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The role of affine connection in theories of gravity

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Introduction

Gravity is one of the four fundamental interactions in nature, among electromagnetism, weak and strong nuclear forces; in particular, it is the weakest of all.
The Greek philosopher Aristotle gave one of the earliest attempts of explanation of gravity by dividing the universe into terrestrial and celestial spheres, with the idea that all bodies move toward their natural place. It was Aristotle who first thought that objects with different masses should fall at different rates. This supposition failed when, between the 16th and 17th century, Galilei conducted his experiments on gravity using inclined planes, pendulums and telescopes, showing experimentally that all the objects accelerate toward the Earth uniformly if we neglect resistant forces. It was the first primordial formulation of the Equivalence Principle.
A consistent theory of gravity was proposed by Newton in 1687 in his treatise "Philosophiae naturalis principia mathematica", where he formulated the inverse-square law of universal gravitation. The most important conceptual ideas introduced by the Newton's theory of gravity are: space and time are two absolute entities, i.e. all the physical phenomena take place in a non dynamical background; gravitational and inertial masses coincide, i.e. the so called Equivalence Principle.
The idea of the absolute space, the failure of the theory to explain the excess precession of Mercury's orbit, together with the incompatibility of Special Theory of Relavity with the Newton's theory were some of the reasons that led Einstein to formulate the General Relativity in 1915 [28].
General Relativity suggests that gravity is no longer a force, as Newton proposed, but instead it is just the effect of geometry, i.e. it is the curvature of spacetime that causes objects to fall down to Earth. Time is not an absolute notion and it depends on the position, in a gravitational field, in which it is measured. Space and time, together, form the notion of space-

time.
Although General Relativity passed numerous tests over the years, for example the three classic tests proposed by Einstein himself (the perihelion precession of Mercury's orbit, the deflection of light by the Sun and the gravitational redshift of light), gravitational lensing and the direct detection of gravitational waves in 2016 [1, 2, 3], it is not free from defects.
A first shortcoming is the inability of General Relativity to be integrated in a more general theory with Quantum Field Theory. The main conceptual problem is that the Quantum Field Theory assumes the spacetime to be non-dynamical, while in General Relativity the spacetime is a dynamical quantity [15].
From a cosmological point of view it exists the so called problem of the dark approach. To explain why the universe is currently undergoing an accelerated expansion phase, a new cosmological model, usually referred to as the Concordance Model or Λ Cold Dark Matter (ΛCDM) model, was introduced. The ΛCDM model assumes that the universe is dominated by an unclustered fluid with negative pressure commonly referred to as dark energy, which drives the accelerated expansion. A candidate for dark energy is the cosmological constant Λ [24, 39]. The ΛCDM fails to explain why the inferred value of Λ is 120 orders of magnitude lower in comparison with the typical value of the vacuum energy density predicted by Standard Model [25], and the coincidence problem, i.e. the energy density of the cosmological constant, as indicated by cosmological observations, has the same order of magnitude with the energy density of the matter content of the universe.
As a possible solution to these problems it was thought to replace the cosmological constant with a scalar field ϕ rolling slowly down a flat section of a potential $V(\phi)$. This model is called quintessence [27, 18], however it still has the coincidence problem since the dark energy and matter densities evolve differently and reach comparable values only during a very short time of history of the universe, coinciding right at present era.
In response to these and other shortcomings of General Relativity, several alternative theories of gravity arose, some of which will be discussed in this thesis. In particular, we will focus on the role that the affine connection plays in various theories of gravity and we will also focus on three quantities dependent on it, which constitute the so called trinity of gravity: the curvature, which measures how much a vector rotates along a closed curve; the torsion, which measures how much two vectors that are par-

allel transported to each other are twisted, and the non-metricity, which instead measures how much the length of a vector varies when it is parallel transported. Depending on what value these three quantities assume we can obtain different theories of gravity. We will concentrate on theories that are obtained by removing two quantities out of curvature, torsion and non-metricity.

Initially we will show basic concepts useful for our study, such as the definition of connection or isometries, and then we will concentrate on how to modify the affine connection in presence of metric and manifold deformations.

Then, we will focus on the theories obtainable from the trinity. First we will deal with the case in which the curvature is the only non-zero quantity and we will study the various approaches that can be obtained depending on which one between the metric and the connection assumes the value of dynamic variable. We will start from General Relativity, a purely metric theory, which leads to the result that the connection must necessarily be the Levi-Civita one. Later we will be interested in theories where the metric is put in the background with respect to the connection, which assumes the value of unique dynamic variable. In this framework we will open a small parenthesis in which we will see that by imposing a non-zero torsion we can naturally recover the concept of cosmological constant. Finally we will discuss an approach to gravitation in which both the metric and the connection have the value of a dynamic variables but relegated to two different roles, the metric is the one that regulates the causal structure while the connection governs the geodesic one.

Afterwards, we will consider "teleparallel" theories, where curvature is null. In particular, we will take in account the Teleparallel Equivalent to General Relativity, which is a gravitational gauge theory where the connection has the torsion as the only non-zero quantity out of the trinity ones. In this theory the geometric character of gravity imposed by General Relativity is lost and gravity is to be considered again as a force. The other teleparallel theory considered is the Symmetric Teleparallel Equivalent to General Relativity, where it is the non-metricity that dominates. This is a theory in which there is ferment in recent years since non-metricity implies the possible invalidity of the Equivalence Principle.

In the last part we will analyse the study of a generic affine connection through which it is possible to catalog, in addition to the theories we considered, all the theories that can be distinguished according to which quantities out of the trinity ones are taken into account.

Chapter 1

The Equivalence Principle

Already known to Galilei thanks to his free fall experiments, the Equivalence Principle(EP) was formulated by Newton in "Philosophiae naturalis principia mathematica": it asserts the equivalence between the inertial mass m_i, i.e. the property of a body to resist to being accelerated by a force $\mathbf{F} = m_i\mathbf{a}$, and the gravitational mass m_g, i.e. the coefficient that appears in the Newtonian gravitational attraction law $\mathbf{F}_g = G_N m_g M_g \mathbf{r}/r^3$:

$$m_i \equiv m_g. \tag{1.1}$$

Today this is known as Weak Equivalence Principle (WEP) and the present accuracy of equivalence is of the order of 10^{-15} [44]. However there are different proposals for new experiments that would lead to the improvement of this estimate, one of them is SAGE [43] where the use of a multi-satellite configuration is considered.

The WEP implies that it is impossible to distinguish, locally, between the effects of a gravitational field from those experienced in uniformly accelerated frames using the simple observation of the free-falling particles behaviour.

Thanks to Special Relativity, Einstein generalised this concept not only for free-falling particles but to any experiment. This principle is called Einstein Equivalence Principle (EEP) and it states [19]:

- Weak Equivalence Principle is valid;
- the outcome of any local non-gravitational test experiment is independent of velocity of free-falling apparatus;

- the outcome of any local non-gravitational test experiment is independent of where and when it is performed.

It is defined as local non-gravitational experiment an experiment performed in a small-size freely falling laboratory. It follows from the EEP that the gravitational interaction must be described in terms of a curved spacetime, i.e. the postulates of the so-called metric theories of gravity have to be satisfied:

- spacetime is endowed with a metric $g_{\mu\nu}$;
- the world lines of test bodies are geodesics of the metric;
- in local freely falling frames, called local Lorentz frames, the non-gravitational laws of physics are those of Special Relativity.

There is another EP that is distinct from the WEP and EEP due to inclusion of self-gravitating bodies and local gravitational experiments, it is called Strong Equivalence Principle (SEP) and it states:

- Weak Equivalence Principle is valid for self-gravitating bodies as well as for test bodies;
- the outcome of any local test experiment is independent of the velocity of the free-falling apparatus;
- the outcome of any local test experiment is independent of where and when it is performed.

Of course WEP is recovered when the gravitational forces are ignored.

1.1 Tests of the Weak Equivalence Principle: the torsion-balance experiments

Among the various tests of the WEP [Table 1.1], the experiments based on the torsion-balance deserve a mention due to the fact that the first high precision experiment was performed by Eötvös precisely by using one of them.

A torsion-balance consists in two masses of different composition connected by a rod and suspended by a thin wire. If the inertial mass was

different from the gravitational one, then the gravity and the centrifugal force would not compensate each other and eventually the rod would rotate.
How to quantify a possible violations of WEP? Let us assume, for example, that the inertial mass m_i differs in a system from the gravitational one according to

$$m_g = m_i + \Sigma_A \eta^A \frac{E^A}{c^2}, \tag{1.2}$$

where E^A is the internal energy of the body generated by interaction A and η^A is a dimensionless parameter quantifying the violation of the WEP induced by this interaction. Thus, the acceleration of a body is given by

$$a = g\left(1 + \Sigma_A \eta^A \frac{E^A}{m_i c^2}\right). \tag{1.3}$$

We define a quantity called Eötvös ratio as the relative difference in acceleration between two different bodies:

$$\eta = 2\frac{|a_1 - a_2|}{|a_1 + a_2|} = \Sigma_A \eta^A \left|\frac{E_1^A}{m_{i1}c^2} - \frac{E_2^A}{m_{i2}c^2}\right|. \tag{1.4}$$

The measured value of η provides information on the WEP-violation parameters η^A.

Researchers	Method	Limit on $\lvert\eta\rvert$
Newton(1686)	Pendulum	10^3
Bessel(1832)	Pendulum	$2 \cdot 10^{-5}$
Eötvös, Pekar and Fekete(1922)	Torsion-balance	$5 \cdot 10^{-9}$
Potter(1923)	Pendulum	$3 \cdot 10^{-6}$
Renner(1935)	Torsion-balance	$2 \cdot 10^{-9}$
Roll, Krotkov and Dicke(1964)	Torsion-balance	$3 \cdot 10^{-11}$
Keiser and Fallen(1981)	Fluid support	$4 \cdot 10^{-11}$
Baessel et all.(1999)	Torsion-balance	$5 \cdot 10^{-14}$
MICROSCOPE(2017)	Earth orbit	10^{-15}

Table 1.1: Tests of the Weak Equivalence Principle.

1.2 The geodesic equation

In this last section we want to show that the free fall motion of a test particle is given by the geodesic equation due to the EP.
If we are in a locally inertial frame, where by the EP we are able to eliminate the gravitational force, the equations of motion would be that of a free particle:

$$\frac{d^2 y^\mu}{ds^2} = 0, \tag{1.5}$$

where

$$ds^2 = \eta_{\alpha\beta} dy^\alpha dy^\beta \tag{1.6}$$

is the line element, with

$$\eta_{\alpha\beta} = diag(1, -1, -1, -1) \tag{1.7}$$

the Minkowski metric. Performing the coordinate transformations

$$y^\mu = y^\mu(x^\nu) \tag{1.8}$$

we get

$$\begin{aligned} \frac{d^2 y^\mu}{ds^2} =& \frac{d}{ds}\left(\frac{dy^\mu}{ds}\right) = \frac{d}{ds}\left(\frac{\partial y^\mu}{\partial x^\lambda}\frac{dx^\lambda}{ds}\right) = \frac{d}{ds}\left(\frac{\partial y^\mu}{\partial x^\lambda}\right)\frac{dx^\lambda}{ds} + \frac{\partial y^\mu}{\partial x^\lambda}\frac{d^2 x^\lambda}{ds^2} \\ =& \frac{\partial^2 y^\mu}{\partial x^\sigma \partial x^\rho}\frac{dx^\sigma}{ds}\frac{dx^\rho}{ds} + \frac{\partial y^\mu}{\partial x^\lambda}\frac{d^2 x^\lambda}{ds^2} = 0. \end{aligned} \tag{1.9}$$

Multiplying by $\partial x^\lambda / \partial y^\mu$ the Eq.(1.9) becomes

$$\frac{\partial x^\lambda}{\partial y^\mu}\frac{\partial^2 y^\mu}{\partial x^\sigma \partial x^\rho}\frac{dx^\sigma}{ds}\frac{dx^\rho}{ds} + \frac{d^2 x^\lambda}{ds^2} = 0, \tag{1.10}$$

which can be written as

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda{}_{\sigma\rho}\frac{dx^\sigma}{ds}\frac{dx^\rho}{ds} = 0, \tag{1.11}$$

with

$$\Gamma^\lambda{}_{\sigma\rho} = \Gamma^\lambda{}_{\rho\sigma} = \frac{\partial x^\lambda}{\partial y^\mu}\frac{\partial^2 y^\mu}{\partial x^\sigma \partial x^\rho}. \tag{1.12}$$

Eq.(1.11) is the geodesic equation and the quantities (1.12) are called affine connections, these ones express the gravitational force that acts on the particle. The geodesic equation show us that the affine connections give the apparent forces, in the absence of a gravitational field, if we perform a transformation from a locally inertial frame to another generic frame. This manifests the equivalence between inertial and gravitational forces.

Chapter 2

Isometries and Killing equations

In this chapter we want to discuss about isometries, transformations which preserve the metric, and the equations satisfied by the generators of these transformations [36]. However, before this, we must set the geometrical framework and define the Levi-Civita connection.

2.1 Geometrical structure

Spacetime is described by a 4-dimensional differentiable manifold $\boldsymbol{M}$. We can define on it a metric

$$\boldsymbol{g} = g_{\mu\nu} dx^{\mu} dx^{\nu}, \quad g_{\mu\nu} = g_{\nu\mu}, \tag{2.1}$$

which is a rank-2 symmetric covariant tensor. The metric defines a scalar product on the manifold between two vectors V and W

$$\boldsymbol{g}(V, W) = g_{\mu\nu} V^{\mu} W^{\nu}. \tag{2.2}$$

By definition, the metric is assumed to be non-degenerate, $g = det(g_{\mu\nu}) \neq 0$, this allows to determine the inverse matrix $g^{\mu\nu}$ such that $g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}_{\lambda}$. Thanks to this we can establish, in every point p of manifold, an isomorphism between the tangent space $\boldsymbol{V}_p$ and the dual one $\boldsymbol{V}^*_p$. We use a Lorentzian metric, i.e. a metric with a signature ± 2.
Independently to the metric, we can define the affine connection ${\Gamma^{\lambda}}_{\mu\nu}$. It introduces a local isomorphism of tangent spaces at different points x of

the manifold, by specifying a rule that maps a vector V at a point x into a vector W at an infinitely near point $x + dx$:

$$\delta V^\lambda = W^\lambda - V^\lambda = -\Gamma^\lambda{}_{\mu\nu} V^\mu dx^\nu. \tag{2.3}$$

Defining a connection structure, we introduce the covariant derivative of a tensor:

$$\begin{aligned}\nabla_\lambda T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} = &\partial_\lambda T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} + \Gamma^{\mu_1}{}_{\lambda\rho} T^{\rho \dots \mu_n}{}_{\nu_1 \dots \nu_m} + \dots \\ &+ \Gamma^{\mu_n}{}_{\lambda\rho} T^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_m} - \Gamma^\rho{}_{\lambda\nu_1} T^{\mu_1 \dots \mu_n}{}_{\rho \dots \nu_m} - \dots \\ &- \Gamma^\rho{}_{\lambda\nu_m} T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \rho}.\end{aligned} \tag{2.4}$$

The connection also defines the notion of parallel transport. Let $\gamma(t)$ be a curve in $\boldsymbol{M}$, specified by the parametric equations $\gamma(t) = \{x^\mu\}$. We define the covariant derivative of a tensor field T along this curve as

$$\frac{dT}{dt} = \frac{dx^\mu}{dt} \nabla_\mu T. \tag{2.5}$$

The tensor T is said to be parallel transported along $\gamma(t)$ when it satisfies the condition:

$$V^\mu \nabla_\mu T = 0. \tag{2.6}$$

with $V^\mu = dx^\mu/dt$. Geodesics are particular curves whose tangent vectors remain parallel when they are transported along them:

$$V^\mu \nabla_\mu V^\nu = \alpha(t) V^\nu. \tag{2.7}$$

Geodesics are said to be affine-parameterized if $\alpha(t) = 0$, with t the affine paramater. In this situation, Eq.(2.7) becomes

$$V^\mu \nabla_\mu V^\nu = 0 = \frac{d^2 x^\nu}{ds^2} + \Gamma^\nu{}_{\mu\alpha} V^\mu V^\alpha. \tag{2.8}$$

Thanks to the connection and the covariant derivative, we define the quantities

$$T^\lambda{}_{\mu\nu} \equiv 2\Gamma^\lambda{}_{[\mu\nu]}, \tag{2.9}$$

$$Q_{\lambda\mu\nu} \equiv \nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} g_{\mu\rho}, \tag{2.10}$$

$$R^\alpha{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\lambda\mu} \Gamma^\lambda{}_{\beta\nu} - \Gamma^\alpha{}_{\lambda\nu} \Gamma^\lambda{}_{\beta\mu}, \tag{2.11}$$

which are respectively the torsion tensor, the non-metricity tensor and the Riemann tensor.
Contraction of the first index with the third one of the Riemann tensor gives the Ricci tensor,

$$R^{\alpha}{}_{\mu\alpha\nu} \equiv R_{\mu\nu}, \tag{2.12}$$

from which, using the metric, we get the Ricci scalar

$$R \equiv g^{\mu\nu} R_{\mu\nu}. \tag{2.13}$$

It is useful to underline that torsion, non-metricity and curvature, which is given by the Riemann tensor, are all properties of the connection and not of the spacetime.

