.$(A.0.2)$ we obtain:

$$disc(\vec{E}_\parallel) = 0 \qquad disc(B_z) = 0 \tag{A.0.12}$$

$$disc(E_z) = 2q(c/\omega)^2 \vec{\nabla}_\parallel \cdot \vec{E}_\parallel \qquad disc(\vec{B}_\parallel) = -2iq(c/\omega)\hat{z} \times \vec{E}_\parallel \tag{A.0.13}$$

where $q = \frac{2\pi n e^2}{mc^2}$. Using

$$disc(\vec{E}_\parallel) = 0 \tag{A.0.14}$$

Being $\vec{E}_\parallel = (f_1, f_2)$ from eq.$(A.0.11)$ we get:

$$disc(\vec{E}_\parallel) = disc(f_1, f_2) \implies disc(g_3) = 0 \tag{A.0.15}$$

Note that the condition eq.$(A.0.15)$ implies:

$$disc(B_z) = disc(g_3) = 0 \tag{A.0.16}$$

since $f_3 = 0$ we have:

$$\vec{\nabla} \cdot \vec{E} = 0 \implies \vec{\nabla}_{\|} \cdot \vec{E}_{\|} = 0 \tag{A.0.17}$$

for this reason the condition eq.$(A.0.13)$ became:

$$disc(E_z) = 0 \tag{A.0.18}$$

finally let us consider the condition:

$$disc(\vec{B}_{\|}) = -2iq(c/\omega)\hat{z} \times \vec{E}_{\|} \tag{A.0.19}$$

where $\hat{z} \times \vec{E}_{\|} = (-f_2, f_1)$, $\vec{B}_{\|}(g_1, g_2)$; using eq.$(A.0.11)$ we obtain:

$$\begin{aligned} \frac{ik_1\partial_z g_3}{|k_{\|}|^2} &= 2iq(c/\omega)\frac{k_1\omega g_3}{c|k_{\|}|^2} \implies disc(\partial_z g_3) = 2qg_3 \\ \frac{ik_2\partial_z g_3}{|k_{\|}|^2} &= 2iq\frac{k_2\omega g_3}{c|k_{\|}|^2} \implies disc(\partial_z g_3) = 2qg_3 \end{aligned} \tag{A.0.20}$$

for TM modes, we have $g_3 = 0$ so that:

$$g_2 = -\frac{\omega k_1 f_3}{c|k_{\|}|^2} \quad g_1 = \frac{\omega k_2 f_3}{c|k_{\|}|^2} \quad f_2 = \frac{ik_2\partial_z f_3}{|k_{\|}|^2} \quad f_1 = \frac{ik_1\partial_z f_3}{|k_{\|}|^2} \tag{A.0.21}$$

by:

$$disc(\vec{E}_{\|}) = 0 \tag{A.0.22}$$

using eq.$(A.0.21)$:

$$disc(\vec{E}_{\|}) = disc(f_1, f_2) \implies disc(\partial_z f_3) = 0 \tag{A.0.23}$$

from the definition of TM modes one found that the condition $disc(B_z) = 0$ is automatically satisfied, while :

$$disc(E_z) = 2q(c/\omega)^2\vec{\nabla}_{\|} \cdot \vec{E}_{\|} \implies disc(f_3) = -2q(c/\omega)^2\partial_z f_3 \tag{A.0.24}$$

where $\partial_z f_3$ derived from Maxwell equation $\vec{\nabla} \cdot E = 0$ and can be demonstrated that the condition $(A.0.24)$ gives the same result :

$$disc(\vec{B}_{\|}) = -2iq(c/\omega)^2\hat{z} \times \vec{E}_{\|}. \tag{A.0.25}$$

Let us now consider Maxwell equations in materials:

