UNIVERSITÀ DEGLI STUDI DI NAPOLI
"FEDERICO II"

Scuola Politecnica e delle Scienze di Base

Area Didattica di Scienze Matematiche Fisiche e Naturali

Dipartimento di Fisica "Ettore Pancini"

Laurea Magistrale in Fisica

Non-metricity in Theories of Gravity and Einstein Equivalence Principle

Relatori:
Prof. Salvatore Capozziello
Dr. Francesco Bajardi

Candidato:
Marcello Miranda
Matr. N94000401

Anno Accademico 2018/2019

Preface

Framework

The research for a unified theory, which describes both the laws of the microscopic world and those of the macroscopic world, reached a deadlock. The progress of physical theories is hampered by theoretical and experimental limits, which influence each other: on the one hand there are no models that help to interpret experimental observations comprehensively, on the other hand there are no experimental decisive observations for a better formulation of existing models. To be precise, problems arise mostly from the cosmological observations and one tries to find an answer, for example, introducing new types of energy and matter.

The assumption from which most start is that General Relativity (GR) is not a complete theory, despite being one of the most beautiful theories of gravity and its many successes (*e.g* classical tests of GR and gravitational waves) [1–4]; GR lacks a quantum counterpart and does not explain in an exhaustive way observations, such as the accelerated expansion of the universe (negative pressure), the analysis of galactic rotation curves [5], acoustic oscillations in the cosmic microwave background (CMB) [6–8], large scale structure formation [9, 10] and gravitational lensing [11–14].

The most used approach is to combine GR at cosmological scales, with a hypothesis of homogeneity and isotropy, and the Standard Model of particle physics (SM), describing non-gravitational interactions, in the "*concordance model*" of cosmology: the $\Lambda-$CDM (Cold Dark Matter) model. This "compromise" is still far from representing an exhaustive description of the Universe, because important puzzle pieces remain missing. Notably, there is a necessity to introduce what are called *dark energy* (DE), described by a cosmological constant Λ, and cold dark matter (DM), that consists of some unknown stable particle.

The existence of the DE and the DM is an open problem that generates potential conflicts between the concordance model of cosmology and the Standard Model of particles. From cosmological observations [15], it is known that the content of matter and energy of the Universe is about 68.3% DE, 26.8% DM, and 4.9% baryonic matter (the "common" visible matter). Thus, in the Universe there is a strong presence of non-baryonic dark matter particles, effects of which should certainly be predicted by some extension of the SM. Unfortunately, experiments at the Large Hadron Collider (LHC) do not produce results that indicate the existence of a new physics beyond the SM. The conflicts concern the DE too. Indeed, its simplest explanation is the existence of a small, but non-zero, cosmological constant Λ. The latter does not undergo a dynamical evolution and is conventionally associated with the energy of the vacuum in a quantum field theory. In other words, the cosmological constant is a constant energy density filling space homogeneously and isotropically, and is physically equivalent to vacuum energy. Here arises another problem: the vacuum energy density associated to the cosmological constant is $\rho_\Lambda = \Lambda/8\pi G \simeq 10^{-47} GeV^4 (\simeq \rho_{critical})$ where $\rho_{critical}$ is the critical density of the universe; from a quantum point of view, the vacuum energy density is the sum of zero point energy of quantum fields with a cut-off determined by the Planck scale ($\mathrm{m}_P \simeq 1.22\times10^{19}$ GeV) giving $\rho_{vacuum} = 10^{74} GeV^4$, which is about 121 orders of magnitude larger than the observed value [16]. This discrepancy has been called "*the worst theoretical prediction in the history of physics!*" [17].

From a theoretical point of view, nowadays, there is no explanation as to why the cosmological constant should assume the "correct" value at the scale of the observed Universe. The only argument we can give is based on the *anthropic principle*, *i.e* the idea that much larger values would not have led to the formation of stars, planets and ultimately humans.

Although there is such strong evidence for the existence of DE and DM, almost nothing is known about their nature and properties.

All the considerations made so far are motivating physicists to seek ways to extend GR so that it is more compatible with experimental observations and it represents a macroscopic limit of some quantum theory.

Therefore, questioning even about what currently seems obvious or necessarily true is legit. What if the Einstein Equivalence Principle (EEP) is violated at some (unknown) energy scale?

This question is not unreasonable since there are many theories that foresee small violations of principles at certain scales. Moreover, what justifies this question is the fact that the principle of equivalence is not a fundamental symmetry of physics, contrary to the principle of local gauge invariance in particle physics, for instance.

Most physicists believe that GR and the SM are only low-energy approximations of a more fundamental theory that remains to be discovered. Several concepts have been proposed and are currently under investigation (*e.g*, String Theory, Loop Quantum Gravity, extra spatial dimensions) to fill this gap and most of them lead to tiny violations of the basic principles of GR.

One of the most desirable attributes of such fundamental theory is the unification of the fundamental interactions of Nature, *i.e* a unified description of gravity and the three other fundamental interactions. There are several attempts at formulating such a theory, but none of them are widely accepted or considered successful. Furthermore, few of their quantitative predictions are precise in a way that could be verified experimentally. One of them is the Hawking radiation of black holes, which is, however, far from being testable experimentally for stellar-size black holes observed in astrophysics.

Therefore, a comprehensive understanding of gravity will require observations or experiments able to determine the relationship of gravity with the quantum world. This topic is a prominent field of activity with repercussions covering the complete range of physical phenomena, from particle and nuclear physics to galaxies and the Universe as a whole, including DM and DE.

Indeed, most attempts at quantum gravity and unification theories lead to a violation of the EEP [18–25]. In general these violations have to be be handled by some tuning mechanism in order to make the theory compatible with existing limits on EEP violation. For example, in String Theory, moduli fields need to be rendered massive (short range) [23] or stabilized by *e.g* cosmological considerations [19] in order to avoid the stringent limits already imposed by EEP tests.

Therefore, rather than asking why the EEP should be violated, the more natural question to ask is why no violation has been yet observed.

Outline

The present work consists of the collection of surveys made to devise a way to violate the EEP using the nonmetricity of connection. The arguments are all purely classical because they are aimed at generalizing or modifying GR. The thesis relies and takes into account the possibility that, at some energy scale, the principle of equivalence could be violated (strictly) due to a nonmetricity of the "physical connection". One of the aims is to show the link between different theories having in common a non-metric connection. In particular, during the progress of the Chapters, the attention focuses on the Weyl non-metricity tensor.

The organization of Chapters to a large extent reflects the process of investigation.

The starting point is intended as an overview of the pillar on which the most elegant and simple theory of gravitation is based: the EEP. Einstein himself initially called it the *hypothesis* of equivalence before elevating it to a *principle*, once it became clear how pivotal it was in including gravitation in the generalization of special relativity. EEP supports all the theories that are called *metrics*, in which the metric tensor $g_{\mu\nu}$ constitutes a dynamic field that determines the lengths of "objects", the causal structure of the universe and the gravitational field [26]. EEP guarantees a geometric interpretation of the gravitational effects.

Then, two metric modifications of general relativity are mentioned. These are part of the large family of Extended Theories of Gravity (ETG), the scalar-tensor theories and $f(R)$-theories [3, 24, 27–30].

Scalar-tensor theories are characterized by the presence of a scalar field ϕ, or more gen-

erally by a coupling function $F(\phi)$, that can be interpreted as the inverse of an effective universal gravitational constant, G_{eff}. This modification is based on Mach's ideas which states: *"the inertia of each system is the result of the interaction of the system with the rest of the universe; in other words, every particle in the cosmos has influence on every other particle"*. Therefore, it is expected that the gravitational constant is not actually constant with respect to the life of the universe.
In $f(R)$-theories the Ricci curvature scalar $R = g^{\mu\nu}R_{\mu\nu}$ inside the Einstein-Hilbert action is replaced by its generic function (*i.e* $f(R)$) which produces an extra gravitational energy-momentum tensor due to higher-order curvature effects. Moreover, $f(R)$-theories can be generalized using what is called the *Palatini formalism* [31], wherein the connection is considered *a priori* independent from the metric, whereas the matter Lagrangian depends on the metric only (and, obviously, on the matter field). This means that while the metric tensor determines the casual structure of the universe, the connection determines the geodesic curves (or better, the autoparallel curves) *i.e* the free fall. In principle, this decoupling enriches the geometric structure of space-time and generalizes the purely metric formalism. By means of the Palatini field equations, this dual structure of space-time is naturally translated into a bi-metric structure of the theory: instead of a metric and an independent connection, the Palatini formalism can be seen as containing two independent metrics $g_{\mu\nu}$ and $h_{\mu\nu} = f'(R)g_{\mu\nu}$. In Palatini $f(R)$-gravity the new metric $h_{\mu\nu}$ determining the geodesics, is related to the connection as the latter turns out to be the Levi-Civita connection of $h_{\mu\nu}$. Moreover, other geometrical invariants, besides R, can be considered in the Palatini formalism, as well as the second-order curvature invariant.
However, the Palatini method will be developed only in the final part of the work, as well as the relationship between $f(R)$-theories and scalar-tensor theories, together with their interpretation in the context of conformal transformations.

The idea of a connection independent of the metric leads to the analysis of the metric-affine structure of the spacetime [32, 33]. A generic connection can be divided into three objects with different proprieties. Their presence causes changes on a metric pseudo-Riemannian manifold (the GR manifold with metric-compatible affine connection).
Connection coefficients can be seen as the sum of the Levi-Civita connection (Christoffel coefficients), contorsion tensor (linked to the torsion of the connection) and the disformation tensor (related to the non-metricity tensor). This allows one to better understand the consequences of the presence of torsion and nonmetricity in a theory. In particular, the general form of Riemann tensor and its contraction, together with the auto-parallel curves equation, will be useful in the ending part of discussion. Moreover, they allow to further generalize what has been done in this thesis.

Hereafter, the study went on to the so called *"Geometrical Trinity of Gravity"* [34]. It is nothing transcendental. The trinity is formed by GR and two other theories that are equivalent to the last one: *Teleparallel Equivalent to General Relativity* (TEGR) [35–37] a theory based on the torsion, the only geometric aspect that determines gravitational effects in a flat world; *Symmetric Teleparallel Equivalent to General Relativity* (STEGR) [32, 34, 37, 38], a theory based on the non-metricity, *i.e* the "non-conservativity" of the parallel transported vectors length [33]. In particular TEGR can be seen as a gauge theory with respect to the group of the translations [39–42]. According to who writes this point of view is a bit precarious because of the "*intimate*" relation between the external space and the internal gauge one (see Appendix). However, it bridges the search for a theory with a non-metric connection that is a gauge potential on the group of conformal transformations[1].

For this reason the next step is devoted to summing up the basic foundations of isometries, conformal transformation and Weyl rescaling (or conformal transformations of the metric) [50–53].

Subsequently, the attention is focused on the Classical Conformal Theories of Gravity that stress out the idea that conformal metric transformations should be a symmetry of gravity (spacetime). In this framework, Weyl's geometry and the Weyl vector W_μ are introduced. Weyl geometry is characterized by a symmetric but non-metric connection, which is invariant under Weyl rescaling. In particular, it is possible to link Weyl's geometry and a Brans-Dicke action by using a conformally invariant scalar field ϕ. In this regard, it is possible to think Weyl vector as generalization of a scalar field because $W = W_\mu dx^\mu$ is, generally, a non-exact form but, by using its equations of motion, W_μ results equal to the derivative of a scalar field function. Therefore, W_μ can be "absorbed" in a particular conformal rescaling which links different actions (in this case Einstein-Hilbert action and Brans-Dicke action).

To stress out this idea, following [27], the relation between non-metricity and Weyl rescaling in Palatini formalism is discussed. Both in $f(R)$-theories and Scalar-Tensor theories, there is a second metric $h_{\mu\nu}$ associated to the connection, which can be seen as conformally related to $g_{\mu\nu}$. Moreover, it is possible to generalize these two theories by copulating a scalar field with $f(R)$. However, GR vacuum equations can be recovered only when the scalar field and the curvature can be decoupled. This leads to think that there may be a kind of "physical equivalence" between theories which can be obtained by performing a conformal transformation. Historically, this issue was born with the Brans-Dicke action which shows two faces: *Jordan frame* and *Einstein frame*. They are linked by a conformal transformation and, in absence of a specific transformation law of the matter Lagrangian, their equivalence produces a violation of the EEP.

In order to have a more comprehensive view of Weyl's geometry, in the last Chapter the free-fall of particles in such geometry is studied. Moreover, we set out to analyse the role of Weyl vector in generic quadratic actions. Here, the similarities and differences between the Weyl field and the electromagnetic field are highlighted – W_μ was introduced by Weyl precisely to unify gravity and electromagnetism in a single formalism. However,

[1]This is not the first time that such an idea emerges [43–49] but it presents several theoretical difficulties. Many obstacles arise, especially if one tries to see Weyl rescaling as local scale transformations caused by tetrad conformal transformation coordinates.

Weyl vector behaves as an electromagnetic field in Bach equations. The presence of Weyl energy-momentum tensor could explain the anomaly of galaxy rotational curves, describing (at least partially) the presence of DM.

On the basis of these correspondences and observations, it is suggested that Weyl's geometry may be a valuable tool for constructing a generalized action and that a conformal breaking symmetry may discriminate against different theories. In this regard, there are several generalizations that can be taken in consideration. Some of these are mentioned in the Conclusions, but no one are intended to concretize the "conformal equivalence" taking into account the possibility of coupling Weyl's geometry, scalar field and quadratic orders in curvature. This could be an interesting starting point for future studies. Moreover, this line of research is very compatible with the current context of high-precision measurements in space which aim to observe violation of EEP [16, 54, 55].

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Chapter 1

Einstein Equivalence Principle

1.1 Towards Metric Theories

The Equivalence Principle is one of the great pillars of GR, defined by Einstein as "*the most beautiful thought of his whole life*".
GR is rooted into three large groups of ideas:

1. Mach's criticism to the Newtonian conception of space and time as absolute entities;
2. Special Relativity (SR), which revolutionized the concept of space and time, making them an inseparable unit, the space-time;
3. Einstein Equivalence Principle, which focused the attention on a "new universal reference motion", the free fall of an uncharged body.

The last idea is based on the works of Galileo and Kepler. In *Principia*, Newton formulated a version of it so that the inertial mass of any body, $\mathrm{m_I}$, *i.e.* that property of a body that governs its response to a given force, $\mathbf{F} = \mathrm{m_I}\mathbf{a}$, is equal to its gravitational mass, $\mathrm{m_g}$, the quantity that dictates its response to gravitation, $\mathbf{F}_\mathrm{g} = \mathrm{m_g}\mathbf{g} = \mathrm{G}_N\mathrm{M_g}\mathrm{m_g}\mathbf{r}/\mathrm{r}^3$. This is known today as the *Weak Equivalence Principle* (WEP) and is better stated as "*if an uncharged test body is placed at an initial trajectory it will be independent of its internal structure and composition*" [1].
The WEP implies that it is impossible to distinguish, locally, the effects of a gravitational field from those experienced in uniformly accelerated frames using the simple observation of the free-falling particles behaviour.

However, Einstein was the only one to think about the universality of free fall and to generalize this idea to all the physics laws, not only the mechanical ones: Physics is the same in any free falling frame. In this sense Einstein introduced a "new universal reference motion".
It was this idea that opened the road to GR, the most famous and fashionable example of metric theory (as well as the founder of all metric theories of gravity).

It is called the *Einstein Equivalence Principle* and it states that: (i) WEP is valid universally, (ii) the outcome of any local non-gravitational test experiment is independent of the velocity of the (freelyfalling) apparatus (which is the Local Lorentz Invariance, LLI) and (iii) the outcome of any local non-gravitational test experiment is independent of where and when in the universe it is performed (which is the Local Position Invariance, LPI).

The validity of EEP in the so-called *metric theories of gravity* constrains gravitation to be a "*curved space-time*" phenomenon. When speaking of a "metric theory", one refers to a theory wherein the metric tensor has a double role to determinate both the geometry and the causal structure of space-time and the manifestation of gravitational field effects on it.

This description of gravity implicates that gravitational field, the metric, determines the space-time. Specifically, gravity is mediated through the curvature, *i.e.* the deviation of space-time from flatness, and this is directly related to the matter and energy content of the Universe [56]. Mathematically, it is always possible to choose a so-called *local inertial frame* wherein $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu} = 0$, *i.e.* the gravitational effects vanish.

Therefore, metric theories have to satisfy the following two postulates [26, 57, 58]:

1. There exists a metric $g_{\mu\nu}$, which is a non-degenerate symmetric rank-2 tensor and determines all the geometric and causal structure of space-time through the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

2. If $T_{\mu\nu}$ is the energy-momentum tensor of all matter fields and ∇_μ a covariant derivative, derived by the Levi-Civita connection of the above metric $g_{\mu\nu}$, then $\nabla_\mu T^{\mu\nu} = 0$.

Levi-Civita conceived a *gedanken experiment*, a way to test a metric theory of gravity (GR in particular), that required three principles, necessary for the physical meaning of the theory and for the consistency with SR.

Let γ a generic curve of the spacetime with tangent vector $\mathbf{T} = T^\mu \partial_\mu = (d\gamma^\mu/d\lambda)\partial_\mu$, then:

i. matter curves are always time-like curves, $\gamma : g(\mathbf{T}, \mathbf{T}) < 0$, *i.e.* the 4-velocity of a matter particle is always smaller than the light one;

ii. under geometrical optics approximation, light rays describe always light-like curves (null-curve), $\gamma : g(\mathbf{T}, \mathbf{T}) = 0$;

iii. a free fall material particle motion is described by the equation $\gamma : \nabla_{\mathbf{T}} \mathbf{T} = \alpha \mathbf{T}$, with $g(\mathbf{T}, \mathbf{T}) < 0$.

Therefore, with a finite number of free fall matter particles and light rays, it is possible to measure the gravitational field and test a metric theory.

From a more practical point of view, Schiff and Dicke were the first who realized [59, 60] that the gravitational experiments would be the way to probe the foundations of gravitational theories, and not only of general relativity itself. Therefore, Dicke formulated a framework in which one can discuss the nature of space-time and gravity [58]. According to him any theory of gravity should satisfy the following assumptions and constraints, summarized below [32]:

- Geometric points have to be associated with physical events. The only geometric properties a space-time should have *a priori* are those of a 4-dimensional differentiable manifold (with a generic affine connection, neither metric-compatible nor symmetric).
- All the mathematical quantities should be expressed in a *coordinate covariant form.*
- Gravitational effects should be described by one or more long-range fields, having a tensorial form (scalar, vector, 2-rank tensor or even higher).
- The dynamical equations will be obtained from an invariant action principle.
- Nature likes things as simple as possible (*requirement of simplicity*).

Throughout the years, theorists formulated a set of fundamental criteria that any viable gravitation theory should respect, not only from a theoretical viewpoint but also from an experimental one [32]. A theory must be:

- complete, *i.e.* the analysis of experimental results should be based on "*first principles*";
- self-consistent, *i.e.* the interpretation of experimental data should not be ambiguous and independent of the calculation method used;
- relativistic, *i.e.* it must reconcile with SR when gravitational effects are negligible;
- compatible with the Newtonian limit when the masses/energies are sufficiently weak.

The last two criteria are based on the great success of both SR and Newtonian theory of gravity at their range of validity.

To conclude, it is worth mentioning the existence of another Equivalent Principle which distinguishes itself from WEP and EEP by including self-gravitating bodies and also local gravitational experiments. It is called the *Strong Equivalence Principle* (SEP) and it states that: (i) WEP is valid not only for test bodies but also for bodies with self-interactions (planets, stars), (ii) the outcome of any local test experiment is independent of the velocity of the (freely falling) apparatus, and (iii) the outcome of any local test experiment is independent of where and when in the universe it is performed.
SEP includes EEP in the limit where self-gravitational forces are ignored. Up to now, there is no other theory satisfying the SEP but GR.

1.1.1 Geodesic Equation

As a consequence of the Equivalence Principle, the free fall motion of a test particle is given by the geodesic equation.
In a locally inertial frame, where the gravitational force is eliminated thanks to the EEP, the equations of motion is that of a free particle:

$$\frac{d^2y^\mu}{ds^2} = 0, \tag{1.1}$$

where

$$ds^2 = \eta_{\alpha\beta}dy^\alpha dy^\beta, \tag{1.2}$$

is the line element, with $\eta_{\alpha\beta} = diag(+1, -1, -1, -1)$ the Minkowski metric. Performing the coordinate transformations $y^\mu = y^\mu(x^\nu)$, the eq. (1.1) becomes

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda{}_{\sigma\rho}\frac{dx^\sigma}{ds}\frac{dx^\rho}{ds} = 0 \tag{1.3}$$

with

$$\Gamma^\lambda{}_{\sigma\rho} = \Gamma^\lambda{}_{\rho\sigma} = \frac{\partial x^\lambda}{\partial y^\mu}\frac{\partial^2 y^\mu}{\partial x^\sigma \partial x^\rho}. \tag{1.4}$$

The eq. (1.3) is the *geodesic equation* and the quantities $\Gamma^\lambda{}_{\sigma\rho}$ are called affine connections; these latter express the gravitational force that acts on the particle. In the absence of a gravitational field, the geodesic equation shows that the affine connections give the apparent forces if a transformation from a locally inertial frame to another generic frame is performed. This manifests the equivalence between inertial and gravitational forces.

1.2 General Relativity

In GR, space-time is described by a manifold M, which is endowed with a metric $g_{\mu\nu}$ and a connection ∇ characterized by the connection coefficients $\Gamma^\lambda{}_{\mu\nu}$. The metric fixes the causal structure of space-time (the light cones) as well as its metric relations (clocks and rods). The connection Γ fixes the free-fall, namely the locally inertial observers. They have, of course, to satisfy a number of compatibility relations which require that photons follow null geodesics of Γ, so that Γ and g can be independent, *a priori*, but constrained, *a posteriori*, by some physical restrictions. In GR however, the connection is assumed to be the Levi-Civita one, *i.e.*

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{1.5}$$

It is easily seen that the above connection is symmetric in its two lower indices, $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$, and the metric is covariantly conserved,

$$\nabla_\alpha g_{\mu\nu} = 0. \tag{1.6}$$

When a connection satisfies eq. (1.6) it is labelled as *metric-compatible*.
The fact that the connection is the Levi-Civita one (*i.e.* it is symmetric) means that the torsion, defined as two times its antisymmetric part, vanishes:

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} = 2\Gamma^{\lambda}{}_{[\mu\nu]} = 0. \tag{1.7}$$

In the eq. (3.20), $\Gamma^{\lambda}{}_{[\mu\nu]}$ is the antisymmetric part of the Levi-Civita connection and the antisymmetrizer of indices is defined with a normalization factor (in this case 1/2).

Hence, in GR, the space-time (or more correctly, the connection) is assumed to be symmetric and metric-compatible. In this context, there is the possibility to consider a local inertial frame where the manifold becomes the Minkowski flat space of SR.

This means that the spacetime is characterized by the curvature, whose expression is given by

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\sigma\nu} - \partial_{\sigma}\Gamma^{\mu}{}_{\rho\nu} + \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\rho\nu}. \tag{1.8}$$

The above Riemann tensor is symmetric in the exchange of the first and last pair of indices and anti-symmetric in the flipping of any pair. In addition, it satisfies the first and second Bianchi identities which respectively read

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0, \tag{1.9}$$

$$\nabla_{\alpha}R^{\mu}{}_{\nu\rho\sigma} + \nabla_{\nu}R^{\mu}{}_{\rho\alpha\sigma} + \nabla_{\rho}R^{\mu}{}_{\alpha\nu\sigma} = 0. \tag{1.10}$$

Furthermore, one can now define uniquely the Ricci tensor as $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = R_{\nu\mu}$ and from this one, the Ricci scalar as $R = g^{\mu\nu}R_{\mu\nu}$.

1.2.1 Parallel Transport and Geodesics in General Relativity

It is known that in a Riemannian space it is necessary to introduce a rule to *transport* a tensor on the manifold. This rule is given by the connection [56, 61]. The derivative of a vector $\mathbf{v} = v^{\mu}\partial_{\mu}$ along a curve γ with tangent vector $\mathbf{T} = T^{\mu}\partial_{\mu} = (d\gamma^{\mu}/d\lambda)\partial_{\mu}$, parameterized with λ, is defined as following

$$\frac{D\mathbf{v}}{d\lambda} \equiv \nabla_{\mathbf{T}}\mathbf{v}. \tag{1.11}$$

Then, it is possible to define a parallel transport rule associated to the connection

$$\frac{D\mathbf{v}}{d\lambda} = 0 \longleftrightarrow \nabla_{\mathbf{T}}\mathbf{v} = 0, \tag{1.12}$$

namely

$$\frac{dv^{\mu}}{d\lambda} + \Gamma^{\mu}{}_{\rho\nu}\frac{d\gamma^{\rho}}{d\lambda}\frac{dv^{\nu}}{d\lambda} = 0. \tag{1.13}$$

This transport rule establishes an isomorphism from tangent vector subjects to the choice of the curve. Thanks to this, a generalization of geodesic is given for a manifold as auto-parallels curve: *geodesic is a curve γ having a parallel transported tangent vector*,

$$\nabla_{\mathbf{T}}\mathbf{T} = \alpha(t)\mathbf{T}, \tag{1.14}$$

and if the parametrization is such that $\alpha(t)$ is null, then it is *affinelly parameterized*. In coordinates, the eq. (1.14) gets the following form

$$\frac{d^2\gamma^\mu}{d\lambda} + \Gamma^\mu{}_{\rho\nu}\frac{d\gamma^\rho}{d\lambda}\frac{d\gamma^\nu}{d\lambda} = 0. \tag{1.15}$$

It is easy to verify that, if $\mathbf{T}$ is a time-like vector, the integral curve of eq. (1.14) is an extremal of *length functional*,

$$\ell = \int_A^B ds, \tag{1.16}$$

indeed

$$\begin{aligned}
0 = \delta\ell = \delta\int_A^B ds &= \int_A^B \delta\left[g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]^{1/2} d\lambda, \\
&= \int_A^B \left[\partial_\rho g_{\mu\nu}\delta x^\rho \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] d\lambda = \\
&= \int_A^B \left[\partial_\rho g_{\mu\nu}\delta x^\rho \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2g_{\mu\nu}\delta x^\mu\frac{d^2x^\nu}{d\lambda^2} - 2\partial_\sigma g_{\mu\nu}\delta x^\mu \frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda}\right] d\lambda \\
&= -2\int_A^B \left[g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{1}{2}\Big(- \partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu}\Big)\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda}\right]\delta x^\mu d\lambda,
\end{aligned}$$

and multiplying by $g^{\mu\sigma}$ one gets

$$\frac{d^2x^\sigma}{d\lambda^2} + \Gamma^\sigma{}_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{1.17}$$

Thus, in GR there are two equivalent definition of geodesic.

1.2.2 Field Equations

GR field equations can be obtained in different ways [3]. Before getting his equations of motion through the variational approach, Einstein reasoned by analogy with Maxwell and Poisson equations, and he looked for an equation that was as simple as possible: from the possibility to choose a locally inertial frame, it is necessary to involve second order derivatives of the metric tensor and the criterion of simplicity leads to consider a linear dependence on them. Moreover, he was looking for tensorial equations so as to ensure the general covariance principle.

From the Riemannian differential geometry [3], it is known that Riemann tensor and its contraction are the only tensors which are linear in second order derivatives of the metric and depending on the first order derivatives, too. Therefore, a natural choice is to consider in vacuum

$$R^{\mu}{}_{\nu\rho\sigma} = 0, \tag{1.18}$$

but these are too restrictive and provide flat solutions. Thus, the choice goes to

$$R_{\mu\nu} = 0. \tag{1.19}$$

Then, in order to introduce the description of the matter dynamic and the influence of the matter on the space-time, the eq. (1.19) is replaced by

$$G_{\mu\nu} = 0, \tag{1.20}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - 1/2 g_{\mu\nu} R$ is the Einstein tensor.
In presence of matter, field equations can be obtained using the energy-momentum tensor of a perfect fluid ($T^{\mu\nu} = \rho u^{\mu} u^{\nu} + p h^{\mu\nu}$) and taking into account the classical Poisson equation for the gravitational field ($\Delta U_g = 4\pi G \rho$). In general, the Einstein field equations are

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.21}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter and the constant term $\frac{8\pi G}{c^4}$ provides the correct Newtonian limit.
Since the Einstein tensor has null divergence $\nabla_{\mu} G^{\mu\nu} = 0$, the equation (1.21) includes the conservation of energy-momentum tensor, $\nabla_{\mu} T^{\mu\nu} = 0$, saving theory from possible internal inconsistencies.

From the variational formalism viewpoint, the eq. (1.21) is given by the Einstein-Hilbert action:

$$S = \frac{c^4}{16\pi G_N} \int d^4x \sqrt{-g} R + S_m, \tag{1.22}$$

where g is the determinant of the metric and R is the Ricci scalar. This action was introduced by Hilbert and it is the simplest action that gives second order covariant equations of motion for the metric, which is the dynamical field.
The second term in the right hand side is given by

$$S_m = \int d^4x \sqrt{-g} \mathscr{L}_m(g_{\mu\nu}, \Psi), \tag{1.23}$$

and it is called matter action. It contains the matter Lagrangian $\mathscr{L}_m$ in which all matter fields (denoted for simplicity by Ψ) directly couple to the metric.

Field equations are obtained by varying with respect to the metric tensor the E-H action. Observing that

$$\delta g_{\rho\sigma} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\mu\nu}, \tag{1.24}$$

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu}, \tag{1.25}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{1.26}$$

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho{}_{\mu\nu} - \nabla_{(\mu|}\delta\Gamma^\rho{}_{\rho|\nu)}, \tag{1.27}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{1.28}$$

then, the variation of the action (1.22) is

$$\begin{aligned}\delta S &= \frac{c^4}{16\pi G_N}\int d^4x[\delta\sqrt{-g}R + \sqrt{-g}\delta R] + \delta S_m \\ &= \frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}[-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}R + R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\rho{}_{\rho\mu})] + \delta S_m \\ &= \frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}[G_{\mu\nu}\delta g^{\mu\nu} + \nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\rho{}_{\rho\mu})] + \delta S_m. \end{aligned} \tag{1.29}$$

The first term in the integral is the Einstein tensor. The second term multiplied by $\sqrt{-g}$ becomes a total derivative and thus, by Stoke's theorem, yields a boundary term when integrated [27]. Finally, the last term in eq. (1.29) will give the energy-momentum tensor of all the matter fields in the universe as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{1.30}$$

Thus, the field equations for the gravitational field reads

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}. \tag{1.31}$$

1.3 Other Metric Theories: ETG

With the aim to answer open questions about Gravity, *non-minimally coupled terms* between matter fields and geometry and *higher-order curvature invariants* can be added into the Einstein-Hilbert Lagrangian. These manipulations of GR theory are part of the so called *Extended Theories of Gravity* (ETG) [24, 27, 29, 30].

1.3.1 Scalar-Tensor Theories (Generalized Brans-Dicke Gravity)

The Brans-Dicke theory of gravity is the prototype of gravitational theories alternative to GR. It was born as an implementation of Mach's Principle [29] which states that the

local inertial frame is determined by the average motion of distant astronomical objects. Before, Dirac hypothesized a gravitational coupling G_N with a temporal dependence, keeping the other fundamental constants fixed. Then, thanks to P. Jordan's ideas, G was promoted to the role of a gravitational scalar field. Finally, Brans and Dicke developed more rigorously the idea of a Scalar-Tensor Theory wherein a non-minimal coupling scalar field, ϕ, describes gravity together with the metric tensor.
Therefore, Scalar-Tensor Theories are characterized by a non-constant gravitational coupling, determined by all matter in the universe, and the Newton constant G_N is replaced by the *effective gravitational coupling*,

$$G_{eff} = \frac{1}{F(\phi)}, \tag{1.32}$$

where $F(\phi)$ is a generic function of the scalar field.
The generic action has the following form:

$$S = \int d^4x\sqrt{-g}\Big[F(\phi)R + \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\Big] + S_m, \tag{1.33}$$

with $V(\phi)$ a generic scalar field potential. Here it is possible to obtain two equations, one by varying the action with respect to the metric ($g^{\mu\nu}$) and the other by varying the action with respect to the scalar field ϕ:

$$G_{\mu\nu} = \frac{1}{F(\phi)}\Big[T^{(m)}_{\mu\nu} + T^{(eff)}_{\mu\nu}\Big] \tag{1.34}$$

where

$$T^{(eff)}_{\mu\nu} = \nabla_\mu\nabla_\nu F(\phi) - g_{\mu\nu}\Box F(\phi) - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{4}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi - \frac{1}{2}g_{\mu\nu}V(\phi), \tag{1.35}$$

is an effective energy-momentum tensor due to the presence of the scalar field; if the action is varied with respect to ϕ, it follows

$$\Box\phi - RF'(\phi) + V'(\phi) = 0, \tag{1.36}$$

that is the invariant Klein-Gordon equation in a curved space-time - see the equations (4.53) and (5.16).
The computation of the additional contributions in eq. (1.34) has been omitted because it is very similar to that of next Section. $T^{(eff)}_{\mu\nu}$ contains the terms coming from the variation of $\sqrt{-g}$ (proportional to the kinetic term and to the potential), from the variation of the squared kinetic term and from the variation $\delta R_{\mu\nu}$ that is no more a boundary term because of the presence of $F(\phi)$.
An analogous situation arises for metric theories, described below.