2.2 The Levi-Civita connection

Recovering the argument of the last chapter, let us take in account the transformations

$$y^{\mu} = y^{\mu}(x^{\nu}). \tag{2.14}$$

Under these transformations the metric becomes

$$\boldsymbol{g} = \eta_{\alpha\beta} dy^{\alpha} dy^{\beta} = \eta_{\alpha\beta} \frac{\partial y^{\alpha}}{\partial x^{\mu}} dx^{\mu} \frac{\partial y^{\beta}}{\partial x^{\nu}} dx^{\nu} = g_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{2.15}$$

where

$$g_{\mu\nu} = g_{\nu\mu} = \eta_{\alpha\beta} \frac{\partial y^{\alpha}}{\partial x^{\mu}} \frac{\partial y^{\beta}}{\partial x^{\nu}}. \tag{2.16}$$

Deriving this equation with respect to x^{λ}, we get

$$\partial_{\lambda} g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial^2 x^{\alpha}}{\partial x^{\mu} \partial x^{\lambda}} \frac{\partial y^{\beta}}{\partial x^{\nu}} + \eta_{\alpha\beta} \frac{\partial y^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 y^{\beta}}{\partial x^{\nu} \partial x^{\lambda}}. \tag{2.17}$$

From Eq.(1.12), we achieve

$$\partial_{\lambda} g_{\mu\nu} = \eta_{\alpha\beta} \Gamma^{\rho}{}_{\mu\lambda} \frac{\partial y^{\alpha}}{\partial x^{\rho}} \frac{\partial y^{\beta}}{\partial x^{\nu}} + \eta_{\alpha\beta} \Gamma^{\rho}{}_{\nu\lambda} \frac{\partial y^{\beta}}{\partial x^{\rho}} \frac{\partial y^{\alpha}}{\partial x^{\mu}} = g_{\rho\nu} \Gamma^{\rho}{}_{\mu\lambda} + g_{\rho\mu} \Gamma^{\rho}{}_{\nu\lambda}. \tag{2.18}$$

Finally, permuting the indices, we write the Levi-Civita connection:

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} \right). \tag{2.19}$$

Thus, if we start from the statement of the WEP, we find a metric dependent affine connection. The relation (2.19) can also be derived by the metric-compatibility relation

$$\nabla^{\lambda} g_{\mu\nu} = 0, \tag{2.20}$$

imposing a null torsion.

2.3 Killing equations

A transformation $y : x^{\mu} \to y^{\nu}(x^{\mu})$ is an isometry if it preserves the metric:

$$g_{\mu\nu}(x) = \frac{\partial y^{\alpha}}{\partial x^{\mu}} \frac{\partial y^{\beta}}{\partial x^{\nu}} \tilde{g}_{\alpha\beta}(y). \tag{2.21}$$

Let us consider the infinitesimal transformation

$$y^{\alpha} = x^{\alpha} + \epsilon \xi^{\alpha} \qquad \epsilon \to 0 \tag{2.22}$$

where ξ is the vector field that generates the transformation. Substituting this transformation in Eq.(2.21), we obtain

$$\begin{aligned} g_{\mu\nu}(x) = &\left(\delta^{\alpha}_{\mu} + \epsilon \frac{\xi^{\alpha}}{\partial x^{\mu}}\right) \left(\delta^{\beta}_{\nu} + \epsilon \frac{\xi^{\beta}}{\partial x^{\nu}}\right) \tilde{g}_{\alpha\beta}(y) \\ \simeq &\tilde{g}_{\mu\nu}(y) + \epsilon \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} g_{\alpha\nu}(y) + \epsilon \frac{\partial \xi^{\beta}}{\partial x^{\nu}} g_{\mu\beta}(y) + o(\epsilon^2). \end{aligned} \tag{2.23}$$

In the last line we have replace $\tilde{g}_{\mu\nu}$ with $g_{\mu\nu}$ since the two tensors differ by an ϵ. Being

$$\tilde{g}_{\mu\nu}(y) \simeq \tilde{g}_{\mu\nu}(x) + \epsilon \xi^{\alpha} \frac{\partial \tilde{g_{\mu\nu}}}{\partial x^{\alpha}} \simeq \tilde{g}_{\mu\nu}(x) + \epsilon \xi^{\alpha} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}, \tag{2.24}$$

we find

$$g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) + \epsilon \xi^{\alpha} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} + \epsilon \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} g_{\alpha\nu}(x) + \epsilon \frac{\partial \xi^{\beta}}{\partial x^{\nu}} g_{\mu\beta}(x) \tag{2.25}$$

In order that the metric tensor is invariant under an infinitesimal transformation, it must be:

$$\xi^{\alpha} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} + \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} g_{\alpha\nu} + \frac{\partial \xi^{\beta}}{\partial x^{\nu}} g_{\mu\beta} = 0. \tag{2.26}$$

Using the Levi-Civita connection, this equation can be rewritten as

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{2.27}$$

These are the Killing equations. From the definition of Lie derivative

$$L_\xi g_{\mu\nu} = \lim_{\epsilon \to 0} \frac{g_{\mu\nu}(x) - \tilde{g}_{\mu\nu}(x)}{\epsilon}, \tag{2.28}$$

the Killing equations assume the form

$$L_\xi g_{\mu\nu} = 0. \tag{2.29}$$

This equation denotes that a transformation is an isometry if the Lie derivative of metric with respect to the generator vector field is null and ξ is called Killing vector field.
Let ξ and $\tilde{\xi}$ be two Killing vector fields. We easily verify that:

1. a linear combination $a\xi + b\tilde{\xi}$ $(a, b \in \mathbb{R})$ is a Killing vector field;

2. the Lie bracket $\left[\xi, \tilde{\xi}\right]$ is a Killing vector field.

(1) is obvious from the linearity of the covariant derivative. To prove (2) we use the propriety

$$L_{\left[\xi,\tilde{\xi}\right]} = L_\xi L_{\tilde{\xi}} - L_{\tilde{\xi}} L_\xi, \tag{2.30}$$

so

$$L_{\left[\xi,\tilde{\xi}\right]} \boldsymbol{g} = L_\xi L_{\tilde{\xi}} \boldsymbol{g} - L_{\tilde{\xi}} L_\xi \boldsymbol{g} = 0, \tag{2.31}$$

since $L_\xi \boldsymbol{g} = 0$ and $L_{\tilde{\xi}} \boldsymbol{g} = 0$.
Thus, all the Killing vector fields form a Lie algebra of the symmetric operations on the manifold $\boldsymbol{M}$.

Chapter 3

The spacetime deformations

Studies concerning the deformations of spacetime [23, 26] are related to physical problems that ranging from the spontaneous breaking of symmetry up to the gravitational waves. In this chapter we use an approach in which the deformations are treated through Lorentz matrices of scalar fields $\Phi^a{}_b$. These fields have a straightforward physical interpretation which could contribute to explain several fundamental issues as the inflation in cosmology and other pictures where scalar fields play a fundamental role in dynamics.

3.1 Metric deformations

Let us consider a 4-dimensional spacetime manifold $\boldsymbol{M}$ endowed with a Lorentzian metric $\boldsymbol{g}$. We work in the tetrad formalism [49] where the tetrads

$$h_a = h_a{}^\mu \partial_\mu \quad and \quad h^a = h^a{}_\mu dx^\mu \tag{3.1}$$

are linear bases that relate $\boldsymbol{g}$ to the tangent space metric

$$\eta = \eta_{ab} dx^a dx^b \tag{3.2}$$

through the relation

$$\eta_{ab} = g_{\mu\nu} h_a{}^\mu h_b{}^\nu . \tag{3.3}$$

This means that a tetrad is a linear frame whose members are pseudo-orthogonal by the pseudo-riemannian metric $\boldsymbol{g}$. Due to the relations

$$h^a{}_\mu h_a{}^\nu = \delta^\nu_\mu \quad and \quad h^a{}_\mu h_b{}^\mu = \delta^a_b , \tag{3.4}$$

we can reverse Eq.(3.3):

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu. \tag{3.5}$$

In the expressions just written we used the Greek alphabet ($\mu, \nu, \rho, ... = 0, 1, 2, 3$) to denote indices related to spacetime and the Latin alphabet ($a, b, c, ... = 0, 1, 2, 3$) to denote indices related to the tangent space.
Defining a new tetrad field with the help of a matrix scalar fields $\Phi^a{}_b \in GL(4)$,

$$\tilde{h}^a = \Phi^a{}_c h^c, \tag{3.6}$$

we introduce a manifold $\tilde{\boldsymbol{M}}$ with a metric

$$\tilde{\boldsymbol{g}} = \eta_{ab} \Phi^a{}_c \Phi^b{}_d h^c h^d = \gamma_{cd}(x) h^c h^d, \tag{3.7}$$

where

$$\gamma_{cd}(x) = \eta_{ab} \Phi^a{}_c(x) \Phi^b{}_d(x) \tag{3.8}$$

is a matrix of fields which are scalars with respect to the coordinate transformations. The manifold $\tilde{\boldsymbol{M}}$ and the metric $\tilde{\boldsymbol{g}}$ are respectively the deformation of $\boldsymbol{M}$ and $\boldsymbol{g}$. If all the functions of $\Phi^a{}_b$ are continuous, then there is a one-to-one correspondence between the points of $\boldsymbol{M}$ and the points of $\tilde{\boldsymbol{M}}$.
The matrices $\Phi^a{}_b(x)$ are known as first deformation matrices, instead $\gamma_{cd}(x)$ are called second deformation matrices.
A particular subset of deformation matrices is given by

$$\Phi^a{}_b(x) = \Omega(x) \delta^a_b. \tag{3.9}$$

This relation defines conformal transformation of the metric:

$$\tilde{\boldsymbol{g}} = \Omega^2(x) \boldsymbol{g}. \tag{3.10}$$

Then Eq.(3.7) can be regarded as a generalization of the conformal transformations.

3.2 The deformed connection

Let us decompose the matrix $\Phi^a{}_b(x)$ in its symmetric and antisymmetric parts:

$$\eta_{ac} \Phi^c{}_b = \Phi_{ab} = \Phi_{(ab)} + \Phi_{[ab]} = \Omega \eta_{ab} + \Theta_{ab} + \varphi_{ab}, \tag{3.11}$$

with $\Omega = \Phi^a{}_a$, Θ_{ab} the traceless symmetric part and φ_{ab} the skew symmetric part of the first deformation matrix. Then the second deformation matrix assumes the form

$$\begin{aligned}\gamma_{ab} =&\eta_{cd}\left(\Omega\delta^c_a + \Theta^c{}_a + \varphi^c{}_a\right)\left(\Omega\delta^d_b + \Theta^d{}_b + \varphi^d{}_b\right)\\ =&\Omega^2\eta_{ab} + 2\Omega\Theta_{ab} + \eta_{cd}\Theta^c{}_a\Theta^d{}_b + \eta_{cd}\left(\Theta^c{}_a\varphi^d{}_b + \varphi^c{}_a\Theta^d{}_b\right) + \\ &+ \eta_{cd}\varphi^c{}_a\varphi^d{}_b.\end{aligned} \tag{3.12}$$

Thanks to this relation the metric can be written as

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} + \gamma_{\mu\nu} \tag{3.13}$$

where

$$\begin{aligned}\gamma_{\mu\nu} =&(2\Omega\Theta_{ab} + \eta_{cd}\Theta^c{}_a\Theta^d{}_b + \eta_{cd}\left(\Theta^c{}_a\varphi^d{}_b + \varphi^c{}_a\Theta^d{}_b\right) + \\ &+ \eta_{cd}\varphi^c{}_a\varphi^d{}_b)h^a{}_\mu h^b{}_\nu.\end{aligned} \tag{3.14}$$

In order to find the deformed connection we need to obtain the contravariant deformed metric. It can be decomposed in the following way

$$\tilde{g}^{\mu\nu} = \alpha^2 g^{\mu\nu} + \lambda^{\mu\nu}. \tag{3.15}$$

The matrices $\gamma_{\mu\nu}$ and $\lambda^{\mu\nu}$ are related by a relation that we get imposing

$$\tilde{g}_{\mu\nu}\tilde{g}^{\nu\sigma} = \delta^\sigma_\mu = \alpha^2\Omega^2\delta^\sigma_\mu + \alpha^2\gamma_\mu{}^\sigma + \Omega^2\lambda_\mu{}^\sigma + \gamma_{\mu\nu}\lambda^{\nu\sigma}, \tag{3.16}$$

with $\alpha = \left(\Phi^{-1}\right)^a{}_a$. Assuming that the deformations are conformal transformations, we have $\alpha = \Omega^{-1}$, therefore

$$\Omega^{-2}\gamma_\mu{}^\sigma + \Omega^2\lambda_\mu{}^\sigma + \gamma_{\mu\nu}\lambda^{\nu\sigma} = 0. \tag{3.17}$$

From Eq.(3.17) we achieve

$$\lambda_\nu{}^\sigma = -\Omega^{-4}\left(\delta^\nu_\mu + \Omega^{-2}\gamma_\mu{}^\nu\right)^{-1}\gamma_\mu{}^\sigma. \tag{3.18}$$

The Levi-Civita connection corresponding to the metric $\tilde{g}_{\mu\nu}$ is related to the original Levi-Civita one by the relation

$$\tilde{\Gamma}^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\mu\nu} + C^\sigma{}_{\mu\nu}. \tag{3.19}$$

where

$$\begin{aligned} C^{\sigma}{}_{\mu\nu} =& 2\Omega\tilde{g}^{\sigma\lambda}g_{\lambda(\mu}\nabla_{\nu)}\Omega - \Omega g_{\mu\nu}\tilde{g}^{\sigma\lambda}\nabla_{\lambda}\Omega + \\ &+ \frac{1}{2}\tilde{g}^{\sigma\lambda}\left(\nabla_{\mu}\gamma_{\lambda\nu} + \nabla_{\nu}\gamma_{\mu\lambda} - \nabla_{\lambda}\gamma_{\mu\nu}\right). \end{aligned} \tag{3.20}$$

The connection (3.19) derives from the relation

$$\tilde{\nabla}_{\lambda}\tilde{g}_{\mu\nu} = \nabla_{\lambda}\tilde{g}_{\mu\nu} - C^{\sigma}{}_{\lambda\mu}\tilde{g}_{\sigma\nu} - C^{\sigma}{}_{\lambda\nu}\tilde{g}_{\mu\sigma} \tag{3.21}$$

imposing the metric compatibility.

The connection deformation acts like a force that deviates the test particles from the geodesic motion in the unperturbed spacetime, in fact the geodesic equation

$$\frac{d^2x^{\sigma}}{ds^2} + \tilde{\Gamma}^{\sigma}{}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0 \tag{3.22}$$

becomes

$$\frac{d^2x^{\sigma}}{ds^2} + \Gamma^{\sigma}{}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = -C^{\sigma}{}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}. \tag{3.23}$$

The deformed Riemann tensor is given by

$$\tilde{R}^{\alpha}{}_{\beta\mu\nu} = R^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu}C^{\alpha}{}_{\beta\nu} - \nabla_{\nu}C^{\alpha}{}_{\beta\mu} + C^{\alpha}{}_{\lambda\mu}C^{\lambda}{}_{\beta\nu} - C^{\alpha}{}_{\lambda\nu}C^{\lambda}{}_{\beta\mu}, \tag{3.24}$$

while the deformed Ricci tensor is

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \nabla_{\alpha}C^{\alpha}{}_{\mu\nu} - \nabla_{\nu}C^{\alpha}{}_{\mu\alpha} + C^{\alpha}{}_{\lambda\alpha}C^{\lambda}{}_{\mu\nu} - C^{\alpha}{}_{\lambda\nu}C^{\lambda}{}_{\mu\alpha} \tag{3.25}$$

and the deformed curvature scalar

$$\tilde{R} = \tilde{g}^{\mu\nu}R_{\mu\nu} + \tilde{g}^{\mu\nu}\left(\nabla_{\alpha}C^{\alpha}{}_{\mu\nu} - \nabla_{\nu}C^{\alpha}{}_{\mu\alpha} + C^{\alpha}{}_{\lambda\alpha}C^{\lambda}{}_{\mu\nu} - C^{\alpha}{}_{\lambda\nu}C^{\lambda}{}_{\mu\alpha}\right). \tag{3.26}$$

We obtain the equation for the deformations in presence of matter From Eq.(3.25):

$$R_{\mu\nu} + \nabla_{\alpha}C^{\alpha}{}_{\mu\nu} - \nabla_{\nu}C^{\alpha}{}_{\mu\alpha} + C^{\alpha}{}_{\lambda\alpha}C^{\lambda}{}_{\mu\nu} - C^{\alpha}{}_{\lambda\nu}C^{\lambda}{}_{\mu\alpha} = k\tilde{\Theta}_{\mu\nu} - \frac{1}{2}k\tilde{g}_{\mu\nu}\tilde{\Theta} \tag{3.27}$$

where $\tilde{\Theta}_{\mu\nu}$ is the deformed energy-momentum tensor.