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \tag{A.0.26}$$

in a situation where two parallel conducting plates are surrounded by two media one with dielectric permittvity $\epsilon_1(\omega)$ and another with dielectric permittivity $\epsilon_2(\omega)$. In this case we find the following boundary condition:

$$disc(\vec{E}_{\parallel}) = 0 \quad disc(B_z) = 0 \tag{A.0.27}$$
$$disc(D_z) = 0 \quad disc(\vec{B}_{\parallel}) = 0 \tag{A.0.28}$$

where $\vec{D} = \epsilon\vec{E}$. Like previously , we use the eq.$(A.0.26)$ and obtain :

$$g_2 = \frac{ick_2\partial_z g_3 - \omega\epsilon k_1 f_3}{c|k_{\parallel}|^2} \quad g_1 = \frac{ick_1\partial_z g_3 + \omega\epsilon k_2 f_3}{c|k_{\parallel}|^2} \tag{A.0.29}$$
$$f_2 = \frac{ick_2\partial_z f_3 + k_1\omega g_3}{c|k_{\parallel}|^2} \quad f_1 = \frac{ick_1\partial_z f_3 - \omega k_2 g_3}{c|k_{\parallel}|^2} \tag{A.0.30}$$

where f_3 and g_3 are solutions of the following equations:

$$\frac{\partial^2 f_3}{\partial z^2} + \left(\frac{\epsilon\omega^2}{c^2} - |k_{\parallel}^2|\right) f_3 = 0 \qquad \frac{\partial^2 g_3}{\partial z^2} + \left(\frac{\epsilon\omega^2}{c^2} - |k_{\parallel}^2|\right) g_3 = 0 \tag{A.0.31}$$

for TE modes we have $f_3 = 0$:

$$g_2 = \frac{ik_2\partial_z g_3}{|k_{\parallel}|^2} \quad g_1 = \frac{ik_1\partial_z g_3}{|k_{\parallel}|^2} \quad f_2 = \frac{k_1\omega g_3}{c|k_{\parallel}|^2} \quad f_1 = -\frac{\omega k_2 g_3}{c|k_{\parallel}|^2} \tag{A.0.32}$$

when considering two plasma sheet with a dielectric in between, we can show (see Barton-Jackson [14], [15]) that the condition eq.$(A.0.27)$, eq.$(A.0.28)$ became:

$$disc(\vec{E}_{\parallel}) = 0 \qquad disc(B_z) = 0 \tag{A.0.33}$$
$$disc(D_z) = 2q(c/\omega)^2\vec{\nabla}_{\parallel} \cdot \vec{E}_{\parallel} \qquad disc(\vec{B}_{\parallel}) = -2iq(c/\omega)\hat{z} \times \vec{E}_{\parallel} \tag{A.0.34}$$

the conditions in g_3 and $\partial_z g_3$ are the same of previous case.

For TM modes we have $g_3 = 0$:

$$f_2 = \frac{ik_2\partial_z f_3}{|k_{\parallel}|^2} \quad f_1 = \frac{ik_1\partial_z f_3}{|k_{\parallel}|^2} \quad g_2 = \frac{\omega\epsilon k_2 f_3}{c|k_{\parallel}|^2} \quad g_1 = -\frac{\omega\epsilon k_1 f_3}{c|k_{\parallel}|^2} \tag{A.0.35}$$

the condition in $\partial_z f_3$ are the same of the previous case, while the condition f_3 are (use the same process of vacuum condition):

$$disc(\epsilon f_3) = -2q\partial_z f_3 \tag{A.0.36}$$

Appendix B

Solution for 4 and 5 cavities

Let us consider five parallel plasma sheet that are put at distance L from each other (see figure B.0.1). We have two continuous media $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ dielectric permettivities where:

$$\begin{array}{rl} \epsilon_1(\omega) & z < 0 \; L < z < 2L \; 3L < z < 4L \\ \epsilon_2(\omega) & \text{otherwise} \end{array} \tag{B.0.1}$$

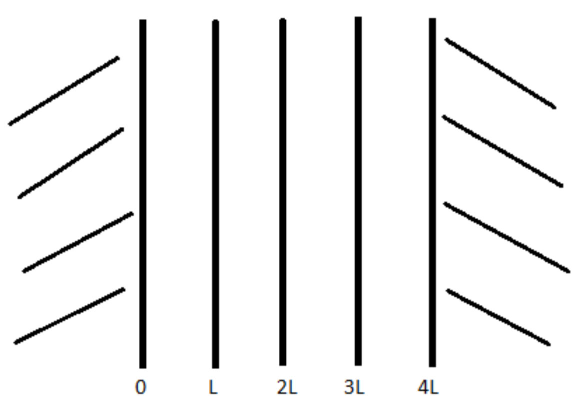

Figure B.0.1: five parallel plasma sheet

Using the same method of section 3.2 we obtain the following combination of exponential:

$$\begin{aligned}
f^{(0)} &= e^{iK_0z} + re^{-iK_0z} \\
f^{(1)} &= t_1e^{iK_1z} + r_1e^{-iK_1z} \\
f^{(2)} &= t_2e^{iK_0z} + r_2e^{-iK_0z} \\
f^{(3)} &= t_3e^{iK_1z} + r_3e^{-iK_1z} \\
f^{(4)} &= t_4e^{iK_0z} + r_4e^{-iK_0z} \\
f^{(5)} &= te^{iK_1z} \qquad \text{(B.0.2)}
\end{aligned}$$

using the boundary condition eq.(3.2.5) in $z = 0$, $z = L$, $z = 2L$,$Z = 3L$ and $Z = 4L$ and solving the system obtain:

$$\begin{aligned}
\Delta_4 = {} & e^{-2i(K_0-K_1)}[A + Be^{2iLK_1} + Ce^{4iLK_1} + De^{2iLK_0} + Ee^{2iL(K_0+K_1)} + Fe^{2iL(K_0+2K_1)} + \\
& Ge^{4iLK_0} + He^{2iL(2K_0+K_1)} + Ie^{4iL(K_0+K_1)}] \qquad \text{(B.0.3)}
\end{aligned}$$

where:

$$\begin{aligned}
A &= -i(K_0 + K_1 + 2i\Omega)^5 \\
B &= -2i(K_0 - K_1 + 2i\Omega)(K_1 - K_0 - 2i\Omega)(K_0 + K_1 + 2i\Omega)^3 \\
C &= -i(K_1 - K_0 - 2i\Omega)^2(K_0 - K_1 + 2i\Omega)^2(K_0 + K_1 + 2i\Omega) \\
D &= -2i(K_1 - K_0 + 2i\Omega)(K_0 - K_1 - 2i\Omega)(K_0 + K_1 + 2i\Omega)^3 \\
E &= -i\{(K_0 - K_1 - 2i\Omega)(K_0 - K_1 + 2i\Omega)[(K_1 - K_0 - 2i\Omega)(K_1 - K_0 + 2i\Omega)(K_0 + K_1 + 2i\Omega) \\
&\quad +2(K_0 + K_1 - 2i\Omega)(K_0 + K_1 + 2i\Omega)^2] + (K_1 - K_0 + 2i\Omega)(K_1 - K_0 - 2i\Omega)* \\
&\qquad (K_1 + K_0 - 2i\Omega)(K_1 + K_0 + 2i\Omega)^2 \\
F &= -i\{(K_0 - K_1 + 2i\Omega)(K_1 - K_0 - 2i\Omega)(K_0 + K_1 - 2i\Omega)[(K_0 - K_1 - 2i\Omega)(K_0 - K_1 + 2i\Omega) \\
&\qquad + (K_0 + K_1 + 2i\Omega)(K_0 + K_1 - 2i\Omega)] \\
G &= -i(K_1 - K_0 + 2i\Omega)^2(K_0 - K_1 - 2i\Omega)^2(K_0 + K_1 + 2i\Omega) \\
H &= -i(K_1 - K_0 + 2i\Omega)(K_0 - K_1 - 2i\Omega)[(K_0 - K_1 + 2i\Omega)(K_0 - K_1 - 2i\Omega) + \\
&\qquad + (K_0 + K_1 - 2i\Omega)^2(K_0 + K_1 + 2i\Omega)] \\
I &= -i(K_0 - K_1 + 2i\Omega)(K_0 - K_1 - 2i\Omega)(K_0 + K_1 - 2i\Omega)^3 \qquad \text{(B.0.4)}
\end{aligned}$$

and:

$$K_\alpha = k_\perp^2 + \frac{\epsilon_\alpha(\omega)\omega^2}{c^2} \qquad \alpha = 0,1,2,3,4,5 \quad R_2 = R_0 = R_4 \quad R_1 = R_3 = R_5 \tag{B.0.5}$$

in according to the rotation along the imaginary axis, we substitute $K_0 \to iK_0$ and $K_1 \to iK_1$ and obtain that the eqs. $(B.0.4)$,$(B.0.3)$ in terms of real coefficient.