1.3.2 $f(R)$ Theories

A second generalization of the Einstein-Hilbert Lagrangian is given by the replacement of R by a function of it, $f(R)$:

$$S = \int d^4x\sqrt{-g}f(R) + S_m. \tag{1.37}$$

It is easy to get the field equations associated to the action (1.37). With this purpose, we observe that

$$\begin{aligned} \delta f(R) &= f'(R)\delta R = f'(R)R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}f'(R)\delta R_{\mu\nu} = \\ &= f'(R)R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}f'(R)\big[\nabla_\rho\delta\Gamma^\rho{}_{\mu\nu} - \nabla_{(\mu|}\delta\Gamma^\rho{}_{\rho|\nu)}\big]. \end{aligned} \tag{1.38}$$

From eq. (1.38) it is possible to see that the presence of $f(R)$ causes additional contributions coming from $\delta R_{\mu\nu}$, which is no longer a boundary term.
Moreover, to simplify the computation of filed equations, it is useful to write the following variations

$$\delta\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\big(2\nabla_{(\mu}\delta g_{\nu)\rho} - \nabla_\rho\delta g_{\mu\nu}\big), \tag{1.39}$$

$$\delta\Gamma^\lambda{}_{\mu\lambda} = \frac{1}{2}g^{\lambda\rho}\nabla_\mu\delta g_{\rho\lambda} = -\frac{1}{2}g_{\lambda\rho}\nabla_\mu\delta g^{\rho\lambda}. \tag{1.40}$$

Then, the variation of the action (1.37) with respect to the metric tensor can be obtained in the following way (integrating by parts and ignoring total derivative contributions):

$$\begin{aligned} \int d^4x\delta[\sqrt{-g}f(R)] &= \int d^4x[\delta(\sqrt{-g})f(R) + (\sqrt{-g})\delta f(R)] = \\ &= \int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + f'(R)\delta R\Big] = \\ &= \int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + f'(R)R_{\mu\nu}\delta g^{\mu\nu} + f'(R)g^{\mu\nu}\delta R_{\mu\nu}\Big] = \\ &= \int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + f'(R)R_{\mu\nu}\delta g^{\mu\nu} + f'(R)g^{\mu\nu}\nabla_\rho(\delta\Gamma^\rho{}_{\mu\nu} - \delta^\rho_\mu\delta\Gamma^\lambda{}_{\lambda\nu})\Big] = \\ &= \int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + f'(R)R_{\mu\nu}\delta g^{\mu\nu} - g^{\mu\nu}\nabla_\rho f'(R)(\delta\Gamma^\rho{}_{\mu\nu} - \delta^\rho_\mu\delta\Gamma^\lambda{}_{\lambda\nu})\Big] = \\ &= \int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \\ &\quad -\frac{1}{2}g^{\mu\nu}\nabla^\rho f'(R)(2\nabla_\mu\delta g_{\rho\nu} - \nabla_\rho\delta g_{\mu\nu} - g_{\rho\mu}g^{\alpha\lambda}\nabla_\nu\delta g_{\alpha\lambda})\Big] = \end{aligned}$$

$$
\begin{aligned}
&= \int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \\
&\qquad - \nabla_\mu\nabla_\nu f'(R)\delta g^{\mu\nu} + \frac{1}{2}\Box f'(R)g_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\Box f'(R)g_{\mu\nu}\delta g^{\mu\nu}\Big] = \\
&= \int d^4x\sqrt{-g}\Big[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) - \nabla_\mu\nabla_\nu f'(R) + \Box f'(R)g_{\mu\nu}\Big]\delta g^{\mu\nu},
\end{aligned} \tag{1.41}
$$

therefore,

$$
-\frac{1}{2}g_{\mu\nu}f(R) + f'(R)R_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = 0. \tag{1.42}
$$

Adding the matter term and putting in evidence the Einstein tensor (adding the null quantity $1/2g_{\mu\nu}f'(R)R - 1/2g_{\mu\nu}f'(R)R$), these field equations take the form

$$
G_{\mu\nu} = \frac{T_{\mu\nu}}{f'(R)} + \frac{1}{f'(R)}\Big\{\frac{1}{2}g_{\mu\nu}[f(R) - f'(R)R] + \nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\Big\}. \tag{1.43}
$$

The second term in the second side of (1.43) can be interpreted as an extra gravitational energy-momentum tensor due to higher-order curvature effects.

Scalar-tensor theories and $f(R)$-theories are only two examples of GR extensions. This thesis does not aim to list all possible generalizations. However, what seems interesting is to analyze the possibility of combining these two approaches, considering both curvature functions and a non-minimal coupling with a scalar field. This possibility will be discussed later, both with the relation between non-metricity and the violation of EEP.

Chapter 2

Affine-Structure of Spacetime

The previous Chapter ends with the description of two examples of metric theories aiming to generalize GR. A different way to obtain interesting changes with respect to GR is to modify the affine connection of the manifold that describes spacetime. What happens when the connection is no longer the Levi-Civita connection?
The most general connection is neither metric-compatible nor symmetric. Spaces characterized by a general connections are broadly known as *non-Riemannian geometries*.

Several alternative theories are based on this geometrical modification of GR. Among the first attempts, there are Weyl's [62] and Cartan's [63]. Weyl's connection is symmetric but not metric-compatible, while Cartan's connection is metric-compatible but has an antisymmetric part. Subsequently, Einstein discovered what is today known as *Palatini's method* [64]. One considers metric tensor $g_{\mu\nu}$ and the torsionless affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ of a manifold to be independent; then the connection acts as a rank-3 gravitational tensor field and all the curvature invariants are defined through the connection and not through the metric. A relation among them may be found only after using the field equations. However, the matter part of the action does not depend on the connection.

A more general formulation of gravitation in such geometry is called Metric-Affine Gravity [33, 65, 66]. In the Metric-Affine general formulation, the metric tensor and the affine connection are treated as independent variables but both the gravity and matter sectors can depend on the affine connection. Hence, the additional contributions in the Metric-Affine theories come from torsion and non-metricity but also from the dependence of the Lagrangian of matter on the connection.

To better understand this point of view, one can start from the Palatini formalism by varying the Einstein-Hilbert action $S \sim \int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}$ with respect to metric tensor $g_{\mu\nu}$ and to the connection $\Gamma^{\lambda}{}_{\mu\nu}$. Then, the following equation are obtained:

$$R_{(\mu\nu)} - \frac{1}{2}g^{\mu\nu}R = 0 \tag{2.1}$$

$$\nabla_{\alpha}(\sqrt{g}g_{\mu\nu}) = 0 \tag{2.2}$$

where $R_{(\mu\nu)}$ is the symmetric part of $R_{\mu\nu}$ and ∇ denotes the covariant derivative with respect to ${\Gamma^\lambda}_{\mu\nu}$. The equation (2.2) constraints the connection ${\Gamma^\lambda}_{\mu\nu}$, which is *a priori* arbitrary, to coincide *a posteriori* with the Levi-Civita connection of the metric $g_{\mu\nu}$: $\Gamma^\lambda_{\mu\nu} = \{^\lambda{}_{\mu\nu}\}$. Notice that the fact that Γ is the Levi-Civita connection of g is no longer an assumption, but it is the outcome of the field equations.
However, this is only a coincidence due to the simple form of the action. Considering other Lagrangians, field equations in metric and Palatini formalisms are, in general, different. Thus, it is possible to obtain field equations equivalent to GR with a different connection.

The situation does not change if matter is present by means of a matter Lagrangian $\mathscr{L}_m$ (independent of Γ but just depending on $g_{\mu\nu}$ and other external matter fields), that generates the energy-momentum tensor $T_{\mu\nu}$. If the total Lagrangian is then assumed to be $\mathscr{L} \equiv \mathscr{L}_{PE} + \mathscr{L}_m$, field equations are replaced by

$$R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{2.3}$$

and again, the eq. (2.2) implies, *a posteriori*, that (2.3) reduces to the Einstein equations.

In the case of Matric-Affine Gravity, the situation changes radically when one allows matter to couple the connection. Here a new tensor needs to be defined

$${\Delta^\lambda}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta {\Gamma^\lambda}_{\mu\nu}}, \tag{2.4}$$

that is called *Hypermomentum-tensor* or energy-momentum tensor associated to ${\Gamma^\lambda}_{\mu\nu}$.

Therefore, GR is based on several assumptions and has some fundamental roots. Both field equations and action are covariant, this means that they do not depend on the coordinates choice. In addition, the only field that mediates gravity is the metric, which contains all the necessary information to describe the gravitational interactions. The connection is symmetric and metric-compatible. Finally, all the matter fields couple directly (and only) to the metric.
If someone changes one or more of the above assumptions, the theory changes completely. This is the reason why is interesting to explore the properties of a metric-affine spacetime, with particular attention to the consequences of a non-metric connection.

2.1 Metric-Affine Geometry

Spacetime is described by a 4-dimensional differentiable manifold M, the metric $g_{\mu\nu}$ (a symmetric rank-2 covariant tensor on M) and the affine connection ${\Gamma^\lambda}_{\mu\nu}$. The affine connection is related to the parallel transport of a tensor and therefore it defines the covariant derivative as follows

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\lambda} v^\lambda, \tag{2.5}$$
$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu} \omega_\lambda, \tag{2.6}$$
$$\nabla_\mu T^\nu{}_\rho = \partial_\mu T^\nu{}_\rho + \Gamma^\nu{}_{\mu\lambda} T^\lambda{}_\rho - \Gamma^\lambda{}_{\mu\rho} T^\nu{}_\lambda. \tag{2.7}$$

It is known from differential geometry [65] that a generic affine connection can be decomposed into the following three parts

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\} + K^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu}, \tag{2.8}$$

where the second term is the contorsion tensor, defined through the torsion tensor

$$K^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(T_{\mu\rho\nu} + T_{\nu\rho\mu} + T_{\rho\mu\nu}) \tag{2.9}$$
$$= -\frac{1}{2} g^{\lambda\rho}(T_{\mu\nu\rho} + T_{\nu\mu\rho} + T_{\rho\nu\mu}) = -K_{\nu\mu}{}^\lambda, \tag{2.10}$$

and the third term is the disformation tensor, defined through the non-metricity tensor as

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(-Q_{\mu\rho\nu} - Q_{\nu\rho\mu} + Q_{\rho\mu\nu}) = L^\lambda{}_{\nu\mu}, \tag{2.11}$$

The torsion and the non-metricity tensor are given by

$$T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}, \tag{2.12}$$
$$Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} g_{\mu\lambda}. \tag{2.13}$$

In addition, through this connection, one can define the Riemann tensor as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\sigma\nu} - \partial_\sigma \Gamma^\mu{}_{\rho\nu} + \Gamma^\mu{}_{\rho\lambda} \Gamma^\lambda{}_{\sigma\nu} - \Gamma^\mu{}_{\sigma\lambda} \Gamma^\lambda{}_{\rho\nu}, \tag{2.14}$$

The contraction of the first index with the third one of the Riemann tensor gives the Ricci tensor, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$, which is not symmetric in general; by contracting $R_{\mu\nu}$ with the (inverse of the) metric, one obtains the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$; the contraction of the first index with the second one of the Riemann tensor provides a new tensor named *homothetic curvature* $\check{\mathrm{R}}_{\mu\nu} = R^\lambda{}_{\lambda\mu\nu}$.

It is important to underline that curvature, torsion and non-metricity are all properties of the connection (which determines the free fall and not the casual-structure). Moreover, the connection can have an arbitrary number of these different geometric entities. To each of these objects a geometric interpretation can be given: the curvature tensor is associated to the variation of a vector when parallelly transported along a closed curve; the torsion tensor describes the *not-closure* of an infinitesimal parallelogram obtained by the parallel transport of one vector along the direction of another vector; the non-metricity tensor leads to the change of a vector's norm when it is parallelly transported.

Mathematically, the Riemann tensor and the torsion compete in the calculation of the covariant derivative components commutator acting on a tensor. For a function, a vector, a covector and, in particular, for the metric tensor, the action of the covariant derivative commutator gives

$$[\nabla_\mu, \nabla_\nu] f = -T^\lambda{}_{\mu\nu} \nabla_\lambda f, \tag{2.15}$$

$$[\nabla_\mu, \nabla_\nu] v^\rho = R^\rho{}_{\lambda\mu\nu} v^\lambda - T^\lambda{}_{\mu\nu} \nabla_\lambda v^\rho, \tag{2.16}$$

$$[\nabla_\mu, \nabla_\nu] \omega_\sigma = -R^\lambda{}_{\sigma\mu\nu} \omega_\lambda - T^\lambda{}_{\mu\nu} \nabla_\lambda \omega_\sigma, \tag{2.17}$$

$$[\nabla_\mu, \nabla_\nu] g_{\rho\sigma} = -R^\lambda{}_{\rho\mu\nu} g_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\nu} g_{\rho\lambda} - 2T^\lambda{}_{\mu\nu} \nabla_\lambda g_{\rho\sigma} = 2\nabla_{[\mu} Q_{\nu]\rho\sigma}. \tag{2.18}$$

It naturally follows that if the torsion and the non-metricity are "switched off", then the connection turns into the Levi-Civita one and the theory reduces to GR.
The remarkable fact is that, working individually with the properties of the connection, it is possible to formulate two other theories equivalent to GR. The three theories go all together under the name of *Geometrical Trinity of Gravity* [34]. In GR, gravitational effects are manifested through the curvature of the spacetime, characterized by the Levi-Civita connection which is symmetric and metric-compatible, completely determined by the metric tensor which represents the dynamical field. Setting the curvature to zero and choosing a metric-compatible but not symmetrical connection, one can obtain the *Teleparallel Equivalent to General Relativity* (TEGR) [35–37, 39, 41], wherein gravity is mediated by torsion on a flat spacetime and the dynamical field are the tetrads which, as will be shown in the next Chapter, can be seen as a "deeper" description of the metric. Choosing a symmetric connection with vanish curvature, one can obtain the *Symmetric Teleparallel Equivalent to General Relativity* (STEGR), wherein gravitational effect are associated to the non-metricity [37, 38, 67].

These two theories will be treated in the next Chapter, together with some of their extensions [68, 69]. However, in literature there are also more generic teleparallel theories with both torsion and non-metricity (*e.g.* [70]).

From this point onwards and <u>only in this Chapter</u>, the quantities relating to a generic ("*complete*") connection will be characterized by a "bar" (*e.g.* $\bar{A}$), excluding torsion and non-metricity tensors whereas it understood, while the traditional symbology will be used for Levi-Civita connection, *i.e.* $\Gamma^\lambda{}_{\mu\nu} \equiv \{^\lambda{}_{\mu\nu}\}$. The choice of this notation seemed to be the most convenient since all quantities will be expressed in terms of Levi-Civita connection.

2.1.1 Geodesic

Unlike what happens in GR, in a general Metric-Affine framework the auto-parallels curves are not the same as the extremal curves of functional length (1.3). Taking the definition

(1.14), in a local chart an affinelly parametrized auto-parallels curve is described by the following equation

$$\frac{d^2x^\mu}{d\lambda} + \Gamma^\mu{}_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} = -(K^\mu{}_{\rho\nu} + L^\mu{}_{\rho\nu})\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{2.19}$$

or, equivalently

$$\frac{d^2x^\mu}{d\lambda} + \Gamma^\mu{}_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} = -(T_\rho{}^\mu{}_\nu + L^\mu{}_{\rho\nu})\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda}. \tag{2.20}$$

For this reason, geodesics of space and auto-parallels curves are, in general, different. It is necessary to emphasize that the affine parameter does not coincide with the proper time due to the presence of non-metricity. Indeed, non-metricity does not allow (generally) the conservation of vector length. Moreover, it is no possible to raise up or to lower down indices of vector under covariant derivative. This because $g_{\mu\nu}$ is not covariantly conserved and it is necessary use the Leibniz rule to link derivatives of covariant components of a tensor with its contravariant components, *i.e.* $g_{\mu\nu}\bar{\nabla}_\alpha v^\nu = \bar{\nabla}_\alpha v_\mu - v^\nu \bar{\nabla}_\alpha g_{\mu\nu} \neq \bar{\nabla}_\alpha v_\mu$. However, this will be shown more comprehensively in the following sections.

2.2 Torsion Tensor

As already seen, the torsion tensor is defined by (3.20). It is possible to define an associated vector by indices contraction:

$$T_\mu = T^\lambda{}_{\mu\lambda}, \tag{2.21}$$

and, from this it is possible to see that

$$K^{\mu\lambda}{}_\lambda = T^\mu, \tag{2.22}$$

$$K^\lambda{}_{\lambda\mu} = -T_\mu, \tag{2.23}$$

$$K^\lambda{}_{\mu\lambda} = 0. \tag{2.24}$$

Moreover, a second pseudo-vector can be defined contracting the torsion tensor with the Levi-Civita tensor,

$$\tilde{T}^\mu = \epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}. \tag{2.25}$$

Torsion has a particular geometrical meaning: it represents the *not-closure* of an infinitesimal parallelogram obtained by parallel transportation of two vectors, one to the direction of the other and *vice versa*. Therefore a vector suffers the influence of the vortices produced by the torsion which causes the rotation of the vector with respect to the parallel transport of Levi-Civita connection.

2.3 Non-Metricity Tensor

Non-metricity is the failure of connection to covariantly conserve the metric. From its definition, $Q_{\alpha\mu\nu} = \bar{\nabla}_\alpha g_{\mu\nu}$ it is possible to define the expression of the non-metricity with last two indices raised,

$$Q_\alpha{}^{\rho\sigma} \equiv g^{\mu\rho} g^{\nu\sigma} \bar{\nabla}_\alpha g_{\mu\nu} = -\bar{\nabla}_\alpha g^{\rho\sigma}. \tag{2.26}$$

Having defined the non-metricity tensor, there exist two independent associated vectors:

$$Q_\alpha = g^{\mu\nu} Q_{\alpha\mu\nu} = Q_{\alpha\lambda}{}^{\lambda} = Q_\alpha{}^{\lambda}{}_{\lambda}, \tag{2.27}$$

$$\tilde{Q}_\nu = g^{\alpha\mu} Q_{\alpha\mu\nu} = Q^{\lambda}{}_{\lambda\nu} = \bar{\nabla}^\mu g_{\mu\nu}. \tag{2.28}$$

From the above considerations it is easy to obtain the following

$$L^{\lambda}{}_{\mu\lambda} = -\frac{1}{2} Q_\mu = L^{\lambda}{}_{\lambda\mu} = L^{\lambda}{}_{\lambda\mu}, \tag{2.29}$$

$$L^{\mu\lambda}{}_{\lambda} = -\tilde{Q}^\mu + \frac{1}{2} Q^\mu. \tag{2.30}$$

Finally, it follows the relation:

$$Q_\mu = g^{-1} \bar{\nabla}_\mu g, \tag{2.31}$$

where $g = det(g_{\mu\nu})$.

The presence of non-metricity has special consequences, starting from the impossibility to raise up or to lower down indices of a vector which is under covariant derivative.
As stated above, generally, the non-metricity does not preserve the length of a vector. To demonstrate such a propriety, let $\mathbf{v} = v^\mu \partial_\mu$ and $\mathbf{w} = w^\nu \partial_\nu$ be two vectors parallel transported along a curve γ with tangent vector $\mathbf{T} = T^\rho \partial_\rho$, *i.e.* $\bar{\nabla}_\mathbf{T} \mathbf{v} = 0$ and $\bar{\nabla}_\mathbf{T} \mathbf{w} = 0$; let consider their scalar product evolution along the curve:

$$T^\rho \bar{\nabla}_\rho (g_{\mu\nu} v^\mu w^\nu) = T^\rho \bar{\nabla}_\rho (\bar{\nabla}_\rho v^\mu) w_\mu + v^\mu T^\rho \bar{\nabla}(w^\mu) + T^\rho \bar{\nabla}(g_{\mu\nu}) v^\mu w^\nu = \tag{2.32}$$

$$= T^\rho \bar{\nabla}(g_{\mu\nu}) v^\mu w^\nu = Q_{\alpha\mu\nu} T^\alpha v^\mu w^\nu, \tag{2.33}$$

and if $\mathbf{w} = \mathbf{v}$, it follows

$$T^\rho \bar{\nabla}_\rho (g_{\mu\nu} v^\mu v^\nu) = Q_{\alpha\mu\nu} T^\alpha v^\mu v^\nu. \tag{2.34}$$

For this reason it is not possible to normalize the norm of a vector, and this causes the impossibility to define a proper time (and a 4-velocity) as in GR, because the length of a vector is not constant. Moreover, let γ be a generic curve with tangent vector $\mathbf{T}$, then it is possible to define two independent "*accelerations*" associated to $\mathbf{T}$, one for its contravariant components (as usual) and another one for its covariants components:

$$a^\mu = T^\rho \bar{\nabla}_\rho T^\mu, \tag{2.35}$$

$$\tilde{a}_\nu = T^\rho \bar{\nabla}_\rho T_\nu = T^\rho \bar{\nabla}_\rho (g_{\mu\nu} T^\mu) = a_\nu + Q_{\rho\mu\nu} T^\rho T^\mu. \tag{2.36}$$

Here $\tilde{a}$ will be called *anomalous acceleration*. As additional consequence, there is no longer perpendicularity between 4-velocity and 4-acceleration:

$$\begin{aligned} T_\mu a^\mu &= T_\mu T^\rho \bar{\nabla}_\rho T^\mu = \\ &= T^\rho \bar{\nabla}_\rho (T^\mu T_\mu) - (T^\rho \bar{\nabla}_\rho T_\mu) T^\mu = \\ &= T^\rho \bar{\nabla}_\rho (g_{\mu\nu} T^\mu T^\nu) - \tilde{a}_\mu T^\mu = \\ &= Q_{\rho\mu\nu} T^\rho T^\mu T^\nu + 2 T_\mu a^\mu - \tilde{a}_\mu T^\mu, \end{aligned} \tag{2.37}$$

hence

$$a^\mu T_\mu = \tilde{a}_\mu T^\mu - Q_{\rho\mu\nu} T^\rho T^\mu T^\nu, \tag{2.38}$$

or equivalently

$$(a^\mu - \tilde{a}^\mu) T^\mu = -Q_{\rho\mu\nu} T^\rho T^\mu T^\nu. \tag{2.39}$$

So, in a metric-affine space, an auto-parallels curve has an acceleration, too, the anomalous one:

$$a^\mu = T^\rho \bar{\nabla}_\rho T^\mu = 0, \tag{2.40}$$

$$\tilde{a}_\nu = T^\rho \bar{\nabla}_\rho T_\nu = T^\rho \bar{\nabla}_\rho (g_{\mu\nu} T^\mu) = Q_{\rho\mu\nu} T^\rho T^\mu. \tag{2.41}$$

Therefore, from this point o view it is worth remarking the two consequences which follow:

1. the length of a vector is not preserved, generally;
2. the auto-parallels curves have an anomalous acceleration.

This two "inconveniences" could be deleted asking for non-metricity:

1. $Q_{(\alpha\mu\nu)} = 0 \quad \Rightarrow \quad T^\rho \bar{\nabla}_\rho (g_{\mu\nu} T^\mu T^\nu) = Q_{\alpha\mu\nu} T^\alpha T^\mu T^\nu = Q_{(\alpha\mu\nu)} T^\alpha T^\mu T^\nu = 0;$
2. $Q_{(\alpha\mu)\nu} = 0 \quad \Rightarrow \quad \tilde{a}_\nu = Q_{\rho\mu\nu} T^\rho T^\mu = Q_{(\rho\mu)\nu} T^\rho T^\mu = 0.$

However, these conditions are too stringent constraints. They do not take into consideration any explicit form of the non-metricity tensor. In the final Chapter, a special non-metricity tensor will be considered and the consequences (1) and (2) will be cancelled by using a deeply different way.

2.4 Riemann Tensor and Curvature

It is easy to verify that under shift of the connection $\bar{\Gamma}^\lambda{}_{\mu\nu}$ by a tensor $N^\lambda{}_{\mu\nu}$, the Riemann tensor associated to the new connection $\hat{\Gamma}^\lambda{}_{\mu\nu} = \bar{\Gamma}^\lambda{}_{\mu\nu} + N^\lambda{}_{\mu\nu}$ transforms [65] as following:

$$\hat{R}^\alpha{}_{\beta\mu\nu} = \bar{R}^\alpha{}_{\beta\mu\nu} + \bar{T}^\lambda{}_{\mu\nu} N^\alpha{}_{\lambda\beta} + 2\bar{\nabla}_{[\mu|} N^\alpha{}_{|\nu]\beta} + 2 N^\alpha{}_{[\mu|\lambda} N^\lambda{}_{|\nu]\beta}. \tag{2.42}$$

In particular, using the relation (2.42) for $\bar{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + N^{\lambda}{}_{\mu\nu}$, where $N^{\lambda}{}_{\mu\nu} = K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}$, one obtains

$$\begin{aligned} \bar{R}^{\mu}{}_{\nu\rho\sigma} &= R^{\mu}{}_{\nu\rho\sigma} + 2\nabla_{[\rho|}N^{\mu}{}_{|\sigma]\nu} + 2N^{\mu}{}_{[\rho|\lambda}N^{\lambda}{}_{|\sigma]\nu} = \\ &= R^{\mu}{}_{\nu\rho\sigma} + K^{\mu}{}_{\nu\rho\sigma} + L^{\mu}{}_{\nu\rho\sigma} + I^{\mu}{}_{\nu\rho\sigma}, \end{aligned} \tag{2.43}$$

where

$$R^{\mu}{}_{\nu\rho\sigma} = 2\nabla_{[\rho}\Gamma^{\mu}{}_{|\sigma]\nu} + 2\Gamma^{\mu}{}_{[\rho|\lambda}\Gamma^{\lambda}{}_{|\sigma]\nu} \tag{2.44}$$

$$K^{\mu}{}_{\nu\rho\sigma} = 2\nabla_{[\rho}K^{\mu}{}_{|\sigma]\nu} + 2K^{\mu}{}_{[\rho|\lambda}K^{\lambda}{}_{|\sigma]\nu} \tag{2.45}$$

$$L^{\mu}{}_{\nu\rho\sigma} = 2\nabla_{[\rho}L^{\mu}{}_{|\sigma]\nu} + 2L^{\mu}{}_{[\rho|\lambda}L^{\lambda}{}_{|\sigma]\nu} \tag{2.46}$$

$$I^{\mu}{}_{\nu\rho\sigma} = 2K^{\mu}{}_{[\rho|\lambda}L^{\lambda}{}_{|\sigma]\nu} + 2L^{\mu}{}_{[\rho|\lambda}K^{\lambda}{}_{|\sigma]\nu}, \tag{2.47}$$

and it is easily verifiable that $K^{\mu\nu}{}_{\rho\sigma}$ is antisymmetric in the first two indices and in the second ones.

From (2.43) it is possible to get the following quantities:

- the Ricci tensor, contracting the first and the third indices,

$$\bar{R}_{\mu\nu} = \bar{R}^{\lambda}{}_{\mu\lambda\nu} = R_{\mu\nu} + K_{\mu\nu} + L_{\mu\nu} + I_{\mu\nu}, \tag{2.48}$$

  where

$$R_{\mu\nu} = 2\nabla_{[\lambda}\Gamma^{\lambda}{}_{|\nu]\mu} + 2\Gamma^{\lambda}{}_{[\lambda|\alpha}\Gamma^{\alpha}{}_{|\nu]\mu} \tag{2.49}$$

$$\begin{aligned} K_{\mu\nu} &= 2\nabla_{[\lambda}K^{\lambda}{}_{|\nu]\mu} + 2K^{\lambda}{}_{[\lambda|\alpha}K^{\alpha}{}_{|\nu]\mu} = \\ &= \nabla_{\lambda}K^{\lambda}{}_{\nu\mu} + \nabla_{\nu}T_{\mu} - T^{\alpha}K_{\alpha\nu\mu} - \frac{1}{4}T_{\mu}{}^{\alpha}{}_{\lambda}(T_{\nu}{}^{\lambda}{}_{\alpha} + 2T_{\alpha}{}^{\lambda}{}_{\nu}) \end{aligned} \tag{2.50}$$

$$\begin{aligned} L_{\mu\nu} &= 2\nabla_{[\lambda}L^{\lambda}{}_{|\nu]\mu} + 2L^{\lambda}{}_{[\lambda|\alpha}L^{\alpha}{}_{|\nu]\mu} \\ &= \nabla_{\lambda}L^{\lambda}{}_{\nu\mu} + \frac{1}{2}\nabla_{\nu}Q_{\mu} - \frac{1}{2}Q_{\alpha}L^{\alpha}{}_{\nu\mu} - \frac{1}{4}(Q_{\nu}{}^{\lambda}{}_{\alpha}Q_{\mu\lambda}{}^{\alpha} + 4Q^{\lambda\alpha}{}_{\nu}Q_{[\alpha\lambda]\mu}) \end{aligned} \tag{2.51}$$

$$\begin{aligned} I_{\mu\nu} &= 2K^{\lambda}{}_{[\lambda|\alpha}L^{\alpha}{}_{|\nu]\mu} + 2L^{\lambda}{}_{[\lambda|\alpha}K^{\alpha}{}_{|\nu]\mu} \\ &= -T_{\alpha}L^{\alpha}{}_{\nu\mu} - K^{\lambda}{}_{\nu\alpha}L^{\alpha}{}_{\lambda\mu} - \frac{1}{2}Q_{\alpha}K^{\alpha}{}_{\nu\mu} - L^{\lambda}{}_{\nu\alpha}K^{\alpha}{}_{\lambda\mu}; \end{aligned} \tag{2.52}$$

- the homothetic curvature is achieved contracting the first two indices,

$$\check{R}_{\mu\nu} = \bar{R}^{\lambda}{}_{\lambda\mu\nu} = 2\bar{\nabla}_{[\mu}\bar{\Gamma}^{\lambda}{}_{\nu]\lambda} = 2\partial_{[\mu}L^{\lambda}{}_{\nu]\lambda} = -\partial_{[\mu}Q_{\nu]}, \tag{2.53}$$

or, in analogous way, using the relation (2.18),

$$
\begin{aligned}
\check{R}_{\rho\sigma} = g^{\mu\nu}\bar{R}_{(\mu\nu)\rho\sigma} &= g^{\mu\nu}(-\bar{\nabla}_{[\rho}Q_{\sigma]\mu\nu} - \frac{1}{2}T^{\lambda}{}_{\rho\sigma}Q_{\lambda\mu\nu}) = \\
&= -\bar{\nabla}_{[\rho}Q_{\sigma]} + Q_{[\sigma|\mu\nu}\bar{\nabla}_{|\rho]}g^{\mu\nu} - \frac{1}{2}T^{\lambda}{}_{\rho\sigma}Q_{\lambda} = \\
&= -\bar{\nabla}_{[\rho}Q_{\sigma]} - \cancel{Q_{[\sigma|\mu\nu}Q^{\mu\nu}{}_{|\rho]}} - \frac{1}{2}T^{\lambda}{}_{\rho\sigma}Q_{\lambda} = \\
&= -\nabla_{[\rho}Q_{\sigma]} + \frac{1}{2}T^{\lambda}{}_{\rho\sigma}Q_{\lambda} - \frac{1}{2}T^{\lambda}{}_{\rho\sigma}Q_{\lambda} = \\
&= -\nabla_{[\rho}Q_{\sigma]} = -\partial_{[\rho}Q_{\sigma]}
\end{aligned}
\tag{2.54}
$$

Therefore a generic connection leads to a new symmetric contribution in the first two indices of Riemann tensor, characterized by the only symmetric object, the non-metricity;

- the scalar curvature takes the form,

$$
\bar{R} = g^{\mu\nu}\bar{R}_{\mu\nu} = R + K + L + I, \tag{2.55}
$$

where

$$
\begin{aligned}
K &= 2\nabla_{\lambda}T^{\lambda} - T_{\lambda}T^{\lambda} + K_{\lambda\mu\nu}K^{\mu\lambda\nu} \\
&= 2\nabla_{\lambda}T^{\lambda} - T_{\lambda}T^{\lambda} + \frac{1}{4}T_{\mu\nu\lambda}(T^{\mu\nu\lambda} + 2T^{\lambda\nu\mu})
\end{aligned}
\tag{2.56}
$$

$$
L = \nabla_{\lambda}(Q^{\lambda} - \tilde{Q}^{\lambda}) - \frac{1}{4}Q_{\lambda}Q^{\lambda} + \frac{1}{2}Q_{\lambda}\tilde{Q}^{\lambda} + \frac{1}{4}Q_{\mu\lambda\alpha}Q^{\mu\lambda\alpha} - \frac{1}{2}Q_{\mu\lambda\alpha}Q^{\lambda\mu\alpha} \tag{2.57}
$$

$$
I = T_{\lambda}(\tilde{Q}^{\lambda} - Q^{\lambda}) + L_{\alpha\lambda\mu}K^{\alpha\lambda\mu}. \tag{2.58}
$$

2.4.1 Generalized Bianchi Identities

In this Subsection we set out to discuss the generalized Bianchi identities for a generic connection, *i.e.* torsionfull and nonmetric-compatible. These identities are also known as *Weitzenböck identities* [33]. By fully anti-symmetrizing the Riemann tensor in its three lower indices, it is possible to obtain the first identity,

$$
\begin{aligned}
\bar{R}^{\mu}{}_{[\nu\rho\sigma]} &= \frac{2}{3!}(\bar{R}^{\mu}{}_{\nu[\rho\sigma]} + \bar{R}^{\mu}{}_{\sigma[\nu\rho]} + \bar{R}^{\mu}{}_{\rho[\sigma\nu]}) = \\
&= \frac{2}{3!}(\bar{R}^{\mu}{}_{\nu\rho\sigma} + \bar{R}^{\mu}{}_{\sigma\nu\rho} + \bar{R}^{\mu}{}_{\rho\sigma\nu}) = \\
&= \bar{\nabla}_{[\nu}T^{\mu}{}_{\rho\sigma]} - T^{\lambda}{}_{[\nu\rho}T^{\mu}{}_{\sigma]\lambda}.
\end{aligned}
\tag{2.59}
$$

The second identity is achieved as

$$
\bar{\nabla}_{[\lambda|}\bar{R}^{\mu}{}_{\nu|\rho\sigma]} = -\bar{R}^{\mu}{}_{\nu\alpha[\lambda}T^{\alpha}{}_{\rho\sigma]}. \tag{2.60}
$$

From eq. (2.60), two other equations can be derived:
by contracting the indices μ and ν, the homothetic curvature satisfies the equation

$$\bar{\nabla}_{[\lambda}\check{R}_{\rho\sigma]} = -2\check{R}_{\alpha[\lambda}T^{\alpha}{}_{\rho\sigma]}, \tag{2.61}$$

while, by contracting μ and λ, it follows

$$\bar{\nabla}_{\lambda}\bar{R}^{\lambda}{}_{\nu\rho\sigma} - 2\bar{\nabla}_{[\rho|}\bar{R}_{\nu|\sigma]} = \bar{R}_{\nu\alpha}T^{\alpha}{}_{\rho\sigma} + 2\bar{R}^{\lambda}{}_{\nu\alpha[\rho}T^{\alpha}{}_{\sigma]\lambda}. \tag{2.62}$$

If one considers a torsionless and nonmetric-compatible connection, it results:

$$\bar{R}^{\mu}{}_{[\nu\rho\sigma]} = 0 \quad \Rightarrow \quad R^{\mu}{}_{[\nu\rho\sigma]} = 0,\ L^{\mu}{}_{[\nu\rho\sigma]} = 0 \tag{2.63}$$

$$\bar{\nabla}_{[\lambda}\check{R}_{\rho\sigma]} = 0 \quad \Rightarrow \quad \partial_{[\lambda}\check{R}_{\rho\sigma]} = 0 \tag{2.64}$$

2.5 Lie Derivative

It is worth including in this framework the Lie derivative: to change the connection of space means to change the relationship between the Lie derivative and the covariant derivative, which allows a practical local formulation.