3.3 The Killing equations for deformed metric

Let us compute the Lie derivative of the deformed metric:

$$\begin{aligned}L_{\tilde{\xi}}\tilde{g}_{\mu\nu} =&\tilde{\xi}^\sigma\tilde{\partial}_\sigma\tilde{g}_{\mu\nu} + \tilde{g}_{\sigma\nu}\tilde{\partial}_\mu\tilde{\xi}^\sigma + \tilde{g}_{\mu\sigma}\tilde{\partial}_\nu\tilde{\xi}^\sigma \\ =&\tilde{\xi}^\sigma\tilde{\nabla}_\sigma\tilde{g}_{\mu\nu} + \tilde{\xi}^\sigma\tilde{\Gamma}^\lambda{}_{\sigma\mu}\tilde{g}_{\lambda\nu} + \tilde{\xi}^\sigma\tilde{\Gamma}^\lambda{}_{\sigma\nu}\tilde{g}_{\lambda\mu} + \\ &+ \tilde{g}_{\sigma\nu}\tilde{\nabla}_\mu\tilde{\xi}^\sigma - \tilde{g}_{\sigma\nu}\tilde{\Gamma}^\sigma{}_{\mu\lambda}\tilde{\xi}^\lambda + \tilde{g}_{\sigma\mu}\tilde{\nabla}_\nu\tilde{\xi}^\sigma - \tilde{g}_{\sigma\mu}\tilde{\Gamma}^\sigma{}_{\nu\lambda}\tilde{\xi}^\lambda,\end{aligned} \tag{3.28}$$

from which, since $\tilde{\Gamma}$ is symmetrical in the last two indices, we get

$$\tilde{\nabla}_{(\mu}\tilde{\xi}_{\nu)} = 0. \tag{3.29}$$

However, so far we have set ourselves in the case of metric compatibility, if we impose that the non-metricity tensor is not null, then the relation just written is no longer valid since we have that the Lie derivative must necessarily be different from zero since the presence of the non-metricity does not allow the preservation of the scalar product. Therefore, what we find is

$$\tilde{\nabla}_{(\mu}\tilde{\xi}_{\nu)} + \frac{1}{2}\left(\tilde{Q}_{\sigma\mu\nu} - \tilde{Q}_{\mu\sigma\nu} - \tilde{Q}_{\nu\sigma\mu}\right)\tilde{\xi}^\sigma \neq 0, \tag{3.30}$$

with $\tilde{Q}_{\sigma\mu\nu}$ the non-metricity tensor of the deformed connection.

Chapter 4

Metric theories of gravity

By the EEP, Einstein formulated a new gravity theory, General Relativity (GR), in terms of the spacetime curvature. In this chapter we present a brief overview of GR and its extensions, while in the next chapters we will discuss other approaches to gravity theory based on the curvature.

4.1 General Relativity: a metric theory

General Relativity [46] is a purely metric theory, where the only field that mediates gravity is the metric and every concept and quantity are linked to it.
The connection is torsionless and metric-compatible, i.e.

$$T^{\lambda}{}_{\mu\nu} = 0 \rightarrow \Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}, \tag{4.1}$$

$$Q_{\lambda\mu\nu} = 0 \rightarrow \nabla_{\lambda} g_{\mu\nu} = 0, \tag{4.2}$$

thus the connection is assumed to be the Levi-Civita one:

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} \right). \tag{4.3}$$

The Riemann tensor is the same of Eq.(2.11) and it satisfies the following Bianchi identities:

$$R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\mu\nu\beta} + R^{\alpha}{}_{\nu\beta\mu} = 0 \tag{4.4}$$

$$\nabla_{\rho} R^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu} R^{\alpha}{}_{\beta\nu\rho} + \nabla_{\nu} R^{\alpha}{}_{\beta\rho\mu} = 0 \tag{4.5}$$

4.1.1 The geodesic in General Relativity

In GR we have two equivalent definitions of geodesic. The first is already given in Chapter 2 by Eq.(2.7). The second one states that the geodesic is the curve that minimises the distance between two points in a manifold, that is

$$\delta S = \delta \int_p^q ds = \delta \int_p^q \left(g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right)^{\frac{1}{2}} ds = 0. \tag{4.6}$$

The variation of the integral leads to

$$\delta S = \int_p^q \frac{1}{2\sqrt{g_{\alpha\beta} v^\alpha v^\beta}} \left[\partial_\lambda g_{\alpha\beta} \delta x^\lambda \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} + 2 g_{\alpha\beta} \frac{d}{ds} (\delta x^\alpha) \frac{dx^\beta}{ds} \right] ds, \tag{4.7}$$

with $v^\alpha = dx^\alpha / ds$. Integrating by parts and cancelling the boundary term, we get

$$\begin{aligned} \delta S = & \int_p^q \frac{1}{2} \left[\partial_\lambda g_{\alpha\beta} \delta x^\lambda \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \right] ds + \\ & - \int_p^q \left[g_{\alpha\beta} \frac{d^2 x^\beta}{ds^2} + \partial_\lambda g_{\alpha\beta} \delta x^\lambda \frac{dx^\lambda}{ds} \frac{dx^\beta}{ds} \right] \delta x^\alpha ds \\ = & \int_p^q \left[\left(\frac{1}{2} \partial_\lambda g_{\alpha\beta} - \partial_\alpha g_{\lambda\beta} \right) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} - g_{\lambda\beta} \frac{d^2 x^\beta}{ds^2} \right] \delta x^\lambda ds = 0, \end{aligned} \tag{4.8}$$

where we used the fact that $g_{\alpha\beta} dx^\alpha dx^\beta = 1$. Since

$$\partial_\alpha g_{\lambda\beta} v^\alpha v^\beta = \partial_\beta g_{\lambda\alpha} v^\alpha v^\beta = \frac{1}{2} \left(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} \right) v^\alpha v^\beta, \tag{4.9}$$

finally we achieve the geodesic equation

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda{}_{\alpha\beta} v^\alpha v^\beta = 0. \tag{4.10}$$

The equivalence of these two definitions is due to the fact that GR is a torsion-free theory.

4.1.2 Field equations

We can derive the field equations from Hilbert-Einstein action:

$$S = k \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_m \left(g_{\mu\nu}, \phi \right), \tag{4.11}$$

where k is a dimensional constant, g is the determinant of the metric, R is the Ricci scalar and $\mathcal{L}_m$ is the matter lagrangian density.
Varying the action with respect to the metric, we achieve

$$\begin{aligned}\delta S &= k\int d^4x\left(\delta\sqrt{-g}R+\sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu}+\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\right)+\delta S_m\\ &= \int d^4x\sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}+\delta S_m.\end{aligned} \tag{4.12}$$

The second integral can be evaluated in the local inertial frame, obtaining

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha}, \tag{4.13}$$

$$\delta R_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha{}_{\mu\nu} - \partial_\nu \delta\Gamma^\alpha{}_{\mu\alpha}, \tag{4.14}$$

$$\begin{aligned}g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\partial_\alpha\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\nu}\partial_\nu\delta\Gamma^\alpha{}_{\mu\alpha}\\ &= \partial_\rho\left(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\alpha{}_{\mu\alpha}\right).\end{aligned} \tag{4.15}$$

Then, we can write

$$g^{\mu\nu}\delta R_{\mu\nu} = \partial_\rho W^\rho, \tag{4.16}$$

$$W^\rho = g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\alpha{}_{\mu\alpha}. \tag{4.17}$$

Therefore the integral is null since its argument is a pure divergence. In fact, in general coordinates it is:

$$\begin{aligned}\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \int d^4x\sqrt{-g}\partial_\rho W^\rho = \int d^4x\sqrt{-g}\nabla_\rho W^\rho\\ &= \int d^4x\partial_\rho\left(\sqrt{-g}W^\rho\right) = 0.\end{aligned} \tag{4.18}$$

Then

$$\delta S = k\int d^4x\sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+\delta S_m, \tag{4.19}$$

from which we find the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k\Theta_{\mu\nu}, \tag{4.20}$$

where

$$\Theta_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}} \tag{4.21}$$

is the matter energy-momentum tensor.
The second Bianchi identity Eq.(4.5) can be rewritten as follows

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{4.22}$$

This identity gives us the energy conservation equation

$$\nabla^{\mu} \Theta_{\mu\nu} = 0. \tag{4.23}$$

4.2 Extended Theories of Gravity

Extended Theories of Gravity (ETG) [22, 20] are an answer to the need to put together a new theory capable of describing the gravitational interaction, need arose with the appearance of the shortcomings of General Relativity (quantum gravity, inflationary paradigm, etc.). ETG consist essentially of adding higher order curvature invariants or non-minimally coupled terms between matter fields and geometry into the Hilbert-Einstein lagrangian. In this section we analyse some of these theories.

4.2.1 $f(R)$ theories

In $f(R)$ theories the lagrangian density is a general function of the Ricci scalar R, therefore the Hilbert-Einstein action takes the form

$$S = \int d^4x \sqrt{-g} f(R) + S_m. \tag{4.24}$$

Let us calculate the field equations from the variational principle (for simplicity, we calculate this variation in an inertial local frame):

$$\delta S = \int d^4x \delta \left[\sqrt{-g} f(R) \right] + \delta S_m. \tag{4.25}$$

The last term gives us the usual matter energy-momentum tensor (4.21), whereas the first term provides:

$$\begin{aligned} \int d^4x \delta \left[\sqrt{-g} f(R) \right] &= \int d^4x \left[\delta \sqrt{-g} f(R) + \sqrt{-g} \delta f(R) \right] \\ &= \int d^4x \sqrt{-g} \left[f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) \right] \delta g^{\mu\nu} \\ &\quad + \int d^4x \sqrt{-g} f'(R) g^{\mu\nu} \delta R_{\mu\nu}. \end{aligned} \tag{4.26}$$

The second integral can be rewritten (Eq.(4.17)) as

$$\begin{aligned}\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} &= \int d^4x\sqrt{-g}f'(R)\partial_\rho W^\rho \\ &= \int d^4x\partial_\rho\left[\sqrt{-g}f'(R)W^\rho\right] - \partial_\rho\left[\sqrt{-g}f'(R)\right]W^\rho \\ &= -\int d^4x\partial_\rho\left[\sqrt{-g}f'(R)\right]W^\rho.\end{aligned} \tag{4.27}$$

From the expression of the Levi-Civita connection (2.19), we obtain

$$g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\partial^\rho\left(g_{\mu\nu}\delta g^{\mu\nu}\right) - \partial^\mu\left(g_{\alpha\mu}\delta g^{\rho\alpha}\right), \tag{4.28}$$

$$g^{\mu\rho}\delta\Gamma^\alpha{}_{\mu\nu} = -\frac{1}{2}\partial^\rho\left(g_{\nu\alpha}\delta g^{\nu\alpha}\right). \tag{4.29}$$

Inserting these equations in Eq.(4.27), the integral becomes

$$\begin{aligned}\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} &= \int d^4x\partial_\rho\left[\sqrt{-g}f'(R)\right]\left[\partial^\mu\left(g_{\mu\nu}\delta g^{\rho\nu}\right) - \partial^\rho\left(g_{\mu\nu}\delta g^{\mu\nu}\right)\right] \\ &= \int d^4x g_{\mu\nu}\partial^\rho\partial_\rho\left[\sqrt{-g}f'(R)\right]\delta g^{\mu\nu} + \\ &\quad - \int d^4x g_{\mu\nu}\partial^\mu\partial_\rho\left[\sqrt{-g}f'(R)\right]\delta g^{\rho\nu}.\end{aligned} \tag{4.30}$$

The variation of the action is then

$$\begin{aligned}\int d^4x\delta\left[\sqrt{-g}f(R)\right] &= \int d^4x\sqrt{-g}\left[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R)\right]\delta g^{\mu\nu} + \\ &\quad + \int d^4x\Big[g_{\mu\nu}\partial^\rho\partial_\rho\left(\sqrt{-g}f'(R)\right) \\ &\quad - g_{\rho\nu}\partial^\rho\partial_\mu\left(\sqrt{-g}f'(R)\right)\Big]\delta g^{\mu\nu}.\end{aligned} \tag{4.31}$$

The vanishing of the variation implies the following field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R). \tag{4.32}$$

Adding the matter term and putting in evidence the Einstein tensor, these field equations take the form

$$G_{\mu\nu} = \frac{\Theta_{\mu\nu}}{f'(R)} + \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\mu\nu}\left[f(R) - f'(R)R\right] + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right\}. \tag{4.33}$$

The term in braces,

$$\Theta_{\mu\nu}^{eff} = \frac{1}{2} g_{\mu\nu} \left[f(R) - f'(R)R \right] + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R), \tag{4.34}$$

is an effective energy-momentum tensor which can be interpreted as an extra gravitational energy-momentum tensor due to higher-order curvature effects.

4.2.2 Scalar-tensor theories

Scalar-tensor theories was born when we tried to incorporate the Mach's principle, which states that the local inertial frame is determined by the average motion of distant astronomical objects, into metric gravity and they are characterized by the presence of a non-minimal coupling scalar field ϕ. These theories exhibit a non-constant gravitational coupling, thus the Newton constant G_N is replaced by the effective gravitational coupling:

$$G_{eff} = \frac{1}{F(\phi)}, \tag{4.35}$$

where $F(\phi)$ is a generic function of the scalar field.
In a situation where gravity is not minimally coupled, a generic action is given by

$$S = \int d^4x \sqrt{-g} \left[F(\phi) R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] + S_m \left(\psi, \nabla \psi \right), \tag{4.36}$$

with $V(\phi)$ a generic scalar field potential.
The variation of the action with respect to metric leads to the equation

$$\begin{aligned} -\frac{1}{2} g_{\mu\nu} F(\phi) R + F(\phi) R_{\mu\nu} + g_{\mu\nu} \Box F(\phi) - \nabla_\mu \nabla_\nu F(\phi) + \\ -\frac{1}{4} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi + + \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} V(\phi) g_{\mu\nu} = \Theta_{\mu\nu}, \end{aligned} \tag{4.37}$$

that we can rewrite as

$$G_{\mu\nu} = \frac{\Theta_{\mu\nu}}{F(\phi)} + \Theta_{\mu\nu}^{eff} \tag{4.38}$$

with

$$\begin{aligned} \Theta_{\mu\nu}^{eff} = & -\frac{1}{2F(\phi)} \Big[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi + g_{\mu\nu} V(\phi) + \\ & + 2 g_{\mu\nu} \Box F(\phi) - 2 \nabla_\mu \nabla_\nu F(\phi) \Big] \end{aligned} \tag{4.39}$$

an effective energy-momentum tensor which includes the contributions of non-minimal coupling, the kinetic terms and the potential of the scalar field.
Varying the action (4.36) with respect to ϕ, instead we find:

$$\Box\phi - RF'(\phi) - V'(\phi) = 0. \tag{4.40}$$

As expected, we have obtained the Klein-Gordon equation.

Chapter 5

Purely affine theories of gravity

The purely affine formulation [33, 42] is an approach to the gravitational theory where the affine connection is the only variable and we express every gravitational quantities in terms of it.
A general lagrangian density depends on the affine connection, its first derivative, a matter field of any nature and on the correspondent derivative:

$$\mathcal{L} = \mathcal{L}\left(\Gamma, \partial\Gamma, \phi, \partial\phi\right), \tag{5.1}$$

where Γ is an arbitrary affine connection and ϕ denotes a general matter field.
The purely affine formulation allows us to solve several shortcomings of the metric formulation of GR. It is important to underline that this formulation is not a modified theory of gravity but GR itself written in terms of the affine connection as a dynamical configuration variable. This equivalence implies that also purely affine gravity is consistent with experimental tests of GR.