Let us now consider the renormalization coefficient $\Delta_4^{ren} = \Delta_4/\Delta_4^\infty$, for obtain this we calculate $\Delta_4^\infty = \lim_{L\to\infty}\Delta_4$ and obtain:

$$\Delta_4^\infty = e^{2(K_0-K_1)L}A \tag{B.0.6}$$

$$\begin{aligned}\Delta_4^{ren} = {} & [1 + B'e^{-2LK_1} + C'e^{-4LK_1} + D'e^{-2LK_0} + E'e^{-2L(K_0+K_1)} + F'e^{-2L(K_0+2K_1)} + \\ & + G'e^{-4LK_0} + H'e^{-2L(2K_0+K_1)} + I'e^{-4L(K_0+K_1)}]\end{aligned} \tag{B.0.7}$$

where $B' = B/A$, $C' = C/A$ and $D' = D/A$,$E' = E/A$,$F' = F/A$,$G' = G/A, H' = H/A, I' = I/A$, substituting $eq.(3.4.3)$ in $eq.(B.0.7)$ obtain that:

$$\begin{aligned} & B' = 2R_{TE}^{01}S_{TE}^{12} \quad C' = (R_{TE}^{01})^2(S_{TE}^{12})^2 \quad D' = 2S_{TE}^{01}R_{TE}^{12} \\ & E' = S_{TE}^{01}R_{TE}^{01}[S_{TE}^{12}R_{TE}^{12} + 2T_{TE}^{12}] + T_{TE}^{01}S_{TE}^{12}R_{TE}^{12} \\ & F' = T_{TE}^{12}S_{TE}^{12}R_{TE}^{01}[S_{TE}^{01}R_{TE}^{01} + T_{TE}^{01}] \quad G' = (R_{TE}^{12})^2(S_{TE}^{01})^2 \\ & H' = R_{TE}^{12}S_{TE}^{01}[R_{TE}^{01}S_{TE}^{01} + T_{TE}^{01}T_{TE}^{12}] \quad I' = R_{TE}^{01}S_{TE}^{01}T_{TE}^{01}(T_{TE}^{12})^2 \end{aligned} \tag{B.0.8}$$

let us define:

$$H_{TE}^{ijk} = S_{TE}^{ij}R_{TE}^{jk} + e^{-2d_jK_j}T_{TE}^{ij}T_{TE}^{jk} \tag{B.0.9}$$

Using $eq.(B.0.9)$, $eq.3.4.6)$,$eq.(3.4.21)$ and $eq.(3.4.34)$ in $eq.(B.0.7)$ we find that:

$$\Delta_4^{ren} = (E_{TE}^{012})^2 + F_{TE}^{012}G_{TE}^{012}e^{-2K_0L} + E_{TE}^{012}F_{TE}^{012}(G'_{TE})^{01} + F_{TE}^{012}e^{-4K_0L}H_{TE}^{012}G'^{01}_{TE} \tag{B.0.10}$$

define:

$$I'_3 = F_{TE}^{012}G'^{01}_{TE}e^{-4K_0L}H_{TE}^{012} \tag{B.0.11}$$

Using $eq.(3.4.8)$, $eq.(3.4.23)$, $eq.(3.4.36)$ and $eq.(B.0.11)$ in $eq.(B.0.10)$ obtain:

$$\Delta_4^{ren} = I_1^2 + I_2 + I_1I'_2 + I'_3 \tag{B.0.12}$$

Let us consider six parallel plasma sheet that are put at distance L from each other (see figure B.0.2). We have two continuous media $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ dielectric permettivities where:

$$\begin{aligned} \epsilon_1(\omega) &\qquad z<0 \ L<z<2L \ 3L<z<4L \ z>5L \\ \epsilon_2(\omega) &\qquad \text{otherwise} \end{aligned} \tag{B.0.13}$$