The Lie derivative is the first method to derivate a tensorial object which is used on a generic manifold where a locally defined vectorial field is present. It maps a tensorial quantity, of l-covariant and m-contravariant type, in an another one of the same type [61]. The defining properties of Lie derivative guarantee its uniqueness, unlike the covariant derivative. There are two equivalent definitions of the Lie derivative: the first one is based on the properties that the Lie derivative must have, like linearity, Leibniz rule, action on vector as commutator, action on a function and commutation with respect to contraction of indices; the second one uses the induced maps by a diffeomorphism on the manifold on the tangent and co-tangent space, respectively push-forward and pull-back.

Let us now outline the properties of the Lie derivative in a local map. Let f be a diffeomorphism, $X = X^{\mu}\partial_{\mu}$, $Y = Y^{\mu}\partial_{\mu}$ vectorial fields and $\omega = \omega_{\mu}dx^{\mu}$ an 1-form, each locally defined on the manifold. It follows that

$$L_X(f) = X(f) = X^{\mu}\partial_{\mu}f, \tag{2.65}$$

$$L_X(Y) = [X, Y] = (X^{\nu}\partial_{\nu}Y^{\mu} - Y^{\nu}\partial_{\nu}X^{\mu})\partial_{\mu}, \tag{2.66}$$

$$L_X(\omega) = (X^{\nu}\partial_{\nu}\omega_{\mu} + \omega_{\nu}\partial_{\mu}X^{\nu})dx^{\mu}, \tag{2.67}$$

$$L_X(g) = (X^{\rho}\partial_{\rho}g_{\mu\nu} + g_{\rho\nu}\partial_{\mu}X^{\rho} + g_{\mu\rho}\partial_{\nu}X^{\rho})dx^{\mu}dx^{\nu}. \tag{2.68}$$

In GR, wherein the connection is symmetric and metric-compatible, eq.s (2.65), (2.66), (2.67) and (2.68) take the form

$$L_X(Y) = (X^\nu\nabla_\nu Y^\mu - Y^\nu\nabla_\nu X^\mu)\partial_\mu, \tag{2.69}$$

$$L_X(\omega) = (X^\nu\nabla_\nu\omega_\mu + \omega_\nu\nabla_\mu X^\nu)dx^\mu, \tag{2.70}$$

$$\begin{aligned} L_X(g) &= (X^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\mu\rho}\partial_\nu X^\rho)dx^\mu dx^\mu = \\ &= (X^\rho\partial_\rho g_{\mu\nu} + \partial_\mu X_\nu - X^\rho\partial_\mu g_{\rho\nu} + \partial_\nu X_\mu - X^\rho\partial_\nu g_{\rho\mu})dx^\mu dx^\nu = \\ &= (\nabla_\mu X_\nu + \nabla_\nu X_\mu)dx^\mu dx^\nu. \end{aligned} \tag{2.71}$$

In GR the Lie derivative can be interpreted, *a posteriori*, as the directional derivative along X of a tenso,r when the local chart $\{x_\mu\}$ is "*adapted*" to the field X, *i.e.* for some $\mu = \bar{\mu}$, $X = \partial_{\bar{\mu}}$. Moreover, it is possible to show that there exists a deep relation with one-parameter transformation groups: if a one-parameter transformation group is a symmetry for a tensor, then the Lie derivative with respect to the infinitesimal generator of the transformations is null (see [61] for a detailed explanation). This allows to express or to recognize a symmetry using the local form of the Lie derivative. For these reasons it can be useful to know how eq.s (2.69), (2.70) and (2.71) change in a non-Riemannian space:

$$L_X(Y) = (X^\rho\bar{\nabla}_\rho Y^\mu - Y^\sigma\bar{\nabla}_\sigma X^\mu - T^\mu{}_{\rho\sigma}X^\rho Y^\sigma)\partial_\mu, \tag{2.72}$$

$$L_X(\omega) = (X^\nu\bar{\nabla}_\nu\omega_\mu + \omega_\nu\bar{\nabla}_\mu X^\nu - T^\lambda{}_{\mu\nu}X^\nu\omega_\lambda)dx^\mu, \tag{2.73}$$

$$\begin{aligned} L_X(g) &= [\bar{\nabla}_\mu X_\nu + \bar{\nabla}_\nu X_\mu + 2(K^\lambda{}_{(\mu\nu)} + L^\lambda{}_{\mu\nu})X_\lambda]dx^\mu dx^\nu = \\ &= \left[\bar{\nabla}_\mu X_\nu + \bar{\nabla}_\nu X_\mu + 2\left(T_\mu{}^\lambda{}_\nu - Q_\mu{}^\lambda{}_\nu + \frac{1}{2}Q^\lambda{}_{\mu\nu}\right)X_\lambda\right]dx^\mu dx^\nu. \end{aligned} \tag{2.74}$$

Chapter 3

The Geometrical Trinity of Gravity

As stated in the previous Chapters, EEP enables a geometric formulation of gravity. This means that gravitational effects are geometric properties of spacetime and are independent of the internal structure of matter.
GR is the most elegant and simple geometric formulation of gravity which is mediated by the curvature, entirely determined by the metric tensor. Actually, gravity can be equivalently described by a "pure torsion space", giving rise to the *Teleparallel Equivalent to General Relativity* (TEGR), or by a "pure non-metricity space", giving rise to the *Symmetric Teleparallel Equivalent to General Relativity* (STEGR). These three formulations of gravity go under the name of *Geometrical Trinity of Gravity* [34].
In the next sections we stress out the main aspects of these two theories.

3.1 Teleparallel Gravity

Using the tetrad formalism [41, 61, 71], it is possible to address gravity exclusively to the torsion, considering a metrical flat space-time with a metric-compatible connection. This connection is named *Weitzenböck*. The term *Teleparallel* is related to the condition of flatness of the space, or *teleparallel condition*. In this theory tetrads are the dynamical fields and, their "field strength" is the torsion. Moreover, this treatment of gravity can be seen as a gauge theory [72] for the translation group. The geometrical framework is the *principal bundle* [61] associated to the *tangent bundle* of spacetime, with translation group $\mathcal{T}^{(1,3)}$ as structure group. While, $SO(1, 3)$ transformations are used to evidence the inertial effects and to work with general Lorentz frames [39–41].

The action of this theory must be built from torsion invariant quadratic forms. The most general action of this type is

$$S_{\mathbb{T}} = \frac{c^4}{16\pi G_N}\int d^4x(\sqrt{-g}\mathbb{T} + \lambda_\mu{}^{\nu\rho\sigma}\bar{R}^\mu{}_{\nu\rho\sigma} + \tilde{\lambda}^{\alpha\mu\nu}Q_{\alpha\mu\nu}), \tag{3.1}$$

where the last two terms are Lagrange multipliers, while $\mathbb{T}$ is the generic torsion scalar defined as

$$\mathbb{T} = -c_1 T_\alpha T^\alpha + c_2 \frac{1}{4} T_{\mu\nu\alpha} T^{\mu\nu\alpha} + c_3 \frac{1}{2} T_{\mu\nu\alpha} T^{\alpha\nu\mu}. \tag{3.2}$$

Notice that the Lagrange multipliers have the obvious symmetries ${\lambda_\mu}^{\nu\rho\sigma} = {\lambda_\mu}^{\nu[\rho\sigma]}$, $\tilde{\lambda}^{\alpha\mu\nu} = \tilde{\lambda}^{\alpha(\mu\nu)}$, and are defined as tensorial densities of weight -1 for convenience.

According to (2.55) and (2.56), if $c_1{=}c_2{=}c_3 = 1$, eq. (3.1) is equivalent to the E-H action (up to a total derivative term) when imposing, from the outset, flatness and symmetry of the connection:

$$\begin{aligned} 0 = \bar{R} = R + K \Longrightarrow R = -K = T_\lambda T^\lambda - \frac{1}{4} T_{\mu\nu\lambda} T^{\mu\nu\lambda} - \frac{1}{2} T_{\mu\nu\lambda} T^{\lambda\nu\mu} - 2\nabla_\lambda T^\lambda = \\ = -\mathbb{T}(c_1 = 1, c_2 = 1, c_3 = 1) - 2\nabla_\alpha T^\alpha \\ = -\mathring{\mathbb{T}} - 2\nabla_\alpha T^\alpha, \end{aligned} \tag{3.3}$$

where $\mathring{\mathbb{T}} \equiv \mathbb{T}(c_1 = 1, c_2 = 1, c_3 = 1)$.
Thus, varying the action with respect to tetrad one gets the same equations as the Einstein theory, proving that the two approaches are equivalent. This can be a problem because, having the same equations of motion, the two theories cannot be experimentally distinguished. However, this "degeneration" of the field equations could be solved by the presence of some little violations of EEP due, for example, to the coupling between torsion and spin particles.

In what follows, useful tools and further developments of the theory will be discussed.

3.1.1 Tetrad Formalism

To better understand the equivalence between GR and TEGR, it is necessary to introduce the tetrad formalism for a generic Riemannian manifold. In order to do this, useful relations will be listed for future calculations (specifically in case of a flat and torsionfull space).

Let M be the differential manifold representing the space-time; tetrad, or vierbein (vielbeins, if it is many dimensional), are the set of coefficients $\{{e^a}_\mu\} \in GL(4, \mathbb{R})$ associated to a non-coordinate basis $\{\hat{e}_a\} \in TM$ and to relative dual basis $\{\hat{\theta}^a\} \in T^*M$, defined as:

$$\hat{e}_a(x) = {e_a}^\mu(x)\partial_\mu, \tag{3.4}$$

$$\hat{\theta}^a(x) = {e^a}_\mu(x) dx^\mu, \tag{3.5}$$

$$g\big(\hat{e}_a(x), \hat{e}_b(x)\big) = {e_a}^\mu(x) {e_b}^\nu(x) g_{\mu\nu}(x) = \eta_{ab}, \tag{3.6}$$

$$g_{\mu\nu}(x) = {e^a}_\mu(x) {e^b}_\nu(x) \eta_{ab}, \tag{3.7}$$

from which the following relation can be obtained,

$$g = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}\hat{\theta}^a(x)\hat{\theta}^b(x), \tag{3.8}$$
$$e^a{}_\mu(x) = g_{\mu\nu}(x)\eta^{ab}e_b{}^\nu(x), \tag{3.9}$$
$$e^a{}_\mu(x)e_a{}^\nu(x) = \delta^\nu_\mu, \tag{3.10}$$
$$e^a{}_\mu(x)e_b{}^\mu(x) = \delta^a_b. \tag{3.11}$$

In this way, the point-like dependence of metric is absorbed into $\{e^a{}_\mu(x)\}$ - from now on, the point-like dependence will no longer be made explicitly.
In the used notation, the Latin indices are relate to the local Lorentz space-time (or local laboratory) coordinates, while the Greek ones to the general space-time coordinate.
The non-coordinate basis has a non-vanishing Lie bracket whose structure constants $f_{ab}{}^c$ read and depend on the point as:

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c\hat{e}_c, \tag{3.12}$$

where

$$f_{ab}{}^c = e^c{}_\nu(e_a{}^\mu\partial_\mu e_b{}^\nu - e_b{}^\mu\partial_\mu e_a{}^\nu) = -e_a{}^\mu e_b{}^\nu(\partial_\mu e^c{}_\nu - \partial_\nu e^c{}_\mu). \tag{3.13}$$

The above equation can be obtained after replacing the quantity $\{\hat{e}_a\}$ with its definition, $\hat{e}_a = e_a{}^\mu\partial_\mu$.
Moreover, it is possible to define the connection coefficient with respect to the non-coordinate basis as following

$$\nabla_a\hat{e}_b = \Gamma^c{}_{ab}\hat{e}_c, \tag{3.14}$$

where

$$\Gamma^c{}_{ab} = e^c{}_\nu e_a{}^\mu(\partial_\mu e_b{}^\nu + \Gamma^\nu{}_{\mu\lambda}e_b{}^\lambda) = e^c{}_\nu e_a{}^\mu\nabla_\mu e_b{}^\nu. \tag{3.15}$$

In this basis the components of torsion and Riemann tensor are

$$T^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} - f_{bc}{}^a \tag{3.16}$$
$$R^a{}_{bcd} = \hat{e}_c(\Gamma^a{}_{db}) - \hat{e}_d(\Gamma^a{}_{cb}) + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - f_{cd}{}^e\Gamma^a{}_{eb}. \tag{3.17}$$

This formalism presents the possibility to introduce a matrix-valued 1-form $\{\omega^a{}_b\}$ called *connection 1-form*,

$$\omega^a{}_b \equiv \Gamma^a{}_{cb}\hat{\theta}^c \tag{3.18}$$
$$= \omega^a{}_{b\mu}dx^\mu, \text{ with } \omega^a{}_{b\mu} = \Gamma^a{}_{cb}e^c{}_\mu. \tag{3.19}$$

The peculiarity of $\omega^a{}_b$ is that it allows to express the curvature and the torsion by what are called *Cartan's structure equations*[1] [61],

$$T^a \equiv \frac{1}{2}T^a{}_{bc}\hat{\theta}^b \wedge \hat{\theta}^c = d\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b \tag{3.20}$$

$$R^a{}_b \equiv \frac{1}{2}R^a{}_{bcd}\hat{\theta}^c \wedge \hat{\theta}^d = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{3.21}$$

[1] *Maurer-Cartan structure equations* are defined in Lie Groups framework.
A group G is a set of elements $\{g\}$, closed with respect to two binary operations, which are called *product*

where d is the usual exterior derivative and $\wedge$ is the anti-symmetrized product[2], therefore:

$$d\hat{\theta}^a = d(e^a{}_\nu dx^\nu) = \partial_\mu e^a{}_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu) dx^\mu \wedge dx^\nu \tag{3.22}$$

$$d\omega^a{}_b = d(e^a{}_\nu dx^\nu) = \partial_\mu d\omega^a{}_{b\nu} dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu d\omega^a{}_{b\nu} - \partial_\nu d\omega^a{}_{b\mu}) dx^\mu \wedge dx^\nu. \tag{3.23}$$

Moreover, the eq.s (3.20, 3.21) provide the Bianchi identities for RG in a easier way. Indeed, taking their exterior derivative, result

$$dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge \hat{\theta}^b \tag{3.24}$$

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0. \tag{3.25}$$

Then, setting $T^a = 0$, it follows

$$R^a{}_b \wedge \hat{\theta}^b = 0 \quad \Rightarrow \quad R^a{}_{bcd} \hat{\theta}^b \wedge \hat{\theta}^c \wedge \hat{\theta}^d = 0 \tag{3.26}$$

$$dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \quad \Rightarrow \quad DR^a{}_b = 0, \tag{3.27}$$

where $D = d + \omega$ is a new exterior derivative such that increases the order of a form according to the rule fixed by the equation (3.27). Therefore, the exterior derivative D allows to express T^a in a more compact form,

$$T^a = D\hat{\theta}^a. \tag{3.28}$$

Computing the eq. (3.20) and eq. (3.21), torsion and curvature have the following "mixed-components":

$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu, \tag{3.29}$$

$$R^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu} \omega^c{}_{b\nu} - \omega^a{}_{c\nu} \omega^c{}_{b\mu}. \tag{3.30}$$

$P(\cdot,\cdot)$ and *inversion* $I(\cdot,\cdot)$, and it satisfies three axiom: 1. associativity of the product; 2. existence of identity e; 3. $P(g, I(g)) = e = P(I(g), g)$, $\forall g \in G$.
A group G is a Lie group if it is possible to give a parametrization of G by using an homeomorphism $\phi : G \ni g \to \phi(g) = (a_1, ..., a_n)$, where $n \equiv \dim G$, which defines a differential manifold structure, and if the composition maps, P and I, are differentiable maps. Therefore, every Lie group is a differentiable manifold.
On a group manifold, it is possible to define a left action $L_a : G \ni g \to ag$ with $a \in G$. To the left action is associated its push-forward on the group manifold. Then, a vector field X is called *left-invariant* if $L_{a*}(X|_g) = X|_{ag}$. The set of all left-invariant field on a group manifold is denoted by $L(G)$. It is possible to show that $L(G)$ equipped with a Lie bracket $[\cdot,\cdot]$ defines a finite-dimensional Lie algebra. Moreover, $L(G)$ is isomorphic to the tangent space of G in e, T_eG. $\{T_eG, [\cdot,\cdot]\}$ is called *Lie algebra of G*.
The Lie algebra of G is characterized by its structure constants $c_{\mu\nu}{}^\lambda : [X_\mu, X_\nu] = c_{\mu\nu}{}^\lambda X_\lambda$. The dual basis $\{\theta^\mu\}$ associated to a basis of left-invariant field $\{X_\mu\}$ satisfies the Maurer-Cartan structure equation: $d\theta^\lambda = -\frac{1}{2} c_{\mu\nu}{}^\lambda \theta^\mu \wedge \theta^\nu$.

[2]Let $\omega = \omega_\nu dx^\nu$ be a 1-form on M, $\omega \in T^*M$. The exterior derivative acts in the following way: $d\omega = \partial_\mu \omega_\nu dx^\mu \wedge dx^\nu$, where $dx^\mu \wedge dx^\nu = \frac{1}{2}(dx^\mu dx^\nu - dx^\nu dx^\mu)$.

Therefore, the relations between the connection 1-form and the affine connection are

$$\Gamma^{\lambda}{}_{\mu\nu} = e_a{}^{\lambda} e^b{}_{\nu} \omega^a{}_{b\mu} + e_c{}^{\lambda} \partial_\mu e^c{}_{\nu}, \tag{3.31}$$
$$\omega^a{}_{b\mu} = e^a{}_{\lambda} e_b{}^{\nu} \Gamma^{\lambda}{}_{\mu\nu} + e^a{}_{\rho} \partial_\mu e_b{}^{\rho}. \tag{3.32}$$

Often, these relations are summarized with the *tetrad postulate*, $\nabla_\mu e^a{}_\nu = 0$:

$$\nabla_\mu e^a{}_\nu \equiv \partial_\mu e^a{}_\nu - \Gamma^{\lambda}{}_{\mu\nu} e^a{}_{\lambda} + \omega^a{}_{b\mu} e^b{}_{\nu} = 0. \tag{3.33}$$

In addition to the above, the introduction of the connection 1-form grants to write a covariant derivative with respect to the coordinate basis of non-coordinate tensors components in a easy way; namely

$$\nabla_\mu v^a = \partial_\mu v^a + \omega^a{}_{b\mu} v^b, \tag{3.34}$$
$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega^a{}_{c\mu} X^c{}_b - \omega^c{}_{b\mu} X^a{}_c, \tag{3.35}$$
$$\nabla_\mu \eta_{ab} = -\omega^c{}_{a\mu} \eta_{cb} - \omega^c{}_{b\mu} \eta_{ac} = -\omega_{ba\mu} - \omega_{ab\mu}. \tag{3.36}$$

The last equation is particularly interesting; if $\Gamma^{\lambda}{}_{\mu\nu}$ is a metric-compatible connection then the associated connection 1-form take the name of *Spin connection* (or *Lorentz connection*), defined by the propriety

$$\omega_{ab} = -\omega_{ba}. \tag{3.37}$$

In particular, ω_{ab} can be seen as a 1-form with values in the Lie algebra of Lorentz group,

$$\omega_\mu = \frac{1}{2} \omega_{ab\mu} S^{ab} \tag{3.38}$$

where S_{ab} is a given representation of the Lorentz generators, and it is used, for example, to describe spinors in a curved space-time.
The validity of the eq. (3.37) follows from the metricity condition:

$$\begin{aligned} 0 = \nabla_\mu g_{\rho\sigma} &= \partial_\mu g_{\rho\sigma} - \Gamma^{\lambda}{}_{\mu\rho} g_{\lambda\sigma} - \Gamma^{\lambda}{}_{\mu\sigma} g_{\lambda\rho} = \\ &= \partial_\mu(\eta_{ab} e^a{}_\rho e^b{}_\sigma) - (e^b{}_\rho \omega^a{}_{b\mu} + \partial_\mu e^a{}_\rho)\eta_{ac} e^c{}_\sigma - (e^b{}_\sigma \omega^a{}_{b\mu} + \partial_\mu e^a{}_\sigma)\eta_{ac} e^c{}_\rho = \\ &= -(\omega_{ab\mu} + \omega_{ba\mu}) e^a{}_\rho e^b{}_\sigma. \end{aligned} \tag{3.39}$$

Moreover, by means of the following relations:

$$\omega^c{}_{ab} \equiv \omega^c{}_{a\mu} e_b{}^\mu = \Gamma^c{}_{da} e^d{}_\mu e_b{}^\mu = \Gamma^c{}_{da} \delta^d_b = \Gamma^c{}_{ba}, \tag{3.40}$$
$$T^c{}_{ab} = \omega^c{}_{ba} - \omega^c{}_{ab} - f_{ab}{}^c \rightarrow \omega^c{}_{ba} - \omega^c{}_{ab} = T^c{}_{ab} + f_{ab}{}^c, \tag{3.41}$$

the last equation for three different combinations of indices, gives

$$\frac{1}{2}(f_a{}^c{}_b + f_b{}^c{}_a + f_{ab}{}^c) = \omega^c{}_{ba} - K^c{}_{ab} = \Gamma^c{}_{ab} - K^c{}_{ab}, \tag{3.42}$$

where $K^c{}_{ab}$ is the contortion tensor associated to $T^c{}_{ab}$.

If the torsion-free condition is satisfied, the first Cartan's equation becomes

$$d\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0, \tag{3.43}$$

and the anholonomy coefficients are the only antisymmetric part of the connection in the non-coordinate basis

$$f_{ab}{}^c \hat{e}_c = [\hat{e}_a, \hat{e}_b] = \nabla_a \hat{e}_b - \nabla_b \hat{e}_a, \tag{3.44}$$
$$f_{ab}{}^c = \Gamma^c{}_{ab} - \Gamma^c{}_{ba} = -(\mathring{\omega}^c{}_{ab} - \mathring{\omega}^c{}_{ba}), \tag{3.45}$$
$$\mathring{\omega}^c{}_{ab} = \frac{1}{2}(f_a{}^c{}_b + f_b{}^c{}_a + f_{ab}{}^c), \tag{3.46}$$

where $\mathring{\omega}^c{}_{ab}$ is the spin connection of General Relativity. Then the Riemann tensor components are

$$R^a{}_{bcd} = \hat{e}_c(\Gamma^a{}_{db}) - \hat{e}_d(\Gamma^a{}_{cb}) + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - (\Gamma^e{}_{cd} - \Gamma^e{}_{dc})\Gamma^a{}_{eb}. \tag{3.47}$$

3.1.1.1 Local Lorentz Transformation

The degrees of freedom of tetrad $e^a{}_\mu$ are $n^2 = 16$ in a 4-dimensional Lorentzian manifold, while $g_{\mu\nu}$ has $n(n+1)/2 = 10$ degrees of freedom. This is not surprising because there are many non-coordinate bases which correspond to the same metric, each of which is related to the other by the local pseudo-orthogonal transformation $\Lambda(x) \in SO(1,3)$:

$$\hat{e}_a \longrightarrow \Lambda_a{}^b \hat{e}_b, \tag{3.48}$$
$$\hat{\theta}^a \longrightarrow \Lambda^a{}_b \hat{\theta}^b, \tag{3.49}$$
$$e^a{}_\mu \longrightarrow \Lambda^a{}_b e^b{}_\mu, \tag{3.50}$$

where $\Lambda_a{}^b = (\Lambda^{-1})^b{}_a$. Requiring that the torsion T^a transforms as a vector under local Lorentz transformations (or, equivalently, $R^a{}_b$ transforms as a (1,1)-tensor), the connection 1-form transforms in the following way

$$\omega'^a{}_b = \Lambda^a{}_c \omega^c{}_d (\Lambda^{-1})^d{}_b + \Lambda^a{}_c (d\Lambda^{-1})^c{}_b. \tag{3.51}$$

This way of transforming characterizes what in the gauge theories are interpreted as *local connection 1-forms on fibre bundles*, which is the geometric interpretation of gauge potentials [61]. Fibre bundles represent a powerful geometric framework which allow to describe gauge theories in an unified way. Gauge potentials, in this description, transform according to eq. (3.51) under transformations of the gauge group.
In other words, fibre bundles are constituted by a *base manifold* and a *fiber* on which a *structure group* acts (the gauge group).

Thanks to the eq. (3.51), it is possible to compare the spin connection and the curvature to the gauge potentials and field strength of gauge theories in particle physics. Common aspects are the existence of an "internal" vector space, which is influenced by gauge transformations, and the description of the interaction. The latter is taken into account by introducing a covariant derivative, which contains the internal degree of freedom through the local connection 1-forms. However, a big difference between tetrad formalism and gauge theory is the *nature of the internal space*. While for a gauge theory the internal space is independent from the base manifold, in Riemannian geometry the internal space is built from tangent space and cotangent one. Consequently, there is the constraint that the tangent space and its related quantities are "*intimately*" associated with the manifold itself, and are naturally defined once the manifold is set up. There is no other gauge theory involving tetrads, which relate orthonormal bases to coordinate bases.

Therefore, it is important to stress out some reasons why GR cannot be considered a gauge theory[3] [41]:

- In a generic gauge theory, the dynamical field (with to respect which variations are taken) is the gauge potential, *i.e.* the connection. In GR the variation is taken with respect the metric tensor.
- In GR, the Einstein-Hilbert action is not quadratic in the curvature and R depends on the metric tensor and on its first and second derivatives.
- There exists always a local inertial frame wherein the gravitational effect can be neglected. Thus, gravity does not appear as a force but as a geometrical effect.
- The group of diffeomorphisms cannot be considered the gauge group of gravitation because any theory can be written covariantly under diffeomorphisms without any dynamical meaning.
- Contrary to gauge theories, the curvature is not given by the external covariant derivative $D = d + \omega$.

With the aim to modifying GR, many attempts to find a gauge theory for gravity that is equivalent to GR in its validity ranges occurred.
In this regard, it can be interesting noting that, while $R^a{}_b \neq D\omega^a{}_b$, for the torsion it results $T^a = D\hat{e}^a$. Furthermore, in case of flat and torsionful spacetime, $R^a{}_b = d\omega^a_b + \omega^a{}_c \wedge \omega^c{}_b = 0$ and, from structure equations, it follows $DT^a = D^2\hat{e}^a = 0$. Therefore, the torsion would

[3]There are two ways to use the term "gauge theory": one simply expresses an internal freedom of theory that has no physical consequences; the other one is strictly related to fibre bundles formalism, used for the particle physics [61]. The latter requires the existence of a connection 1-form ω, which defines a covariant derivative on the fibre bundle D, and a field strength $\Omega = D\omega$. When we say that GR is not a gauge theory, we mean that any internal freedom cannot be expressed in the same way as particle physics. However, one can refer to the GR as a gauge theory taking into account the local Lorentz invariance of the tetrad formalism or the invariance under diffeomorphisms of Einstein linearised field equations.

seem to behave like a gauge field strength but, again, there is a difference: the covariant derivative, containing the gauge potential, must act on the gauge potential itself to provide the field; here, $D = d + \omega$ acts on the tetrad, $\hat{e}^a$.

However, before dealing with teleparallel Lagrangian, let us consider *Palatini action*,

$$S = \int \Omega_V \, R_{ab} \wedge *(\hat{\theta}^a \wedge \hat{\theta}^b), \quad \text{with } \Omega_V = \hat{\theta}^0 \wedge \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3 = d^4x\sqrt{-g} \tag{3.52}$$

which is equivalent to Einstein-Hilbert action but in tetrad formalism. The $*$ in eq. (3.52) is called Hodge star. It is a linear map which decreases the order of a r-form on M, $* : \Omega^r(M) \to \Omega^{r-1}$(M), defined by

$$*(dx^{\mu_1} \wedge .. \wedge dx^{\mu_r}) = \frac{\sqrt{-g}}{(m-r)!} \epsilon^{\mu_1\mu_2...\mu_r}{}_{\nu_{r+1}...\nu_m} dx^{\nu_{r+1}} \wedge ... \wedge dx^{\nu_m}, \tag{3.53}$$

where ϵ is the totally anti-symmetric tensor [61].

Without going into the details of calculations, considering tetrad $e^a{}_\mu$ and the 1-form $\omega^a{}_\mu$ as independent quantities, it is possible to obtain two equations: the equation obtained varying with respect to $e^a{}_\mu$ represents GR field equations in tetrad formalism; the equation obtained varying with respect to $\omega^a{}_\mu$ leads to the torsion-free condition.
Therefore, even using the Palatini action and tetrad, there are some differences from a gauge theory: the action is not quadratic in the field strength and the connection 1-form is not the dynamical field, which gives the curvature, but tetrads (which are a deeper description of the metric). This aspect will be clarified in the subsequent paragraphs.