5.1 The affine variational principle

Let us calculate the dynamical equations for a purely affine theory. To do this it is required to define the canonical momenta conjugate to Γ,

$$\pi_{\lambda}{}^{\mu\nu\rho} \equiv \frac{\partial\mathcal{L}\left(\Gamma, \partial\Gamma, \phi, \partial\phi\right)}{\partial\left(\partial_{\rho}\Gamma^{\lambda}{}_{\mu\nu}\right)}, \tag{5.2}$$

and the canonical momenta conjugate to the field ϕ,

$$p^{\rho} \equiv \frac{\partial \mathcal{L}\left(\Gamma, \partial\Gamma, \phi, \partial\phi\right)}{\partial\left(\partial_{\rho}\phi\right)}. \tag{5.3}$$

The total differential of the affine lagrangian density is equal to

$$\begin{aligned} d\mathcal{L} =& \pi_{\lambda}{}^{\mu\nu\rho} d\left(\partial_{\rho}\Gamma^{\lambda}{}_{\mu\nu}\right) + \frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\mu\nu}} d\Gamma^{\lambda}{}_{\mu\nu} + p^{\rho} d\left(\partial_{\rho}\phi\right) + \frac{\partial\mathcal{L}}{\partial\phi} d\phi \\ =& \left(\frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}{}_{\mu\nu}} - \partial_{\rho}\pi_{\lambda}{}^{\mu\nu\rho}\right) d\Gamma^{\lambda}{}_{\mu\nu} + \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\rho}p^{\rho}\right) d\phi + \\ &+ \partial_{\rho}\left(\pi_{\lambda}{}^{\mu\nu\rho} d\Gamma^{\lambda}{}_{\mu\nu} + p^{\rho} d\phi\right). \end{aligned} \tag{5.4}$$

Thus we obtain the following Eulero-Lagrange equations:

$$\partial_{\rho}\pi_{\lambda}{}^{\mu\nu\rho} = \frac{\partial\mathcal{L}\left(\Gamma, \partial\Gamma, \phi, \partial\phi\right)}{\partial\Gamma^{\lambda}{}_{\mu\nu}}, \tag{5.5}$$

$$\partial_{\rho}p^{\rho} = \frac{\partial\mathcal{L}\left(\Gamma, \partial\Gamma, \phi, \partial\phi\right)}{\partial\phi}. \tag{5.6}$$

Due to diffeomorphism invariance of General Relativity, to recover the Einstein equations, the affine lagrangian must depend on the only invariant quantity that can be construct in terms of $\partial\Gamma$, the Riemann tensor. In particular, the affine lagrangian must assume the form

$$\mathcal{L} = \mathcal{L}\left(\Gamma, P, \phi, \partial\phi\right), \tag{5.7}$$

where P is the symmetric part of Ricci tensor. Using this lagrangian density the differential becomes

$$d\mathcal{L} = \pi^{\mu\nu} dP_{\mu\nu} + \mathcal{J}_{\lambda}{}^{\mu\nu} d\Gamma^{\lambda}{}_{\mu\nu} + p^{\rho} d\left(\partial_{\rho}\phi\right) + \frac{\partial\mathcal{L}}{\partial\phi} d\phi, \tag{5.8}$$

where we have defined the two quantities:

$$\pi^{\mu\nu} \equiv k\sqrt{-g} g^{\mu\nu} = \frac{\partial\mathcal{L}\left(\Gamma, P, \phi, \partial\phi\right)}{\partial P_{\mu\nu}}, \tag{5.9}$$

which represents the controvariant density of the metric associated to the manifold and it is a function of $\partial\Gamma$ being the canonical momentum conjugate to Γ, and

$$\mathcal{J}_{\lambda}{}^{\mu\nu} \equiv \frac{\partial\mathcal{L}\left(\Gamma, P, \phi, \partial\phi\right)}{\partial\Gamma^{\lambda}{}_{\mu\nu}}. \tag{5.10}$$

Using the relation

$$\pi_\lambda{}^{\mu\nu\rho} = \delta^\rho_\lambda \pi^{\mu\nu} - \delta^{(\mu}_\lambda \pi^{\nu)\rho}, \tag{5.11}$$

after some manipulations, we achieve

$$\begin{aligned} d\mathcal{L} = &\left(\mathcal{J}_\lambda{}^{\mu\nu} - \nabla_\rho \pi_\lambda{}^{\mu\nu\rho}\right) d\Gamma^\lambda{}_{\mu\nu} + \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\rho p^\rho\right) d\phi + \\ &+ \partial_\rho \left(\pi_\lambda{}^{\mu\nu\rho} d\Gamma^\lambda{}_{\mu\nu} + p^\rho d\phi\right), \end{aligned} \tag{5.12}$$

from which the Eulero-Lagrange equations derive:

$$\nabla_\lambda \pi^{\mu\nu} = \mathcal{J}_\lambda{}^{\mu\nu} - \frac{2}{3}\delta^{(\mu}_\lambda \mathcal{J}_\rho{}^{\nu)\rho}, \tag{5.13}$$

$$\partial_\rho p^\rho = \frac{\partial \mathcal{L}\left(\Gamma, P, \phi, \partial\phi\right)}{\partial \phi}. \tag{5.14}$$

The Levi-Civita connection is recovered when the lagrangian density depends on affine connection only through the symmetric part of the Ricci tensor, being $\mathcal{J}$ null by definition:

$$\nabla_\lambda \pi^{\mu\nu} = 0. \tag{5.15}$$

To analyse the relation between the purely affine theory and metric theory, it is useful to subtract from the lagrangian density $\mathcal{L}$ the Hilbert-Einstein one

$$\mathcal{L}_{H-E} = k\sqrt{-g}R = \pi^{\mu\nu} P_{\mu\nu} \tag{5.16}$$

and then define the affine matter lagrangian density:

$$\mathcal{L}^{matt}\left(\pi, \Gamma, \phi, \partial\phi\right) \equiv \mathcal{L}\left(\Gamma, P, \phi, \partial\phi\right) - \pi^{\mu\nu} P_{\mu\nu}\left(\pi, \Gamma, \phi, \partial\phi\right). \tag{5.17}$$

The Eq.(5.8) implies the following expression for the total differential of affine matter lagrangian density

$$d\mathcal{L}^{matt} = P_{\mu\nu} d\pi^{\mu\nu} + \mathcal{J}_\lambda{}^{\mu\nu} d\Gamma^\lambda{}_{\mu\nu} + p^\rho d\left(\partial_\rho \phi\right) + \partial_\rho p^\rho d\phi, \tag{5.18}$$

with

$$P_{\mu\nu} = -\frac{\partial \mathcal{L}^{matt}\left(\pi, \Gamma, \phi, \partial\phi\right)}{\partial \pi^{\mu\nu}} \tag{5.19}$$

$$\mathcal{J}_\lambda{}^{\mu\nu} \equiv \frac{\partial \mathcal{L}^{matt}\left(\pi, \Gamma, \phi, \partial\phi\right)}{\partial \Gamma^\lambda{}_{\mu\nu}} \tag{5.20}$$

$$p^{\rho} \equiv \frac{\partial \mathcal{L}^{matt}\left(\pi, \Gamma, \phi, \partial\phi\right)}{\partial\left(\partial_{\rho}\phi\right)}. \tag{5.21}$$

From Eq.(5.9) we can derive $\boldsymbol{g}$ as a function of π,

$$g_{\mu\nu} = k\sqrt{-\pi}\left(\pi^{-1}\right)_{\mu\nu}. \tag{5.22}$$

Inserting Eq.(5.22) into Eq.(5.19), we obtain

$$\begin{aligned} P_{\mu\nu} =& -\frac{\partial g_{\rho\sigma}}{\partial \pi^{\mu\nu}} \frac{\partial \mathcal{L}^{matt}\left(\boldsymbol{g}, \Gamma, \phi, \partial\phi\right)}{\partial g_{\rho\sigma}} \\ =& k\sqrt{-\pi}\left[\left(\pi^{-1}\right)_{\rho(\mu}\left(\pi^{-1}\right)_{\nu)\sigma} - \frac{1}{2}\left(\pi^{-1}\right)_{\mu\nu}\left(\pi^{-1}\right)_{\rho\sigma}\right]\frac{\partial \mathcal{L}^{matt}}{\partial g_{\rho\sigma}} \\ =& \frac{k}{\sqrt{-g}}\left(g_{\rho(\mu}g_{\nu)\sigma} - \frac{1}{2}g_{\mu\nu}g_{\rho\sigma}\right)\frac{\partial \mathcal{L}^{matt}}{\partial g_{\rho\sigma}}. \end{aligned} \tag{5.23}$$

Inverting this relation we get the Einstein equations

$$G^{\mu\nu} = k\Theta^{\mu\nu}, \tag{5.24}$$

where

$$G^{\mu\nu} \equiv \sqrt{-g}\left(P^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) \tag{5.25}$$

is the density of the Einstein tensor expressed in terms of the symmetric part of the Ricci tensor, and

$$\Theta^{\mu\nu} \equiv \frac{\partial \mathcal{L}^{matt}\left(\boldsymbol{g}, \Gamma, \phi, \partial\phi\right)}{\partial g_{\mu\nu}} \tag{5.26}$$

is the affine energy-momentum tensor density.
In the case of $J = 0$ we have, as seen above, the connection Γ equal to the Levi-Civita one and Θ reduces to the standard metric energy-momentum tensor.

5.2 The cosmological constant Λ in a purely affine approach

In this section we want to show that we can write a lagrangian density, in a purely affine formalism, which leads to Einstein equations with a cosmological constant [41] as we consider a non null torsion tensor T:

$$\mathcal{L}_{\Lambda} = \sqrt{k}k^{\mu\nu}R_{\mu\nu} + \alpha\mathcal{L}_m\left(k, \phi, \partial\phi\right), \tag{5.27}$$

where $R_\mu\nu$ is the Ricci tensor, $k^{\mu\nu}$ is a symmetric tensor defined as

$$k_{\mu\nu} = T^\rho{}_{\lambda\mu} T^\lambda{}_{\rho\nu}, \quad k^{\mu\rho} k_{\nu\rho} = \delta^\mu_\nu, \tag{5.28}$$

with $T_\mu = T^\nu{}_{\mu\nu}$, k is the absolute value of the determinant of $k_{\mu\nu}$ and $\mathcal{L}_m$ is the matter lagrangian density that depends on the connection only through the torsion tensor.
The variation of the action corresponding to Eq.(5.27) is

$$\begin{aligned} \delta S &= \delta \int d^4x \sqrt{k} k^{\mu\nu} R_{\mu\nu} \\ &= \int d^4x \sqrt{k} k^{\mu\nu} \delta R_{\mu\nu} + \int d^4x \sqrt{k} \left(R_{\mu\nu} - \frac{1}{2} R_{\rho\sigma} k^{\rho\sigma} k_{\mu\nu} \right) \delta k^{\mu\nu} + \alpha \delta S_m. \end{aligned} \tag{5.29}$$

Field equations result from the variation (5.29) calculated under $\delta\Gamma^\lambda{}_{\mu\nu}$. Since the variation $\delta\Gamma^\lambda{}_{\mu\nu}$ can be divided into the symmetric part $\delta\Gamma^\lambda{}_{(\mu\nu)}$ and the antisymmetric part $\delta\Gamma^\lambda{}_{[\mu\nu]} = \delta T^\lambda{}_{\mu\nu}$, we can vary the action under $\delta\Gamma^\lambda{}_{(\mu\nu)}$, substitute the resulting field equations into the action, and then vary the action under $\delta T^\lambda{}_{\mu\nu}$.
Varying under $\delta\Gamma^\lambda{}_{(\mu\nu)}$, we find the equations

$$\nabla_\rho k_{\kappa\lambda} = \partial_\rho k_{\kappa\lambda} - \Gamma^\sigma{}_{\rho\kappa} k_{\sigma\lambda} - \Gamma^\sigma{}_{\rho\lambda} k_{\kappa\sigma} = -Q_{\rho\kappa\lambda}, \tag{5.30}$$

where

$$Q_{\rho\kappa\lambda} = -\left(\frac{2}{3} T_\rho k_{\kappa\lambda} + \frac{1}{3} T_\kappa k_{\rho\lambda} + \frac{1}{3} T_\lambda k_{\rho\kappa} + T^\sigma{}_{\rho\kappa} k_{\sigma\lambda} + T^\sigma{}_{\rho\lambda} k_{\sigma\kappa} \right). \tag{5.31}$$

Permuting the indices in Eq.(5.30) we get the symmetric part of the connection:

$$\Gamma^\rho{}_{\mu\nu} = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}_k + T^\sigma{}_{\mu\lambda} k^{\rho\lambda} k_{\nu\sigma} + T^\sigma{}_{\nu\lambda} k^{\rho\lambda} k_{\mu\sigma} - \frac{1}{2} k^{\rho\lambda} \left(Q_{\lambda\mu\nu} - Q_{\mu\lambda\nu} - Q_{\nu\lambda\mu} \right), \tag{5.32}$$

where

$$\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}_k = \frac{1}{2} k^{\rho\sigma} \left(\partial_\mu k_{\sigma\nu} + \partial_\nu k_{\sigma\mu} - \partial_\sigma k_{\mu\nu} \right) \tag{5.33}$$

is the Levi-Civita connection constructed from the rank-2 tensor $k_{\mu\nu}$. Using Eq.(5.30) and Eq.(5.32), we obtain the affine connection

$$\Gamma^\rho{}_{\mu\nu} = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}_k + T^\rho{}_{\mu\nu} - \frac{1}{3} \left(\delta^\rho_\mu T_\nu + \delta^\rho_\nu T_\mu \right). \tag{5.34}$$

Defining the quantity

$$C^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}_k = T^{\rho}{}_{\mu\nu} - \frac{1}{3}\left(\delta^{\rho}_{\mu} T_{\nu} + \delta^{\rho}_{\nu} T_{\mu}\right), \tag{5.35}$$

we can decompose the Riemann tensor as

$$R^{\lambda}{}_{\rho\mu\nu} = R^{\lambda(k)}_{\rho\mu\nu} + \nabla^{\{\}}_{\mu} C^{\lambda}{}_{\rho\nu} - \nabla^{\{\}}_{\nu} C^{\lambda}{}_{\rho\mu} + C^{\lambda}{}_{\sigma\mu} C^{\sigma}{}_{\rho\nu} - C^{\lambda}{}_{\sigma\nu} C^{\sigma}{}_{\rho\mu} \tag{5.36}$$

where $R^{\lambda(k)}_{\rho\mu\nu}$ is the Riemann tensor constructed form the Levi-Civita connection (5.33) and $\nabla^{\{\}}$ is the covariant derivative defined using this connection.
Substituting Eq.(5.36) in Eq.(5.27), after some manipulation, we find the action:

$$S = \int d^4x \sqrt{k} \left(R^{(k)}_{\mu\nu} k^{\mu\nu} + 4 - \frac{1}{3} T_{\mu} T_{\nu} k^{\mu\nu} \right) + \alpha S_m. \tag{5.37}$$

The last term in the parentheses can be set to zero by one arbitrary coordinate transformation, instead the first two terms have the form of the Einstein-Hilbert lagrangian for the gravitational field with a cosmological constant if we identify, up to a multiplicative constant factor, the tensor $k_{\mu\nu}$ with the metric tensor $g_{\mu\nu}$:

$$g_{\mu\nu} = \frac{2}{\Lambda} k_{\mu\nu}, \tag{5.38}$$

where the constant Λ has the dimension of inverse square length. Thus Eq.(5.37) becomes the Hilbert-Einstein action with the cosmological constant:

$$S \propto \int d^4x \sqrt{g} \left(R^{(g)}_{\mu\nu} g^{\mu\nu} + 2\Lambda \right). \tag{5.39}$$

Now, considering action (5.37) and varying it with respect to $\delta T^{\rho}{}_{\mu\nu}$, we obtain the equations

$$T_{\mu} = 0 \tag{5.40}$$

$$R^{(k)}_{\mu\nu} - \frac{1}{2} R^{(k)}_{\rho\sigma} k^{\rho\sigma} k_{\mu\nu} = 2k_{\mu\nu} - \frac{\alpha}{\sqrt{k}} \frac{\delta \mathcal{L}_m}{\delta k^{\mu\nu}} \tag{5.41}$$

Inserting (5.38) in (5.41), the Einstein equations of General Relativity with cosmological constant are reproduced

$$R^{(g)}_{\mu\nu} - \frac{1}{2} R^{(g)}_{\rho\sigma} g^{\rho\sigma} g_{\mu\nu} = \Lambda g_{\mu\nu} - \frac{\alpha}{\Lambda} \Theta_{\mu\nu}. \tag{5.42}$$

Therefore we have shown that the cosmological constant occurs naturally in the case in which we have a purely affine theory and a non-zero torsion, unlike what happens in General Relativity.