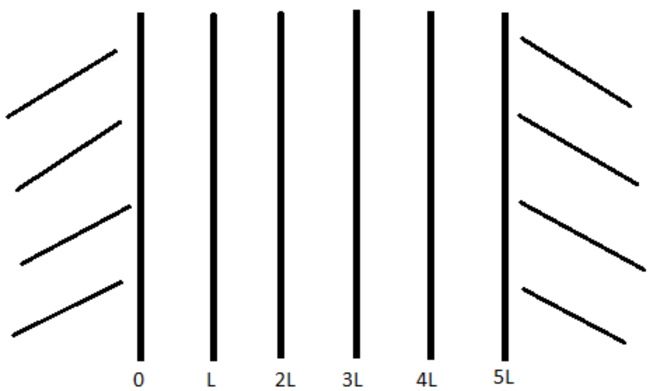

Figure B.0.2: six parallel plasma sheet

Using the same method of section 3.2 we obtain the following combination of exponential:

$$\begin{aligned} f^{(0)} &= e^{iK_0 z} + re^{-iK_0 z} \\ f^{(1)} &= t_1 e^{iK_1 z} + r_1 e^{-iK_1 z} \\ f^{(2)} &= t_2 e^{iK_0 z} + r_2 e^{-iK_0 z} \\ f^{(3)} &= t_3 e^{iK_1 z} + r_3 e^{-iK_1 z} \\ f^{(4)} &= t_4 e^{iK_0 z} + r_4 e^{-iK_0 z} \\ f^{(5)} &= t_5 e^{iK_1 z} + r_5 e^{-iK_1 z} \\ f^{(6)} &= te^{iK_0 z} \end{aligned} \tag{B.0.14}$$

using the boundary condition eq.(3.2.5) in $z=0$, $z=L$, $z=2L$,$z=3L$,$z=4L$

and $z = 5L$ and solving the system obtain:

$$\Delta_5 = e^{3iL(K_0-K_1)}[A + Be^{2iLK_1} + Ce^{4iLK_1} + De^{6iLK_1} + Ee^{2iLK_0} + Fe^{2iL(K_0+K_1)} + Ge^{2iL(K_0+2K_1)} + He^{2iL(K_0+3K_1)} + Ie^{4iK_0L} + Je^{2i(2K_0+K_1)L} + Ke^{2iL(2K_0+2K_1)} + Le^{2iL(2K_0+3K_1)}] \tag{B.0.15}$$

where:

$$\begin{aligned}
A &= -(K_0 + K_1 + 2i\Omega)^6 \\
B &= 3(K_0 - K_1 + 2i\Omega)^2(K_0 + K_1 + 2i\Omega)^4 \\
C &= -3(K_0 - K_1 + 2i\Omega)^4(K_0 + K_1 + 2i\Omega)^2 \\
D &= (K_0 - K_1 + 2i\Omega)^6 \\
E &= 2(-K_0 + K_1 + 2i\Omega)^2(K_0 + K_1 + 2i\Omega)^4 \\
F &= -2(K_0 - K_1 + 2i\Omega)(K_0 - K_1 - 2i\Omega)(K_0 + K_1 + 2i\Omega)^2[(K_1 - K_0 - 2i\Omega)(K_1 - K_0 + 2i\Omega)+ \\
&\quad (K_0 + K_1 - 2i\Omega)(K_0 + K_1 + 2i\Omega)] - (K_1 - K_0 - 2i\Omega)(K_1 - K_0 + 2i\Omega)(K_0 + K_1 - 2i\Omega)* \\
&\quad (K_0 + K_1 + 2i\Omega)^3 \\
G &= -2(K_0 + K_1 + 2i\Omega)(K_0 + K_1 - 2i\Omega)(K_1 - K_0 - 2i\Omega)(K_0 - K_1 + 2i\Omega)[(K_0 - K_1 + 2i\Omega)* \\
&\quad (K_0 - K_1 - 2i\Omega) + (K_1 - K_0 - 2i\Omega)(K_1 - K_0 + 2i\Omega) + (K_0 + K_1 - 2i\Omega)(K_0 + K_1 + 2i\Omega)] \\
H &= -2(K_0 + K_1 - 2i\Omega)^2(K_0 - K_1 + 2i\Omega)^4 \\
I &= -(-K_0 + K_1 + 2i\Omega)^4(K_0 + K_1 + 2i\Omega)^2 \\
J &= -(K_1 - K_0 + 2i\Omega)(K_0 - K_1 - 2i\Omega)(K_0 + K_1 - 2i\Omega)(K_0 + K_1 + 2i\Omega)[(K_0 - K_1 - 2i\Omega)* \\
&\quad (K_0 - K_1 + 2i\Omega) + (K_1 - K_0 - 2i\Omega)(K_1 - K_0 + 2i\Omega)+(K_0 + K_1 - 2i\Omega)(K_0 + K_1 + 2i\Omega)] \\
K &= -(K_0 + K_1 - 2i\Omega)^2(K_0 - K_1 + 2i\Omega)(K_0 - K_1 - 2i\Omega)[(K_1 - K_0 - 2i\Omega)(K_1 - K_0 + 2i\Omega)+ \\
&\quad +(K_1 + K_0 - 2i\Omega)(K_1 + K_0 + 2i\Omega)] + (K_1 - K_0 + 2i\Omega)(K_1 - K_0 + 2i\Omega)(K_0 + K_1 - 2i\Omega)^3* \\
&\quad (K_0 + K_1 + 2i\Omega) \\
L &= (K_1 + K_0 - 2i\Omega)^4(K_0 - K_1 + 2i\Omega)^2
\end{aligned} \tag{B.0.16}$$