3.1.2 Teleparallel Field Equations

Once clarified the role and the features of tetrad fields, it is possible to rewrite the Lagrangian (3.1) and to solve the constraints:

$$\begin{cases} \bar{R}^\mu{}_{\nu\rho\sigma} = 0 \quad \Rightarrow \quad R^a{}_{bcd} = 0 \quad \Rightarrow \quad \Gamma^a{}_{bc} = 0 & (3.54) \\ \bar{\nabla}_\lambda g_{\mu\nu} = 0 & (3.55) \end{cases}$$

In order to solve the first equation, note that the affine curvature is the gauge field strength of the connection for general linear transformations $GL(4,\mathbb{R})$. We can use this fact to argue that, since eq. (3.54) is trivially solved by vanishing connection, then it must be satisfied by any connection obtained by a general linear transformation. In the tetrad language the general linear transformation is $e^a{}_b$, so that the connection reads as:

$$\bar{\Gamma}^\lambda{}_{\mu\nu} = e_c{}^\lambda \partial_\mu e^c{}_\nu, \tag{3.56}$$

then,

$$T^\lambda{}_{\mu\nu} = 2e_c{}^\lambda \partial_{[\mu} e^c{}_{\nu]}. \tag{3.57}$$

Here, torsion is given exclusively by the anholonomy coefficient of tetrad since $T^c{}_{ab} = \Gamma^c{}_{ab} - \Gamma^c{}_{ba} - f_{ab}{}^c = -f_{ab}{}^c$. The second constraint (3.55) provides the relation between the derivatives of the metric and of the tetrad field

$$\begin{aligned} 0 = \bar{\nabla}_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - 2\bar{\Gamma}^\rho{}_{\lambda(\mu} g_{\nu)\rho} = \\ &= \partial_\lambda g_{\mu\nu} - 2e_c{}^\rho \partial_\lambda e^c{}_{(\mu} g_{\nu)\rho}, \end{aligned} \tag{3.58}$$

thus

$$\partial_\lambda g_{\mu\nu} = 2e_c{}^\rho \partial_\lambda e^c{}_{(\mu} g_{\nu)\rho}. \tag{3.59}$$

Now, the connection is fixed and totally determinate by tetrad. Therefore we can calculate the variation with respect to the tetrad and to get the field equations. For this reason, the following variation will be useful:

$$\delta e_b{}^\nu = -e_b{}^\mu e_a{}^\nu \delta e^a{}_\mu; \tag{3.60}$$
$$\delta g_{\mu\nu} = \delta(\eta_{ab} e^a{}_\mu e^b{}_\nu) = \eta_{ab}(e^a{}_\mu \delta e^b{}_\nu + e^b{}_\nu \delta e^a{}_\mu); \tag{3.61}$$
$$\delta\|\mathrm{e}\| = \delta\sqrt{-g} = 1/2\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu} = \|\mathrm{e}\| e_a{}^\mu \delta e^a{}_\mu; \tag{3.62}$$
$$\delta g^{\mu\nu} = -g^{\mu\sigma} g^{\rho\nu} \delta g_{\rho\sigma} = -g^{\mu\sigma} e_a{}^\nu \delta e^a{}_\sigma - g^{\rho\nu} e_a{}^\mu \delta e^a{}_\rho = -(g^{\mu\lambda} e_a{}^\nu + g^{\lambda\nu} e_a{}^\mu)\delta e^a{}_\lambda; \tag{3.63}$$
$$\delta T^\lambda{}_{\mu\nu} = \delta(2e_c{}^\lambda \partial_{[\mu} e^c{}_{\nu]}) = -e_a{}^\lambda T^\gamma{}_{\mu\nu} \delta e^a{}_\gamma + 2e_c{}^\lambda \partial_{[\mu} \delta e^c{}_{\nu]}. \tag{3.64}$$

It is quite simple to find the first order variations of the following quadratic combinations of torsion component:

$$\begin{aligned} \delta(T_\mu T^\mu) &= \delta(g^{\mu\nu} T_\mu T_\nu) = T_\mu T_\nu \delta g^{\mu\nu} + 2T^\mu \delta T_\mu = \\ &= -(g^{\mu\lambda} e_a{}^\nu + g^{\lambda\nu} e_a{}^\mu)\delta e^a{}_\lambda T_\mu T_\nu + 2T^\mu(-e_a{}^\lambda T^\gamma{}_{\mu\lambda} \delta e^a{}_\gamma + 2e_c{}^\lambda \partial_{[\mu} \delta e^c{}_{\lambda]}) = \\ &= -2(T^\lambda T_\mu + T^\rho T^\lambda{}_{\rho\mu}) e_a{}^\mu \delta e^a{}_\lambda + 2(T^\mu e_c{}^\lambda - T^\lambda e_c{}^\mu)\partial_\mu \delta e^c{}_\lambda; \end{aligned} \tag{3.65}$$

$$\begin{aligned} \delta(T^{\lambda\mu\nu} T_{\nu\mu\lambda}) &= \delta(T^\lambda{}_{\mu\nu} T^\nu{}_{\rho\lambda} g^{\rho\mu}) = 2T^{\nu\mu}{}_\lambda \delta T^\lambda{}_{\mu\nu} + T^\lambda{}_{\mu\nu} T^\nu{}_{\rho\lambda} \delta g^{\rho\mu} = \\ &= 2T^{\nu\mu}{}_\lambda(-e_a{}^\lambda T^\gamma{}_{\mu\nu} \delta e^a{}_\gamma + 2e_c{}^\lambda \partial_{[\mu} \delta e^c{}_{\nu]}) + \\ &\quad + T^\lambda{}_{\mu\nu} T^\nu{}_{\rho\lambda}(-(g^{\mu\sigma} e_a{}^\rho + g^{\sigma\rho} e_a{}^\mu)\delta e^a{}_\sigma) = \\ &= -2T^{\nu\mu}{}_\lambda T^\gamma{}_{\mu\nu} e_a{}^\lambda \delta e^a{}_\gamma + 2(T^{\nu\mu}{}_\lambda - T^{\mu\nu}{}_\lambda) e_c{}^\lambda \partial_\mu \delta e^c{}_\nu = \\ &= 2(T_\mu{}^\gamma{}_\nu - T^\gamma{}_{\mu\nu}) T^{\nu\mu}{}_\lambda e_a{}^\lambda \delta e^a{}_\gamma + 2(T^{\nu\mu}{}_\lambda - T^{\mu\nu}{}_\lambda) e_c{}^\lambda \partial_\mu \delta e^c{}_\nu; \end{aligned} \tag{3.66}$$

$$\begin{aligned} \delta(T^{\lambda\mu\nu} T_{\lambda\mu\nu}) &= (T^\lambda{}_{\mu\nu} T^\gamma{}_{\rho\sigma} g_{\lambda\gamma} g^{\mu\rho} g^{\nu\sigma}) = \\ &= 2T_\lambda{}^{\mu\nu} \delta T^\lambda{}_{\mu\nu} + T^\lambda{}_{\mu\nu} T^{\gamma\mu\nu} \delta g_{\lambda\gamma} + 2T^\lambda{}_{\mu\nu} T_{\lambda\rho}{}^\nu \delta g^{\mu\rho} = \\ &= 2T_\lambda{}^{\mu\nu}(-e_a{}^\lambda T^\gamma{}_{\mu\nu} \delta e^a{}_\gamma + 2e_c{}^\lambda \partial_{[\mu} \delta e^c{}_{\nu]}) + T^\lambda{}_{\mu\nu} T^{\gamma\mu\nu} \eta_{ab}(e^a{}_\lambda \delta e^b{}_\gamma + e^b{}_\gamma \delta e^a{}_\lambda) + \\ &\quad + 2T^\lambda{}_{\mu\nu} T_{\lambda\rho}{}^\nu(-(g^{\mu\sigma} e_a{}^\rho + g^{\sigma\rho} e_a{}^\mu)\delta e^a{}_\sigma) = \\ &= -4T_{\lambda\mu\nu} T^{\gamma\mu\nu} e_a{}^\lambda \delta e^a{}_\gamma + 4T_\lambda{}^{\mu\nu} e_a{}^\lambda \partial_\mu \delta e^c{}_\nu + 2T_{\lambda\mu\nu} T^{\gamma\mu\nu} e_a{}^\lambda \delta e^a{}_\gamma + \\ &\quad - 4T^{\lambda\sigma\nu} T_{\lambda\mu\nu} e_a{}^\mu \delta e^a{}_\sigma = \\ &= -4T^{\lambda\sigma\nu} T_{\lambda\mu\nu} e_a{}^\mu \delta e^a{}_\sigma + 4T_\lambda{}^{\mu\nu} e_a{}^\lambda \partial_\mu \delta e^c{}_\nu, \end{aligned} \tag{3.67}$$

where, in the last line, we used the antisymmetry of ${e_a}^{\lambda}\delta{e^a}_{\gamma}$.
Therefore, the variation of TEGR action is:

$$\delta S_{\mathbb{T}} = \int d^4x \Big(\delta\|\mathrm{e}\|\mathring{\mathbb{T}} + \|e\|\delta\mathring{\mathbb{T}} \Big) = 0. \tag{3.68}$$

Replacing the above calculated terms, it follows

$$\begin{aligned}\delta S_{\mathbb{T}} = \int d^4x \|\mathrm{e}\| \Big[& \mathring{\mathbb{T}}{e_a}^{\lambda}\delta{e^a}_{\lambda} + \\ & + 2(T^{\lambda}T_{\rho} + T^{\mu}{T^{\lambda}}_{\mu\rho}){e_a}^{\rho}\delta{e^a}_{\lambda} - 2(T^{\mu}{e_a}^{\lambda} - T^{\lambda}{e_a}^{\mu})\partial_{\mu}\delta{e^a}_{\lambda} + \\ & + ({T_{\mu}}^{\lambda}{}_{\nu} - {T^{\lambda}}_{\mu\nu}){T^{\nu\mu}}_{\rho}{e_a}^{\rho}\delta{e^a}_{\lambda} + ({T^{\lambda\mu}}_{\nu} - {T^{\mu\lambda}}_{\nu}){e_a}^{\nu}\partial_{\mu}\delta{e^a}_{\lambda} + \\ & - T^{\mu\lambda\nu}T_{\mu\rho\nu}{e_a}^{\rho}\delta{e^a}_{\lambda} + {T_{\nu}}^{\mu\lambda}{e_a}^{\nu}\partial_{\mu}\delta{e^a}_{\lambda} \Big]. \end{aligned} \tag{3.69}$$

Now it is useful introduce the *superpotential* $S^{\lambda\mu\nu}$ such that $\mathring{\mathbb{T}} = 1/2T_{\lambda\mu\nu}S^{\lambda\mu\nu}$:

$$S^{\lambda\mu\nu} \equiv K^{\mu\lambda\nu} + g^{\lambda\mu}T^{\nu} - g^{\lambda\nu}T^{\mu} = -S^{\lambda\nu\mu}. \tag{3.70}$$

In this way, integrating by part the second terms of every line, the variation turns out to be

$$\delta S_{\mathbb{T}} = \int d^4x \Big[\|\mathrm{e}\|\mathring{\mathbb{T}}{e_a}^{\lambda} - 2\|\mathrm{e}\|S^{\mu\lambda\nu}T_{\mu\rho\nu}{e_a}^{\rho} + 2\partial_{\mu}({S_{\nu}}^{\lambda\mu}{e_a}^{\nu}\|\mathrm{e}\|) \Big] \delta{e^a}_{\lambda}. \tag{3.71}$$

Thus, for the arbitrariness and not-degeneracy of tetrad, the equations got are

$$\|\mathrm{e}\|\mathring{\mathbb{T}}{e_a}^{\lambda} - 2\|\mathrm{e}\|S^{\mu\lambda\nu}T_{\mu\rho\nu}{e_a}^{\rho} + 2\partial_{\mu}({S_{\rho}}^{\lambda\mu}{e_a}^{\rho}\|\mathrm{e}\|) = 0, \tag{3.72}$$

Finally, it is possible to multiply by ${e^a}_{\sigma}$, and to rewrite last term in the following way, using the Levi-Civita connection and the antisymmetry of $S^{\lambda\mu\nu}$:

$$\begin{aligned} {e^a}_{\sigma}\partial_{\mu}({S_{\rho}}^{\lambda\mu}{e_a}^{\rho}\|\mathrm{e}\|) &= {e^a}_{\sigma}\partial_{\mu}({S_{\rho}}^{\lambda\mu}){e_a}^{\rho}\|\mathrm{e}\| + {e^a}_{\sigma}{S_{\rho}}^{\lambda\mu}\partial_{\mu}({e_a}^{\rho})\|\mathrm{e}\| + {e^a}_{\sigma}{S_{\rho}}^{\lambda\mu}{e_a}^{\rho}\partial_{\mu}(\|\mathrm{e}\|) = \\ &= \partial_{\mu}({S_{\sigma}}^{\lambda\mu})\|\mathrm{e}\| - {\bar{\Gamma}^{\rho}}_{\mu\sigma}{S_{\rho}}^{\lambda\mu}\|\mathrm{e}\| + {\Gamma^{\rho}}_{\rho\mu}{S_{\sigma}}^{\lambda\mu}\|\mathrm{e}\| = \\ &= \|\mathrm{e}\|(\nabla_{\mu}{S_{\sigma}}^{\lambda\mu} - {\Gamma^{\lambda}}_{\mu\rho}{S_{\sigma}}^{\rho\mu} + {\Gamma^{\rho}}_{\mu\sigma}{S_{\rho}}^{\lambda\mu} - {\Gamma^{\mu}}_{\mu\rho}{S_{\sigma}}^{\lambda\rho} + \\ &\quad - {\Gamma^{\rho}}_{\mu\sigma}{S_{\rho}}^{\lambda\mu} - {K^{\rho}}_{\mu\sigma}{S_{\rho}}^{\lambda\mu} + \\ &\quad + {\Gamma^{\rho}}_{\rho\mu}{S_{\sigma}}^{\lambda\mu}) \\ &= \|\mathrm{e}\|(\nabla_{\mu}{S_{\sigma}}^{\lambda\mu} - {K^{\rho}}_{\mu\sigma}{S_{\rho}}^{\lambda\mu}), \end{aligned} \tag{3.73}$$

where, $\partial_{\mu}\|\mathrm{e}\| = \partial_{\mu}\sqrt{-g} = 1/2\sqrt{-g}g^{\rho\sigma}\partial_{\mu}g_{\rho\sigma} = -1/2\|\mathrm{e}\|g^{\rho\sigma}({\Gamma^{\lambda}}_{\mu\rho}g_{\lambda\sigma} + {\Gamma^{\lambda}}_{\mu\sigma}g_{\lambda\rho}) = \|\mathrm{e}\|{\Gamma^{\lambda}}_{\mu\lambda}$.
Therefore, the field equations read

$$\nabla_{\mu}{S_{\rho}}^{\lambda\mu} - S^{\mu\lambda\nu}(T_{\mu\rho\nu} + K_{\mu\nu\rho}) + \frac{1}{2}\mathring{\mathbb{T}}\delta^{\lambda}_{\rho} = 0. \tag{3.74}$$

The equation (3.74) can be written as

$$-K_{\mu\nu} + 1/2Kg_{\mu\nu} = 0, \tag{3.75}$$

having in mind the relations (2.50) and (2.56). Then, from the teleparallel condition, $0 = \bar{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + K^{\mu}{}_{\nu\rho\sigma} \Rightarrow R_{\mu\nu} = -K_{\mu\nu}$ and $R = -K$, the equivalence between the eq. (3.74) and the Einstein equations in vacuum is immediate.

It is necessary, at this level, to make some comments regarding the approach used to obtain the field equations. The procedure consists in resolving first the constraints in the Lagrangian (3.1). In this way, the field equations are obtained by varying with respect to $e^{a}{}_{\mu}$ and the connection results fixed (entirely determined by tetrad).
The connection defined by eq. (3.56) is called *Weitzenböck connection*. It has the property of null curvature, but non-vanishing torsion, and it characterizes any frames with null spin connection.
However, it is possible to consider the connection as further degrees of freedom and to vary the action with respect to $\omega^{a}{}_{b\mu}$. This further degrees of freedom is associated with inertial effects. Indeed, unlike GR wherein the spin connection represents both gravitation and inertial effects, Teleparallel Gravity inherits from SR the interpretation of spin connections as pure inertial effects.
To understand how natural this association is, let us just consider the equations of a free particle, in a Minkowski space and in a general frame:

$$\frac{du^a}{ds} = 0 \;\; \longrightarrow \;\; \frac{du^a}{ds} + \Gamma^{a}{}_{cb} u^c u^b = 0, \tag{3.76}$$

$$\frac{du^a}{ds} + \omega^{a}{}_{\mu b} u^{\mu} u^b = 0. \tag{3.77}$$

The free particle equations (3.77) guarantees the local Lorentz invariance thanks to the spin connection property of transformation (3.51). Therefore, the choice of a specific frame breaks this invariance:

$$\frac{du^a}{ds} = 0 \;\; \longrightarrow, \;\; \frac{d(\Lambda^{a}{}_{b} u^b)}{ds} = 0, \tag{3.78}$$

$$\Lambda^{a}{}_{b} \frac{du^b}{ds} + \partial_{\mu}(\Lambda^{a}{}_{b}) u^{\mu} u^b = 0$$

and, multiplying by $\Lambda_{\mathrm{a}}{}^{\mathrm{c}}$, it results

$$\frac{du^c}{ds} - \overset{\bullet}{\omega}{}^{c}{}_{b\mu} u^{\mu} u^b = 0, \tag{3.79}$$

where

$$\overset{\bullet}{\omega}{}^{c}{}_{b\mu} \equiv \Lambda^{a}{}_{b} \partial_{\mu} \Lambda_{a}{}^{c} = -\Lambda_{a}{}^{c} \partial_{\mu} \Lambda^{a}{}_{b}, \tag{3.80}$$

that is called *purely inertial/gauge connection.*
This also happens when we choose the Weitzenböck connection. A local Lorentz transformation leads to the appearance of other terms due to the inertial connection, which

breaks the invariance. Thus, although one starts from the Lagrangian (3.2), which is invariant under local Lorentz transformations (due to the propriety of the torsion to change covariantly $T^a{}_{\mu\nu} \to \Lambda^a{}_b T^b{}_{\mu\nu}$), the choice of the vanishing spin-connection frame in TEGR breaks this invariance, changing the transformation rule of the torsion

$$T^a{}_{\mu\nu} \to \Lambda^a{}_b T^b{}_{\mu\nu} + (e^b{}_\nu \partial_\mu \Lambda^a{}_b - e^b{}_\mu \partial_\nu \Lambda^a{}_b), \tag{3.81}$$

that is no longer local Lorentz covariant. This means that the torsion scalar is Lorentz violating. However, this is not a problem because the Lorentz violating term in the torsion scalar gives just a boundary term [32, 35, 36], which does not contribute when integrated,

$$\mathring{\mathbb{T}}(e^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}) = \mathring{\mathbb{T}}(e^a{}_\mu, 0) + \frac{4}{\|e\|}\partial^\mu(\|e\|\omega^a{}_{b\nu} e_a{}^\nu e^b{}_\mu). \tag{3.82}$$

This means that any linear combination of the torsion scalar in the action will be Lorentz invariant (up to a total derivative term).

3.1.2.1 The spin connection of Special Relativity

Before concluding the Subsection, some clarification about the spin connection in SR is worth. In absence of gravity, the space-time is the Minkowski one, fully described by the metric $\eta_{\mu\nu} = \mathrm{diag}(+1, -1 -1 -1)$.Here the Lorentz connection represents inertial effects of a given frame; if the frame is inertial the Lorentz connection vanishes identically. Let $\{x^\mu\}$ be the chart on the base space and $\{x'^a\}$ an inertial (holonomic) frame of the fiber, *i.e.* $[e'_a, e'_b] = f'{}_{ab}{}^c = 0$, in a general coordinate system it results

$$d\theta'^c = -\frac{1}{2} f'{}_{ab}{}^c\, \theta'^a \wedge \theta'^b = 0 \tag{3.83}$$

$$e'^a{}_\mu = \partial_\mu x'^a, \tag{3.84}$$

while in the specific case of Cartesian coordinates

$$e'^a{}_\mu = \delta^a_\mu. \tag{3.85}$$

Under a local Lorentz transformation $\Lambda_a{}^b = \Lambda_a{}^b(x^\mu)$, $\{e'_a\}$ becomes $\{e_a\}$, which is no more holonomic but anholonomic, as follows

$$\begin{aligned} e_a &= \Lambda_a{}^b e'_b = \Lambda_a{}^b e'{}_b{}^\mu \partial_\mu \\ &= e_a{}^\mu \partial_\mu, \end{aligned} \tag{3.86}$$

$$x^a = \Lambda^a{}_b x'^b = \Lambda^a{}_b e'^b{}_\mu x^\mu = e^a{}_\mu x^\mu, \tag{3.87}$$

$$e^a{}_\mu = \Lambda^a{}_b e'^b{}_\mu. \tag{3.88}$$

Now it is possible to compute the last equation using the first two,

$$\begin{aligned} e^a{}_\mu &= \Lambda^a{}_b(x^\nu) e'^b{}_\mu = \\ &= \Lambda^a{}_b(x^\nu)\partial_\mu x'^b = \\ &= \partial_\mu(\Lambda^a{}_b(x^\nu)x'^b) - x'^b\partial_\mu\Lambda^a{}_b(x^\nu) = \\ &= \partial_\mu x^a - \Lambda_c{}^b(x^\nu)x^c\partial_\mu\Lambda^a{}_b(x^\nu) = \\ &= \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{c\mu}x^c \\ &\equiv \overset{\bullet}{\mathscr{D}}_\mu x^a, \end{aligned} \tag{3.89}$$

where the quantity

$$\overset{\bullet}{\omega}{}^a{}_{b\mu} = \Lambda^a{}_e\partial_\mu\Lambda_b{}^e, \tag{3.90}$$

is called purely inertial connection, just because it represents the inertial effects, and $\overset{\bullet}{\mathscr{D}}_\mu$ is the associated covariant derivative.
Another way to get the purely inertial connection is to use the Lorentz symmetry of the internal space and to compute the Spin connection using its transformation rule[4], starting from an holonomic frame:

$$\omega^a{}_{b\mu} = 0, \tag{3.91}$$

$$\overset{\bullet}{\omega}{}^a{}_{b\mu} = \Lambda^a{}_e\omega^e{}_{d\mu}\Lambda_b{}^d + \Lambda^a{}_e\partial_\mu\Lambda_b{}^e = \Lambda^a{}_e\partial_\mu\Lambda_b{}^e. \tag{3.92}$$

Now, it is possible to use the definition of $f_{ab}{}^c$ in terms of $e^a{}_\mu$ to get the relations between the anholonomy coefficient and the purely inertial connection:

$$f_{ab}{}^c = \overset{\bullet}{\omega}{}^c{}_{ba} - \overset{\bullet}{\omega}{}^c{}_{ab}, \tag{3.93}$$

$$\overset{\bullet}{\omega}{}^c{}_{ba} = \frac{1}{2}(f_a{}^c{}_b + f_b{}^c{}_a + f_{ab}{}^c) \equiv \overset{\circ}{\omega}{}^c{}_{ab}, \tag{3.94}$$

where $\overset{\bullet}{\omega}{}^c{}_{ab} = \overset{\bullet}{\omega}{}^c{}_{a\mu}e_b{}^\mu$. Furthermore it results[5]:

$$\overset{\bullet}{\Gamma}{}^\lambda{}_{\mu\nu} = e_c{}^\lambda\partial_\mu e^c{}_\nu + e_c{}^\lambda\overset{\bullet}{\omega}{}^c{}_{b\mu}e^b{}_\nu \equiv e_c{}^\lambda\overset{\bullet}{\mathscr{D}}_\mu e^c{}_\nu, \tag{3.95}$$

$$\overset{\bullet}{\omega}{}^a{}_{b\mu} = e^a{}_\lambda e_b{}^\nu\overset{\bullet}{\Gamma}{}^\lambda{}_{\mu\nu} + e^a{}_\rho\partial_\mu e_b{}^\rho \equiv e^a{}_\rho\overset{\bullet}{\nabla}_\mu e_b{}^\rho, \tag{3.96}$$

$$\overset{\bullet}{R}{}^a{}_{b\mu\nu} = \partial_\mu\overset{\bullet}{\omega}{}^a{}_{b\nu} - \partial_\nu\overset{\bullet}{\omega}{}^a{}_{b\mu} + \overset{\bullet}{\omega}{}^a{}_{e\mu}\overset{\bullet}{\omega}{}^e{}_{b\nu} - \overset{\bullet}{\omega}{}^a{}_{e\nu}\overset{\bullet}{\omega}{}^e{}_{b\mu} = 0, \tag{3.97}$$

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{e\mu}e^e{}_\nu - \overset{\bullet}{\omega}{}^a{}_{e\nu}e^e{}_\mu = 0. \tag{3.98}$$

[4]The spin connection can be seen as a connection 1-form with values in the Lie algebra of Lorentz group; indeed, considering an infinitesimal Lorentz transformation, then it results $\Lambda^a{}_e\partial_\mu\Lambda_b{}^e = (\delta + 1/2\epsilon_{ij}S^{ij})^a{}_e\partial_\mu(\delta - 1/2\epsilon_{lm}S^{lm})_b{}^e = -1/2\partial_\mu(\epsilon_{lm})(S^{lm})^a{}_b$, where $(S^{lm})^a{}_b$ are the Lorentz transformations generators.

[5]It is possible to observe that $\overset{\bullet}{T}{}^a{}_{\mu\nu} = 2\overset{\bullet}{\mathscr{D}}_{[\mu}e^a{}_{\nu]}$ but $\overset{\bullet}{R}{}^a{}_{b\mu\nu} \neq 2\overset{\bullet}{\mathscr{D}}_{[\mu|}\overset{\bullet}{\omega}{}^a{}_{b|\nu]}$.

In SR, therefore, Lorentz connections represent inertial effects only, and are crucial for the local Lorentz invariance of relativistic physics. Equations (3.90) and (3.96) highlight the correspondence between the inertial connection and the affine connection. The first one is obtained by performing a local Lorentz transformation, while the second one is related to a generic coordinate transformation. Therefore, there is an equivalence between local Lorentz and general coordinate transformation (or diffeomorphism) [41] and, since diffeomorphisms are empty of dynamical meaning, local Lorentz transformations are consequently empty of dynamical meaning as well.

3.1.3 Extended TEGR Theories

A first extension of TEGR is the $f(\mathring{\mathbb{T}})$ -theories of gravity [68, 73, 74]. Similarly to what happens for GR and $f(R)$-theories, the scalar torsion $\mathring{\mathbb{T}}$ is replaced by its function, $f(\mathring{\mathbb{T}})$. The action of an $f(\mathring{\mathbb{T}})$-theory reads

$$S_{f(\mathring{\mathbb{T}})} = \int d^4x \|e\| f(\mathring{\mathbb{T}}) + S_m. \tag{3.99}$$

Before describing this family of theories, it is important to stress out that TEGR Lagrangian and related field equations are invariant under local Lorentz transformations, although the vanishing spin-connection frame. However, when one moves to generalizations of the form of $f(\mathring{\mathbb{T}})$, things change. What for TEGR is a boundary term, now, is no longer such. This happens in $f(R)$-theories too, but in $f(\mathring{\mathbb{T}})$-theories there is the problem of the local Lorentz symmetry breaking. To overcome this problem, there are two possible alternatives [32]: the first one is to keep working in Weitzenböck frames, with null spin connection, and selecting, *a posteriori*, those tetrads that provide field equations with matching solutions with GR, in the limit of $f(\mathring{\mathbb{T}}) \to \mathring{\mathbb{T}}$; the second one is to work in a general frame, with a non-vanishing spin connection, and varying the action with respect $e^a{}_\mu$ and $\omega^a{}_{b\nu}$. The latter guarantees the local Lorentz invariance, but greatly complicates the calculations. Coming back to the action (3.99) with respect to the tetrad, tthe variational principle with respect to the tetrad provides

$$\delta S_{f(\mathring{\mathbb{T}})} = \int d^4x \left(\delta\|\mathrm{e}\| f(\mathring{\mathbb{T}}) + \|\mathrm{e}\| f'(\mathring{\mathbb{T}})\delta\mathring{\mathbb{T}}\right). \tag{3.100}$$

Then, integrating by part, the variation results to be

$$\delta S_{\mathbb{T}} = \int d^4x \left[\|\mathrm{e}\| f(\mathring{\mathbb{T}}) e_a{}^\lambda - 2\|\mathrm{e}\| f'(\mathring{\mathbb{T}}) S^{\mu\lambda\nu} T_{\mu\rho\nu} e_a{}^\rho + 2\partial_\mu (f'(\mathring{\mathbb{T}}) S_\nu{}^{\lambda\mu} e_a{}^\nu \|\mathrm{e}\|)\right] \delta e^a{}_\lambda. \tag{3.101}$$

In this way, the field equations are

$$\nabla_\mu \big(f'(\mathring{\mathbb{T}}) S_\rho{}^{\lambda\mu}\big) - f'(\mathring{\mathbb{T}}) S^{\mu\lambda\nu}(T_{\mu\rho\nu} + K_{\mu\nu\rho}) + \frac{1}{2} f(\mathring{\mathbb{T}}) \delta^\lambda_\rho = \kappa T_\rho{}^\lambda, \tag{3.102}$$

where $T_\rho{}^\lambda \equiv -\frac{1}{2\|e\|} e^a{}_\rho \delta S_m / \delta e^a{}_\lambda$. Obviously, replacing $f(\mathring{\mathbb{T}}) \to \mathring{\mathbb{T}}$ the (3.102) becomes (3.74) adding the contribute of the matter.

Finally, it is interesting to notice that, despite the Teleparallel theory is completely equivalent to GR, generally, the same does not happen for $f(R)$ and $f(\mathring{\mathbb{T}})$ theories. This happens because the boundary term becomes completely arbitrary, for non-linear terms of the torsion tensor. Anyhow, often, one finds that the correspondence between TEGR and $f(\mathring{\mathbb{T}})$ is more natural than that between GR and $f(R)$-theories, because in the latter the field equations are of the fourth order, while in GR, TEGR and $f(\mathring{\mathbb{T}})$-theories the equations are the of second order.

3.1.4 Gravity as Gauge Theory of Translation Group

As already mentioned, TEGR can be seen as a gauge theory for the translation group. The base space is a general flat pseudo-Riemannian space-time M with metric $g_{\mu\nu}$. The fiber is a Minkowski tangent-space $TM = \mathbb{R}^{1,3}$ with metric η_{ab}, on which the gauge transformations, translations, take place.

Let $\{x^a\}$ be a local chart of the tangent bundle. Let us consider the following infinitesimal local translation

$$x^a \to \tilde{x}^a = x^a + \epsilon^a(x^\mu). \tag{3.103}$$

The generators of infinitesimal translations are the differential operators $P_a = \partial/\partial x^a = \partial_a$ which satisfy the commutation relations

$$[P_a,\, P_b] = [\partial_a,\, \partial_b] = 0. \tag{3.104}$$

The corresponding infinitesimal transformation can then be written in the form

$$\delta_\epsilon x^a = \epsilon^b P_b x^a \ \left(= \epsilon^b \delta^a_b = \epsilon^a \right). \tag{3.105}$$

Let B_μ be the potential gauge of theory which belongs to the Lie algebra of the translation group

$$B_\mu = B^a{}_\mu P_a. \tag{3.106}$$

This gauge potential allows to obtain the right covariant derivative of the theory (similarly to what happens with electromagnetic potential). Let ψ be a general source field (which represents a local Section of the associate fiber bundle with $G = \mathscr{T}^{1,3}$). Under an infinitesimal tangent space translation, it transforms according [6] to

$$\delta_\epsilon \psi = \epsilon^a(x^\mu)\partial_a \psi \tag{3.107}$$

[6] Generally, this variation is called "*total variation*". In case of a function f, under translation $x \to x'^\mu = x^\mu + \delta x^\mu$ the total variation is defined as $\delta_0 f = \tilde{f}(x) - f(x) = -\delta x^\mu \partial_\mu f$.

while its partial derivative does not transform in the same way,

$$\delta_\epsilon(\partial_\mu\psi) = \partial_\mu(\delta_\epsilon\psi) = \partial_\mu(\epsilon^a(x)\partial_a\psi) = \partial_\mu(\epsilon^a)\partial_a\psi + \epsilon^a\partial_a\partial_\mu\psi \neq \epsilon^a\partial_a\partial_\mu\psi. \tag{3.108}$$

The gauge covariant derivative such that the derivative of field transform covariantly is

$$\partial_\mu\psi \to h_\mu\psi = \partial_\mu\psi + B^a{}_\mu\partial_a\psi = h^a{}_\mu\partial_a\psi, \tag{3.109}$$

where

$$h^a{}_\mu = \partial_\mu x^a + B^a{}_\mu = e'^a{}_\mu + B^a{}_\mu, \tag{3.110}$$

and the translational potential transforms as follows

$$\delta_\epsilon B^a{}_\mu = -\partial_\mu\epsilon^a(x). \tag{3.111}$$

In fact,

$$\begin{aligned}\delta_\epsilon(h_\mu\psi) &= \delta_\epsilon(\partial_\mu\psi + B^b{}_\mu\partial_b\psi) = \\ &= \partial_\mu(\epsilon^a)\partial_a\psi + \epsilon^a(x)\partial_a\partial_\mu\psi + \delta_\epsilon B^b{}_\mu\partial_b\psi + B^b{}_\mu\partial_b\delta_\epsilon\psi = \\ &= \partial_\mu(\epsilon^a)\partial_a\psi + \epsilon^a(x)\partial_a\partial_\mu\psi + \delta_\epsilon B^b{}_\mu\partial_b\psi + B^b{}_\mu\partial_b(\epsilon^a\partial_a\psi) = \\ &= \epsilon^a\partial_a(\partial_\mu\psi + B^b{}_\mu\partial_b\psi) + \partial_\mu(\epsilon^a)\partial_a\psi + \delta_\epsilon B^b{}_\mu\partial_b\psi = \\ &= \epsilon^a\partial_a(h_\mu\psi),\end{aligned} \tag{3.112}$$

where $B^a{}_\mu = B^a{}_\mu(x^\nu)$ and, since $\epsilon^a = \epsilon^a(x^\mu)$, it results $\delta_\epsilon\partial_b\psi = \partial_b\delta_\epsilon\psi$.