Chapter 6

Metric-affine theories of gravity

Until now we have discussed about theories where the main variable is either the metric or the connection. In this chapter we analyse a mixed approach: the Palatini formalism.
The Palatini approach to gravitational theories is based on the idea that the affine connection Γ is a variable independent of the spacetime metric $\boldsymbol{g}$. From a physical point of view, this is equivalent to decouple the casual structure of spacetime and its geodesic structure.
The Palatini formalism applied to the Hilbert-Einstein action is completely equivalent to the purely metric theory since it leads to the same field equations, however, the situation is different when we consider the Extended Theories of Gravity as we will see [35, 9, 38, 21].

6.1 Extended Theories of Gravity in Palatini formalism

Let us begin with the study of the $f(R)$ theories:

$$S = \int d^4x\sqrt{-g}f(R) + S_m\left(\boldsymbol{g}, \phi\right) \tag{6.1}$$

where $R \equiv R(\boldsymbol{g}, \Gamma) = g^{\mu\nu}R_{\mu\nu}\left(\Gamma\right)$ (it will be so for the whole section), i.e. the Ricci scalar R depends on both $\boldsymbol{g}$ and Γ, instead the Ricci tensor is a function of the only affine connection. Varying the action with respect to the metric tensor, we obtain

$$\begin{aligned}\delta_g S &= \int d^4x \delta_g \left[\sqrt{-g} f(R)\right] + \delta_g S_m \\ &= \int d^4x \left[\delta_g \sqrt{-g} f(R) + \sqrt{-g}\delta_g f(R)\right] + \delta_g S_m \\ &= \int d^4x \left[\delta_g \sqrt{-g} f(R) + \sqrt{-g} f'(R)\delta_g R\right] + \delta_g S_m \\ &= \int d^4x \left[\delta_g \sqrt{-g} f(R) + \sqrt{-g} f'(R)\delta_g g^{\mu\nu} R_{\mu\nu}\right] + \delta_g S_m.\end{aligned} \tag{6.2}$$

from which we determine the field equations

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = \Theta_{\mu\nu}. \tag{6.3}$$

Varying, instead, with respect to the connection, we get

$$\nabla^\Gamma_\lambda \left(\sqrt{-g} f'(R) g^{\mu\nu}\right) = 0, \tag{6.4}$$

with ∇^Γ_λ the covariant derivative defined using the independent connection Γ. From Eq.(6.4) the bimetric structure of spacetime emerges, indeed we can introduce a new metric $h_{\mu\nu}$ conformally related to $g_{\mu\nu}$ by

$$\sqrt{-g} f'(R) g^{\mu\nu} = \sqrt{-h} h^{\mu\nu}. \tag{6.5}$$

This implies that the connection Γ is the Levi-Civita one of the metric $h_{\mu\nu}$, with the only restriction that the conformal factor relating $g_{\mu\nu}$ and $h_{\mu\nu}$ is non-degenerate. The General Relativity case is recovered for $f'(R) = 1$.
It is useful to consider the trace of the field equations (6.3) :

$$f'(R)R - 2f(R) = g^{\mu\nu}\Theta_{\mu\nu} \equiv \Theta. \tag{6.6}$$

We refer to this scalar equation as the structural equation of spacetime. In vacuo, this scalar equation admits constant solutions. In this case, Palatini $f(R)$ gravity reduces to GR with a cosmological constant [29].
Now, let us extend the Palatini formalism to non-minimally coupled scalar-tensor theories, with the aim of understanding the bimetric structure of spacetime in these theories.
We find the following field equations for the metric and the connection from the scalar-tensor action (4.36):

$$F(\phi)\left(R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R\right) = \Theta^\phi_{\mu\nu} + \Theta_{\mu\nu}, \tag{6.7}$$

$$\nabla^{\Gamma}_{\lambda}\left(\sqrt{-g}F(\phi)g^{\mu\nu}\right)=0. \tag{6.8}$$

The equations of motion of the matter fields are

$$\Box\phi=V(\phi)+F(\phi)R, \tag{6.9}$$

$$\frac{\delta\mathcal{L}_m}{\delta\psi}=0. \tag{6.10}$$

In this case, the structural equation of spacetime implies that

$$R=-\frac{\left(\Theta^{\phi}+\Theta\right)}{F(\phi)}. \tag{6.11}$$

The bimetric structure of spacetime is defined by the ansatz

$$\sqrt{-g}F(\phi)g^{\mu\nu}=\sqrt{-h}h^{\mu\nu}. \tag{6.12}$$

It follows from Eq.(6.11) that in vacuo, $\Theta^{\phi}=0$ and $\Theta=0$, this theory is equivalent to vacuum GR.
As a further step, let us generalise the previous results to the case of non-minimal coupling in framework of $f(R)$ theories. The action can be written as

$$S=\int d^4x\sqrt{-g}\left[F(\phi)f(R)+\frac{1}{2}g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi-V(\phi)\right]+S_m\left(\psi,\nabla\psi\right). \tag{6.13}$$

The Palatini field equations are

$$F(\phi)\left[f'(R)R_{(\mu\nu)}-\frac{f(R)}{2}g_{\mu\nu}\right]=\Theta^{\phi}_{\mu\nu}+\Theta_{\mu\nu}, \tag{6.14}$$

$$\nabla^{\Gamma}_{\lambda}\left(\sqrt{-g}F(\phi)f'(R)g^{\mu\nu}\right)=0, \tag{6.15}$$

instead, the equations of motion for the scalar and the matter fields are

$$\Box\phi-F'(\phi)f(R)-V'(\phi)=0, \tag{6.16}$$

$$\frac{\delta\mathcal{L}_m}{\delta\psi}=0. \tag{6.17}$$

In this case, the structural equation of spacetime implies that

$$f'(R)R-2f(R)=\frac{\Theta^{\phi}+\Theta}{F(\phi)}. \tag{6.18}$$

The bimetric structure of spacetime is given by

$$\sqrt{-g}F(\phi)f'(R)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu}. \tag{6.19}$$

In vacuo, we obtain from Eq.(6.18) that the theory reduces again to Einstein gravity as for minimally interacting $f(R)$ theories.
Finally, let us discuss the situation in which the gravitational lagrangian is a general function of ϕ and R:

$$S = \int d^4x\sqrt{-g}\left[K(\phi, R) + \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right] + S_m(\psi, \nabla\psi), \tag{6.20}$$

which yields the gravitational field equations

$$\frac{\partial K(\phi, R)}{\partial R}R_{\mu\nu} - \frac{1}{2}K(\phi, R)\, g_{\mu\nu} = \Theta^\phi_{\mu\nu} + \Theta_{\mu\nu}, \tag{6.21}$$

$$\nabla^\Gamma_\lambda\left(\sqrt{-g}\frac{\partial K(\phi, R)}{\partial R}g^{\mu\nu}\right) = 0, \tag{6.22}$$

while the scalar and matter fields obey

$$\Box\phi - \frac{K(\phi, R)}{\partial\phi} - V'(\phi) = 0, \tag{6.23}$$

$$\frac{\delta\mathcal{L}_m}{\delta\psi} = 0. \tag{6.24}$$

The structural equation of spacetime can be expressed as

$$\frac{\partial K(\phi, R)}{\partial R}R - 2K(\phi, R) = \Theta^\phi + \Theta. \tag{6.25}$$

The bimetric structure of spacetime is defined by

$$\sqrt{-g}\frac{\partial K(\phi, R)}{\partial R}g^{\mu\nu} = \sqrt{-h}h^{\mu\nu}. \tag{6.26}$$

In this case, in general, we do not recover GR, as it is evident from Eq.(6.25) in which the strong coupling between R and ϕ prevents, even in vacuo, the possibility of obtaining constant solutions.

Chapter 7

Teleparallel theories

We want to focus on those theories that are equivalent to the General Relativity whose main peculiarity is to work in situations with null curvature, in particular we deal with the Teleparallel Equivalent to General Relativity and the Symmetric Teleparallel Equivalent to General Relativity.

7.1 Teleparallel Gravity

The Teleparallel Equivalent to General Relativity (TEGR) [7, 34] is an equivalent description of General Relativity but with different conceptual basis. In particular, TEGR, in contrast to GR, is well integrated within a gauge theory context and can be defined as the gauge theory for the translation group.
In this theory, gravity is mediated by torsion on a flat spacetime and the dynamical field is the tetrad, differently from the theories seen so far where the spacetime is curved and hence gravity is viewed as a purely geometric effect.

7.1.1 Geometrical setting

The geometrical setting of TEGR is the tangent bundle, in which spacetime is the base space and the tangent space at each point of the base space, on which the gauge transformations take place, is the fiber of the bundle. Spacetime is assumed to be a metric spacetime with a general metric $g_{\mu\nu}$

and the tangent space is by definition a Minkowski spacetime with tangent space metric η_{ab}.
We will use the Greek alphabet ($\mu, \nu, \rho, ... = 0, 1, 2, 3$) to denote indices related to spacetime and the Latin alphabet ($a, b, c, ... = 0, 1, 2, 3$) to denote indices related to the tangent space.
Since the base and the fiber are both four-dimensional spacetimes, the bundle is said to be soldered. This means that the metrics are related by the relation:

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu, \tag{7.1}$$

where $h^a{}_\mu$ is the tetrad field, i.e. the components of the solder 1-form. Differently from other gauge theories, in TEGR the soldering property ensures the presence of torsion tensor, being by definition the covariant derivative of the solder 1-form.
The tetrad fields, or vierbiens,

$$h^a = h^a{}_\mu dx^\mu \quad and \quad h_a = h_a{}^\mu \partial_\mu, \tag{7.2}$$

as already seen in Chapter 3, are general linear bases on the spacetime manifold that satisfy the relations

$$h^a{}_\mu h_a{}^\nu = \delta^\nu_\mu \qquad h^a{}_\mu h_b{}^\mu = \delta^a_b, \tag{7.3}$$

$$[h_a, h_b] = f^c{}_{ab} h_c, \tag{7.4}$$

where $f^c{}_{ab}$ are the so called structure coefficients, or coefficients of anholonomy, of frame $\{h_a\}$. The whole set of such bases constitutes the bundle of linear frames.
Because of the soldered character of the tangent bundle, a tetrad field relates tangent space (or internal) tensors with spacetime (or external) tensors. For example, if v^a is an internal vector, therefore

$$v^\mu = h_a{}^\mu v^a \tag{7.5}$$

is an external vector, conversely

$$v^a = h^a{}_\mu v^\mu. \tag{7.6}$$

In the presence of gravitation, the coefficients of anholonomy $f^c{}_{ab}$ includes both inertial and gravitational effects. In this case, the spacetime metric $g_{\mu\nu}$ represents a general pseudo-riemannian spacetime. Instead, in absence of gravitation, the anholonomy of the frames is entirely related to the inertial forces that are present in these frames. In this case $h^a{}_\mu$ becomes trivial and it is indicated with $e^a{}_\mu$, while $g_{\mu\nu}$ is the Minkowski metric in a general coordinate system:

$$\eta_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu. \tag{7.7}$$

A preferred class of inertial frames, that we denote as h'_a, is characterized by

$$f'^c{}_{ab} = 0 \tag{7.8}$$

They are called, for this reason, holonomic frames.
A connection form ω, it also called spin connection or Lorentz connection, of a given connection in the bundle of linear frames, i.e. a linear connection, is a 1-form with values in the Lie algebra of Lorentz group:

$$\omega = \omega_\mu dx^\mu = \frac{1}{2}\omega^a{}_b S_a{}^b = \frac{1}{2}\omega^a{}_{b\mu} S_a{}^b dx^\mu, \tag{7.9}$$

with $S_a{}^b$ a given representation of the Lorentz generators and $\omega^a{}_b \in \Omega^1(\boldsymbol{M})$, where $\omega^a{}_b$ are matrices of 1-form and $\Omega^1(\boldsymbol{M})$ is the space of all 1-form.
A general linear connection $\Gamma^\rho{}_{\mu\nu}$ is related to the corresponding spin connection $\omega^a{}_b$ by

$$\Gamma^\rho{}_{\nu\mu} = h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho \omega^a{}_{b\mu} h^b{}_\nu. \tag{7.10}$$

This equation in nothing else but the tetrad postulate:

$$\nabla_\mu h^a{}_\rho = \partial_\mu h^a{}_\rho - \Gamma^\sigma{}_{\rho\mu} h^a{}_\sigma + \omega^a{}_{b\mu} h^b{}_\rho = 0, \tag{7.11}$$

where ∇_μ is the standard covariant derivative in the connection $\Gamma^\rho{}_{\nu\mu}$. The postulate states that the tetrads are parallel vector fields.
Defining the covariant exterior derivative of a tensor, valued on the p-form $B^a{}_b$, as the operator $\mathcal{D} : \Omega^p(\boldsymbol{M}, \boldsymbol{T}^r_s) \to \Omega^{p+1}(\boldsymbol{M}, \boldsymbol{T}^r_s)$, we get

$$\mathcal{D}B^a{}_b = dB^a{}_b + \omega^a{}_c \wedge B^c{}_b - \omega^d{}_b \wedge B^a{}_d, \tag{7.12}$$

where d is the exterior derivative of a p-form. Through the Cartan structure equations,

$$T^a = \mathcal{D}h^a = \frac{1}{2} T^a{}_{bc} h^b \wedge h^c \tag{7.13}$$

and

$$R^a{}_b = \mathcal{D}\omega^a{}_b = \frac{1}{2} R^a{}_{bcd} h^c \wedge h^d, \tag{7.14}$$

we can define the torsion of a spin connection as a 2-form assuming values in the Lie algebra of the translation group

$$\mathcal{T} = T^a P_a = \frac{1}{2} T^a{}_{\mu\nu} P_a dx^\mu dx^\nu, \tag{7.15}$$

with $P_a = \partial_a$ the translation generators, and the curvature of a spin connection as a 2-form assuming values in the Lie algebra of the Lorentz group

$$\mathcal{R} = \frac{1}{2} R^a{}_b S_a{}^b = \frac{1}{4} R^a{}_{b\mu\nu} S_a{}^b dx^\mu \wedge dx^\nu. \tag{7.16}$$

The torsion and curvature components have the form

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + \omega^a{}_{c\nu} h^c{}_\mu - \omega^a{}_{c\mu} h^c{}_\nu, \tag{7.17}$$

$$R^a{}_{b\nu\mu} = \partial_\nu \omega^a{}_{b\mu} - \partial_\mu \omega^a{}_{b\nu} + \omega^a{}_{c\nu} \omega^c{}_{b\mu} - \omega^a{}_{c\mu} \omega^c{}_{b\nu}. \tag{7.18}$$

These relations can be expressed in purely spacetime forms if we use the Eq.(7.10):

$$T^\rho{}_{\nu\mu} \equiv h_a{}^\rho T^a{}_{\nu\mu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}, \tag{7.19}$$

$$R^\rho{}_{\lambda\nu\mu} \equiv h_a{}^\rho h^b{}_\lambda R^a{}_{b\nu\mu} = \partial_\nu \Gamma^\rho{}_{\lambda\mu} - \partial_\mu \Gamma^\rho{}_{\lambda\nu} + \Gamma^\rho{}_{\sigma\nu} \Gamma^\sigma{}_{\lambda\mu} - \Gamma^\rho{}_{\sigma\mu} \Gamma^\sigma{}_{\lambda\nu}. \tag{7.20}$$

It is easily verified, using the relation

$$\omega^a{}_{bc} = \omega^a{}_{b\mu} h_c{}^\mu, \tag{7.21}$$

that in the anholonomic basis $\{h_a\}$ the torsion and curvature components are given respectively by

$$T^a{}_{bc} = \omega^a{}_{cb} - \omega^a{}_{bc} - f^a{}_{bc}, \tag{7.22}$$

$$R^a{}_{bcd} = h_c(\omega^a{}_{bd}) - h_d(\omega^a{}_{bc}) + \omega^a{}_{ec} \omega^e{}_{bd} - \omega^a{}_{ed} \omega^e{}_{bc} - f^e{}_{cd} \omega^a{}_{be}. \tag{7.23}$$

7.1.2 Gravity as a gauge theory

A gauge transformation in Teleparallel Gravity is defined as a local translation of the tangent space coordinates

$$x'^a = x^a + \epsilon^a(x^\mu) \tag{7.24}$$

with $\epsilon^a(x^\mu)$ the infinitesimal transformation parameters. An arbitrary source field $\psi \equiv \psi(x^a(x^\mu))$ transforms under this transformation as

$$\delta\psi = \epsilon^a \partial_a \psi. \tag{7.25}$$

However, the ordinary derivative does not transform covariantly under such transformation:

$$\delta(\partial_\mu \psi) = \epsilon^a \partial_a(\partial_\mu \psi) + (\partial_\mu \epsilon^a)\partial_a \psi. \tag{7.26}$$

As is usual in gauge theories, to recover the covariance, we have to replace ordinary derivatives by covariant derivatives involving a connection. Thus, we introduce a gauge potential B_μ, a 1-form assuming values in the Lie algebra of the translation group:

$$B_\mu = B^a{}_\mu P_a. \tag{7.27}$$

It is easy to see, that the covariant derivative we construct with B_μ,

$$h_\mu \psi = \partial_\mu \psi + B^a{}_\mu \partial_a \psi, \tag{7.28}$$

transforms covariantly, i.e.

$$\delta(h_\mu \psi) = \epsilon^a \partial_a (h_\mu \psi), \tag{7.29}$$

if the gauge potential transforms as

$$\delta B^a{}_\mu = -\partial_\mu \epsilon^a. \tag{7.30}$$

Due to the soldered property of the frame bundle, the covariant derivative can be rewritten in the form

$$h_\mu \psi = h^a{}_\mu \partial_a \psi, \tag{7.31}$$

where

$$h^a{}_\mu = \partial_\mu x^a + B^a{}_\mu, \tag{7.32}$$

is a non-trivial tetrad field, which means a tetrad with

$$B^a{}_\mu \neq \partial_\mu \epsilon^a, \tag{7.33}$$

otherwise it would be just a translational gauge transformation of the trivial tetrad $e^a{}_\mu = \partial_\mu e^a$.