and:

$$K_\alpha = k_\perp^2 + \frac{\epsilon_\alpha(\omega)\omega^2}{c^2} \qquad \alpha = 0, 1, 2, 3, 4, 5, 6 \quad R_2 = R_0 = R_4 = R_6 \quad R_1 = R_3 = R_5 \tag{B.0.17}$$

in according to the rotation along the imaginary axis, we substitute $K_0 \to iK_0$ and $K_1 \to iK_1$ and obtain that the eqs. $(B.0.16)$,$(B.0.15)$ in terms of real coefficient.

Let us now consider the renormalization coefficient $\Delta_5^{ren} = \Delta_5/\Delta_5^{\infty}$, for obtain this we calculate $\Delta_5^{\infty} = \lim_{L\to\infty}\Delta_5$ and obtain:

$$\Delta_5^{\infty} = e^{-3L(K_0-K_1)}A \tag{B.0.18}$$

$$\Delta_5^{ren} = [1 + B'e^{-2LK_1} + C'e^{-4LK_1} + D'e^{-6LK_0} + E'e^{-2LK_0} + F'e^{-2L(K_0+K_1)} + \\ +G'e^{-2L(K_0+2K_1)} + H'e^{-2L(K_0+3K_1)} + I'e^{-4LK_0} + J'e^{-2(2K_0+K_1)L} + K'e^{-2L(2K_0+2K_1)} + Le^{-2L(2K_0+3K_1)}] \tag{B.0.19}$$

where $B' = B/A$, $C' = C/A$ and $D' = D/A$, $E' = E/A$, $F' = F/A$, $G' = G/A$,$H' = H/A$,$I' = I/A$,$J' = J/A$,$K' = K/A$,$L' = L/A$ substituting $eq.(3.4.3)$ in $eq.(B.0.19)$ obtain that:

$$\begin{aligned}
&B' = 3R^{01}_{TE}S^{12}_{TE} \quad C' = 3(R^{01}_{TE})^2(S^{12}_{TE})^2 \quad D' = (R^{01}_{TE})^3(S^{12}_{TE})^3\\
&E' = 2R^{12}_{TE}S^{01}_{TE} \quad F' = 2[R^{01}_{TE}S^{12}_{TE}S^{01}_{TE}R^{12}_{TE} + R^{01}_{TE}T^{12}_{TE}S^{01}_{TE} + S^{12}_{TE}T^{01}_{TE}R^{12}_{TE}]\\
&G' = 2[(R^{01}_{TE})^2(S^{12}_{TE})T^{12}_{TE}S^{01}_{TE} + R^{01}_{TE}(S^{12}_{TE})^2T^{01}_{TE}R^{12}_{TE} + S^{01}_{TE}T^{01}_{TE}T^{12}_{TE}R^{12}_{TE}]\\
&H' = 2[(R^{01}_{TE})^2(S^{12}_{TE})^2T^{01}_{TE}T^{12}_{TE}] \quad I' = (R^{12}_{TE})^2(S^{01}_{TE})^2\\
&J' = R^{01}_{TE}T^{12}_{TE}(S^{01}_{TE})^2R^{12}_{TE} + S^{12}_{TE}T^{01}_{TE}(R^{12}_{TE})^2S^{01}_{TE} + T^{01}_{TE}T^{12}_{TE}R^{12}_{TE}S^{01}_{TE}\\
&K' = R^{01}_{TE}T^{12}_{TE}T^{01}_{TE}S^{12}_{TE}S^{01}_{TE}R^{12}_{TE} + R^{01}_{TE}(T^{12}_{TE})^2T^{01}_{TE}S^{01}_{TE} + S^{12}_{TE}(T^{01}_{TE})^2T^{12}_{TE}R^{12}_{TE}
\end{aligned} \tag{B.0.20}$$