Now it is possible to compute the field strength of the theory by the commutator of the covariant derivative: using the gauge viewpoint (not explaining internal degrees of freedom), it follows

$$\begin{aligned}[h_\mu,\, h_\nu]\psi =& (\partial_\mu + B_\mu)(\partial_\nu + B_\nu)\psi - (\mu \leftrightarrow \nu) = \\ &= \partial_\mu\partial_\nu\psi + \partial_\mu B_\nu\psi + B_\nu\partial_\mu\psi + B_\mu\partial_\nu\psi + B_\mu B_\nu - (\mu \leftrightarrow \nu) = \\ &= (\partial_\mu B_\nu - \partial_\nu B_\mu)\psi = (\partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu)\partial_a\psi \equiv T^a{}_{\mu\nu}.\partial_a\end{aligned} \tag{3.113}$$

What happens if a local Lorentz transformation $x^a \to \tilde{x}^a = \Lambda^a{}_b x^b$ is performed on the internal space? In literature is present the following treatment [41].
A new tetrad is defined, assuming that $B^a{}_\mu \to \Lambda_a{}^b B^b{}_\mu$

$$h^a{}_\mu = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu}x^b + B^a{}_\mu = \overset{\bullet}{\mathscr{D}}_\mu x^a + B^a{}_\mu, \tag{3.114}$$

where $\overset{\bullet}{\omega}{}^a{}_{b\mu} = \Lambda^a{}_c\partial_\mu\Lambda_b{}^c$ is the usual purely inertial Lorentz connection. In this class of frames, the internal coordinates $\{x^a\}$ are related to an holonomic frame of the internal space. In this class of frame, the gauge potential $B^a{}_\mu$ transforms under the group of translations according to

$$\delta_\epsilon B^a{}_\mu = -\overset{\bullet}{\mathscr{D}}_\mu\epsilon^a. \tag{3.115}$$

Then, the field strength is archived as

$$[h_\mu,\, h_\nu] = T^a{}_{\mu\nu} P_a, \tag{3.116}$$

where

$$T^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} B^b{}_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} B^b{}_\mu = \overset{\bullet}{\mathscr{D}}_\mu B^a{}_\nu - \overset{\bullet}{\mathscr{D}}_\nu B^a{}_\mu \tag{3.117}$$

Moreover, using the relation

$$[\overset{\bullet}{\mathscr{D}}_\mu,\, \overset{\bullet}{\mathscr{D}}_\nu] x^a = 0. \tag{3.118}$$

it is possible to write

$$T^a{}_{\mu\nu} = \overset{\bullet}{\mathscr{D}}_\mu h^a{}_\nu - \overset{\bullet}{\mathscr{D}}_\nu h^a{}_\mu = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} h^b{}_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} h^b{}_\mu. \tag{3.119}$$

The tetrad $h^a{}_\mu$ defined by (3.114) allows to obtain the translational (gravitational) coupling prescription and it is consistent to the so-called *general covariance principle* [41] thanks to the presence of the inertial spin connection.
In this way, the connection is given by

$$\Gamma^\lambda{}_{\mu\nu} = h_a{}^\lambda \partial_\mu h^a{}_\nu + h_a{}^\lambda \overset{\bullet}{\omega}{}^a{}_{b\mu} h^b{}_\nu, \tag{3.120}$$

while the torsion is

$$T^\lambda{}_{\mu\nu} = h_a{}^\lambda T^a{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}, \tag{3.121}$$

and the relation between the connection (3.120) and the Levi-Civita one is

$$\Gamma^\lambda{}_{\mu\nu} = \{^\lambda{}_{\mu\nu}\} + K^\lambda{}_{\mu\nu}. \tag{3.122}$$

In this framework the torsion plays the role of gravitational force and its form is really similar to the electromagnetic one (but with an universal coupling). In fact, the spacetime interval of a particle trajectory can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \to ds = g_{\mu\nu} u^\mu dx^\nu = \eta_{ab} u^a h^b, \tag{3.123}$$

where $u^\mu = dx^\mu/ds$ is the holonomic 4-velocity and $u^a \equiv h^a{}_\mu u^\mu$ is its anholonomic version. This allows to generalize the description of a free particle in SR to the description of a free particle in teleparallel gravity:

$$S = -mc \int_A^B u_a h^a = -mc \int_A^B u_a (dx^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b dx^\mu + B^a{}_\mu dx^\mu). \tag{3.124}$$

Now, varying with respect to x^μ, the equations of motion are

$$\frac{du_a}{ds} - \overset{\bullet}{\omega}{}^b{}_{a\nu} u_b u^\nu = T^b{}_{a\nu} u_b u^\nu = -K^b{}_{a\nu} u_b u^\nu. \tag{3.125}$$

Finally, contracting with tetrads and using the relation between the tetrad connection and the connection of the spacetime, the (3.125) becomes

$$\frac{du_\mu}{ds} - \Gamma^\lambda{}_{\mu\nu} u_\lambda u^\nu = -K^\lambda{}_{\mu\nu} u_\lambda u^\nu, \tag{3.126}$$

which is nothing but the force equation.

3.2 Symmetric Teleparallel Gravity

General relativity can be presented in terms of other geometries besides Riemannian, as already discussed. In this Section, the attention is payed on another alternative: the *Symmetric Teleparallel Equivalent to General Relativity* (STEGR) [67, 75–77], in which the torsion vanishes but the non-metricity ∇g does not, carrying the "gravitational force". In this framework there is the special possibility to globally remove the connection by choosing appropriate coordinates, so that the spacetime is trivially connected.This choice of coordinate represents a preferred call of frames wherein the computations are greatly simplified.

The action of this theory must be built from the non-metricity even-parity second order quadratic forms. The most general action of this type is

$$S_{\mathbb{Q}} = \frac{c^4}{16\pi G_N}\int d^4x(-\sqrt{-g}\mathbb{Q} + \lambda_\mu{}^{\nu\rho\sigma}\bar{R}^\mu{}_{\nu\rho\sigma} + \tilde{\lambda}_\alpha{}^{\mu\nu}T^\alpha{}_{\mu\nu}), \tag{3.127}$$

where $\lambda_\mu{}^{\nu\rho\sigma} = -\lambda_\mu{}^{\nu\sigma\rho}$, $\tilde{\lambda}_\alpha{}^{\mu\nu} = -\tilde{\lambda}_\alpha{}^{\nu\mu}$ are the Lagrangian multipliers and $\mathbb{Q}$ is the generic non-metricity scalar, defined as

$$\mathbb{Q} = \frac{c_1}{4}Q_{\mu\nu\alpha}Q^{\mu\nu\alpha} - \frac{c_2}{2}Q_{\mu\nu\alpha}Q^{\nu\mu\alpha} - \frac{c_3}{4}Q_\lambda Q^\lambda + (c_4-1)\tilde{Q}_\sigma\tilde{Q}^\sigma + \frac{c_5}{2}Q_\rho\tilde{Q}^\rho. \tag{3.128}$$

In this case there are a 5-parameter family of quadratic theories.
Obviously, for $c_i = 1$ and by constraining the connection to be torsionless and with vanishing curvature from the outset, (3.127) is equivalent to the E-H action since the eq.s (2.55) and (3.129), but they differ by a total derivative term:

$$\begin{aligned}
0 = \bar{R} = R + L \Longrightarrow R = -L = &\\
&= \frac{1}{4}Q_\lambda Q^\lambda - \frac{1}{2}Q_\lambda\tilde{Q}^\lambda - \frac{1}{4}Q_{\mu\lambda\alpha}Q^{\mu\lambda\alpha} + \frac{1}{2}Q_{\mu\lambda\alpha}Q^{\lambda\mu\alpha} - \nabla_\lambda(Q^\lambda - \tilde{Q}^\lambda)\\
&= -\mathbb{Q}(c_i=1) - \nabla_\lambda(\tilde{Q}^\lambda - Q^\lambda)\\
&= -\mathring{\mathbb{Q}} - \nabla_\lambda(\tilde{Q}^\lambda - Q^\lambda),
\end{aligned} \tag{3.129}$$

where $\mathring{\mathbb{Q}} \equiv \mathbb{Q}(c_i = 1)$.
As for TEGR it is more comfortable solving the constraints before getting the field equation. Again, the flatness condition requires the connection to be by a general element $e^a{}_\mu \in GL(4, \mathbb{R})$,

$$\bar{\Gamma}^\lambda{}_{\mu\nu} = e_a{}^\lambda\partial_\mu e^a{}_\nu. \tag{3.130}$$

The torsion related to the connection (3.130) is, again,

$$T^\lambda{}_{\mu\nu} = e_a{}^\lambda\partial_{[\mu}e^a{}_{\nu]}. \tag{3.131}$$

Requiring the torsionless condition, from eq. (3.131) one obtains the following constraint on the tetrads

$$T^\lambda{}_{\mu\nu} = e_a{}^\lambda\partial_{[\mu}e^a{}_{\nu]} = 0 \Rightarrow \partial_{[\mu}e^a{}_{\nu]} = 0. \tag{3.132}$$

Reversing the argument, $\{\hat{e}_a\}$ must be an holonomic frame. Therefore, the general element $e^a{}_\mu \in GL(4,\mathbb{R})$ can be parametrized by a set of functions $\xi^a = \xi^a(x^\mu)$ as follows

$$\bar{\Gamma}^\lambda{}_{\mu\nu} = e_a{}^\lambda \partial_\mu e^a{}_\nu = \frac{\partial x^\lambda}{\partial \xi^a}\partial_\mu\partial_\nu\xi^a. \tag{3.133}$$

The equation (3.133) shows that the connection can be trivialized by a coordinate transformation – see eq. (1.5). The possibility to choose ξ^a can be interpreted as a gauge freedom. The particular gauge where $\xi^\mu \equiv x^\mu$ is called *coincident gauge* and it corresponds to choose the origin of the tangent space coincident with the space-time origin. Therefore, one obtains:

$$0 = \bar{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu} \quad \Rightarrow \quad \Gamma^\lambda{}_{\mu\nu} = -L^\lambda{}_{\mu\nu} \quad \text{and} \quad Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}, \tag{3.134}$$

and then the action (3.127) reads [34, 37, 38]:

$$S_{CGR} = S_{\mathbb{Q}}(\bar{\Gamma}=0) = -\frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}g^{\mu\nu}(L^\alpha{}_{\alpha\lambda}L^\lambda{}_{\mu\nu} - L^\alpha{}_{\mu\lambda}L^\lambda{}_{\nu\alpha}) \tag{3.135}$$

$$= \frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}g^{\mu\nu}(\Gamma^\alpha{}_{\alpha\lambda}\Gamma^\lambda{}_{\mu\nu} - \Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\alpha}). \tag{3.136}$$

Coming back to STEGR, let us work with the following action

$$S_{STEGR} = \frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}\mathring{\mathbb{Q}}. \tag{3.137}$$

It is possible introduce a superpotential (non-metricity conjugate) $P^\alpha{}_{\mu\nu}$ s.t. $\mathring{\mathbb{Q}} = P^\alpha{}_{\mu\nu}Q_\alpha{}^{\mu\nu}$, defined as

$$P^\alpha{}_{\mu\nu} \equiv \frac{1}{4}\Big[-Q^\alpha{}_{\mu\nu} + 2Q_{(\mu}{}^\alpha{}_{\nu)} + Q^\alpha g_{\mu\nu} - \tilde{Q}^\alpha g_{\mu\nu} - \delta^\alpha_{(\mu}Q_{\nu)}\Big] = \tag{3.138}$$

$$= -\frac{1}{2}L^\alpha{}_{\mu\nu} + \frac{1}{4}\Big(Q^\alpha - \tilde{Q}^\alpha\Big)g_{\mu\nu} - \frac{1}{4}\delta^\alpha_{(\mu}Q_{\nu)}. \tag{3.139}$$

In order to calculate the variation of action with respect to the metric tensor, there are several useful equations:

$$L^\lambda{}_{\lambda\nu} = -\frac{1}{2}Q_\nu; \tag{3.140}$$

$$L^{\lambda\mu}{}_\mu = -\tilde{Q}^\lambda + \frac{1}{2}Q^\lambda; \tag{3.141}$$

$$\delta Q_{\alpha\mu\nu} = \bar{\nabla}_\alpha\delta g_{\mu\nu} = \nabla_\alpha\delta g_{\mu\nu} - 2L^\lambda{}_{\alpha(\mu|}\delta g_{\lambda|\nu)}; \tag{3.142}$$

$$\begin{aligned}
\delta Q_\alpha &= \delta(g^{\mu\nu}Q_{\alpha\mu\nu}) = Q_{\alpha\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta Q_{\alpha\mu\nu} = \\
&= -Q_\alpha{}^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\nabla_\alpha \delta g_{\mu\nu} - 2L^\mu{}_\alpha{}^\nu \delta g_{\mu\nu}; \qquad (3.143)\\
\delta\tilde{Q}_\nu &= \delta(g^{\alpha\mu}Q_{\alpha\mu\nu}) = -Q^{\alpha\mu}{}_\nu \delta g_{\alpha\mu} + g^{\alpha\mu}\nabla_\alpha \delta g_{\mu\nu} - 2g^{\alpha\mu}L^\lambda{}_{\alpha(\mu|}\delta g_{\lambda|\nu)}; \qquad (3.144)\\
\delta(Q_{\alpha\mu\nu}Q^{\alpha\mu\nu}) &= 2Q^{\alpha\mu\nu}\delta Q_{\alpha\mu\nu} - Q^{\mu\alpha\beta}Q^\nu{}_{\alpha\beta}\delta g_{\mu\nu} - 2Q^{\alpha\beta\mu}Q_{\alpha\beta}{}^\nu \delta g_{\mu\nu} = \\
&= 2\nabla_\alpha(Q^{\alpha\mu\nu}\delta g_{\mu\nu}) + \\
&+ (-2\nabla_\alpha Q^{\alpha\mu\nu} - 4Q^{\alpha\beta\mu}L^\nu{}_{\alpha\beta} - Q^{\mu\alpha\beta}Q^\nu{}_{\alpha\beta} - 2Q^{\alpha\beta\mu}Q_{\alpha\beta}{}^\nu)\delta g_{\mu\nu}; \qquad (3.145)\\
\delta(Q_{\alpha\mu\nu}Q^{\mu\alpha\nu}) &= 2Q^{\mu\alpha\nu}\delta Q_{\alpha\mu\nu} + (-Q^{\alpha\beta\mu}Q^\nu{}_{\alpha\beta} - Q^{\mu\alpha\beta}Q_{\alpha\beta}{}^\nu - Q^{\alpha\beta\mu}Q_{\beta\alpha}{}^\nu)\delta g_{\mu\nu} = \\
&= 2\nabla_\alpha(Q^{\mu\alpha\nu}\delta g_{\mu\nu}) + \\
&+ (-2\nabla_\alpha Q^{\mu\nu\alpha} - 2Q^{\alpha\beta\mu}L^\nu{}_{\alpha\beta} - 2Q^{\mu\alpha\beta}L^\nu{}_{\alpha\beta} + \\
&- 2Q^{\alpha\beta\mu}Q^\nu{}_{\alpha\beta} - Q^{\alpha\beta\mu}Q_{\beta\alpha}{}^\nu)\delta g_{\mu\nu}; \qquad (3.146)\\
\delta(Q_\alpha Q^\alpha) &= 2Q^\alpha \delta Q_\alpha - Q^\mu Q^\nu \delta g_{\mu\nu} = \\
&= 2\nabla_\alpha(Q^\alpha g^{\mu\nu}\delta g_{\mu\nu}) + \\
&+ (-2Q^\alpha Q_\alpha{}^{\mu\nu} - 2\nabla_\alpha(Q^\alpha g^{\mu\nu}) - 4Q^\alpha g^{\mu\beta}L^\nu{}_{\alpha\beta} - Q^\mu Q^\nu)\delta g_{\mu\nu}; \qquad (3.147)\\
\delta(\tilde{Q}^\alpha Q_\alpha) &= Q^\alpha \delta\tilde{Q}_\alpha + \tilde{Q}^\alpha \delta Q_\alpha - Q^\mu \tilde{Q}^\nu \delta g_{\mu\nu} = \\
&= \nabla_\alpha(Q^\nu g^{\alpha\mu}\delta g_{\mu\nu}) + \nabla_\alpha(\tilde{Q}^\alpha g^{\mu\nu}\delta g_{\mu\nu}) + \\
&+ (-Q^\alpha Q^{\mu\nu}{}_\alpha - \nabla_\alpha(Q^\nu g^{\alpha\mu}) - Q^\nu g^{\alpha\beta}L^\mu{}_{\alpha\beta} - Q^\beta g^{\alpha\mu}L^\nu{}_{\alpha\beta} + \\
&- \tilde{Q}^\alpha Q_\alpha{}^{\mu\nu} - \nabla_\alpha(\tilde{Q}^\alpha g^{\mu\nu}) - 2\tilde{Q}^\alpha g^{\mu\beta}L^\nu{}_{\alpha\beta} - Q^\mu \tilde{Q}^\nu)\delta g_{\mu\nu}. \qquad (3.148)
\end{aligned}$$

In this way, the variation of the action gives

$$2\nabla_\alpha P^{\alpha\mu\nu} - Q^{[\alpha\beta](\mu}Q_{\alpha\beta}{}^{\nu)} + \frac{1}{4}Q^{\mu\alpha\beta}Q^\nu{}_{\alpha\beta} + \frac{1}{4}Q^\alpha Q_\alpha{}^{\mu\nu} - \frac{1}{2}Q^\alpha Q^{(\mu\nu)}{}_\alpha + \frac{1}{2}g^{\mu\nu}\mathring{\mathbb{Q}} = 0, \quad (3.149)$$

which are equivalent to the E-H field equations by imposing the teleparallel condition:

$$-L_{(\mu\nu)} + 1/2 L g_{\mu\nu} = 0. \quad (3.150)$$

3.2.1 Extending STEGR

The most straightforward modification is the generalization of the action to an arbitrary function of the non-metricity scalar $\mathring{\mathbb{Q}}$, [69] just as with $f(R)$ and $f(\mathbb{T})$. The action of the theory reads

$$S_{f(\mathring{\mathbb{Q}})} = \int d^4x\sqrt{-g}f(\mathring{\mathbb{Q}}) + S_m. \quad (3.151)$$

Varying the action (3.151) with respect to the metric ($\delta g_{\mu\nu}$), it results

$$\delta S_{f(\mathring{\mathbb{Q}})} = \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}f(\mathring{\mathbb{Q}})\delta g_{\mu\nu} + f'(\mathring{\mathbb{Q}})\delta\mathring{\mathbb{Q}}\Big) + \delta S_m. \quad (3.152)$$

With the prime accounting for the derivative with respect to $Q_{\alpha\mu\nu}$, similarly to what happens in $f(R)$ and $f(\mathbb{T})$, the boundary terms written in the precedent variations are no longer such, due to the presence of the factor $f'(\mathring{\mathbb{Q}})$ in the second term of variation (3.152). Therefore, taking into account the variations used for the non-metricity scalar, one has to do a further integration by parts. Consequently, with respect to eq. (3.149), one can obtain the new field equations multiplying by $f'(\mathring{\mathbb{Q}})$ the superpotential $P^{\alpha\mu\nu}$ and the terms of the variations proportional to $\delta g_{\mu\nu}$, and replacing $\mathring{\mathbb{Q}}$ with $f(\mathring{\mathbb{Q}})$. Then, adding the matter contribute and dividing all by $f'(\mathring{\mathbb{Q}})$, the field equations read:

$$\begin{aligned}\frac{2\nabla_\alpha(f'(\mathring{\mathbb{Q}})P^{\alpha\mu\nu})}{f'(\mathring{\mathbb{Q}})} - Q^{[\alpha\beta](\mu}Q_{\alpha\beta}{}^{\nu)} + \frac{1}{4}Q^{\mu\alpha\beta}Q^{\nu}{}_{\alpha\beta} +\\ +\frac{1}{4}Q^\alpha Q_\alpha{}^{\mu\nu} - \frac{1}{2}Q^\alpha Q^{(\mu\nu)}{}_\alpha + \frac{1}{2}g^{\mu\nu}\frac{f(\mathring{\mathbb{Q}})}{f'(\mathring{\mathbb{Q}})} = \frac{T^{\mu\nu}}{f'(\mathring{\mathbb{Q}})},\end{aligned} \tag{3.153}$$

where $T^{\mu\nu} \equiv -\frac{1}{\sqrt{-g}}\delta S_m/\delta g_{\mu\nu}$. It seems that there are some interesting models, at least in cosmology that need further investigation.

In addition to $f(\mathring{\mathbb{Q}})$-theories, there are two other interesting extensions of the STEGR that it worth to mention in this Subsection [32].
It is possible to consider a function of every independent quadratic invariant form present inside the non-metricity scalar, getting the most general $f(\mathbb{Q})$ theory:

$$S = \int d^4x\sqrt{-g}f(A,\, B,\, C,\, D,\, E) + S_m, \tag{3.154}$$

where

$$A = Q_{\mu\nu\alpha}Q^{\mu\nu\alpha}, \quad B = Q_{\mu\nu\alpha}Q^{\nu\mu\alpha}, \quad C = Q_\lambda Q^\lambda, \quad D = \tilde{Q}_\sigma\tilde{Q}^\sigma, \quad E = Q_\rho\tilde{Q}^\rho. \tag{3.155}$$

Finally, similarly to the treatment of Scalar-Tensor theories, one can consider the possibility to include scalar fields [78, 79]. In particular, one can consider the action [32]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\,(\mathscr{L}_g + \mathscr{L}_\ell) + S_m(g_{\mu\nu},\, \psi), \tag{3.156}$$

where the gravitational Lagrangian density is given by

$$\mathscr{L}_g = -\mathscr{A}(\phi)\mathring{\mathbb{Q}} - \mathscr{B}(\phi)\partial_\mu\phi\partial^\mu\phi - 2\mathscr{V}(\phi), \tag{3.157}$$

the Lagrange multiplier terms are

$$\mathscr{L}_\ell = 2\lambda_\mu{}^{\nu\rho\sigma}\bar{R}^\mu{}_{\nu\rho\sigma} + 2\tilde{\lambda}_\alpha{}^{\mu\nu}T^\alpha{}_{\mu\nu}, \tag{3.158}$$

and the matter action depends only on the metric and matter fields coupled on it.
When $\mathscr{A} = 1$ and $\mathscr{B} = 0 = \mathscr{V}$ the theory (3.156) reduces to STEGR.

Chapter 4

Isometries, Conformal Transformations and Weyl Rescaling

In physics, to understand a system, *i.e.* to solve a problem, the starting point is to analyze the symmetries of the system under consideration. Symmetries help to examine and to simplify physical problems that otherwise would be unsolvable. One of the main objects of physics is metric tensor, particularly in GR. For this reason, physicists are interested in the transformations which leave unchanged the metric, or "almost". These transformations are linked to diffeomorphisms on the spacetime and they identify an equivalence class of frames and define a subgroup of diffeomorphisms.
In this Chapter, the attention is focused on the transformations called isometries and on conformal transformations of the metric [50, 80]. Then the dissertation will be focused on what is called Weyl rescaling, or Weyl transformation, or also conformal transformation of metric. The interest for this argument turns around the research of *conformally invariant theories*.

4.1 Isometries

Just to make the notation unambiguous, here some definitions are recalled.
Let $(M,\, g)$ be a pseudo-Riemannian manifold, under a general coordinate transformation, the components of metric tensor change in analogue way to any $(0,\, 2)$-type tensor:

$$\begin{aligned} g(p) &= g_{\mu\nu}(x_p)dx^\mu dx^\nu \\ &= \tilde{g}_{\mu\nu}(y_p)dy^\mu dy^\nu, \\ g_{\mu\nu}(x_p) &= \tilde{g}_{\rho\sigma}(y_p)\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}. \end{aligned} \tag{4.1}$$

Let $f: M \to M$, $\{x^\mu\}_p \to \{y^\alpha\}_{f(p)}$ be a diffeomorphism, then it induces on vectors and co-vectors the maps called respectively push-forward and pull-back, which act on the components of tensor like coordinate transformation. Let $g \in \mathscr{F}(M)$, $V \in T_pM$ and $\omega \in T^*_{f(p)}M$, then

$$f_* : T_pM \to T_{f(p)}M, \qquad (f_*V)[g] \equiv V[g \circ f], \qquad \tilde{V}^\alpha\big(f(p)\big) = V^\mu(p)\partial y^\alpha/\partial x^\mu, \tag{4.2}$$
$$f^* : T^*_{f(p)}M \to T^*_pM, \qquad (f^*\omega)[V] = \omega[f_*V], \qquad \tilde{\omega}_\mu(p) = \omega_\alpha\big(f(p)\big)\partial y^\alpha/\partial x^\mu. \tag{4.3}$$

According to the notation used, it is possible to define an *isometry* as a diffeomorphism, $f : M \to M$ which preserves the metric:

$$f^* g_{f(p)} = g_p, \tag{4.4}$$

or equivalently,

$$g_{f(p)}(f_*X,\, f_*Y) = g_p(X,\, Y) \qquad \text{for } X,\, Y \in T_pM. \tag{4.5}$$

Explicating the coordinate dependence, the condition (4.4) becomes

$$g_{\mu\nu}(p) = g_{\rho\sigma}\big(f(p)\big)\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}, \tag{4.6}$$

where $\{x^\mu\}$ and $\{y^\nu\}$ are the local charts in p and $f(p)$, respectively.
Easily, it is possible to observe that the set of all isometries forms a group [61]. From a more practical point of view, the eq. (4.4) states that an isometry preserves the *length* of a vector.
Let us consider an infinitesimal displacement, $f : x^\mu \mapsto x^\mu + \varepsilon X^\mu$, which preserves the metric. Then, according to (4.6) it satisfies

$$\frac{\partial(x^\rho + \varepsilon X^\rho)}{\partial x^\mu}\frac{\partial(x^\sigma + \varepsilon X^\sigma)}{\partial x^\nu} g_{\rho\sigma}(x + \varepsilon X) = g_{\mu\nu}(x). \tag{4.7}$$

Expanding $g_{\rho\sigma}(x + \varepsilon X)$ with respect to x at the first order in ε, then what follow are the *Killing equations*:

$$X^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu X^\lambda + g_{\lambda\nu}\partial_\mu X^\lambda = 0, \tag{4.8}$$

and X is called *Killing vector field*, infinitesimal generator of the isometry. Taking into account the eq. (2.68), Killing equations can be expressed in terms of the Lie derivative of the metric tensor

$$\big(L_X g\big)_{\mu\nu} = 0, \tag{4.9}$$

where $\big(L_X g\big)_{\mu\nu}$ are the component of $L_X g$. Moreover, as seen in Chapter 2, by using eq.s (2.71) and (2.74), it is possible to rearrange the Lie derivative and then the eq. (4.9) splits in the two alternatives form

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0, \tag{4.10}$$
$$\bar{\nabla}_\mu X_\nu + \bar{\nabla}_\nu X_\mu + 2(T_{\mu\ \ \nu}^{\ \ \lambda} - Q_{\mu\ \ \nu}^{\ \ \lambda} + \frac{1}{2}Q^\lambda{}_{\mu\nu)})X_\lambda = 0, \tag{4.11}$$

where ∇ the Levi-Civita connection and $\bar{\nabla}$ a general connection.
As already mentioned, the Lie derivative $L_X(\,\cdot\,)$ can be seen as a directional derivative along X direction if one uses the adapted frame to X. Then, from eq. (4.9) it follows that the metric does not depend on X. To be more precise, the metric tensor is unchanged along the integral curves of the vector field X, *i.e.* along the flow of X [61]. Moreover, setting a parametrization of the integral curves, the flow of X defines a 1-parameter group, $\phi_t : M \to M$ which generates the Killing vector field X [61]. This correspondence allows to interpret eq. (4.9) as an invariance of the local geometry along ϕ_t.
A metric can have different Killing vectors fields. However the the maximum number of linearly independent Killing vectors are limited by the dimension of the metric tensor [50]. To see that, it is necessary to consider the action of commutator of Levi-Civita connection on X_σ,

$$\nabla_\mu \nabla_\nu X_\rho - \nabla_\nu \nabla_\mu X_\rho = -R^\sigma{}_{\rho\mu\nu} X_\sigma. \tag{4.12}$$

Using the eq. (4.10), it reads

$$\nabla_\mu \nabla_\nu X_\rho + \nabla_\nu \nabla_\rho X_\mu = -R^\sigma{}_{\rho\mu\nu} X_\sigma. \tag{4.13}$$

Then, let us consider the following cyclic permutation of the eq. (4.13):

$$(+) \qquad \nabla_\mu \nabla_\nu X_\rho + \nabla_\nu \nabla_\rho X_\mu = -R^\sigma{}_{\rho\mu\nu} X_\sigma, \tag{4.14}$$

$$(-) \qquad \nabla_\nu \nabla_\rho X_\mu + \nabla_\rho \nabla_\mu X_\nu = -R^\sigma{}_{\mu\nu\rho} X_\sigma, \tag{4.15}$$

$$(+) \qquad \nabla_\rho \nabla_\mu X_\nu + \nabla_\mu \nabla_\nu X_\rho = -R^\sigma{}_{\nu\rho\mu} X_\sigma. \tag{4.16}$$

Summing these three permutation and using the first Bianchi identity, it provides

$$\nabla_\mu \nabla_\nu X_\rho = R^\sigma{}_{\mu\nu\rho} X_\sigma. \tag{4.17}$$

This identity allows to define a second order Cauchy problem for X^μ, with $n(n+1)/2$ initial conditions, by using the tensor $L_{\mu\nu} \equiv \nabla_\mu X_\nu$ [50]:

$$\begin{cases} \dfrac{D}{d\lambda} L_{\nu\rho} = T^\mu R^\sigma{}_{\mu\nu\rho} X_\sigma & (4.18) \\[2ex] \dfrac{D}{d\lambda} X_\nu = T^\mu L_{\mu\nu} & (4.19) \end{cases}$$

Thus, the number of Killing vector fields for a metric $g_{\mu\nu}$ corresponds to the number of possible linearly independent initial condition sets. A metric with $n(n+1)/2$ Killing vector fields is called *maximally symmetric metric.*

If $g_{\mu\nu} = \eta_{\mu\nu}$, it is immediate to see $\{X^\mu\}$ satisfy the equation

$$\partial_\mu X_\nu + \partial_\nu X_\mu = 0. \tag{4.20}$$

It easy to see that X^μ is, at most, of the first order in x, and the solutions are the generators of Poincaré group [50, 61].

4.2 Conformal Transformations

Another interesting set of transformations is that of conformal transformations. These transformations preserve the scalar product *up to a scale*, which implies that the angles between the vectors are unchanged.
The formal definition of a conformal transformation is a diffeomorphism $f : M \to M$ such that

$$f^* g_{f(p)} = e^{2\omega} g_p \qquad \omega \in \mathscr{F}(M). \tag{4.21}$$

In components, the condition (4.21) becomes

$$g_{\rho\sigma}\big(f(p)\big)\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} = e^{2\omega(p)} g_{\mu\nu}(p). \tag{4.22}$$

As for isometries, the set of conformal transformations is a group, called the *conformal group* $Conf(M)$. From eq. (4.22) it is possible to notice that the angle between two vectors is preserved; in other words, a conformal transformation changes the *scale* but not the *shape*:

$$\cos\theta = \frac{g_{\mu\nu}X^\mu Y^\nu}{\sqrt{g_{\rho\sigma}X^\rho X^\sigma g_{\lambda\eta}Y^\lambda Y^\eta}} = \frac{e^{2\omega}g_{\mu\nu}X^\mu Y^\nu}{\sqrt{e^{2\omega}g_{\rho\sigma}X^\rho X^\sigma e^{2\omega}g_{\lambda\eta}Y^\lambda Y^\eta}} = \cos\theta'. \tag{4.23}$$

Therefore, the isometries can be viewed as a special case of conformal transformations, with $\omega(p) = 0\ \forall p \in M$.
If $f : x^\mu \mapsto x^\mu + \varepsilon X^\mu$, ε being infinitesimal, is a conformal transformation, according to (4.22) it satisfies

$$\frac{\partial(x^\rho + \varepsilon X^\rho)}{\partial x^\mu}\frac{\partial(x^\sigma + \varepsilon X^\sigma)}{\partial x^\nu} g_{\rho\sigma}(x + \varepsilon X) = e^{2\sigma} g_{\mu\nu}(x), \tag{4.24}$$

where X is called *conformal Killing vector field.*
Making a first order expansion in ε with respect to x, then what follows is the conformal Killing equation:

$$\varepsilon\big[X^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu X^\lambda + g_{\lambda\nu}\partial_\mu X^\lambda\big] = (1 - e^{2\omega})g_{\mu\nu}, \tag{4.25}$$

hence,

$$\big(L_{\varepsilon X} g\big)_{\mu\nu} = (1 - e^{2\omega})g_{\mu\nu}. \tag{4.26}$$

Now, multiplying both the sides by $g^{\mu\nu}$, it results

$$(1 - e^{2\omega}) = \frac{\varepsilon}{n}\big[X^\lambda g^{\mu\nu}\partial_\lambda g_{\mu\nu} + 2\partial_\lambda X^\lambda\big] \tag{4.27}$$

where $n = \mathrm{dim}M$. If $g_{\mu\nu} = \eta_{\mu\nu}$, it is immediate to see $\{X^\mu\}$ satisfies the equation

$$\partial_\mu X_\nu + \partial_\nu X_\mu = \frac{2}{n}\eta_{\mu\nu}\partial \cdot X. \tag{4.28}$$

Multiplying by $\partial^\mu\partial_\lambda$ and relabelling the indices, it follows

$$\big[\eta_{\mu\nu}\Box + (n-2)\partial_\mu\partial_\nu\big](\partial\cdot X) = 0, \tag{4.29}$$

which is a "wave equation" for Killing conformal vector fields [61].
Due to the presence of third derivatives of X, the solution are quadratic in x, at most. For $n > 2$, X^μ corresponds to the generators of $SO(p+1,\, q+1)$, where $p+q=n$. In particular, for $n=4$ the solutions are the generators of the conformal group[1] $SO(2,\, 4)$:

- constant, translation, $X^\mu = a^\mu$;
- linear proportional, scale transformation, $X^\mu = cx^\mu$;
- linear, Lorentz transformation, $X^\mu = \omega^\mu{}_\nu x^\nu$;
- quadratic, conformal boost/special conformal transformation, $X^\mu = b^\mu x^2 - 2x^\mu b\cdot x$.