When the translational covariant derivative assumes the form (7.31), the translational coupling prescription acquires the simple form:

$$\partial_\mu \psi = e^a{}_\mu \partial_a \psi \to h_\mu \psi = h^a{}_\mu \partial_a \psi. \tag{7.34}$$

Consequently, the spacetime metric changes according to

$$\eta_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \to g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu. \tag{7.35}$$

To obtain relations valid in a general Lorentz frame, it is necessary performing a local Lorentz transformation:

$$x^a \to \Lambda^a{}_b x^b. \tag{7.36}$$

The scalar field transforms under this transformation as

$$\psi \to U(\Lambda)\psi, \tag{7.37}$$

with $U(\Lambda)$ an element of the Lorentz group in the representation appropriate for the field ψ. Since $B^a{}_\mu \to \Lambda^a{}_b B^b{}_\mu$, it is immediate to see that the translational covariant derivative assumes the form

$$h_\mu \psi = \partial_\mu \psi + \dot{\omega}^a{}_{b\mu} x^b \partial_a \psi + B^a{}_\mu \partial_a \psi, \tag{7.38}$$

where $\dot{\omega}^a{}_{b\mu}$ is the purely inertial Lorentz connection

$$\dot{\omega}^a{}_{b\mu} = \Lambda^a{}_e(x) \partial_\mu \Lambda_b{}^e(x), \tag{7.39}$$

i.e. it is the connection obtained from a Lorentz transformation of the vanishing spin connection $\dot{\omega}'^e{}_{d\mu} = 0$:

$$\dot{\omega}^a{}_{b\mu} = \Lambda^a{}_e(x) \dot{\omega}'^e{}_{d\mu} \Lambda_b{}^d(x) + \Lambda^a{}_e(x) \partial_\mu \Lambda_b{}^e(x). \tag{7.40}$$

Defining the Lorentz covariant derivative

$$\dot{\mathcal{D}}_\mu x^a = \partial_\mu x^a + \dot{\omega}^a{}_{b\mu} x^b, \tag{7.41}$$

the tetrad becomes

$$h^a{}_\mu = \dot{\mathcal{D}}_\mu x^a + B^a{}_\mu. \tag{7.42}$$

In this class of frames, the gauge potential $B^a{}_\mu$ transforms according to

$$\delta B^a{}_\mu = -\dot{\mathcal{D}}_\mu \epsilon^a. \tag{7.43}$$

In addition to being invariant under local translations, any theory must also be invariant under local Lorentz transformations. This second invariance is related to the fact that physics must be the same, independently of the Lorentz frame used to describe it. The Lorentz coupling prescription can be obtained from the so-called general covariance principle. It states: an equation valid in Special Relativity can be made to hold in the presence of gravitation if it is written in a generally covariant form, i.e. if it preserves its form under general coordinate transformations.
The general covariance principle can be seen as an active version of the Strong Equivalence Principle, in the sense that, by making a special relativistic equation covariant and using the SEP, it is possible to obtain its form in the presence of gravitation. This vision of the general covariance principle is opposed to the usual (or passive) SEP, which says that, given an equation valid in the presence of gravitation, the corresponding special relativistic equation is recovered locally.
The process of obtaining this coupling prescription comprises two steps: the first is to pass to a general anholonomic frame, where inertial effects are present in the form of a compensating term; then, by using the SEP, the compensating term is replaced by a connection representing a gravitational field.
Let us consider a vector field ϕ'^c on Minkowski spacetime whose ordinary derivative, in trivial holonomic frame, is

$$e'_a\phi'^c = \partial_a\phi'^c = \delta^\mu_a\partial_\mu\phi'^c. \tag{7.44}$$

Under a local Lorentz transformation ${\Lambda_d}^c(x)$, a vector field transforms as

$$\phi^c = {\Lambda_d}^c(x)\phi'^d. \tag{7.45}$$

Thus, the relation between the Lorentz-transformed derivative and the previous one is given by

$$e'_a\phi'^c = {\Lambda^b}_a{\Lambda_d}^c\mathcal{D}_b\phi^d, \tag{7.46}$$

where

$$\mathcal{D}_b\phi^d = e_b\phi^d + {\Lambda^d}_e e_b\left({\Lambda_c}^e\right)\phi^c \tag{7.47}$$

and

$$e_b = {\Lambda_b}^a e'_a \tag{7.48}$$

is the transformed frame, which, due to the locality of the Lorentz transformation, is anholonomic:

$$[e_b, e_c] = f^a{}_{bc} e_a. \tag{7.49}$$

Obtaining the Lorentz group element from the Eq.(7.48), the covariant derivative becomes

$$\mathcal{D}_a \phi^c = e_a \phi^c + \frac{1}{2}\left(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ab}\right)\phi^b \tag{7.50}$$

In the right-hand side, the second term is the announced compensating term which represents the inertial effects inherent to the chosen frame.
In the presence of gravitation, according to the translational coupling prescription , the trivial tetrad is replaced by the non-trivial one and the coupling prescription assumes the form:

$$\mathcal{D}_a \phi^c = h_a \phi^c + \frac{1}{2}\left(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ab}\right)\phi^b. \tag{7.51}$$

Let us consider the Eq.(7.22). With a permutation of the indices, we find the relation

$$\frac{1}{2}\left(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ab}\right) = \overset{\bullet}{\omega}{}^c{}_{ba} - \overset{\bullet}{K}{}^c{}_{ba}, \tag{7.52}$$

where

$$\overset{\bullet}{K}{}^c{}_{ba} = \frac{1}{2}\left(\overset{\bullet}{T}_b{}^c{}_a + \overset{\bullet}{T}_a{}^c{}_b - \overset{\bullet}{T}{}^c{}_{ba}\right) \tag{7.53}$$

is the contortion tensor in the tetrad frame. The left-hand side is the usual expression of the General Relativity Lorentz connection in terms of the coefficients of anholonomy:

$$\overset{\circ}{\omega}{}^c{}_{ab} = \frac{1}{2}\left(f_b{}^c{}_a + f_a{}^c{}_b - f^c{}_{ab}\right). \tag{7.54}$$

In the Eq.(7.52) we have both inertial and gravitational effects.
According to the general covariance principle, inserting Eq.(7.52) in Eq.(7.50), we achieve the so-called full gravitational coupling prescription in Teleparallel Gravity.:

$$\mathcal{D}_a \phi^c = h_a \phi^c + \left(\overset{\bullet}{\omega}{}^c{}_{ba} - \overset{\bullet}{K}{}^c{}_{ba}\right)\phi^b. \tag{7.55}$$

This equation is also valid when we consider a general source field

$$\psi \rightarrow U(\Lambda)\psi, \tag{7.56}$$

with

$$U(\Lambda) = exp\left(\frac{i}{2}\epsilon_{bc}S^{bc}\right) \tag{7.57}$$

the element of the Lorentz group in the arbitrary representation S^{bc}. In fact, after some calculations we get

$$\mathcal{D}_a\psi = h_a\psi - \frac{i}{2}\left(\dot{\omega}^{bc}{}_a - \dot{K}^{bc}{}_a\right)S_{bc}\psi. \tag{7.58}$$

7.1.3 Translational field strength

In gauge theories, we can find the field strength by the commutation relation of gauge covariant derivatives. In the case of Teleparallel Gravity, what we get is

$$[h_\mu, h_\nu] = \dot{T}^a{}_{\mu\nu}P_a, \tag{7.59}$$

where

$$\dot{T}^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \dot{\omega}^a{}_{b\mu}B^b{}_\nu - \dot{\omega}^a{}_{b\nu}B^b{}_\mu \tag{7.60}$$

is the translational field strength. Using the relation

$$\dot{\mathcal{D}}_\mu\left(\dot{\mathcal{D}}_\nu x^a\right) - \dot{\mathcal{D}}_\nu\left(\dot{\mathcal{D}}_\mu x^a\right) = 0, \tag{7.61}$$

it follows

$$\dot{T}^a{}_{\mu\nu} = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu + \dot{\omega}^a{}_{b\mu}h^b{}_\nu - \dot{\omega}^a{}_{b\nu}h^b{}_\mu. \tag{7.62}$$

Contracting with a tetrad, we find that the translational field strength has the form of the torsion tensor

$$\dot{T}^\rho{}_{\mu\nu} \equiv h_a{}^\rho\dot{T}^a{}_{\mu\nu} = \dot{\Gamma}^\rho{}_{\nu\mu} - \dot{\Gamma}^\rho{}_{\mu\nu}. \tag{7.63}$$

where we have defined the Weitzenböck connection:

$$\dot{\Gamma}^\rho{}_{\nu\mu} = h_a{}^\rho\partial_\mu h^a{}_\nu + h_a{}^\rho\dot{\omega}^a{}_{b\mu}h^b{}_\nu. \tag{7.64}$$

This connection is the spacetime indexed connection corresponding to the inertial Lorentz connection. The Weitzenböck connection has an identically null curvature tensor:

$$R^\lambda{}_{\rho\nu\mu} = \partial_\nu\dot{\Gamma}^\lambda{}_{\rho\mu} - \partial_\mu\dot{\Gamma}^\lambda{}_{\rho\nu} + \dot{\Gamma}^\lambda{}_{\sigma\nu}\dot{\Gamma}^\sigma{}_{\rho\mu} - \dot{\Gamma}^\lambda{}_{\sigma\mu}\dot{\Gamma}^\sigma{}_{\rho\nu} = 0. \tag{7.65}$$

Therefore, in Teleparallel Gravity, gravitation is represented by torsion, not by curvature, conversely from what happens in General Relativity.
The Weitzenböck connection $\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu}$ is related to the Levi-Civita connection of General Relativity $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}$ by

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + \overset{\bullet}{K}{}^{\rho}{}_{\mu\nu}. \tag{7.66}$$

7.1.4 Teleparallel force equation

Let us consider a particle of mass m in presence of gravitation. Along the particle trajectory the spacetime interval can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \;\rightarrow\; ds = g_{\mu\rho} u^\mu dx^\rho = \eta_{ab} u^a h^b, \tag{7.67}$$

where

$$u^\mu = \frac{dx^\mu}{ds} \tag{7.68}$$

is the holonomic four-velocity, which is related to the anholonomic four-velocity u^a by

$$u^a \equiv h^a{}_\mu u^\mu = h^a \left(\frac{d}{ds}\right). \tag{7.69}$$

Then, the teleparallel version of the action of the particle, described from a general Lorentz frame, is

$$S = -mc \int_p^q u_a h^a = -mc \int_p^q u_a \left(dx^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b dx^\mu + B^a{}_\mu dx^\mu\right). \tag{7.70}$$

The first term represents the free particle, the second one represents the interaction with the inertial effects and the last one represents the gravitational interaction. The variation of the action under a general spacetime transformation,

$$x^\mu \;\rightarrow\; x^\mu + \delta x^\mu, \tag{7.71}$$

assumes the form

$$\begin{aligned} \delta S = &- mc \int_p^q (u_a d\delta x^a + u_a \delta \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b dx^\mu + u_a \overset{\bullet}{\omega}{}^a{}_{b\mu} \delta x^b dx^\mu + \\ &+ u_a \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b d\delta x^\mu + u_a \delta B^a{}_\mu dx^\mu + u_a B^a{}_\mu d\delta x^\mu). \end{aligned} \tag{7.72}$$

Integrating by parts, substituting the expressions

$$\delta x^a = \partial_\mu x^a \delta x^\mu, \quad \delta \overset{\bullet}{\omega}{}^a{}_{b\mu} = \partial_\rho \overset{\bullet}{\omega}{}^a{}_{b\mu} \delta x^\rho, \quad \delta B^a{}_\mu = \partial_\rho B^a{}_\mu \delta x^\rho, \tag{7.73}$$

and considering that the curvature tensor of the Lorentz connection is null, we find

$$\delta S = mc \int_p^q \left[h^a{}_\mu \left(\frac{du_a}{ds} - \overset{\bullet}{\omega}{}^b{}_{a\rho} u_b u^\rho \right) - \overset{\bullet}{T}{}^b{}_{\mu\rho} u_b u^\rho \right] \delta x^\mu ds, \tag{7.74}$$

from which we have the equation of motion:

$$\frac{du_a}{ds} - \overset{\bullet}{\omega}{}^b{}_{a\rho} u_b u^\rho = \overset{\bullet}{T}{}^b{}_{a\rho} u_b u^\rho. \tag{7.75}$$

Using the identity

$$\overset{\bullet}{T}{}^b{}_{a\rho} u_b u^\rho = -\overset{\bullet}{K}{}^b{}_{a\rho} u_b u^\rho, \tag{7.76}$$

and contracting with the tetrads, we can write the equation of motion in a purely spacetime form:

$$\frac{du_\mu}{ds} - \overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} u_\rho u^\nu = -\overset{\bullet}{K}{}^\rho{}_{\mu\nu} u_\rho u^\nu, \tag{7.77}$$

this is the force equation. As already mentioned, it is the torsion that plays the role of gravitational force.
From the relation (7.52), we achieve that the teleparallel force equation coincide with the geodesic equation of General Relativity:

$$\frac{du^a}{ds} + \overset{\circ}{\omega}{}^a{}_{b\rho} u^b u^\rho = 0. \tag{7.78}$$

Then, we showed that the teleparallel description of the gravitational interaction is completely equivalent to the description of General Relativity although they are conceptually different theories.