$$L' = R^{01}_{TE}S^{12}_{TE}(T^{01}_{TE})^2(T^{12}_{TE})^2$$

Using $eq.(B.0.9)$, $eq.3.4.6)$,$eq.(3.4.21)$ and $eq.(3.4.34)$ in $eq.(B.0.19)$ we find that:

$$\Delta_5^{ren} = (E^{012}_{TE})^3 + 2E^{012}_{TE}F^{012}_{TE}G^{012}_{TE}e^{-2LK_0} + F^{012}_{TE}G^{012}_{TE}H^{012}_{TE}e^{-4LK_0} \tag{B.0.21}$$

let us define:

$$I_3 = F^{012}_{TE}G^{012}_{TE}H^{012}_{TE}e^{-4LK_0} \tag{B.0.22}$$

using $eq.(3.4.8)$, $eq.(3.4.23)$, $eq.(3.4.36)$, $eq.(B.0.22)$ and $eq.(B.0.11)$ in $eq.(B.0.21)$ and obtain:

$$\Delta_5^{ren} = I_1^3 + 2I_2I_1 + I_3 \tag{B.0.23}$$

Bibliography

[1] P.W Milonni The Quantum Vacuum: an introduction to quantum electrodynamics, Academy Press, New York 1994

[2] Klein M.J., "Max Planck and the Beginning of Quantum Theory", Arch. Hist. Exact Sciences 1,459 (1962)

[3] Klein M.J. "Thermodynamics and Quanta in Planck's Work", Physics Today (November 1966),23

[4] Kuhn, T.S. , Black-Body Theory and the Quantum Discontinuity, 1894-1912 (Oxford University Press, New York , 1978)

[5] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (Dover Books, New York, 1971)

[6] E.T. Whittaker and G.N. Watson, A course of Modern Analysis 4th ed. (Cambridge University Press, New York, 1969)

[7] M.Bordag, U.Mohideen, V.M. Mostepanenko, New Developments in the Casimir Effect (2001)

[8] E.M. Lifshitz, Sov. Phys.-JETP (USA) 2 (1956) 73

[9] N.G. van Kampen , B.R.A Nijiboer, K.Sehram, Phys. Lett. A 26 (1968) 307

[10] K.Schram, Phys. Lett. A 43 (1973) 283

[11] I.Belyanitsekii-Birula, J.B. Brojan, Phys. Rev. D5 (1972) 485

[12] G.L. Klimehitsknyn, U.Mohideen, V.M. Mostepanenko, Phys. Rev. A 61 (2000) 062107

[13] H.G.B Casimir, Proe. Kon. Nederl. Akad. Wet. 51 (1948) 793

[14] G.Barton, Casimir Effect for a flat plasma sheet: I. Energies (2005)

[15] J.D. Jackson, Classical Electrodynamics Second edition, John Wiley & Sons, New York (1975)

[16] Barton G. 2004a J. Phys. A: Math Gen 37 1011

[17] M.Bordag J.Phys. A: Math. Gen. 39 (2006) 6173-6185

[18] Gerlach E. 1971 Equivalence of the Van der Waals forces between solids and surface-plasmon interaction Phys. Rev. B4 393-6

[19] M.Laine A.Vuorinen, Basics of Thermal Field Theory (2018)

[20] A.Kempf J.Phys A 41, 164038 (2008)

[21] L.Rosa et all Eur. Phys. J. Plus (2017) 132 : 478