Again, it is possible to see the isometries like a special case of conformal transformations, as Poincaré group is a subgroup of the conformal one.
For $n=2$, the conformal algebra is infinite-dimensional (even if $SO(p+1,\, q+1)$ is again obtained obtained by exponentiation). This algebra is called *Witt algebra* or, if there is another object which commutates with every element of the algebra (*central charge*), it is named *Virasoro* algebra (the unique central extension of the Witt algebra).

4.3 Weyl Rescalings

A Weyl Rescaling is a transformation of the metric tensor such that

$$g_p \;\; \rightarrow \;\; \tilde{g}_p = e^{2\omega(p)} g_p. \tag{4.30}$$

The metric $\tilde{g}$ is said to be conformally related to g. Therefore, the eq. (4.30) defines an equivalence relation among the set of metrics on a manifold M. The equivalence class is called *the conformal structure*. The set of Weyl rescalings on M is a group denoted by $Weyl(M)$.
What is interesting is that a conformal transformation on a Lorentz manifold $(M,\, g)$ preserves the local light cone structure; therefore, if two metric manifold $(M,\, g)$ and $(M,\, \tilde{g})$ have identical causal structure, consequently $\tilde{g}_{\mu\nu}$ must be related to $g_{\mu\nu}$ by a Weyl rescaling, and *vice versa*.

Often, saying "conformal transformations" physicists means what we here called here Weyl rescalings. In the same way, most call "conformal isometries" what we here called

[1] The generators of the conformal group are $P_\mu = -i\partial_\mu$, $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, $D = -ix^\mu\partial_\mu$, $K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$.

here conformal transformations [50]. This sometimes makes some confusion about the two transformations. The ambiguity is "broken" when the conformal transformation is related to the metric, instead of coordinates. For this reason, in this Section no doubts occur, as only transformations of the metric tensor are considered.
However, before proceeding with the analysis of the Weyl rescalings, it is necessary to stress the difference that exists between them.
A Weyl rescaling is not a coordinate transformation on the spacetime; it is a physical change of the metric, $g_{\mu\nu}(x) \mapsto e^{2\omega(x)} g_{\mu\nu}(x)$ which changes the proper distances at each point by a local factor. It should be emphasized that a Weyl transformation is not, in general, associated with a diffeomorphism of M [50].
This transformation is far from being a metric symmetry, as it stands, in fact the metric and every objects connected to it change.
However, these transformations are taken into account due to the possibility that they could represent a new symmetry of the Universe. Therefore, there is a great interest in actions invariant under Weyl rescaling, which could give a more complete description of gravity.

The first susceptible object to Weyl transformation is GR connection, $\nabla \to \tilde{\nabla}$; to know how Christoffel symbols change it is enough to put the transformation $g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu} = e^{2\omega(x)} g_{\mu\nu}(x)$ in the eq. (1.4). Then, it results

$$\Gamma^{\lambda}{}_{\mu\nu} \to \tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + \left(\delta^{\lambda}_{\mu}\partial_{\nu}\omega + \delta^{\lambda}_{\nu}\partial_{\mu}\omega - g_{\mu\nu}\partial^{\lambda}\omega\right) = \Gamma^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}. \tag{4.31}$$

The additive term, $L^{\lambda}{}_{\mu\nu}$, can be seen as the disformation tensor with respect to the original metric $g_{\mu\nu}$. In this way, the nonmetricity tensor associated to $L^{\lambda}{}_{\mu\nu}$ is

$$Q_{\alpha\mu\nu} = \tilde{\nabla}_{\alpha} g_{\mu\nu} = -L_{\nu\alpha\mu} - L_{\mu\alpha\nu} = -2g_{\mu\nu}\partial_{\alpha}\omega \tag{4.32}$$

$$\Rightarrow Q_{\alpha} = g^{\mu\nu} Q_{\alpha\mu\nu} = -2n\partial_{\alpha}\omega \tag{4.33}$$

$$\Rightarrow \tilde{Q}_{\mu} = g^{\alpha\nu} Q_{\alpha\mu\nu} = -2\partial_{\mu}\omega \Rightarrow \tilde{Q}_{\mu} = Q_{\mu}/n. \tag{4.34}$$

It is possible use equation (4.31) to compare the geodesic with respect to ∇ with those with respect to $\tilde{\nabla}$.
Let γ be an affinely parameterized geodesic with respect to ∇, with tangent vector T,

$$\gamma : T^{\mu}\nabla_{\mu}T^{\nu} = 0. \tag{4.35}$$

Then, it follows

$$T^{\mu}\tilde{\nabla}_{\mu}T^{\nu} = 2T^{\nu}T^{\mu}\partial_{\mu}\omega - T^{\lambda}T_{\lambda}\partial^{\nu}\omega. \tag{4.36}$$

Thus, in general Weyl transformations does not preserve geodesics. However, in the case of null geodesic, $g_{\mu\nu}T^{\mu}T^{\nu} = T^{\mu}T_{\mu} = 0$, the equation (4.36) is just non-affinely parameterized geodesic equation (1.14) with $\alpha = 2T^{\mu}\partial_{\mu}\omega$. Taking into account (4.31) it is possible also to derive the Riemann and Ricci tensors transformation law using what we have seen in Chapter 2:

$$\begin{aligned}
\tilde{R}^{\mu}{}_{\nu\rho\sigma} &= R^{\mu}{}_{\nu\rho\sigma} - 2g_{\nu[\sigma}\nabla_{\rho]}\nabla^{\mu}\omega + 2\delta^{\mu}_{[\sigma}\nabla_{\rho]}\nabla_{\nu}\omega + \\
&\quad + 2\delta^{\mu}_{[\rho}\nabla_{\sigma]}\omega\nabla_{\nu}\omega - 2g_{\nu[\rho}\nabla_{\sigma]}\omega\nabla^{\mu}\omega - 2\nabla_{\lambda}\omega\nabla^{\lambda}\omega\delta^{\mu}_{[\rho}g_{\sigma]\nu}; \qquad (4.37)\\
\tilde{R}_{\mu\nu} &= R_{\mu\nu} - (n-2)\nabla_{\mu}\nabla_{\nu}\omega - g_{\mu\nu}\Box\omega + (n-2)\nabla_{\mu}\omega\nabla_{\nu}\omega - (n-2)g_{\mu\nu}\nabla_{\lambda}\omega\nabla^{\lambda}\omega; \qquad (4.38)\\
\tilde{R} &= \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} = e^{-2\omega}\big[R - 2(n-1)\Box\omega - (n-1)(n-2)\nabla_{\lambda}\omega\nabla^{\lambda}\omega\big]. \qquad (4.39)
\end{aligned}$$

where $\Box = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$, and $\partial_{\mu}\omega = \nabla_{\mu}\omega$ ($[\nabla_{\mu},\, \nabla_{\nu}]\omega = 0$).
If $n > 3$ it is possible to define the "trace-free part" of Riemann tensor, called *Weyl tensor* which satisfies some key proprieties of the Riemann tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{n-2}\big(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\big) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu}. \qquad (4.40)$$

Now, using the above written transformation laws, it is possible to verify that the Weyl tensor is unchanged by a conformal transformation of the metric, *i.e.*

$$\tilde{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}. \qquad (4.41)$$

It is important to notice that the equality crucially depends on the index position, indeed

$$\tilde{C}_{\mu\nu\rho\sigma} = \tilde{g}_{\mu\lambda}\tilde{C}^{\lambda}{}_{\nu\rho\sigma} = e^{2\omega}g_{\mu\lambda}\tilde{C}^{\lambda}{}_{\nu\rho\sigma} = e^{2\omega}C_{\mu\nu\rho\sigma} \qquad (4.42)$$

The Weyl tensor associated to Levi-Civita connection has the following key proprieties:

i. $C^{\mu}{}_{\nu\rho\sigma} = -C^{\mu}{}_{\nu\sigma\rho}$;

ii. $C^{\mu}{}_{[\nu\rho\sigma]} = 0$;

iii. $C_{\mu\nu\rho\sigma} = -C_{\nu\mu\rho\sigma}$;

iv. $\nabla_{\lambda}C^{\lambda}{}_{\nu\rho\sigma} = 2\frac{(n-3)}{n-2}\nabla_{[\rho}\Big(R_{\sigma]\nu} - \frac{R}{2(n-1)}g_{\sigma]\nu}\Big)$;

v. in vacuum, according to GR, $C^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma}$.

The property (*iv*) can be easily proved by using a contraction of the second Bianchi identity: $\nabla_{\lambda}R^{\lambda}{}_{\nu\rho\sigma} = 2\nabla_{[\rho}R_{\sigma]\nu}$.
At this point, it is interesting to consider the transformation of the quadratic contractions of the curvature tensors (looking for what could be a "*papabile*" quadratic conformally

invariant action):

$$
\begin{aligned}
(\tilde{R}^{\mu}{}_{\nu\rho\sigma})^2 = e^{-4\omega}\big[&(R^{\mu}{}_{\nu\rho\sigma})^2 + 8R^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega - 8R^{\mu\nu}\nabla_\mu\nabla_\nu\omega - 4R(\nabla_\lambda\omega)^2 + \\
&+ 4(n-2)(\nabla_\mu\nabla_\nu\omega)^2 - 8(n-2)(\nabla_\mu\nabla_\nu\omega)(\nabla^\mu\omega)(\nabla^\nu\omega) + \\
&+ 8(n-2)(\Box\omega)(\nabla_\lambda\omega)^2 + 2(n-2)(n-1)(\nabla_\lambda\omega)^4 + 4(\Box\omega)^2\big],
\end{aligned} \tag{4.43}
$$

$$
\begin{aligned}
(\tilde{R}_{\mu\nu})^2 = e^{-4\omega}\big[&(R_{\mu\nu})^2 - 2(n-2)R^{\mu\nu}\nabla_\mu\nabla_\nu\omega + 2(n-2)R^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega + \\
&- 2R\Box\omega - 2(n-2)R(\nabla_\lambda\omega)^2 + (3n-4)(\Box\omega)^2 + \\
&+ (n-1)(n-2)^2(\nabla_\lambda\omega)^4 + (n-2)^2(\nabla_\mu\nabla_\nu\omega)^2 + \\
&- 2(n-2)^2(\nabla_\mu\nabla_\nu\omega)(\nabla^\mu\omega)(\nabla^\nu\omega) + 2(n-2)(2n-3)(\Box\omega)(\nabla_\lambda\omega)^2\big],
\end{aligned} \tag{4.44}
$$

$$
\tilde{R}^2 = e^{-4\omega}\big[R - 2(n-1)\Box\omega - (n-1)(n-2)\nabla_\lambda\omega\nabla^\lambda\omega\big]^2, \tag{4.45}
$$

then, from the invariance of the Weyl tensor, we have

$$
(\tilde{C}^{\mu}{}_{\nu\rho\sigma})^2 = e^{-4\omega}(C^{\mu}{}_{\nu\rho\sigma})^2, \tag{4.46}
$$

$$
(C^{\mu}{}_{\nu\rho\sigma})^2 = (R^{\mu}{}_{\nu\rho\sigma})^2 - \frac{4}{n-2}(R_{\mu\nu})^2 + \frac{2}{(n-1)(n-2)}R^2. \tag{4.47}
$$

For reasons that will be clear later, it is interesting to observe the transformation law of the following quadratic term:

$$
\mathscr{G} = (R^{\mu}{}_{\nu\rho\sigma})^2 - 4(R_{\mu\nu})^2 + R^2, \tag{4.48}
$$

$$
\begin{aligned}
\tilde{\mathscr{G}} = e^{-4\omega}\big[&\mathscr{G} - 8(n-3)R^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega + 8(n-3)R^{\mu\nu}\nabla_\mu\nabla_\nu\omega + \\
&- 2(n-3)(n-4)R(\nabla_\lambda\omega)^2 - 4(n-3)R\Box\omega + 4(n-3)(n-2)(\Box\omega)^2 + \\
&+ 8(n-2)(n-3)(\nabla_\mu\nabla_\nu\omega)(\nabla^\mu\omega)(\nabla^\nu\omega) - 4(n-2)(n-3)(\nabla_\mu\nabla_\nu\omega)^2 \\
&+ 4(n-2)(n-3)^2(\Box\omega)(\nabla_\lambda\omega)^2 + (n-1)(n-2)(n-3)(n-4)(\nabla_\lambda\omega)^4\big].
\end{aligned} \tag{4.49}
$$

For the specific case of a 4-dimensional manifold, $\mathscr{G}$ transforms as

$$
\begin{aligned}
\tilde{\mathscr{G}} =& e^{-4\omega}\big[\mathscr{G} - 8R^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega + 8R^{\mu\nu}\nabla_\mu\nabla_\nu\omega - 4R\Box\omega + \\
&+ 8(\Box\omega)^2 + 16(\nabla_\mu\nabla_\nu\omega)(\nabla^\mu\omega)(\nabla^\nu\omega) - 8(\nabla_\mu\nabla_\nu\omega)^2 + 8(\Box\omega)(\nabla_\lambda\omega)^2\big].
\end{aligned} \tag{4.50}
$$

All these terms emerge frequently as quantum corrections to GR, in particular in the frameworks of semiclassical approach to gravity and in string theory [81].

4.3.1 Conformal Invariance

An equation for a field ψ is said to be *conformally invariant* (with respect to Weyl) if there exists a number $s \in \mathbb{R}$ (called the *conformal weight* of the field) such that ψ is a solution with metric $g_{\mu\nu}$ iff $\tilde{\psi} = e^{s\omega}\psi$ is a solution with the metric $\tilde{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu}$.
The two traditional examples of conformally invariant equations are the generalization of

massless Klein-Gordon scalar field and the electromagnetic field.

Let ϕ be a scalar field, solution of generalised Laplace's equation in a curved spacetime:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0. \tag{4.51}$$

It is easy show the above equation can be conformally invariant only for $n = 2$ choosing $s = 0$:

$$\begin{aligned}
\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\phi} &= e^{-2\omega}g^{\mu\nu}\tilde{\nabla}_\mu(\nabla_\nu e^{s\omega}\phi) = \\
&= e^{-2\omega}g^{\mu\nu}\big[\nabla_\mu\nabla_\nu(e^{s\omega}\phi) - L^\lambda{}_{\mu\nu}\nabla_\lambda(e^{s\omega}\phi)\big] = \\
&= e^{-2\omega}g^{\mu\nu}\big[\nabla_\mu(e^{s\omega}s\nabla_\nu\omega\phi + e^{s\omega}\nabla_\nu\phi) - se^{s\omega}\phi L^\lambda{}_{\mu\nu}\nabla_\lambda\omega - e^{s\omega}L^\lambda{}_{\mu\nu}\nabla_\lambda\phi\big] = \\
&= e^{(s-2)\omega}g^{\mu\nu}\big[s^2\nabla_\mu\omega\nabla_\nu\omega\phi + s\nabla_\mu\nabla_\nu\omega\phi + 2s\nabla_\mu\omega\nabla_\nu\phi + \nabla_\mu\nabla_\nu\phi + \\
&\quad - s\phi\big(2\delta^\lambda_\mu\nabla_\nu\omega - g_{\mu\nu}\nabla^\lambda\omega\big)\nabla_\lambda\omega - \big(2\delta^\lambda_\mu\nabla_\nu\omega - g_{\mu\nu}\nabla^\lambda\omega\big)\nabla_\lambda\phi\big] = \\
&= e^{(s-2)\omega}g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \\
&\quad + e^{(s-2)\omega}\big[s\phi\Box\omega + s\phi(n-2+s)\nabla_\lambda\omega\nabla^\lambda\omega + (n-2+2s)\nabla_\lambda\omega\nabla^\lambda\phi\big],
\end{aligned} \tag{4.52}$$

noting that $\tilde{\nabla}_\mu\phi = \partial_\mu\phi = \nabla_\mu\phi$. Thus, if $n = 2$ it is possible to choose $s = 0$, then $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\phi} = 0$ iff $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0$; if $n \neq 2$, there is no choice of s that makes conformally invariant the equation. However, it is possible to modify the eq. (4.51) for $n > 1$, so that it becomes conformally invariant: choosing $s = 1 - n/2$ and adding $\alpha R\phi$ with $\alpha = -(n-2)/4(n-1)$, namely

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{(n-2)}{4(n-1)}R\phi = 0, \tag{4.53}$$

then

$$\begin{aligned}
-\frac{(n-2)}{4(n-1)}\tilde{R}e^{(s-2)\omega}\phi &= -\frac{(n-2)}{4(n-1)}e^{(s-2)\omega}\big[R - 2(n-1)\Box\omega - (n-1)(n-2)\nabla_\lambda\omega\nabla^\lambda\omega\big]\phi = \\
&= e^{(s-2)}\Big[-\frac{(n-2)}{4(n-1)}R\phi - s\phi\Box\omega + s^2\phi\nabla_\lambda\omega\nabla^\lambda\omega\Big].
\end{aligned} \tag{4.54}$$

The equation (4.53) is the conformally invariant generalization to curved spaces of the Laplace and Klein-Gordon equation in flat spaces.

Unlike the scalar field ϕ, the Maxwell equations are conformally invariant only in $n = 4$ assuming a conformal weight $s = 0$:

$$\begin{aligned}
g^{\mu\lambda}\nabla_\lambda F_{\mu\nu} = 0 \to \tilde{g}^{\mu\lambda}\nabla_\lambda\tilde{F}_{\mu\nu} &= e^{-2\omega}g^{\mu\lambda}\tilde{\nabla}_\lambda(e^{s\omega}F_{\mu\nu}) = \\
&= e^{(s-2)\omega}g^{\mu\lambda}\big[\nabla_\lambda F_{\mu\nu} - L^\rho{}_{\lambda\mu}F_{\rho\sigma} - L^\rho{}_{\lambda\nu}F_{\mu\rho} + sF_{\mu\nu}\nabla_\lambda\omega\big] \\
&= e^{(s-2)\omega}g^{\mu\lambda}\big[\nabla_\lambda F_{\mu\nu} + (n-4+s)F_{\mu\nu}\nabla_\lambda\omega\big];
\end{aligned} \tag{4.55}$$

$$\nabla_{[\lambda}F_{\mu\nu]} = 0 \to \tilde{\nabla}_{[\lambda}(e^{s\omega}F_{\mu\nu]}) = e^{s\omega}\big[\nabla_{[\lambda}F_{\mu\nu]} + sF_{[\mu\nu}\nabla_{\lambda]}\omega\big]. \tag{4.56}$$

Moreover, it is easy to note that the Einstein tensor is not invariant under Weyl transformations and this has the consequence that Einstein's field equations are not conformally invariant. There are only two exceptions: the energy-momentum tensor transforms so as to counter the anomalous transformation of the Einstein tensor, or the case in which the conformal transformation is an isometry of the metric, acting as diffeomorphism and leaving all tensor equations invariant.
On the other hand, it is necessary to observe that the energy-momentum tensor conservation $\nabla_\mu T^{\mu\nu} = 0$ is conformally invariant only under special restrictions. To show that, let us evaluate the covariant derivative of the conformally transformed Energy-Momentum tensor:

$$\begin{aligned}\tilde{\nabla}_\mu(e^{s\omega}T^{\mu\nu}) &= e^{s\omega}\big[\nabla_\mu T^{\mu\nu} + L^\mu{}_{\mu\lambda}T^{\lambda\nu} + L^\nu{}_{\mu\lambda}T^{\mu\lambda} + sT^{\mu\nu}\nabla_\mu\omega\big] = \\ &= e^{s\omega}\big[\nabla_\mu T^{\mu\nu} + (n+2+s)\nabla_\mu\omega T^{\mu\nu} - T\nabla^\nu\omega\big] = \\ &= e^{s\omega}\big[\nabla_\mu T^{\mu\nu} + \big((n+2+s)T^{\mu\nu} - Tg^{\mu\nu}\big)\nabla_\mu\omega\big],\end{aligned} \tag{4.57}$$

where $T = g_{\mu\nu}T^{\mu\nu}$. Thus, the conservation of energy-momentum tensor is conformally invariant iff $T = 0$ and $s = -n-2$, or conversely, if $T^{\mu\nu} = Tg^{\mu\nu}/(n+2+s) \Rightarrow s = -2$. This problem can be possibly overcame by introducing the cosmological constant (asking for a peculiar transformation law for $\Lambda g_{\mu\nu}$).

Chapter 5

Classical Conformally Invariant Gravity

With the aim to unify GR and electromagnetism, Weyl replaced the geometrical view of GR with a formalism such that the electromagnetic field plays a geometric role. In this way the equations of motions are invariant under a gauge/conformal transformations.
Weyl received the biggest criticism from Einstein: he believed that it was unacceptable that Weyl's covariant derivative did not preserve the metric and, as a consequence, the lengths of parallel transported vectors.
However, Weyl's work led to think conformal invariance as a more general physical symmetry, a *fundamental symmetry of spacetime*. Over time, the accumulation of studies and works has pushed more and more towards this direction, and here are some of them:

- the analysis of Maxwell's equations which are invariant both under the Lorentz group and under the conformal group (only for n=4), together with invariance of the causal structure under conformal transformations;
- the I.E. Segal's observation about the conformal group that does not result as a limit of other Lie algebra, while the Galilei group is a limiting case of the Poincaré group, and the Poincaré group comes from a contraction of the conformal group [82];
- the Penrose's conformal treatment of infinity which shows how to use a conformal transformation to "*compactify*" an infinite spacetime to a finite region [83];
- the development of the Conformal Field Theory;
- the link between the conformal invariance in a flat spacetime and the invariance under Weyl rescaling in a curved spacetime;
- the Hawking's study of a new topology of curved space–time which incorporates the causal, differential, and conformal structures [84];

These are just some of the steps taken in the search for a theory based on the invariance under conformal transformations. Therefore, the interest in conformal invariance was fueled by different ideas and characters from the world of physics.
Moreover, the study of conformal invariance can foster progress, not only in classical

gravitation, but can open new roads to quantum theories of gravity. In this regard, a conformal theory could give a mass scaling law for high-energy (ultrarelativistic) systems. Because of the dimensionless of coupling constants [85], a conformal field theory could be the right renormalizable theory. Moreover, a conformal quantum theory of gravity could bypass any singularities and/or horizons, pushing them onto infinity in particular classes of coordinates systems [86]. Additionally, new conformally invariant actions could have interesting counterpart in String Theory and in the AdS/CFT correspondence (acronym of "*Anti-de Sitter/Conformal Field Theory*"). Finally, conformal invariance could be the theoretical constraint which is necessary to compensate for the lack of experimental evidences in quantum gravity domain.
On the other hand, a conformal theory of gravity has two main problems: the presence of the massive particles in the Universe which breaks the conformal symmetry and the *conformal anomaly*, namely the difficulty in preserving conformal symmetry in the quantization process (the correction can give a non-null trace of the energy-momentum tensor).

However, under the assumption that GR and the SM are only low-energy approximations of a "*more fundamental theory*", a legitimate suspicion is that the conformal symmetry characterizes this "more fundamental theory". Its spontaneous breaking could produce the today observed phenomenology, in accordance with GR and SM. Obviously the breaking of symmetry can take place in different ways and give rises to different phenomena (in presence of anisotropies and/or non-homogeneities). This argument could be linked to the disparity between matter and anti-matter and the observations associated with the presence of DE and DM.

In this Chapter, two important classical theories of conformal gravity will be quickly developed. They preserve the "*pleasant*" characteristics necessary in a theory of gravity[1]: *scalar-tensor Weyl gauge gravity* and *squared-Weyl tensor gravity*.

5.1 Weyl's Conformally Invariant Geometry

Starting from GR, Weyl defined a covariant derivative under conformal transformation of metric, using the so called *Weyl vector*[2], which transforms like a gauge potential under

[1]Gravity as covariant metric theory in which the metric describes the gravitational field (EEP) and the geometry in the proximity of the Sun is given by the Schwarzschild metric. The differences are the way in which the immediate presence of mass-energy in some region influences the gravitational field.

[2]This nomenclature is a little bit improper; it would be better to call it *Weyl covector* W_μ, because the definition of the Weyl vector is from its "covariant" components, *i.e.* $W_\mu = g_{\mu\nu}W^\nu$, in analogy with the e.m. potential field A_μ, associated to the 1-form $A = A_\mu dx^\mu$.

Weyl rescaling:

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + \big(g_{\mu\nu}W^{\lambda} - \delta^{\lambda}_{\mu}W_{\nu} - \delta^{\lambda}_{\nu}W_{\mu}\big) = \Gamma^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}; \tag{5.1}$$

$$Q_{\alpha\mu\nu} = \tilde{\nabla}g_{\mu\nu} = 2W_{\alpha}g_{\mu\nu}. \tag{5.2}$$

Then, under conformal rescaling

$$g_{\mu\nu} \to g'_{\mu\nu} = e^{2\omega}g_{\mu\nu} \tag{5.3}$$

$$W_{\mu} \to W'_{\mu} = W_{\mu} + \partial_{\mu}\omega \tag{5.4}$$

the connection coefficients are unchanged, due to $\delta\Gamma = -\delta L$, while nonmetricity changes covariantly

$$\tilde{\Gamma}'^{\lambda}{}_{\mu\nu} \to \tilde{\Gamma}^{\lambda}{}_{\mu\nu}, \qquad \tilde{\nabla}_{\alpha}g'_{\mu\nu} = 2W'_{\alpha}g'_{\mu\nu}. \tag{5.5}$$

Weyl vector, similarly to electromagnetic potential, is not trivial, i.e. it is not an exact differential form $W_{\mu} \neq \partial_{\mu}f$.
However, if $W = W_{\mu}dx^{\mu}$ is an exact differential form (*pure gauge case*), it is possible to perform a conformal transformation on the metric such that $W'_{\mu} = 0$, by means of the choice $\omega = -f$; then $\tilde{\nabla}$ turns out to be the Levi-Civita connection associated to $e^{-2f}g_{\mu\nu}$.
If the Weyl vector is a closed form, *i.e.* $W : dW = 0$, then $\partial_{[\mu}W_{\nu]} = 0 \Rightarrow \nabla_{[\mu}W_{\nu]} = \tilde{\nabla}_{[\mu}W_{\nu]} = 0$, for the Poincaré lemma W is locally exact on any contractible domain. Therefore, $\tilde{\nabla}$ can be locally metric-compatible but two observers in different simply connected neighbourhoods disagree on the form of such a metric and the two metrics turn out to be conformally related. Moreover, in this case W can be related to the metric itself as $\nabla_{(\mu}W_{\nu)} \propto g_{\mu\nu}$ and if $\nabla_{(\mu}W_{\nu)} = 1/n\big[W^{\lambda}\partial_{\lambda}\ln|g| + 2\partial_{\lambda}W^{\lambda}\big]g_{\mu\nu}$; the Weyl's connection can be transformed in the Levi-Civita one by performing an *infinitesimal* conformal transformation.
Using the relations obtained in Chapter 2, it is possible to get the expression of the Riemann tensor associated to the covariant Weyl connection:

$$\tilde{R}^{[\mu\nu]}{}_{\rho\sigma} = R^{[\mu\nu]}{}_{\rho\sigma} + 4\delta^{[\mu}_{[\rho}\nabla_{\sigma]}W^{\nu]} + 4\delta^{[\mu}_{[\rho}W_{\sigma]}W^{\nu]} - 2W^2\delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}; \tag{5.6}$$

$$\tilde{R}^{(\mu\nu)}{}_{\rho\sigma} = -2g^{\mu\nu}\nabla_{[\rho}W_{\sigma]} = -2g^{\mu\nu}\mathcal{W}_{\rho\sigma}; \tag{5.7}$$

$$\tilde{R}_{(\mu\nu)} = R_{\mu\nu} + (n-2)\nabla_{(\mu}W_{\nu)} + (n-2)W_{\mu}W_{\nu} + g_{\mu\nu}\big(\nabla_{\lambda}W^{\lambda} - (n-2)W^2\big); \tag{5.8}$$

$$\tilde{R}_{[\mu\nu]} = -n\nabla_{[\mu}W_{\nu]} = -n\mathcal{W}_{\mu\nu}; \tag{5.9}$$

$$\tilde{R} = R - (n-1)(n-2)W^2 + 2(n-1)\nabla_{\lambda}W^{\lambda}; \tag{5.10}$$

where $\mathcal{W}_{\mu\nu} = \nabla_{[\mu}W_{\nu]} = \partial_{[\mu}W_{\nu]}$ is proportional to the *homothetic curvature.* Note the consistency with the equations (4.37), (4.38) and (4.39), substituting $W_{\mu} = -\nabla_{\mu}\omega$.

While Riemann and Ricci tensors end up being unchanged under Weyl rescaling (due to the conformal invariance of the connection), the curvature scalar changes in the following way

$$\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu} \to \tilde{R}' = e^{-2\omega}\tilde{R}. \tag{5.11}$$

This shows that the Weyl connection is not sufficient to get a conformally invariant version of E-H action, because under conformal transformation of metric $\sqrt{-g} \to e^{n\omega}\sqrt{-g}$. Strictly thinking to the electromagnetic potential, for $n = 4$,the only way to have covariantly invariant action is to consider quadratic terms in curvature.

5.2 Scalar-Tensor Weyl Gauge Gravity

A different approach is to consider a scalar-tensor theory of gravity, introducing a field ϕ of conformal weight $1 - n/2$ and coupling it quadratically to the curvature [24, 27, 29]:

$$S = \int d^n x \sqrt{-g}\big(\tilde{R}\phi^2\big). \tag{5.12}$$

It is useful to explicit the form of $\tilde{R}$:

$$S = \int d^n x \sqrt{-g}\big[R\phi^2 + \big(-(n-1)(n-2)W^2 + 2(n-1)\nabla_\lambda W^\lambda\big)\phi^2\big]. \tag{5.13}$$

Varying with respect to W_μ it is easy to derive its equations of motion,

$$W_\mu = -\frac{2}{n-2}\frac{\partial_\mu \phi}{\phi}, \tag{5.14}$$

which obviously is conformally invariant. This equation says that the Weyl vector field is exact in this theory. Hence, varying the action (5.13) with respect to metric we get the field equations,

$$\begin{aligned} G_{\mu\nu} = \frac{1}{\phi^2}\Big[&- g_{\mu\nu}\Box\phi^2 + \nabla_\mu\nabla_\nu\phi^2 + (n-1)(n-2)W_\mu W_\nu \phi^2 + \\ &- (n-1)\big(\phi\nabla_\lambda W^\lambda + 2\phi W^\lambda \nabla_\lambda \phi\big)g_{\mu\nu} + 4(n-1)\phi W_{(\mu}\nabla_{\nu)}\phi \\ &- \frac{\phi^2}{2}(n-1)(n-2)W^2 g_{\mu\nu} + (n-1)\nabla_\lambda W^\lambda \phi^2 g_{\mu\nu}\Big]. \end{aligned} \tag{5.15}$$

Now, substituting the equations of motion of W_μ inside the field equations, it is possible to cancel out the contribution of the Weyl vector,

$$G_{\mu\nu} = \frac{1}{F(\phi)}\Big[\nabla_\mu\nabla_\nu F(\phi) - g_{\mu\nu}\Box F(\phi) - \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{4}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi\Big], \tag{5.16}$$

where

$$F(\phi) = \frac{n-2}{8(n-1)}\phi^2. \tag{5.17}$$

Therefore, the action (5.12) is equivalent to the Brans-Dicke action (1.33). The same result is achieved if, starting from E-H action, the following conformal transformation is performed

$$g_{\mu\nu} \to \phi^2 g_{\mu\nu}. \tag{5.18}$$

Then, using the equation (4.39), it results

$$S = \int d^n x \sqrt{-g} R \rightarrow \int d^n x \sqrt{-g} \phi^{n-2} \Big[R - \frac{n-1}{\phi^2} \big((n-4) \nabla_\mu \phi \nabla^\mu \phi + 2 \phi \Box \phi \big) \Big] \tag{5.19}$$

$$\simeq \int d^n x \sqrt{-g} \Big[\phi^{n-2} R + (n-1)(n-2) \phi^{n-4} \nabla_\mu \phi \nabla^\mu \phi \Big], \tag{5.20}$$

which, after the redefinition of the scalar field $\phi \rightarrow \phi^{2/(n-2)}$, turns out to be proportional to (1.33). Then, the action exactly coincide with the one in (1.33), if we consider the transformation $\phi \rightarrow \phi \sqrt{\frac{(n-2)}{8(n-1)}}$. However, the issue of any conformally invariant theory is to explain the appearance of a mass. Even in this case, the conservation of energy-momentum tensor leads to a matter source with $T = 0$, as seen in the previous Chapter. Considering the field equations with energy–momentum tensor and the equations of motion associated to the scalar field ϕ,

$$G_{\mu\nu} = \frac{T_{\mu\nu}}{F(\phi)} + \frac{T^{(\phi)}_{\mu\nu}}{F(\phi)} \tag{5.21}$$

$$\Box \phi - \frac{(n-2)}{4(n-1)} R \phi = 0 \tag{5.22}$$

(where $F(\phi) = \phi^2 (n-2)/8(n-1)$ and $T^{(\phi)}_{\mu\nu} = \big[\nabla_\mu \nabla_\nu F(\phi) - g_{\mu\nu} \Box F(\phi) - \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{4} g_{\mu\nu} \nabla_\rho \phi \nabla^\rho \phi \big]$) and taking the trace, it results

$$\frac{2-n}{2} R = \frac{T}{F(\phi)} - \frac{n-2}{2} R \quad \Rightarrow \quad T = 0. \tag{5.23}$$

This result holds in all dimension (and not just for $n = 4$).
Therefore, the theory requires some modifications to overcome the problem of energy-momentum tensor null trace. However, any modification causes conformal symmetry breaking. A simple example is the addition of a massive therm for the scalar field, $m\phi^2/2 \Rightarrow T \propto -m^2 \phi^2$ which allows to eliminate ϕ from the theory.