7.1.5 Field equations

Due to the fact that TEGR is a gauge theory, the action is given by

$$S = k \int Tr \left(\overset{\bullet}{\mathcal{T}} \wedge \star \overset{\bullet}{\mathcal{T}} \right) + S_m, \tag{7.79}$$

with

$$\star \dot{\mathcal{T}} = \frac{1}{2} \left(\star \dot{T}^{a}{}_{\rho\sigma} \right) P_a dx^\rho \wedge dx^\sigma \tag{7.80}$$

the Hodge dual of the torsion tensor. After some manipulations, we can write the action in terms of the scalar torsion $\dot{T}$:

$$S = k \int d^4 x h \dot{T} + S_m, \tag{7.81}$$

with $h = \sqrt{-g}$ and

$$\dot{T} \equiv \dot{S}^{\rho\mu\nu} \dot{T}_{\rho\mu\nu} = \dot{K}^{\mu\nu\rho} \dot{K}_{\rho\nu\mu} - \dot{K}^{\mu\rho}{}_{\mu} \dot{K}^{\nu}{}_{\rho\nu}, \tag{7.82}$$

where we have defined the superpotential as

$$\dot{S}^{\rho\mu\nu} = -\dot{S}^{\rho\nu\mu} \equiv \dot{K}^{\mu\nu\rho} - g^{\rho\nu} \dot{T}^{\sigma\mu}{}_{\sigma} + g^{\rho\mu} \dot{T}^{\sigma\nu}{}_{\sigma}. \tag{7.83}$$

The action just written differs from the GR one only because of a divergence. We can achieve this result remembering that the curvature of the Weitzenböck connection vanishes identically. Let us consider Eq.(7.65), substituting the relation (7.66), we find

$$\begin{aligned} 0 = \dot{R}^{\rho}{}_{\theta\mu\nu} = & \mathring{R}^{\rho}{}_{\theta\mu\nu} + \partial_\mu \dot{K}^{\rho}{}_{\theta\nu} - \partial_\nu \dot{K}^{\rho}{}_{\theta\mu} + \\ & + \dot{\Gamma}^{\rho}{}_{\sigma\mu} \dot{K}^{\sigma}{}_{\theta\nu} - \dot{\Gamma}^{\rho}{}_{\sigma\nu} \dot{K}^{\sigma}{}_{\theta\mu} - \dot{\Gamma}^{\sigma}{}_{\theta\mu} \dot{K}^{\rho}{}_{\sigma\nu} + \\ & + \dot{\Gamma}^{\sigma}{}_{\theta\nu} \dot{K}^{\rho}{}_{\sigma\mu} + \dot{K}^{\rho}{}_{\sigma\nu} \dot{K}^{\sigma}{}_{\theta\mu} - \dot{K}^{\rho}{}_{\sigma\mu} \dot{K}^{\sigma}{}_{\theta\nu}, \end{aligned} \tag{7.84}$$

where

$$\mathring{R}^{\rho}{}_{\theta\mu\nu} = \partial_\mu \mathring{\Gamma}^{\rho}{}_{\theta\nu} - \partial_\nu \mathring{\Gamma}^{\rho}{}_{\theta\mu} + \mathring{\Gamma}^{\rho}{}_{\sigma\mu} \mathring{\Gamma}^{\sigma}{}_{\theta\nu} - \mathring{\Gamma}^{\rho}{}_{\sigma\nu} \mathring{\Gamma}^{\sigma}{}_{\theta\mu} \tag{7.85}$$

is the curvature of the Levi-Civita connection. Contracting the first and the third index and multiplying by the metric tensor $g^{\theta\nu}$, Eq.(7.84) becomes

$$\mathring{R} = -\dot{T} - \frac{2}{h} \partial_\mu \left(h \dot{T}^{\nu\mu}{}_{\nu} \right), \tag{7.86}$$

with $\mathring{R}$ the scalar curvature of the Levi-Civita connection. Thanks to this relation we got what we anticipated, that is, up to a divergence, the action of TEGR is equivalent to the action of the GR.
To compute the field equations it is necessary to vary the action (7.81) with respect to the tetrad field $h^{a}{}_{\mu}$. What we get is

$$\partial_\sigma \left(h \dot{S}_a{}^{\rho\sigma} \right) - k h \dot{J}_a{}^{\rho} = k h \chi_a{}^{\rho}. \tag{7.87}$$

The first term

$$h\dot{S}_a{}^{\rho\sigma} = -k\frac{\partial\dot{\mathcal{L}}}{\partial\left(\partial_\sigma h^a{}_\rho\right)} = \dot{K}^{\rho\sigma}{}_a - h^a{}_\rho \dot{T}^{\theta\rho}{}_\theta + h_a{}^\rho \dot{T}^{\theta\sigma}{}_\theta, \tag{7.88}$$

where $\dot{\mathcal{L}}$ stands for the teleparallel lagrangian density, is the superpotential, the second one

$$h\dot{J}_a{}^\rho = -\frac{\partial\dot{\mathcal{L}}}{\partial h^a{}_\rho} = h\left(\frac{1}{k}h_a{}^\mu \dot{S}_c{}^{\nu\rho}\dot{T}_c{}^{\nu\mu} - \frac{h_a{}^\rho}{h}\dot{\mathcal{L}} + \frac{1}{k}\dot{\omega}^c{}_{a\sigma}\dot{S}_c{}^{\rho\sigma}\right) \tag{7.89}$$

represents the gauge current and the last one

$$h\chi_a{}^\rho = h\chi^\rho{}_a = -\frac{\partial\delta\mathcal{L}_m}{\delta h^a{}_\rho} \equiv -\left(\frac{\partial\mathcal{L}_m}{\partial h^a{}_\rho} - \partial_\mu\frac{\partial\mathcal{L}_m}{\partial\left(\partial_\mu h^a{}_\rho\right)}\right) \tag{7.90}$$

is the matter energy-momentum tensor. Due to the antisymmetry of the superpotential in the last two indices, the total energy-momentum density is conserved in the ordinary sense:

$$\partial_\rho\left(h\dot{J}_a{}^\rho + h\chi_a{}^\rho\right) = 0. \tag{7.91}$$

7.1.6 Gravitation without Equivalence Principle

In this section we will show that as well as the Maxwell theory, a gauge theory for the unitary group $U(1)$, is able to describe the non-universal electromagnetic interaction, i.e. every particle feels electromagnetic field in different way, then the TEGR is able to describe the gravitational interaction in the lack of universality, i.e. in the absence of the Weak Equivalence Principle [8].

Analogously to the electromagnetic case, where we consider the fine structure constant

$$\sqrt{\alpha_e} = \frac{q}{q_p}, \tag{7.92}$$

which represents how much the particle electric charge differs with respect to the Planck charge, in Teleparallel Gravity we take in account a dimensionless coupling constant which takes into consideration the gravitational mass m_g in relation to the inertial mass m_i:

$$\sqrt{\alpha_g} = \frac{m_g}{m_i}. \tag{7.93}$$

The translational gauge transformation of a field ψ, representing a particle with $m_g \neq m_i$, is

$$\psi' = \tilde{U}\psi, \tag{7.94}$$

where

$$\tilde{U} = \exp\left(\sqrt{\alpha_g}\epsilon^a \partial_a\right) \tag{7.95}$$

is an element of the translational group. The infinitesimal transformation is given by

$$\tilde{\delta}\psi = \tilde{\delta}x^a \partial_a \psi, \tag{7.96}$$

with

$$\tilde{\delta}x^a = \sqrt{\alpha_g}\epsilon^a \tag{7.97}$$

the non-universal gauge transformation of the tangent space coordinates. From the general definition of covariant derivative

$$h_\mu = \partial_\mu + B^a{}_\mu \frac{\delta}{\delta\epsilon^a}, \tag{7.98}$$

in a general Lorentz frame, we derive the translational gauge covariant derivative of ψ:

$$\tilde{h}_\mu \psi = \tilde{h}^a{}_\mu \partial_a \psi, \tag{7.99}$$

where

$$\tilde{h}^a{}_\mu \equiv \tilde{h}_\mu x^a = \overset{\bullet}{\mathcal{D}}_\mu x^a + \sqrt{\alpha_g} B^a{}_\mu \tag{7.100}$$

is the translational covariant derivative of x^a. Now we are able to write the action of the particle in presence of gravitation:

$$\begin{aligned} S = &- m_i c \int_p^q u_a \left(\overset{\bullet}{\mathcal{D}}_\mu x^a + \sqrt{\alpha_g} B^a{}_\mu\right) dx^\mu \\ = &- m_i c \int_p^q \left[u_a h^a{}_\mu + \left(\sqrt{\alpha_g} - 1\right) u_a B^a{}_\mu\right] dx^\mu \end{aligned} \tag{7.101}$$

It is important to underline that $\tilde{h}^a{}_\mu$ is not a tetrad, unlike $h^a{}_\mu$, because, by definition, a tetrad cannot depend on any property of the particle.
Using the relations

$$\delta u_a = u_\mu \frac{du_a}{ds} \delta x^\mu, \tag{7.102}$$

$$\delta dx^\mu = d\delta x^\mu, \quad \delta B^a{}_\mu = \partial_\rho B^a{}_\mu \delta x^\rho \tag{7.103}$$

and taking into account that the variation of the first term of the action leads to Eq.(7.75), we find

$$\frac{du_a}{ds} - \overset{\bullet}{\omega}{}^b{}_{a\rho} u_b u^\rho = -\overset{\bullet}{K}{}^b{}_{a\rho} u_b u^\rho + F_a, \tag{7.104}$$

where

$$F_a = -\left(\sqrt{\alpha_g} - 1\right) h_a{}^\mu \left[P^\rho{}_\mu B^b{}_\rho \frac{du_b}{ds} - \left(\partial_\mu B^b{}_\rho - \partial_\rho B^b{}_\mu\right) u_b u^\rho \right], \tag{7.105}$$

with

$$P^\rho{}_\mu = \delta^\rho_\mu - u^\rho u_\mu \tag{7.106}$$

a velocity-projection tensor, is a new gravitational force coming from the lack of universality.
Let us note that although the equation of motion depends explicitly on the properties of the particle due to $\sqrt{\alpha_g}$, this does not apply to the gauge potential $B^a{}_\mu$. This means that the teleparallel field equations (7.87) can be consistently solved for $B^a{}_\mu$, independently of the validity or not of WEP. Then, we have achieved the important result that the TEGR is able to describe the motion of a particle even in lack of universality.

7.1.7 Extended TEGR theories

Just as we need to extend General Relativity, in the same way, being an equivalent theory, we need to modify the TEGR. There are many modifications of TEGR, the most straightforward is the generalization of the action to an arbitrary function of the torsion scalar $f(T)$ [17], just like $f(\mathring{R})$ in GR:

$$S_{f(T)} = k \int d^4x f(T) + S_m. \tag{7.107}$$

Varying the action with respect to the tetrad we obtain the field equations

$$\begin{aligned} 4hf''(T)\left(\partial_\mu T\right) S_\nu{}^{\mu\lambda} + 4h^a{}_\nu \partial_\mu \left(h S_a{}^{\mu\lambda}\right) f'(T) + \\ -4hf'(T) T^\sigma{}_{\mu\nu} S_\sigma{}^{\lambda\mu} - hf(T)\delta^\lambda_\nu = k\Theta_\nu{}^\lambda. \end{aligned} \tag{7.108}$$

Replacing $f(T)$ with T we recover the TEGR field equations. It is interesting to notice that, even though the TEGR is completely equivalent to GR, since the Ricci scalar and the torsion scalar differ only by a total derivative

term, the same does not happen for $f(\mathring{R})$ and $f(T)$ theories because the boundary term behaves completely arbitrarily for non-linear terms of the torsion tensor.
Other theories are the teleparallel scalar-tensor theories [13], theories including couplings between the torsion scalar and the boundary term, $f(T, B)$ [12], theories with decomposition of the torsion tensor to its axial, tensorial and vectorial parts, $f(T_{ax}, T_{ten}, T_{vec})$ [11], and more.

7.2 Symmetric Teleparallel Gravity

In this section we continue our study on teleparallel theories, however, this time focusing on the one called Symmetric Teleparallel Equivalent of General Relativity (STEGR)[37, 5, 6, 4]. Symmetric because the torsion is zero and teleparallel because the curvature is also zero. However, STEGR is not metric compatible:

$$\nabla_\lambda g_{\mu\nu} = Q_{\lambda\mu\nu}. \tag{7.109}$$

From this we can compute the following connection

$$\Gamma^\lambda{}_{\mu\nu} = \mathring{\Gamma}^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu}, \tag{7.110}$$

with

$$\mathring{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{7.111}$$

the Levi-Civita connection and

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(Q_{\sigma\mu\nu} - Q_{\mu\sigma\nu} - Q_{\nu\sigma\mu}\right) \tag{7.112}$$

the disformation tensor. It is evident that this connection is symmetric in the last two indices.

7.2.1 Field equations

Let us define the non-metricity scalar:

$$\mathcal{Q} = Q_\alpha{}^{\mu\nu} P^\alpha{}_{\mu\nu} = -\frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\alpha\nu} + \frac{1}{4} Q_\alpha Q^\alpha - \frac{1}{2} Q_a \tilde{Q}^\alpha \tag{7.113}$$

with $Q_\alpha = Q_\alpha{}^\mu{}_\mu$, $\tilde{Q}^\alpha = Q_\mu{}^{\alpha\mu}$. The quantity $P^\alpha{}_{\mu\nu}$ is the non-metricity conjugate and it has the following form:

$$P^\alpha{}_{\mu\nu} \equiv -\frac{1}{4}Q^\alpha{}_{\mu\nu} + \frac{1}{2}Q_{(\mu}{}^\alpha{}_{\nu)} + \frac{1}{4}Q^\alpha g_{\mu\nu} - \frac{1}{2}\left(\tilde{Q}^\alpha g_{\mu\nu} + \delta^\alpha_{(\mu}Q_{\mu)}\right). \tag{7.114}$$

Now we can write the STEGR action:

$$S = k\int d^4x\sqrt{-g}\mathcal{Q} + S_m. \tag{7.115}$$

The variation with respect to the metric leads to the equations

$$\frac{2}{\sqrt{-g}}\nabla_\alpha\left(\sqrt{-g}P^\alpha{}_{\mu\nu}\right) - q_{\mu\nu} - \mathcal{Q}g_{\mu\nu} = \Theta_{\mu\nu} \tag{7.116}$$

where $q_{\mu\nu}$ stands for

$$\begin{aligned} q_{\mu\nu} = &-\frac{1}{4}\left(2Q_{\alpha\beta\mu}Q^{\alpha\beta}_\nu - Q_{\mu\alpha\beta}Q_\nu{}^{\alpha\beta}\right) + \frac{1}{2}Q_{\alpha\beta\mu}Q^{\beta\alpha}{}_\nu \\ &+\frac{1}{4}\left(2Q_\alpha Q^\alpha{}_{\mu\nu} - Q_\mu Q_\nu\right) - \tilde{Q}_\alpha Q^\alpha{}_{\mu\nu}. \end{aligned} \tag{7.117}$$

As in the case of Teleparallel Gravity, the STEGR action also differs from that of General Relativity due to a divergence. Let us consider the curvature tensor and the connection (7.110), we get

$$R^\alpha{}_{\mu\beta\nu} = \mathring{R}^\alpha{}_{\mu\beta\nu} + \mathring{\nabla}_\beta L^\alpha{}_{\nu\mu} - \mathring{\nabla}_\nu L^\alpha{}_{\beta\mu} + L^\rho{}_{\nu\mu}L^\alpha{}_{\beta\rho} - L^\rho{}_{\beta\mu}L^\alpha{}_{\nu\rho}, \tag{7.118}$$

where $\mathring{\nabla}$ is the covariant derivative of the Levi-Civita connection. Contracting the first index with the third one and applying the teleparallel condition $R^\alpha{}_{\mu\beta\nu} = 0$, we obtain the relation

$$\mathring{R} = \mathcal{Q} - \mathring{\nabla}_\alpha\left(Q^\alpha - \tilde{Q}^\alpha\right). \tag{7.119}$$

This relation proves the equivalence between STEGR and General Relativity.

7.2.2 The coincident gauge

Both the teleparallel condition and the torsionless one lead to an important property of STEGR. The teleparallel condition restricts the connection to be purely inertial so that it can be parameterized by a general element $\Lambda^b{}_a$ of $GL(4,\mathbb{R})$:

$$\Gamma^\alpha{}_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\beta \partial_{[\mu}\Lambda^\beta{}_{\nu]}. \tag{7.120}$$

In addition, the torsionless condition constrains the transformation matrix to satisfy $\partial_{[\mu}\Lambda^\beta{}_{\nu]} = 0$, so the general element of $GL(4,\mathbb{R})$ determining the connection can be parameterized by a set of functions ξ^α as $\Lambda^\alpha{}_\mu = \partial_\mu \xi^\alpha$, then

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\lambda}\partial_\mu\partial_\nu\xi^\lambda. \tag{7.121}$$

Thanks to Eq.(7.121) we achieve the result that we can completely remove the connection by means of a diffeomorphism. The gauge where the connection vanishes, i.e.

$$\xi^\alpha = x^\alpha, \tag{7.122}$$

can be interpreted as the gauge where the origin of the tangent space parameterized by ξ^α coincides with the spacetime origin. For this reason the gauge is called the coincident gauge [31].