5.3 Weyl-Squared Theory

To find a conformally invariant action of gravity, the natural suggestion would be to consider an action made of squares of the Weyl tensor (4.40), which is conformally invariant, as seen in the previous Chapter. From the joint work of Rudolf Bach and Hermann Weyl, we known that there exists a unique conformally invariant action constructed from the Weyl tensor in four dimensions, the Weyl-Squared action:

$$S = \int d^4x \sqrt{-g} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}, \tag{5.24}$$

where

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{4}{n-2} R_{\mu\nu} R^{\mu\nu} + \frac{2}{(n-2)(n-1)} R^2. \tag{5.25}$$

For $n = 4$,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}\right) + \frac{1}{3} R g_{\mu[\rho} g_{\sigma]\nu}, \tag{5.26}$$

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2. \tag{5.27}$$

Equations of motion associated to the action (5.24) are called *Bach equation*.
It can be obtained with the help of the usual relations, here reported in a more practical form,

$$\delta g_{\rho\sigma} = -g_{\rho\mu} g_{\sigma\nu} \delta g^{\mu\nu}, \tag{5.28}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \tag{5.29}$$

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\nabla_\mu \delta g_{\rho\nu} + \nabla_\nu \delta g_{\rho\mu} - \nabla_\rho \delta g_{\mu\nu}\right), \tag{5.30}$$

$$\delta R^{\mu}{}_{\nu\rho\sigma} = \nabla_\rho \delta\Gamma^{\mu}{}_{\sigma\nu} - \nabla_\sigma \delta\Gamma^{\mu}{}_{\rho\nu}, \tag{5.31}$$

$$\delta R_{\mu\nu} = \nabla^\rho \nabla_{(\mu} \delta g_{\nu)\rho} - \frac{1}{2} \nabla_\rho \nabla^\rho \delta g_{\mu\nu} - \frac{1}{2} g^{\rho\sigma} \nabla_{(\mu} \nabla_{\nu)} \delta g_{\rho\sigma}, \tag{5.32}$$

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu} + g_{\mu\nu} \nabla_\rho \nabla^\rho \delta g^{\mu\nu}, \tag{5.33}$$

$$\delta(R^{\mu}{}_{\nu\rho\sigma} R_{\mu}{}^{\nu\rho\sigma}) = 4 R_{(\mu|\rho|\nu)\sigma} \nabla^\sigma \nabla^\rho \delta g^{\mu\nu} + 2 R^{\alpha}{}_{\beta\lambda\mu} R_{\alpha}{}^{\beta\lambda}{}_{\nu} \delta g^{\mu\nu}, \tag{5.34}$$

$$\delta(Ric^2) = R_{\mu\nu} \nabla^\rho \nabla_\rho \delta g^{\mu\nu} + R^{\rho\sigma} g_{\mu\nu} \nabla_\rho \nabla_\sigma \delta g^{\mu\nu} - 2 R_{\rho(\mu} \nabla_{\nu)} \nabla^\rho \delta g^{\mu\nu} + 2 R_{\mu\rho} R^{\rho}{}_{\nu} \delta g^{\mu\nu}, \tag{5.35}$$

$$\delta(R^2) = 2 R R_{\mu\nu} \delta g^{\mu\nu} - 2 R \nabla_{(\mu} \nabla_{\nu)} \delta g^{\mu\nu} + 2 R g_{\mu\nu} \nabla^\rho \nabla_\rho \delta g^{\mu\nu}, \tag{5.36}$$

together with the Bianchi identities and their derivatives/contractions,

$$\nabla_\lambda R^\lambda{}_\mu = \frac{1}{2}\nabla_\mu R, \tag{5.37}$$

$$\nabla_\mu \nabla_\nu R^{\mu\nu} = \frac{1}{2}\Box R, \tag{5.38}$$

$$\nabla_\mu R^\mu{}_{\nu\rho\sigma} = \nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho}, \tag{5.39}$$

$$\nabla^\rho \nabla^\sigma R_{\mu\rho\nu\sigma} = \Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu \nabla_\nu R + R_{\mu\rho\nu\sigma}R^{\rho\sigma} - R_{\mu\rho}R^\rho{}_\nu. \tag{5.40}$$

To be more general, instead of calculating the field equations associated with (5.24), let us consider the following Lagrangian

$$\mathscr{L} = \sqrt{-g}\Big(aR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + bR_{\mu\nu}R^{\mu\nu} + cR^2\Big). \tag{5.41}$$

The eq. (5.41) represent a 3-parameter Lagrangian family[3]. It provides back the Weyl-Squared scalar if the coefficients are set as ($a = 1$, $b = -2$, $c = 1/3$). The field equations of ((5.41) have the form:

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}\big(aR_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} + bR_{\rho\sigma}R^{\rho\sigma} + cR^2\big)+\\
&+4aR_{\mu\rho\nu\sigma}R^{\rho\sigma} + 2aR^\alpha{}_{\beta\lambda\mu}R_\alpha{}^{\beta\lambda}{}_\nu - 2b\nabla^\rho\nabla_{(\mu}R_{\nu)\rho} + 2cRR_{\mu\nu} + (-4a+2b)R_{\mu\rho}R^\rho{}_\nu+\\
&+(4a+b)\Box R_{\mu\nu} + (-2a-2c)\nabla_\mu\nabla_\nu R + \Big(\frac{1}{2}b + 2c\Big)\Box R g_{\mu\nu} = 0.
\end{aligned} \tag{5.42}$$

Then, substituting ($a = 1$, $b = -2$, $c = 1/3$), Bach equations turn out to be

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}\big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\big)+\\
&+4R_{\mu\rho\nu\sigma}R^{\rho\sigma} + 2R^\alpha{}_{\beta\lambda\mu}R_\alpha{}^{\beta\lambda}{}_\nu + 4\nabla^\rho\nabla_{(\mu}R_{\nu)\rho} + \frac{2}{3}RR_{\mu\nu}+\\
&-8R_{\mu\rho}R^\rho{}_\nu + 2\Box R_{\mu\nu} - \frac{8}{3}\nabla_\mu\nabla_\nu R - \frac{1}{3}\Box R g_{\mu\nu} = 0.
\end{aligned} \tag{5.43}$$

However, eq. (5.43) is not the final form of the Bach equation. To archive it, we need to introduce the *Gauss-Bonnet* Lagrangian [24, 52, 88–90], which belongs to the three-parameters family of (5.41):

$$\mathscr{L}_{GB} = \sqrt{-g}\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\Big). \tag{5.44}$$

[3] The generic quadratic Lagrangian (5.24) was, and still is, an interesting subject of study.It worth mentioning K.S. Stelle's work. In [87], he shows that actions including quadratic terms in the curvature tensor are renormalizable in Quantum Gravity.

The peculiarity of this Lagrangian is that the corresponding action is a topological term which can be written as a divergence (boundary term); this term is called *Gauss-Bonnet* (or *Chern-Gauss-Bonnet*) term

$$S_{GB} = \int d^4\sqrt{-g}\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\Big). \tag{5.45}$$

The equation (5.45) is an application, for $n = 4$ of the *Chern–Gauss–Bonnet (CGB) theorem* [91]. It states that the Euler characteristic[4] of a closed[5] oriented even-dimensional Riemannian manifold is equal to the integral of a certain polynomial of its curvature. It is a direct generalization of the Gauss–Bonnet theorem (for $2n$ surfaces) to higher dimensions and was first published by Shiing-Shen Chern, connecting global topology with local geometry[6].

It is easy to verify that the action (5.45) is invariant under conformal rescaling of the metric (more precisely, it acquires an additive boundary term); indeed, as seen in the last Chapter,

$$\begin{aligned} S_{GB} \to \tilde{S}_{GB} = &\int d^4x\sqrt{-g}\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\Big)+\\ &+\int d^4x\sqrt{-g}\big[-\underline{8R^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega} + 8R^{\mu\nu}\nabla_\mu\nabla_\nu\omega - 4R\Box\omega+\\ &+\underline{8(\Box\omega)^2} + \underline{\underline{16(\nabla_\mu\nabla_\nu\omega)(\nabla^\mu\omega)(\nabla^\nu\omega)}} - \underline{8(\nabla_\mu\nabla_\nu\omega)^2} + \underline{\underline{8(\Box\omega)(\nabla_\lambda\omega)^2}}\big], \end{aligned} \tag{5.46}$$

where the second integral can be deleted, *unless boundary terms*, using both the contracted Bianchi identity and the following relations

$$2(\nabla_\mu\nabla_\nu\omega)(\nabla^\mu\omega)(\nabla^\nu\omega) \simeq -(\Box\omega)(\nabla_\lambda\omega)^2, \tag{5.47}$$

$$(\nabla_\mu\nabla_\nu\omega)^2 \simeq (\Box\omega)^2 - R^{\mu\nu}\nabla^\mu\omega\nabla^\nu\omega. \tag{5.48}$$

Requiring the variation of (5.45) to be null, a new topological (and dimensional-dependent) equation is obtained[7]:

$$\begin{aligned} &4R_{\mu\rho\nu\sigma}R^{\rho\sigma} + 2R^{\alpha}{}_{\beta\lambda\mu}R_{\alpha}{}^{\beta\lambda}{}_{\nu} + 8\nabla^\rho\nabla_{(\mu}R_{\nu)\rho} + 2RR_{\mu\nu} - 12R_{\mu\rho}R^{\rho}{}_{\nu}+\\ &-\frac{1}{2}g_{\mu\nu}\big(R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - 4R_{\rho\sigma}R^{\rho\sigma} + R^2\big) = 0. \end{aligned} \tag{5.49}$$

[4]Euler characteristic is a topological invariant number number that describes a topological space [61].

[5]The hypothesis of closed manifold is strong. It can be considered true for spacetime using Penrose's argumentations of conformal treatment of infinity [83]; moreover there are a lots of works that aim to enlarge the validity hypotheses of CGB theorem [92, 93].

[6]Here, the term Gauss-Bonnet terms is simply considered a boundary term for the variety that describes the spacetime.

[7]Here, one can just replace the parameters ($a = 1$, $b = -4$, $c = 1$) inside the generic field equations (5.42).

Given that the introduction of the Gauss-Bonnet term into the action does not provide any new contribution to the field equations, it is possible to define a new conformally invariant quadratic action which does not contain the squared Riemann tensor:

$$S = 2\int d^4x\sqrt{-g}\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big). \tag{5.50}$$

By extremizing the action (5.50), we get the "*new version*" of the Bach equation,

$$\begin{aligned} &2\nabla^\rho\nabla_{(\mu}R_{\nu)\rho} + \frac{2}{3}RR_{\mu\nu} - 2R_{\mu\rho}R^\rho{}_\nu - \Box R_{\mu\nu} - \frac{2}{3}\nabla_\mu\nabla_\nu R + \frac{1}{6}\Box R g_{\mu\nu} + \\ &+ \frac{1}{2}g_{\mu\nu}\big(R_{\rho\sigma}R^{\rho\sigma} - \frac{1}{3}R^2\big) = 0. \end{aligned} \tag{5.51}$$

Moreover, the equation (5.51) can be written in a more compact form using the Weyl tensor,

$$2\nabla_\sigma\nabla_\rho C^{\mu\rho\sigma\nu} + R_{\sigma\rho}C^{\mu\rho\sigma\nu} = 0. \tag{5.52}$$

Indeed:

$$\nabla_\rho C^{\rho\mu\nu\sigma} = \nabla^{[\nu}\big(R^{\sigma]\mu} - \frac{1}{6}g^{\sigma]\mu}R\big), \tag{5.53}$$

$$2\nabla_\sigma\nabla_\rho C^{\rho\mu\nu\sigma} = -\Box R^{\mu\nu} + \nabla_\sigma\nabla^\nu R^{\mu\sigma} + \frac{1}{6}\Box R g^{\mu\nu} - \frac{1}{6}\nabla^\mu\nabla^\nu R, \tag{5.54}$$

$$R_{\rho\sigma}C^{\rho\mu\nu\sigma} = R_{\rho\sigma}R^{\rho\mu\nu\sigma} - R^{\mu\rho}R_\rho{}^\nu + \frac{2}{3}RR^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - \frac{1}{6}R^2 g^{\mu\nu}, \tag{5.55}$$

$$R_{\rho\sigma}R^{\rho\mu\nu\sigma} = \nabla^\sigma\nabla^\mu R^\nu{}_\sigma - \frac{1}{2}\nabla^\mu\nabla^\nu R - R^{\rho\mu}R^\nu{}_\rho. \tag{5.56}$$

With the definition of the *Bach tensor*

$$B_{\mu\nu} \equiv 2\nabla^\sigma\nabla^\rho C_{\mu\rho\nu\sigma} + R^{\sigma\rho}C_{\mu\rho\nu\sigma}, \tag{5.57}$$

the eq. (5.52) in presence of matter becomes

$$B_{\mu\nu} = \kappa T_{\mu\nu}. \tag{5.58}$$

$B_{\mu\nu}$ is conformally invariant, with conformal weight -2, and divergence-free. The equation (5.58) is the final form of the Bach equation. Again, the trace of the equation vanishes, as usual in a conformal theory.

5.3.1 Schwarzschild-like Solution of the Bach Equation

In developing any alternative theory to GR, what is always taken into account is the need to find the Schwarzschild solution, at least as a limit of something remarkably close to it. This is because, to date, it is the metric that describes more precisely the geometry of the spaces in proximity of the Sun[8].

In GR, Schwarzschild solution comes from the vacuum Einstein field equations, $R_{\mu\nu} = 0$. A possible approach to get the solution is to start from a general spherically symmetric metric [3],

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2 d\Omega^2, \tag{5.59}$$

calculate the components of the Ricci tensor and impose $R_{\mu\nu} = 0$. Proceeding in this way, *Birkoff's theorem* is automatically demonstrated: *any spherically symmetric solution do the vacuum field equations must be static*[9] *and asymptotically flat.*

The Schwarzschild solution reads

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega^2, \tag{5.60}$$

where

$$A(r) = c^2\left(1 - \frac{r_S}{r}\right) \qquad \text{with } r_S = \frac{2GM}{c^2}. \tag{5.61}$$

The Bach equations (5.52) is quite far from $R_{\mu\nu} = 0$. However, since eq. (5.51) depends only on the Ricci tensor, its contraction and derivative, $B_{\mu\nu} = 0$ admits the Schwarzschild solutions. Obviously, this is not enough to say that the Bach equation admits the Schwarzschild metric as the only spherically symmetric solution, because of the lack of a generalized version Birkhoff's theorem for higher-derivative theories[10] (a study that takes into account contractions of Weyl tensor and its linear combinations is for example [94]).

It is possible, however, to consider a general static, spherically symmetric metric, and to determinate the form of the functions such that the Bach equations are satisfied. This is the approach used by Mannheim and Kazanas [95]: a general static, spherically symmetric metric

$$ds^2 = -a(\rho)dt^2 + b(\rho)d\rho^2 + \rho^2 d\Omega^2 \tag{5.62}$$

[8] The Sun is approximated to a non-rotating spherical source.

[9] It may be useful to recall the difference between stationary spacetime and static spacetime. A gravitational field is stationary if admit a time-like Killing vector, *i.e.* $\xi : L_\xi g = 0$ and $g(\xi,\xi) < 0$; therefore, it is possible to consider the adapted frame to $\xi = \partial_t$ and, as consequence, metric tensor components are independent of time. A gravitational field is static if it is stationary and admits a foliation of space-like hypersurfaces orthogonal to ξ; namely, there exists a frame such that the components $g_{0i} = 0$.

[10] The lack of a generalized version Birkhoff's theorem means that there are spherically symmetric solutions of the Bach's equations which are not related to the Schwarzschild solution by a transformation of coordinates. Therefore, the request of static and spherical symmetry metric constraints to a specific class of solution. This allows to obtain a Schwarzschild.like solution.

can be written as

$$ds^2 = \frac{p(r)^2}{r^2}\Big[-A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2\Big], \tag{5.63}$$

under the general coordinate transformation

$$\rho = p(r), \qquad A(r) = \frac{r^2 a(r)}{p(r)^2}, \qquad B(r) = \frac{r^2 b(r) p'^2(r)}{p(r)}. \tag{5.64}$$

Choosing the arbitrary function $p(r)$ such that

$$-\frac{1}{p(r)} = \int \frac{dr}{r^2[a(r)b(r)]^{1/2}} \tag{5.65}$$

then it results

$$ds^2 = \frac{p(r)^2}{r^2}\Big[-A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega^2\Big]. \tag{5.66}$$

The metric (5.66) is conformally related to a Schwarzschild-like solution, by the identification $e^{2\omega} = p(r)^2/r^2$. Therefore, from the invariance of the theory under conformal transformations, it is possible to consider the line element of the eq. (5.60) with a generic function[11] $A(r)$.
In order to find the function $A(r)$, we need to proceed in the calculation of the Bach tensor components and imposing $B_{\mu\nu} = 0$. From [95], it turns out that

$$A(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \kappa r^2, \tag{5.67}$$

where β, γ and κ are constants of integration.
The eq. (5.67) defines the functional form of $A(r)$ such that the Bach equations are satisfied by a Schwarzschild-like metric. By playing with the constants of integration, it is possible to observe that:

- If $\gamma = 0 = \kappa$ and $\beta > 0$, the metric is the Schwarzschild solution (and provides the asymptotic flatness).
- if $\gamma = 0 = \beta$ and $\kappa > 0$, the metric is the de Sitter solution. In the Einstein theory we have this solution only in presence of a cosmological constant. Here, it is possible to have the de Sitter solution without involving a cosmological constant.
- If $\gamma = 0$, $\kappa > 0$ and $\beta > 0$, the metric is the de Sitter–Schwarzschild solution.

[11] The conditions of static and spherically symmetric solution lead to a Schwarzschild-like solution by using a coordinate transformation and the invariance under Weyl rescaling of Bach equation. Therefore, we find only a particular class of spherically symmetric solution which is “spanned” by any possible Weyl rescaling and coordinate transformation.

- If β, γ, κ are positive, then we can have a solution with a Newtonian $\propto 1/r$ term which should dominate at small distances, a term $\propto r$ which becomes the more dominant one at larger distances and a term $\propto r^2$ which should become important at cosmological distances (working as cosmological constant).

Moreover, it is interesting to analize the case where $\kappa = 0$ and $\gamma \to 0$, *i.e.* γ is a really small number with respect to the cosmological distance. In this case, the function (5.67) can be approximated as

$$A(r) = 1 - \frac{2\beta}{r} + \gamma r, \tag{5.68}$$

requiring $2\beta/r >> \gamma r \Rightarrow r << \sqrt{2\beta/\gamma}$. Therefore, it is possible to choose γ small enough, so that the solution of Bach equations is sufficiently in agreement with the GR regime and, moreover, γ can explain anomalies galactic rotation curves[12] without DM [96].
Unfortunately, by choosing a value of γ high enough to explain galactic rotation curves, several problems arise: the solution provides excessive non-Newtonian perturbations in the motions of the local galactic group, which are not observed [97]; the theory does not reduce to Newtonian gravity, but to some fourth-order theory in the weak-field limit [52, 98], and much of our intuition about Newton gravity is lost (some deviations are measurable in the lab); the β and γ disagree with any interior solution which obeys the weak energy condition; the temperature profile in primordial nucleosynthesis does not provides the observed elemental abundances.
Other interesting details, with relative references, about the solution of Bach's equation are reported in [52]. A more recent discussion about the explanation of flat galactic rotation curves by using Weyl's action is present in [99].

In conclusion, these are criticisms about this solution but the study of the Weyl-squared theory is still open.

[12] The velocity of orbiting objects should decrease as the distance from the potential center increases. From experimental observations, it results that galactic rotation curves remain flat as one approaches the "edge" of galaxies: the orbital velocity does not decline with distance, it remain unchanged or sometimes it increase. This means that there should exist other matter, besides that associated with luminosity. Therefore, the mysterious and invisible massive contribution is called dark matter, which does not interact with baryonic matter, or rather, it only interacts gravitationally [5].

Chapter 6

Palatini Formalism and Conformal Frames

The link between conformal transformations and theories with non-metric connections emerges continuously [24, 27, 29, 100]. In Chapters 5, we showed how the Weyl's geometry coupled to a non-dynamic scalar field, reproduces a particular case of scalar-tensor theory.
Deepening Palatini's method, it is possible to see how the presence of a symmetric but not metric-compatible connection provides a link between different theories. For specific scalar potentials, $f(R)$-theories and the scalar-tensor theories are related by a conformal transformation in vacuum.
This leads to think that there may be a kind of "physical equivalence" among theories which can be obtained by performing a conformal transformation.
Historically, this issue was born with the Brands-Dicke action which shows two faces: *Jordan frame* and *Einstein frame*. They are linked by a conformal transformation and, in absence of a specific transformation law of the matter Lagrangian, their equivalence produces a violation of the EEP.

Following what is done in [27], the aim of this Chapter is to show the links between theories that are, *a priori*, completely different. Therefore, we want to stimulate the idea that there may be a formalism that includes Weyl connection, $f(R)$-theories and Scalar-Tensor theories. In this context, it would be interesting to find a mechanism that breaks the equivalence and produces the differences between the three approaches.

6.1 The Palatini Formalism

As mentioned in Chapter 2, the Palatini formalism is based on the independence between the (usually torsion-free) connection ${\Gamma^\lambda}_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$. As consequence, the Riemann tensor and the Ricci tensor depend only on the connection. Despite the

presence of new degrees of freedom due to the connection, if we consider the E-H action, the field equations obtained by varying with respect to the connection require that this is metric-compatible. Then, for the fundamental theorem of Riemannian geometry, the connection must be the Levi-Civita one.
This leads to think that there is no need to constraint the connection, *a priori*.
However, the fact that the Palatini formalism applied to the E-H action returns GR is only a coincidence due to the extreme simplicity of the action itself. The situation is completely different if one replaces R with a generic function $f(R)$ or if one introduces a non-minimal coupling with a scalar field. This is the case of *Extended Theories of Gravity* (ETG).
To consider the metric $g_{\mu\nu}$ and the connection $\Gamma^{\lambda}{}_{\mu\nu}$ as independent fields, means to de-coupling the metric structure of space-time and its geodesic structure with the connection $\Gamma^{\lambda}{}_{\mu\nu}$ being distinct from the Levi-Civita connection of $g_{\mu\nu}$. Then, the symmetry condition provides field equations for the connection which recast the dual structure of spacetime into a a bi-metric structure of the theory. $\Gamma^{\lambda}{}_{\mu\nu}$ turns out to be the Levi-Civita connection of the metric $h_{\mu\nu} = f'(R)g_{\mu\nu}$. Therefore, there are two independent metrics, $g_{\mu\nu}$ and $h_{\mu\nu}$: the new metric $h_{\mu\nu}$ determines the geodesics and $g_{\mu\nu}$ determines the causal structure.
A further generalization can be obtained if we consider other geometric invariants, besides R, as well as the second-order curvature invariant. Therefore, another possibility is to take in consideration $f(R,\, R_{\mu\nu}R^{\mu\nu})$-theory [101, 102]. A specific case is the conformal Lagrangian given by the Weyl-squared scalar, subtracting the Gauss-Bonnet term.
Moreover, it is possible to show that in Scalar-Tensor gravity, the second metric $h_{\mu\nu}$ is related to the non-minimal coupling of the Brans-Dicke-like scalar. Therefore, making some hypothesis on $f(R)$ and $F(\phi)$, we can link the two families of theories by conformal transformations.

It is important to stress out that in this approach the matter Lagrangian does not depend on the connection (unlike metric-affine theories). Therefore, there is no coupling even between matter and the scalar field and, as consequence, the presence of matter breaks the possibility of passing from one theory to another one.

6.1.1 The Palatini Approach and the Conformal Structure

The action of $f(R)$-theories is

$$S = \int d^4x\sqrt{-g}f(R) + S_m, \tag{6.1}$$

where, now, $R_{\mu\nu} = R_{\mu\nu}(g_{\mu\nu},\, \Gamma^{\lambda}{}_{\mu\nu})$, with $\Gamma^{\lambda}{}_{\mu\nu}$ a generic connection (and ∇_{μ} related to $\Gamma^{\lambda}{}_{\mu\nu}$). Therefore, varying with respect to the metric, $g^{\mu\nu}$, the following field equations are obtained:

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}g_{\mu\nu}f(R) = T_{\mu\nu}, \tag{6.2}$$

where, as usual, $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$.
Varying with respect to the connection[1], it results

$$-\nabla_\lambda(\sqrt{-g}f'(R)g^{\mu\nu}) + \nabla_\sigma(\sqrt{-g}f'(R)g^{\sigma(\mu})\delta^{\nu)}_\lambda = 0. \tag{6.3}$$

Taking the trace of the last equation, it follows

$$\nabla_\sigma(\sqrt{-g}f'(R)g^{\sigma\nu}) = 0. \tag{6.4}$$

Then, the field equations of $f(R)$-theories turn out to be

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}g_{\mu\nu}f(R) = T_{\mu\nu} \tag{6.5}$$

$$\nabla_\lambda[\sqrt{-g}f'(R)g^{\mu\nu}] = 0 \tag{6.6}$$

It is easy to see [100, 104] from the equation (6.6) that $\sqrt{-g}f'(R)g^{\mu\nu}$ is a symmetric tensor density of weight 1, which naturally leads to the introduction of a new metric $h_{\mu\nu}$ conformally related to $g_{\mu\nu}$ by

$$\sqrt{-g}f'(R)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu}. \tag{6.7}$$

As a consequence, $\Gamma^\lambda{}_{\mu\nu}$ turns out to be the Levi-Civita connection of the metric $h_{\mu\nu}$, with the only restriction on the "conformal factor" ($\Omega^2 = e^{2\omega} = f'(R)$) relating $g_{\mu\nu}$ and $h_{\mu\nu}$ to be non-degenerate.
In case of strictly positive $f'(R)$, it is possible to consider the conformal transformation given by

$$g_{\mu\nu} \to h_{\mu\nu} = f'(R)g_{\mu\nu} \tag{6.8}$$

and that implies

$$R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h). \tag{6.9}$$

Therefore $f(R)$-theories in Palatini formalism correspond to $f(R)$-theories in metric formalism with $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = h_{\mu\nu}$. Obviously, this equivalence only holds in presence of conformally invariant matter ($T = g_{\mu\nu}T^{\mu\nu} = 0$).
Moreover, taking the trace of the eq. (6.5) we obtain the so-called *structural equation of space-time* which control the solution of field equations:

$$f'(R)R - 2f(R) = g^{\mu\nu}T_{\mu\nu} \equiv T. \tag{6.10}$$

[1] Here there is the constraint of symmetric connection. Relaxing it, one has: $\delta R^\mu{}_{\nu\rho\sigma} = \nabla_\rho\delta\Gamma^\mu{}_{\sigma\nu} - \nabla_\sigma\delta\Gamma^\mu{}_{\rho\nu} + T^\lambda{}_{\rho\sigma}\delta\Gamma^\mu{}_{\lambda\nu}$. Then, a relation between torsion vector and nonmetricity vectors follows : they are proportional to each other with proportional constants depending on size (here different conventions for indices of the connection are used [33]). This "allows" the further step to a generalized Weyl geometry (*e.g.* [103]).

In vacuum or in presence of conformally invariant matter, this scalar equation admits constant solutions. In these cases, Palatini $f(R)$-gravity reduces to GR with a cosmological constant [104]. In case of interaction with matter fields, the structural equation (6.10), if explicitly solvable, provides the expression $R = F(T)$ and, as a result, both $f(R)$ and $f'(R)$ can be expressed in terms of T. Then, matter rules both the bimetric structure of space-time and the geodesic and metric structures, which are intrinsically different. This behaviour generalizes the vacuum case.

The Palatini formalism can be extended to non-minimally coupled scalar-tensor theories.
The first step is to consider the generalized scalar-tensor action (seen in Chapter 1)

$$S = \int d^4x\sqrt{-g}\Big[F(\phi)R - \frac{\epsilon}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \mathscr{L}_m\big(\psi,\, \overset{g}{\nabla}_\mu\psi\big)\Big], \tag{6.11}$$

where $\epsilon = \pm 1$ refers to the case of ordinary scalar or a phantom field, respectively. The field equations are

$$R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu} = \frac{1}{F(\phi)}\Big(T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}\Big) \tag{6.12}$$

$$\nabla_\lambda[\sqrt{-g}F(\phi)g^{\mu\nu}] = 0 \tag{6.13}$$

$$\epsilon\Box\phi + F'(\phi)R - V'(\phi) = 0 \tag{6.14}$$

Taking the trace of the first field equations, the structural equation of spacetime takes the form

$$R = -\frac{1}{F(\phi)}\Big(T^{(\phi)} + T^{(m)}\Big). \tag{6.15}$$

The bimetric structure of space-time is provided by field equations of the connection

$$\sqrt{-g}F(\phi)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu} \tag{6.16}$$

Therefore, $h_{\mu\nu}$ is conformally related to $g_{\mu\nu}$ if we require $F(\phi) > 0$:

$$g_{\mu\nu} \to h_{\mu\nu} = F(\phi)g_{\mu\nu}. \tag{6.17}$$

In vacuum , $T^{(\phi)} = 0 = T^{(m)}$, so that this theory is equivalent to GR.

As a further step, it is possible to consider a more generic framework wherein $f(R)$ is non-minimally coupled to a scalar field. The action of this theory is

$$S = S = \int d^4x\sqrt{-g}\Big[F(\phi)f(R) - \frac{\epsilon}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \mathscr{L}_m\big(\psi,\, \overset{g}{\nabla}_\mu\psi\big)\Big], \tag{6.18}$$

which provides the following filed equations

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}g_{\mu\nu}f(R) = \frac{1}{F(\phi)}\left(T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}\right) \tag{6.19}$$

$$\nabla_\lambda[\sqrt{-g}F(\phi)f'(R)g^{\mu\nu}] = 0 \tag{6.20}$$

$$\epsilon\Box\phi + F'(\phi)f(R) - V'(\phi) = 0 \tag{6.21}$$

and the structural equation of space-time

$$f'(R)R - 2f(R) = \frac{1}{F(\phi)}\left(T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu}\right). \tag{6.22}$$

In this case, the bimetric structure is given by

$$\sqrt{-g}F(\phi)f'(R)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu} \tag{6.23}$$

with $g_{\mu\nu}$ and $h_{\mu\nu}$ again conformally related if $F(\phi)f'(R) > 0$:

$$h_{\mu\nu} = F(\phi)f'(R)g_{\mu\nu}. \tag{6.24}$$

In vacuum, the structural equation implies that the theory reduces again to Einstein gravity. With the next generalization, it is possible to show that the possibility to recover GR in vacuum is related to the decoupling of the scalar field from the metric.

The final generalization of previous cases is the introduction of a general function $K(\phi,\, R)$:

$$S = \int d^4x\sqrt{-g}\Big[K(\phi,\, R) - \frac{\epsilon}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \mathscr{L}_m\big(\psi,\, \overset{g}{\nabla}_\mu\psi\big)\Big] \tag{6.25}$$

which provides the following filed equations

$$\partial_R K(\phi,\, R)R_{(\mu\nu)} - \frac{1}{2}Kg_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu} \tag{6.26}$$

$$\nabla_\lambda[\sqrt{-g}\partial_R K(\phi,\, R)g^{\mu\nu}] = 0 \tag{6.27}$$

$$\epsilon\Box\phi + \partial_\phi K(\phi,\, R) - V'(\phi) = 0 \tag{6.28}$$

and the structural equation of space-time

$$\partial_R K(\phi,\, R)R - 2K(\phi,\, R) = T^{(\phi)} + T^{(m)}. \tag{6.29}$$

The bimetric structure of space-time is given by

$$\sqrt{-g}\partial_R K(\phi,\, R)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu} \tag{6.30}$$

and, for $K(\phi,\, R) > 0$, the the conformal transformation is

$$g_{\mu\nu} \to h_{\mu\nu} = \partial_R K(\phi,\, R) g_{\mu\nu}. \tag{6.31}$$

Form the eq. (6.29), the conformal factor turns out to depend on the matter fields only through the traces of their stress-energy tensors. In general, in vacuum, GR is not recovered because the strong coupling between R and ϕ prevents the possibility of obtaining constant solutions.
However, an interesting case is the limit for $R \to 0$. The linear expansion of the analytic function $K(\phi,\, R)$ reads

$$K(\phi,\, R) = K_0(\phi) + K_1(\phi)R + o(R^2), \qquad \text{with} \quad \begin{array}{l} K_0(\phi) = K(\phi,\, 0) \\ K_1(\phi) = [\partial_R K(\phi,\, R)]_{R=0} \end{array}. \tag{6.32}$$

In this case, the structural equation of space-time reads

$$R = -\frac{1}{K_1(\phi)}\Big[T^{(\phi)} + T^{(m)} + 2K_0(\phi)\Big]. \tag{6.33}$$

Therefore, the value of the Ricci scalar is always determined, in the linear approximation, in terms of $T^{(\phi)}$, $T^{(m)}$ and ϕ. The bimetric structure is defined by the first term of the Taylor expansion and if $K_1(\phi) > 0$ we can consider the conformal transformation

$$g_{\mu\nu} \to h_{\mu\nu} = K_1(\phi) g_{\mu\nu} \tag{6.34}$$

reproducing the scalar-tensor case (6.17).