7.2.3 Extended STEGR theories

As for the TEGR, since the STEGR is equivalent to General Relativity, it is useful to modify it. The most intuitive modification is the $f(\mathcal{Q})$ theory:

$$S_{f(\mathcal{Q})} = k\int d^4x\sqrt{-g}f(\mathcal{Q}) + S_m. \tag{7.123}$$

Varying with respect to the metric we find the equation of motion

$$\begin{aligned} k\Theta_{\mu\nu} =& \frac{1}{2}f''(\mathcal{Q})\partial_\alpha\left(Q_{(\mu}{}^\alpha{}_{\nu)} - \delta^\alpha_{(\mu}\tilde{Q}_{\nu)} - \frac{1}{2}Q^\alpha{}_{\mu\nu} + \frac{1}{2}Q^\alpha g_{\mu\nu}\right) + \\ &+ f'(\mathcal{Q})\left[\left(\mathring{\nabla}_\alpha - \frac{1}{2}Q_\alpha\right)L^\alpha{}_{\mu\nu} + \frac{1}{2}\mathring{\nabla}_\mu Q_\nu - L^\alpha{}_{\beta\mu}L^\beta{}_{\alpha\nu}\right] + \\ &- \frac{1}{2}g_{\mu\nu}\left[f(\mathcal{Q}) + f'(\mathcal{Q})\mathring{\nabla}_\alpha\left(Q^\alpha - \tilde{Q}^\alpha\right)\right]. \end{aligned} \tag{7.124}$$

An extension of non-metricity theories including scalar fields can be constructed if we consider the following action [30]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(\mathcal{L}_g - \mathcal{L}_l\right) + S_m\left(g_{\mu\nu},\psi\right), \tag{7.125}$$

where the gravitational lagrangian density is equal to

$$\mathcal{L}_g = \mathcal{A}(\phi)\,\mathcal{Q} - \mathcal{B}(\phi)\,\partial_\mu\phi\partial^\mu\phi - 2\mathcal{V}(\phi) \tag{7.126}$$

and the Lagrange multiplier terms are

$$\mathcal{L}_l = 2\lambda_\mu{}^{\beta\alpha\gamma}R^\mu{}_{\beta\alpha\gamma} + 2\lambda_\mu{}^{\alpha\beta}T^\mu{}_{\alpha\beta}. \tag{7.127}$$

The theory reduces to STEGR when $\mathcal{A} = 1$ and $\mathcal{B} = 0 = \mathcal{V}$.

Chapter 8

The trinity of gravity

So far our study has focused on theories whose fulcrum is given by connections characterized by particular peculiarities (flat, torsionless, etc.), now we want to show that taking the most generic connection possible, all other theories can be derived from them. The general affine connection $\Gamma^{\alpha}{}_{\mu\nu}$ [32] that fulfills this task is

$$\Gamma^{\alpha}{}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}. \tag{8.1}$$

The different terms that appear in the equation represent quantities that we have already learned about in the previous chapters:

- the disformation tensor

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(Q^{\alpha}{}_{\mu\nu} - Q_{\mu}{}^{\alpha}{}_{\nu} - Q_{\nu}{}^{\alpha}{}_{\mu}\right), \tag{8.2}$$

  that is related to the non-metricity

$$Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu}; \tag{8.3}$$

- the contortion tensor

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(T^{\alpha}{}_{\mu\nu} + T_{\mu}{}^{\alpha}{}_{\nu} + T_{\nu}{}^{\alpha}{}_{\mu}\right), \tag{8.4}$$

  from which emerges the anti-symmetrical part of the connection by the torsion tensor

$$T^{\alpha}{}_{\mu\nu} = 2\Gamma^{\alpha}{}_{[\mu\nu]}; \tag{8.5}$$

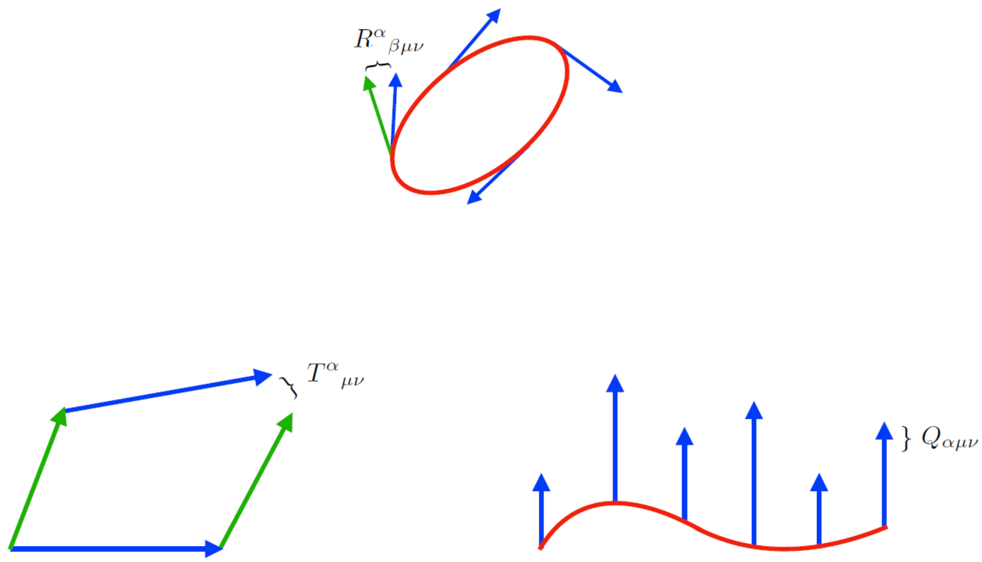

Figure 8.1: This figure illustrates the geometrical meaning of the curvature, the torsion and the non-metricity.

- the Levi-Civita connection

$$\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\alpha\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right), \tag{8.6}$$

  which is the unique connection that is symmetric and compatible with metric.

The curvature is determined by the usual Riemann tensor:

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\lambda\mu} \Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\lambda\nu} \Gamma^{\lambda}{}_{\beta\mu}. \tag{8.7}$$

Let us describe the quantities that represent well-defined geometrical properties of the connection:

- Non-metricity, $Q_{\alpha\mu\nu}$, measures the variation of the length of a vector as it is parallel transported. In metric spaces, i.e. $Q_{\alpha\mu\nu} = 0$, the length of vectors is conserved.

- Torsion, $T^{\alpha}{}_{\mu\nu}$, measures the non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.
- Curvature, $R^{\alpha}{}_{\beta\mu\nu}$, measures the rotation of a vector parallel transported along a closed curve.

Depending on which of these quantities are cancelled, we obtain the different theories we have analysed so far, for example General Relativity is based on a metric and torsionless connection and imputed gravity to the curvature, on the other hand the TEGR is founded on a metric and flat connection with the torsion that plays the role of gravity generator. A list of possible obtainable theories is given by the Table 8.1.
Due to the fact that GR can be equivalently described in terms of the non-metricity, torsion and curvature, we refer to these three seemingly unrelated elements as the trinity of gravity.

8.1 The Killing equations for general connections

We are interested in the study of the possible isometries that a theory presents when it is characterized by a connection of the type (8.1). To do this we need to calculate the Lie derivative of the metric $L_\xi g_{\mu\nu}$, where ξ is a possible generator vector field. By definition of Lie derivative, we get

$$\begin{aligned} L_\xi g_{\mu\nu} =&\xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu}\partial_\mu \xi^\sigma + g_{\mu\sigma}\partial_\nu \xi^\sigma \\ =&\xi^\sigma \nabla_\sigma g_{\mu\nu} + \xi^\sigma \Gamma^\lambda{}_{\sigma\mu} g_{\lambda\nu} + \xi^\sigma \Gamma^\lambda{}_{\sigma\nu} g_{\lambda\mu} + \\ &+ g_{\sigma\nu}\nabla_\mu \xi^\sigma - g_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda}\xi^\lambda + g_{\sigma\mu}\nabla_\nu \xi^\sigma - g_{\sigma\mu}\Gamma^\sigma{}_{\nu\lambda}\xi^\lambda \\ =&\xi^\sigma \left(Q_{\sigma\mu\nu} - Q_{\mu\sigma\nu} - Q_{\nu\mu\sigma}\right) + 2\nabla_{(\mu}\xi_{\nu)} + \\ &+ \xi^\sigma \Gamma^\lambda{}_{\sigma\mu} g_{\lambda\nu} + \xi^\sigma \Gamma^\lambda{}_{\sigma\nu} g_{\lambda\mu} - g_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda}\xi^\lambda - g_{\sigma\mu}\Gamma^\sigma{}_{\nu\lambda}\xi^\lambda . \end{aligned} \tag{8.8}$$

Inserting Eq.(8.1), we obtain

$$\begin{aligned} L_\xi g_{\mu\nu} =&\xi^\sigma \left(Q_{\sigma\mu\nu} - Q_{\mu\sigma\nu} - Q_{\nu\mu\sigma}\right) + 2\nabla_{(\mu}\xi_{\nu)} + \\ &+ \xi^\sigma K^\lambda{}_{\sigma\mu} g_{\lambda\nu} + \xi^\sigma K^\lambda{}_{\sigma\nu} g_{\lambda\mu} - g_{\sigma\nu} K^\sigma{}_{\mu\lambda}\xi^\lambda - g_{\sigma\mu} K^\sigma{}_{\nu\lambda}\xi^\lambda \\ =&2\xi^\sigma L_{\sigma\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)} + \\ &+ \xi^\sigma \left(K^\lambda{}_{\sigma\mu} g_{\lambda\nu} + K^\lambda{}_{\sigma\nu} g_{\lambda\mu} - K^\lambda{}_{\mu\sigma} g_{\lambda\nu} - K^\lambda{}_{\nu\sigma} g_{\lambda\mu}\right) . \end{aligned} \tag{8.9}$$

Geometrical objects	Type of space	Example of theories
$R = 0\ T = 0\ Q = 0$	Minkowski space	Special Relativity theory
$R \neq 0\ T = 0\ Q = 0$	Riemann space	General Relativity theory
$R = 0\ T \neq 0\ Q = 0$	Weitzenböck space	Translational gauge gravity theory
$R \neq 0\ T = 0\ Q \neq 0$	Weyl space	Weyl's gravity theory [47]
$R \neq 0\ T \neq 0\ Q = 0$	Riemann-Cartan space	Einstein-Cartan gravity theory [45]
$R \neq 0\ T \neq 0\ Q \neq 0$	Generalised metric-affine space	Einstein-Schrödinger theory

Table 8.1: Classification of gravity theories [40].

The terms containing the disformation tensor and the Levi-Civita connection, that would have come from the expression (8.1), cancel each other because of their symmetry in the last two indices.
The last equation can be simplified utilising the contorsion tensor definition (8.4), indeed

$$K^{\lambda}{}_{\sigma\mu}g_{\lambda\nu} + K^{\lambda}{}_{\sigma\nu}g_{\lambda\mu} - K^{\lambda}{}_{\mu\sigma}g_{\lambda\nu} - K^{\lambda}{}_{\nu\sigma}g_{\lambda\mu} = -2T_{(\mu\nu)\sigma}. \tag{8.10}$$

Finally we find

$$L_{\xi}g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} - 2\xi^{\sigma}T_{(\mu\nu)\sigma} + 2\xi^{\sigma}L_{\sigma\mu\nu} \tag{8.11}$$

To represent the isometries, this equation should be set equal to zero, however this is not possible due to the non-metricity condition (8.3), which ensures the non conservation of the vector length, therefore

$$L_{\xi}g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} - 2\xi^{\sigma}T_{(\mu\nu)\sigma} + 2\xi^{\sigma}L_{\sigma\mu\nu} \neq 0. \tag{8.12}$$

Now, let us suppose the connection is compatible with the metric, then

$$\nabla_{(\mu}\xi_{\nu)} - \xi^{\sigma}T_{(\mu\nu)\sigma} = 0. \tag{8.13}$$

To obtain the Killing equations Eq.(2.29), the torsion must be totally antisymmetric, $T_{(\mu\nu)\sigma} = 0$. This is exactly the necessary condition for the two concepts of geodesic, that is the curves that minimise the distance between two points in the manifold or a curve whose tangent vectors remain parallel if they are transported along it, to be equivalent in the case of connections with non-zero torsion [14]. In fact, when we have

$$\Gamma^{\alpha}{}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + K^{\alpha}{}_{\mu\nu}, \tag{8.14}$$

the two different definitions of geodesics lead to

$$\delta S = \delta \int_a^b \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = 0 \rightarrow \frac{d^2 x^\lambda}{ds^2} + \mathring{\Gamma}^{\lambda}{}_{\sigma\rho} \frac{dx^\sigma}{ds} \frac{dx^\rho}{ds} = 0 \tag{8.15}$$

and

$$V^\nu \nabla_\nu V^\mu = 0 \rightarrow \frac{d^2 x^\lambda}{ds^2} + \Gamma^{\lambda}{}_{\sigma\rho} \frac{dx^\sigma}{ds} \frac{dx^\rho}{ds} = 0, \tag{8.16}$$

where $\mathring{\Gamma}^{\lambda}{}_{\sigma\rho}$ is the Levi-Civita connection. These differential equations are equivalent if, and only if, $T_{(\mu\nu)\sigma} = 0$.
If in addition to the metric compatibility we impose that the connection is torsionless we obtain the Killing equations again:

$$\nabla_{(\mu} \xi_{\nu)} = 0, \tag{8.17}$$

with ξ the Killing vector field.

Discussion and Conclusions

Let us summarise what we have presented in this thesis. In Chapter 1 we focused on one of the milestones that led to the formulation of gravitational theories, that is the Equivalence Principle. In Chapter 2 we first deduced the Levi-Civita connection from the EP and then we showed which relations must the transformations that leave the metric unchanged satisfy, that is, the Killing equations. In Chapter 3 we presented how the Levi-Civita connection behaves when the metric is subject to deformations. In Chapters 4, 5 and 6 we described theories where gravity is the effect of the spacetime curvature. In the first of these chapters we dealt with metric theories where the dynamic variable is precisely the metric, in particular General Relativity and its extensions. In the second one we regarded the purely affine theories where the dynamic variable is the connection and in the last one we considered metric-affine theories where both the metric and the connection play the role of dynamic variables. In Chapter 7 we presented the teleparallel theories, namely the Teleparallel Equivalent to General Relativity and the Symmetric Teleparallel Equivalent to General Relativity, where the gravity is mediated through the torsion and non-metricity of spacetime respectively. In the last chapter we focused on the so called trinity of gravity (curvature tensor ${R^\lambda}_{\rho\mu\nu}$, torsion tensor ${T^\lambda}_{\mu\nu}$ and non-metricity tensor ${Q^\lambda}_{\mu\nu}$) and what happens when we want to find the Killing vectors for a generic affine connection.
Now we could ask ourselves what is the reason that led us to the study of these different theories of gravity: we find the main answer in the EP. The GR is based on the universality of free fall, which is the result of the equivalence between inertia and gravity due to the EEP. If for some reasons the EP will be disproved, then GR becomes unsuitable. Moreover the EP is not reconcilable with quantum mechanics because controversies arise if a particle is allowed to be in superposition states of different masses. It is

in this scenario that the teleparallel theories come into play, which can be formulated both with the EP and without it, as we showed. Let us stress that although the Teleparallel Gravity is not a usual gauge theory, in the sense that the tangent bundle is soldered and not internal, it keeps every property of a gauge theory. This means that it is more appropriate to create a unified theory with the other three fundamental interactions, in contrast with GR. Furthermore, through its extensions, it is able to provide an explanation for the present cosmic acceleration.

The analysis we did in Chapter 8 on the Lie derivative occurs precisely in a framework where the validity of the EP is not ensured by the presence of the non-metricity. From this analysis some important remarks follow: first, from a general situation we can bring ourselves back to the results of GR and therefore to the Levi-Civita connection, simply by annulling the two elements of the trinity that are not relevant to GR, i.e. torsion and non-metricity. This could be a clue to interpret the changes that involve the introduction of torsion and non-metricity as a deformation of the gravitational field. Secondly, we could interpret the loss of a symmetry by a system, in different points of spacetime, as a consequence of the presence of non-metricity with a consequent loss of importance for the EP. Given the deep importance in understanding the validity of the EP and therefore testing it with the highest possible accuracy, a space race was born that led to the realization and proposition of space experiments such as MICROSCOPE [44], ACES [16], STE-QUEST [10], QTEST [48] and in the near future SAGE [43]. All these experiments aim to test the three, or at least one, of the sub-principles on which the EEP is based, namely the Universality of Free Fall, the Local Position Invariance and the Local Lorentz Invariance. The proof or the denial of one of these principles will tell us if the moment of the decline of GR has come, which will then lead to the dawn of new theories of gravity.

As a continuation to this thesis we will broaden the discussion to the theories that extend the TEGR and the STEGR, that is the $f(T)$ and $f(\mathcal{Q})$ theories of which just a hint was given in Chapter 7, and how possible Noether symmetries are related to the non-conservation of the metric.

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