Finally, there exist also bimetric theories which cannot be conformally related [26] and torsion will also appear in the most general framework [105]. These more general theories will not be discussed here.

6.2 Conformal Frames of Brans-Dicke Gravity

Starting from the E-H action, it is possible to choose a conformal factor $\Omega = \sqrt{G\phi}$ [106] and, performing the transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, to obtain the Brands-Dicke action[2] with parameter $\omega = -3/2$:

$$S_{BD} = \int d^4x\sqrt{-g}\Big[\phi R + \frac{3}{2}g^{\mu\nu}\frac{1}{\phi}\nabla_\mu\phi\nabla_\nu\phi\Big]. \tag{6.35}$$

This form is called *Jordan frame.* Its standard form is

$$S_{BD} = \int d^4x\sqrt{-g}\Big[\phi R - \frac{\omega}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\Big] + S^{(m)}. \tag{6.36}$$

[2]It is necessary to integrate by parts.

Then, by redefining the scalar field

$$\tilde{\phi}(\phi) = \sqrt{\frac{2\omega+3}{16\pi G}} \ln\left(\frac{\phi}{\phi_0}\right) \tag{6.37}$$

(where $\phi \neq 0$, $\omega > -3/2$ and $\phi_0^{-1} = G$), we can rewrite the Brans-Dicke action in the *Einstein frame*,

$$S_{BD} = \int d^4x \left\{ \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - U(\tilde{\phi}) \right] + \exp\left[-8\sqrt{\frac{\pi G}{2\omega+3}} \tilde{\phi} \right] \mathscr{L}^{(m)}[\tilde{g}] \right\} \tag{6.38}$$

where $\tilde{\nabla}$ is the covariant derivative operator of the rescaled metric $\tilde{g}_{\mu\nu}$ and

$$U(\tilde{\phi}) = V[\phi(\tilde{\phi})] \exp -8\sqrt{\frac{\pi G}{2\omega+3}} \tilde{\phi} = V(\phi)/(G\phi)^2 \tag{6.39}$$

is the Einstein frame potential. In the Einstein frame, the action has the first term and the kinetic term of the scalar field in a usual form.

The redefinition (6.37) impose the constraint $\omega > -3/2$, otherwise it is no possible to perform the conformal transformation. Alternatively, taking the absolute value $|2\omega + 3|$, the Brans-Dicke theory in the Einstein frame is still pathologic for $\omega = -3/2$.

However $\omega \leq -3/2$ is not necessary a pathology using particular potential. Then, the value $\omega = -3/2$ can be seen as a frontier between a standard scalar field and a phantom field, because for $\omega < -3/2$ the kinetic term of the scalar field has the "wrong" sign in front of it.

Although the first part of the eq. (6.38) recalls the E-H action, there are two substantial differences between them. From the interpretation of ϕ in Jordan frame as the contribute of all distant astronomical objects, $\tilde{\phi}$ cannot be removed and acts as a source of gravity. Therefore, the vacuum GR equations as $\tilde{R}_{\mu\nu} = 0$ cannot be obtained. The matter Lagrangian $\mathscr{L}^{(m)}$ inside (6.38) is multiplied by the exponential factor, an *anomalous coupling of matter to the scalar* $\tilde{\phi}$ which has no counterpart in GR.

The latter difference causes a change of energy-momentum conservation, leading to a violation of EEP. This can be shown using the transformation law of energy-momentum tensor under conformal transformation (4.57): for $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, setting the conformal weight $s = -6$, it results

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = -\tilde{T} \tilde{g}^{\mu\nu} \tilde{\nabla}_\nu (\ln \Omega) \tag{6.40}$$

in case of non conformal matter ($T^{(m)} \neq 0$).

In particular, for the Jordan frame, $\Omega = \sqrt{G\phi}$ then

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = -\frac{1}{\phi} \tilde{T} \tilde{g}^{\mu\nu} \tilde{\nabla}_\nu (\phi) \tag{6.41}$$

while, in terms of the Einstein frame scalar (6.37), $\phi = \phi_0 \exp\tilde{\phi}\sqrt{16\pi G/(2\omega+3)}$,

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = -\sqrt{\frac{4\pi G}{2\omega+3}}\tilde{T}\tilde{g}^{\mu\nu}\tilde{\nabla}_\nu(\tilde{\phi}). \tag{6.42}$$

Now, considering a dust fluid, *i.e.* a pressureless perfect fluid, from (6.42) it is possible to notice how geodesic equations change. Therefore, let $\tilde{T}^{(m)}_{\mu\nu}$ be the energy-momentum tensor of a dust fluid,

$$\tilde{T}^{(m)}_{\mu\nu} = \tilde{\rho}^{(m)}\tilde{u}_\mu\tilde{u}_\nu \tag{6.43}$$

where $\tilde{\rho}^{(m)}$ is the density and $\tilde{u}_\mu$ the 4-velocity.
Then, substituting (6.43) in (6.42) equation splits into the two equations [24, 27, 29], one for the density an the other for the 4-velocity,

$$\frac{D\tilde{\rho}^{(m)}}{d\lambda} + \tilde{\rho}^{(m)}\tilde{\nabla}_\mu\tilde{u}^\mu = 0 \tag{6.44}$$

$$\frac{D\tilde{u}^\mu}{d\lambda} = \sqrt{\frac{4\pi G}{2\omega+3}}\tilde{\nabla}^\mu\tilde{\phi} \Rightarrow \frac{d^2x^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \sqrt{\frac{4\pi G}{2\omega+3}}\tilde{\nabla}^\mu\tilde{\phi} \tag{6.45}$$

Thus, in the Einstein frame, geodesic equations are modified with respect to GR equations, due to the presence the scalar field. Often, the right side of eq. (6.45) is called *fifth force* that couples universally to all massive test particles. However, this provides a violation of the WEP, which is satisfied by all metric theories of gravity [26]. Therefore, the validity of EEP becomes a property depending on the conformal frame representation.
However, it is important to notice that null geodesics are unchanged by the conformal transformation and this is consistent with the fact that the equation of null geodesics can be derived from the Maxwell equations, using the *eikonal approximation*[3]. There is another way to verify the conformal invariance for null geodesics, by computing the electromagnetic stress-energy tensor:

$$T^{(em)}_{\mu\nu} = 2\Big(F_{\mu\rho}F^\rho{}_\nu - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\Big). \tag{6.46}$$

Since the trace of $T^{(em)}_{\mu\nu}$ is null, it turns out that conservation equation $\nabla^\mu T_{\mu\nu} = 0$ does not change under conformal.
However, the low-energy limit of String Theory provides a correction to the time-like geodesic equations similar to the fifth force and the scalar field is replaced by a *dilaton*. An interesting difference is that, while the dilatonic charge can be setted to zero, the field scalar cannot be eliminated from the Einstein frame.

[3]For simplicity, the common procedure is to consider a scalar function $\phi(x^\mu)$ (which will be the prototype of the electromagnetic potential) and imposes the following *ansatz*: $\phi(x^\mu) = \mathfrak{Re}\big[A(x^\mu)\exp iS(x/\varepsilon)\big]$, where A is the slowly variable amplitude of the wave, the phase S is the eikonal function and the parameter $\varepsilon \to 0$ states that S is rapidly variable. Thus, it is possible to expand S with respect to ε powers. Then, by imposing the D'Alambert equation, one obtains the *eikonal equation* must be satisfied by S. The same procedure may be followed for electromagnetic components and it provides null-geodesics for the wavevector.

6.2.1 The Problem of Conformal Frames

The possibility to work with the Jordan and the Einstein frame (and with all conformally related frames) leads to wonder whether there is a physical correspondence, besides that mathematical one, between the two. Namely: does conformal transformations preserve the "physical meaning" of the theory?
According to Dicke [106], since physics is invariant for to units of measurement, it must also be invariant for local rescaling. As consequence, to Dicke the two frames are equivalent. However, this viewpoint was strongly debated but two main difficulties arised, which identify the frame issue as a *pseudo-problem*:

1. Conformal equivalence, understood in the sense of a gauge theory, must be explicitly verified using the equations which describe physics. It is also necessary to understand if this equivalence is due to the properties of specific fields, what processes/interactions are in favour of it and what equations describe it. In this sense, the concept of "physical equivalence" is too vague.

2. Dicke's point of view is purely classical. It is unknown if conformal equivalence plays some roles to quantum levels and in systems that have extreme characteristics (*e.g.* black holes).

Moreover, there is the problem of EEP which is violated in presence of non-conformal matter and in absence of a rescaling law for the matter Lagrangian. In addition, the conformal factor makes frames energetically inequivalent and provides difference in the acceleration of the universe [24].
It is necessary a definition of "physical equivalence" because in some cases the physical equivalence is verified by considering the coupling of the Brans-Dicke-like scalar field to matter and a point-like of units in the Einstein frame, but this is not always obvious and in some cases it does not seem to be consistent [27].
Therefore, the conformal invariance could involve only specific physical aspects and it could be manifest only at particular energy scales.

Chapter 7

Considerations on the Weyl Vector

7.1 Non-Metricity of the Weyl Geometry

What discussed in previous Chapter justifies the research of a manner such that the non-metricity brings to EEP violation and provides back GR in the metric limit. Moreover, the link between the scalar-tensor action and the Weyl geometry pushes the idea that a possible non-metric part of the theory could be the Weyl vector.
However, there are immediate differences between the contribution of the fifth force and contribution of the Weyl vector. In the scalar-tensor theory, W_μ is an exact form while in general it is not. The fact that the Weyl vector is not an exact differential form causes several problems, but in some sense it can generalize the concept of scalar field.

As seen in Chapter 2, the presence of the non-metricity tensor changes the autoparallel curves equation, which become different from the geodesics of a Riemannian space (unlike what happens in RG). Let $T^\mu = dx^\mu/d\lambda = \dot{x}^\mu$ be the tangent vector component of an affinely parametrized autoparallel curve; then, in a local chart, it results in

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 2W_\nu\frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda} - W^\mu\frac{dx^\nu}{d\lambda}\frac{dx_\nu}{d\lambda}. \tag{7.1}$$

Obviously, the eq. (7.1) is conformally invariant by definition but it is different from the geodesics of GR. However, for a light-like particle, the eq. (7.1) can be seen as a non-affinely parametrized geodesic, with affine parameter $\alpha = 2W_\nu\dot{x}^\nu$.
In a locally inertial frame, *i.e.* $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu}(p) = 0$, the equation (7.1) becomes

$$\begin{aligned} \ddot{x}^\mu &= 2W_\nu\dot{x}^\nu\dot{x}^\mu - W^\mu\dot{x}_\nu\dot{x}^\nu = && (7.2)\\ &= (2W^\nu\dot{x}^\mu - W^\mu\dot{x}^\nu)\dot{x}_\nu = \\ &= (\dot{x}^{(\mu}W^{\nu)} + 3\dot{x}^{[\mu}W^{\nu]})\dot{x}_\nu. && (7.3)\end{aligned}$$

Now, to respect the limit of the geometric optics for the light, it is necessary to require that

$$W_\mu\dot{x}^\mu = 0, \tag{7.4}$$

which looks like what happens by fixing the transversal gauge for the light polarization vector. The request (7.4) could be valid for both light-like and time-like vectors (it would be convenient). In this case, for a "free" massive particle, the eq. (7.2) can be recast as

$$\ddot{x}^\mu = -\dot{x}^2 W^\mu. \tag{7.5}$$

The equation (7.5) could vaguely recall some sort of viscous friction force. Moreover, the condition (7.4), for massive and massless particle, provides two other pleasant advantages: the 4-velocity is normalizable (taking into account the eq. (2.34)) and the *anomalous acceleration*, which does not have interpenetration, becomes equal to the 4-acceleration, vanishing in auto-parallel curves:

$$\tilde{a}_\nu \equiv T^\rho \bar{\nabla}_\rho T_\nu = T^\rho \bar{\nabla}_\rho (g_{\mu\nu} T^\mu) = Q_{\rho\mu\nu} T^\rho T^\mu = 2W_\rho \dot{x}^\rho \dot{x}_\nu \xrightarrow{W_\rho \dot{x}^\rho = 0} 0. \tag{7.6}$$

Therefore, this situation is very far from what happens with the fifth force due to the explicit dependence on $\dot{x}^\mu$.

7.2 Weyl Vector as Electromagnetic Counterpart of Non-metricity?

In this Section, the two ideas of the Weyl connection and Curvature-Squared actions will be merged.
In Weyl geometry, for $n = 4$, every quadratic term of the Riemann tensor and its contractions give a conformally invariant Lagrangian. The aim is to build a generic Lagrangian that provides the "Gauss-Bonnet-Weyl-Squared" action (5.50) for vanishing Weyl vector. For this reason only specific products will be considered:

$$S = \int d^4x \sqrt{-g} \Big(a_1 \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} + a_2 \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\nu\mu\rho\sigma} + a_3 \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\rho\sigma\mu\nu} + \\ + b_1 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + b_2 \tilde{R}_{\mu\nu} \tilde{R}^{\nu\mu} + b_3 \mathcal{W}_{\mu\nu} \mathcal{W}^{\mu\nu} + c\tilde{R}^2 \Big). \tag{7.7}$$

Having in mind the discussion of Chapter 5, there are the following relations for $n = 4$:

$$(\tilde{R}^{[\mu\nu]}{}_{\rho\sigma})^2 = (R_{\mu\nu\rho\sigma})^2 + 8R^{\mu\nu}\nabla_\mu W_\nu + 8R^{\mu\nu} W_\mu W_\nu - 4RW^2 + \\ + 8(\nabla_\mu W_\nu)^2 + 4(\nabla_\lambda W^\lambda)^2 + 12W^4 + 16W^\mu W^\nu \nabla_\mu W_\nu - 16W^2(\nabla_\lambda W^\lambda); \tag{7.8}$$

$$(\tilde{R}^{(\mu\nu)}{}_{\rho\sigma})^2 = 16(\mathcal{W}_{\mu\nu})^2; \tag{7.9}$$

$$\tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} = (\tilde{R}_{[\mu\nu]\rho\sigma})^2 + (\tilde{R}_{(\mu\nu)\rho\sigma})^2; \tag{7.10}$$

$$\tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\nu\mu\rho\sigma} = -(\tilde{R}_{[\mu\nu]\rho\sigma})^2 + (\tilde{R}_{(\mu\nu)\rho\sigma})^2; \tag{7.11}$$

$$\tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\rho\sigma\mu\nu} = (\tilde{R}^{[\mu\nu]}{}_{\rho\sigma})^2 - (\tilde{R}_{(\mu\nu)\rho\sigma})^2; \tag{7.12}$$

$$
\begin{aligned}
(\tilde{R}_{(\mu\nu)})^2 &= (R_{\mu\nu})^2 + 4R^{\mu\nu}\nabla_\mu W_\nu + 4R^{\mu\nu}W_\mu W_\nu + 2R\nabla_\lambda W^\lambda - 4RW^2 + 2(\nabla_\mu W_\nu)^2 + \\
&\quad + 2(\nabla_\mu W_\nu)(\nabla^\nu W^\mu) + 12W^4 + 8(\nabla_\lambda W^\lambda)^2 + 8W^\mu W^\nu \nabla_\mu W_\nu - 20W^2\nabla_\lambda W^\lambda; \qquad (7.13)\\
(\tilde{R}_{[\mu\nu]})^2 &= 16(\mathcal{W}_{\mu\nu})^2; \qquad (7.14)\\
\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} &= (\tilde{R}_{[\mu\nu]})^2 + (\tilde{R}_{(\mu\nu)})^2; \qquad (7.15)\\
\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} &= -(\tilde{R}_{[\mu\nu]})^2 + (\tilde{R}_{(\mu\nu)})^2; \qquad (7.16)\\
\tilde{R}^2 &= R^2 - 12RW^2 + 12R\nabla_\lambda W^\lambda + 36W^4 + 36(\nabla_\lambda W^\lambda)^2 - 72W^2\nabla_\lambda W^\lambda. \qquad (7.17)
\end{aligned}
$$

If for vanishing Weyl vector the (7.7) has to be like the (5.50), then there are the following constraint on the parameters of the action:

$$
\begin{cases}
a_1 - a_2 + a_3 = 0 \Rightarrow a_1 + a_3 = a_2 & (7.18)\\
b_1 + b_2 + 3c_3 = 0 \Rightarrow b_1 + b_2 = -3c & (7.19)
\end{cases}
$$

Therefore, b_3 is manifestly a free parameter of the theory. This means that it acts as a "*sponge*" for terms $\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}$. However, someone may want to consider only "genuine terms" in $\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}$, *i.e.* contributions coming from the contractions of tilde-curvature. Thus, the possibility to have a single proportional constant for $(\mathcal{W}_{\mu\nu})^2$ will not be exploited.
Now, by imposing the above constraints, the action (7.7) becomes:

$$
\begin{aligned}
\int d^4x\sqrt{-g}\Big[&- 3c(R_{\mu\nu})^2 + cR^2 + (32a_1 + 32b_1 + 48c + b_3)\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}\\
&- 3c\Big(4R^{\mu\nu}\nabla_\mu W_\nu + 4R^{\mu\nu}W_\mu W_\nu - 2R\nabla_\lambda W^\lambda + 2(\nabla_\mu W_\nu)^2 +\\
&+ 2(\nabla_\mu W_\nu)(\nabla^\nu W^\mu) - 4(\nabla_\lambda W^\lambda)^2 + 8\underline{W^\mu W^\nu\nabla_\mu W_\nu} + 4\underline{W^2\nabla_\lambda W^\lambda}\Big)\Big].
\end{aligned}
\qquad (7.20)
$$

Then, integrating by part, using the contracted Bianchi identity and the following equalities (*unless boundary terms*),

$$
\begin{aligned}
\nabla_\mu\nabla_\nu W^\nu &= \nabla_\nu\nabla_\mu W^\nu - R_{\mu\nu}W^\nu, \qquad (7.21)\\
W^2\nabla_\lambda W^\lambda &\simeq -2W^\mu W^\nu \nabla_\mu W_\nu, \qquad (7.22)\\
(\nabla_\lambda W^\lambda)^2 &\simeq \nabla_\mu W_\nu \nabla^\nu W^\mu + R_{\mu\nu}W^\mu W^\nu, \qquad (7.23)
\end{aligned}
$$

the second two lines of the eq. (7.20) contribute to the action as $-24c(\mathcal{W}_{\mu\nu})^2$. In this way, the action (7.20) becomes:

$$
\int d^4x\sqrt{-g}\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 + \lambda\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}\right], \qquad (7.24)
$$

where $\lambda = (32a_1 + 32b_1 + 24c + b_3)/(-3c)$. It is interesting noting that, by setting $b_3 = 0$, it results $\lambda = 24$ for any considered action, such as Weyl-squared, Gauss-Bonnet and Bach

action.
Consequently, field equations and the equations of motion are respectively:

$$B_{\mu\nu} = -2\lambda\Big(\mathcal{W}_{\mu\rho}\mathcal{W}_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}\mathcal{W}_{\rho\sigma}\mathcal{W}^{\rho\sigma}\Big) \tag{7.25}$$

$$\Box W_{\mu} - \nabla_{\mu}(\nabla_{\lambda}W^{\lambda}) - R_{\mu\nu}W^{\nu} = 0 \tag{7.26}$$

We now wonder whether any action that allows to interpret Weyl vector as the electrodynamic potential, provides a Bach-like equation for vanishing Weyl vector. The procedure is to impose that the Weyl vector satisfies the equation (7.26). To do this, it is necessary to rewrite the action (7.7), without imposing constraints on the parameters:

$$\begin{aligned} S \simeq \int d^4x\sqrt{-g}\,\Big[&(a_1 - a_2 + a_3)(R_{\mu\nu\rho\sigma})^2 + (b_1 + b_2)(R_{\mu\nu})^2 + cR^2 + \\ &+ 16(a_1 + a_2 - a_3 + b_1 - b_2 + b_3)(\mathcal{W}_{\mu\nu})^2 + \\ &+ (8a_1 - 8a_2 + 8a_3 + 2b_1 + 2b_2)(\nabla_{\mu}W_{\nu})^2 + \\ &+ (4a_1 - 4a_2 + 4a_3 + 10b_1 + 10b_2 + 36c)(\nabla_{\mu}W_{\nu})(\nabla^{\nu}W^{\mu}) + \\ &+ (4a_1 - 4a_2 + 4a_3 + 4b_1 + 4b_2 + 12c)R\nabla_{\lambda}W^{\lambda} + \\ &+ (12a_1 - 12a_2 + 12a_3 + 12b_1 + 12b_2 + 36c)R^{\mu\nu}W_{\mu}W_{\nu} + \\ &- 4(a_1 - a_2 + a_3 + b_1 + b_2 + 3c)RW^2 + \\ &+ 12(a_1 - a_2 + a_3 + b_1 + b_2 + 3c)W^4 + \\ &- (24a_1 - 24a_2 + 24a_3 + 24b_1 + 24b_2 + 72c)W^2\nabla_{\lambda}W^{\lambda}\Big]. \end{aligned} \tag{7.27}$$

Thus, varying with respect to W_{μ} and imposing the equation (7.26), the following constraint results:

$$a_1 - a_2 + a_3 + b_1 + b_2 + 3c = 0. \tag{7.28}$$

The equation (7.28) means that every Lagrangian like the (5.41), which is quadratic in the Riemann tensor and its contractions, allows to interpret the Weyl vector as a electromagnetic potential in a Weyl space, if

$$a + b + 3c = 0. \tag{7.29}$$

Obviously, the equation (7.29) is satisfied by both the Weyl-squared Lagrangian and Gauss-Bonnet one.
This would seem a geometric way to take into account the presence of an electromagnetic field in space. However, there are some difficulties with this interpretation.
Looking at the equation (7.2), the Weyl vector can be interpreted as a electromagnetic potential iff the second term of (7.2) can be interpreted as a electromagnetic force, $F^{\mu\nu}\dot{x}_{\nu}$ where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is the Maxwell-Faraday tensor. This means that

$$\dot{x}_{(\mu}W_{\nu)} = 0 \tag{7.30}$$

$$\dot{x}_{[\mu}W_{\nu]} \propto \partial_{[\mu}A_{\nu]} \tag{7.31}$$

The first equation (7.30) provides the conservation of the 4-velocity norm. Indeed, for an autoparallel curve, the evolution of the scalar product of tangent vector components is

$$T^\rho \bar{\nabla}_\rho (g_{\mu\nu} T^\mu T^\nu) = 2T^\mu T^\rho (\bar{\nabla}_\rho T^\mu) + T^\rho (\nabla_\rho g_{\mu\nu}) T^\mu T^\nu = Q_{\rho\mu\nu} T^\rho T^\mu T^\nu = 2W \cdot T T^2 \quad (7.32)$$

and, if the equation (7.30) is valid, it follows

$$T^\mu T^\nu (T_\mu W_\nu + T_\nu W_\mu) = 0 \;\Rightarrow\; 2W \cdot T T^2 = 0 \;\Rightarrow\; W \cdot T = 0. \quad (7.33)$$

Therefore, the equation (7.33) provides the condition $W_\mu \dot{x}^\mu = 0$ for both light-like and time-like vectors, allowing to obtain the limit of the geometric optics and deleting the anomalous acceleration.

Another point of view, it is to observe that the equation (7.30) has the shape of a Killing equation for a Riemannian space, if one performs a transformation of the form

$$\dot{x}_\mu \longrightarrow \hat{\nabla}_\mu, \quad (7.34)$$

which is coherent with the second equation (7.31).
Moreover, it is interesting to notice what happens multiplying by $\dot{x}^\mu$ and $\hat{\nabla}^\mu$ the two sides of the eq. (7.31), respectively:

$$\dot{x}^\mu \dot{x}_{[\mu} W_{\nu]} = \dot{x}^2 W_\nu - \dot{x}^\mu W_\mu \dot{x}^\nu \xrightarrow{W_\mu \dot{x}^\mu = 0} \dot{x}^2 W_\nu \quad (7.35)$$

$$\hat{\nabla}^\mu \hat{\nabla}_{[\mu} A_{\nu]} \equiv \hat{\nabla}^\mu F_{\mu\nu}. \quad (7.36)$$

Therefore, a light-vector ($\dot{x}^2 = 0$) corresponds to a free electromagnetic field while a time-vector corresponds to an electromagnetic source in the “hat-space”.
However, the eq. (7.30) results to be dependent on the parametrization, and this can lead to inconsistencies.

So far, W_μ cannot be interpreted as a electromagnetic potential. In this sense, supposing a violation of the EEP, the existence of Weyl-nonmetricy could improve the Bach theory, by adding more degrees of freedom, *e.g.* in a way that the Schwarzschild-like solution matches both with the Solar system model and the galactic rotation curves. The problems are: what is the energy range in which violations of EEP occur? How the Weyl vector could couple the matter?

Discussion and Conclusions

The thesis relies and takes into account the possibility that, at some energy scale, the principle of equivalence could be violated (strictly) due to a nonmetricity of the "physical connection".

In order to do this, it is inevitable to end up studying the conformal transformations of the metric tensor, $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\omega(x)} g_{\mu\nu}$. Conformal transformations allow to link GR with other Extended Theories of Gravity, as well as $f(R)$-theories and Scalar-Tensor gravity in Palatini formalism. These can be seen as possible extensions of GR because each of them contains Einstein field equations as particular limits. Although these theories are very different, they share a common aspect: the presence of a modification of the geodesic equations caused by a symmetric but not-metric connection.

It is natural, therefore, to wonder whether these theories, despite being different, could be considered as equivalent. The answer is difficult and so far not accessible.

As stated several times, many physicists have thought of conformal symmetry as a new principle of universal invariance. Despite the different theoretical formulations, there is still no experimental evidence to support this line of research. It is a strong constraint on a theory of gravity. If the conformal invariance is a necessary ingredient of quantum gravity, the space for possible theories would be considerably limited. At same time, such constraint would be very welcome, as the the quantum gravity domain lacks experimental constraints.

Among the various classical conformal theories of gravity there is the Weyl-Squared gravity, $\sqrt{-g}\, C^{\mu}{}_{\nu\rho\sigma} C_{\mu}{}^{\nu\rho\sigma}$. Using the Gauss-Bonnet term it is possible to simplify the action, $2\sqrt{-g}\, (R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)$, and to obtain the Bach's equations, $B_{\mu\nu} = 0$. These equations admit a Schwarzschild-like solution with some free parameters that can be tailored to fit the galaxy rotation curves. This solution does not provide the Newtonian gravity in the weak-field limit, but instead some fourth-order theory which is not observed. Another conformal theory is the Brans-Dicke one with a conformally invariant scalar field with conformal weight $s = 1 - n/2$. The action of this theory can be obtained by using the Weyl geometry, $Q_{\alpha\mu\nu} = 2W_{\alpha} g_{\mu\nu}$.

These two points of view can be coupled in order to rewrite a general Weyl-Squared in a Weyl space. Starting from a conformally invariant action, the result is an additive term proportional to $(\mathcal{W}_{\mu\nu})^2$ where $\mathcal{W}_{\mu\nu} = \nabla_{[\mu} W_{\nu]}$. The presence of the Weyl vector produces a source term. This could explain the modification of galaxy rotation curves with respect to the Schwarzschild metric. Inevitably, this would mean that the Weyl vector describes the presence (at least partially) of the dark matter. In fact, it is easily possible to see from the geodesic equations that W_μ cannot be interpreted as an electrodynamic potential, despite the similarity between the two actions and between the equations of motion. Moreover, the Weyl non-metricity cannot reproduce the fifth force that emerges from the Brans-Dicke gravity in the Einstein frame.

Based on the above statements, it would be interesting to study other possible relations that may exist between $f(R)$-theories, Scalar-Tensor theories and Weyl geometries. Actually, the Weyl-Squared gravity in a Weyl space can be seen under the point of view of the $f(R,\, R_{\mu\nu}R^{\mu\nu})$-theories with a free Weyl field [107]. Another Lagrangian belonging to the $f(R,\, R_{\mu\nu}R^{\mu\nu})$-theories is $\sqrt{-g}\,[R + \alpha(R_{\mu\nu}R^{\mu\nu} - 1/3R^2)]$ which considers the Weyl-squared action as a higher order correction to the Einstein-Hilbert action (therefore, does not appear to be conformally invariant). Then, the next step[1] is to consider $\sqrt{-g}\,[\phi^2(R - 6W^2 + 6\nabla_\lambda W^\lambda) + \alpha(R_{\mu\nu}R^{\mu\nu} - 1/3R^2) + \beta \mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu} + V(\phi)]$, with ϕ being the usual conformally invariant scalar field, or an even more general action of the same order (*e.g.* [108] but in view of Weyl's geometry and a coupling with ϕ).

The research could be further expanded by considering an even more general connection, which is also not symmetrical. However, this would significantly increase the number of quadratic terms into the action. It is possible to consider a Weyl-Cartan-Weitzenböck space [70, 109] or an Einstein-Cartan-Weyl one [110, 111]. Another option is to consider a class of geometries, which extend the Weyl geometry [103], wherein the connection contains the trace vector part of the torsion: $\bar{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + b_1 W^\lambda g_{\mu\nu} - b_2\delta^\lambda_{(\mu}W_{\nu)} + b_3\delta^\lambda_{[\mu}W_{\nu]}$. Those geometries are defined by the most general connection linearly determined by a vector field (and without derivatives). By studying quadratic (conformally invariant) actions, it is possible to display that these vector-tensor theories may arise significant cosmological implications. The effects of the vector field can be important both in the early universe or at late times depending on the values of parameters. Therefore, they can be used to build dark energy/dark matter models or inflationary scenarios. Moreover, the natural presence of upper bounds for the energy density and/or the curvature might help evading singularities without resorting to quantum effects. This framework admits a class of anisotropic solutions that deserves attention.

This type of study could fit into a precise perspective: the universe is described by a *more fundamental theory* than GR and SM, and they are a low-energies limit. Thus, the conformal invariance is elevated to universal symmetry which, during the process of expansion of the universe, has been broken in different ways. The breaking symmetry gives rise to regions of space (due to some non-homogeneities/anisotropies of the early spacetime) dominated by baryon matter and others regions characterized by dark matter

[1] Of course, an immediate generalization of Brans-Dicke action (in Jordan frame) is to add W_μ dynamic, *i.e.* $\mathscr{L} = \sqrt{-g}\big[\phi^2(R - 6W^2 + 6\nabla_\lambda W^\lambda) + \lambda\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}\big]$; varying with respect to W_μ the equations of motion is $\lambda\nabla_\nu\mathcal{W}^{\nu\mu} + 6W^\mu + 6\nabla^\mu\phi^2 = 0$ and, if W is a closed form, it results $W_\mu = -\nabla_\mu\phi^2$ and the action becomes the usual Brand-Dicke theory in Jordan frame.

and dark energy. Perhaps, such a more fundamental theory could also come close to explaining the disparity between the amount of matter and antimatter we observe in the universe.

One of the reasons why the DM has not yet been detected at the LHC might be due to the still low energy scale. An alternative way to test the high-energy physics is by doing high-precision experiment. This would allow to probe Standard Model Extension (SME)[2] and, in general, quantum gravity effects.

The state-of-the-art of high-precision measurements[3] achieves its ultimate performance in space [55]. Small fleets of satellites may probe EEP in different ways, by searching for tiny deformations of spacetime, which are not classically expected, indicating the presence of a new phenomenology. An example of technology designed for such investigative purposes is the satellite STE-QUEST (Space-Time Explorer and QUantum Equivalence Space Test) [16, 54]. The main purpose of STE-QUEST is testing the different aspects of EEP[4] and searching for its violation with high precision quantum sensors. By using differential atom interferometer, it is possible to compare the free-falling of atoms, linking the possible differences in the acceleration with internal differences (*e.g.* atomic number, mass number, spin, *etc.*). In STE-QUEST specific case, the test masses are two isotopes of the Rubidium atoms (^{85}Rb and ^{87}Rb). The subject of the investigation is the so-called *Eöstöv ratio*, which denotes the correlation between inertial mass and gravitational one[5]. Moreover, STE-QUEST can test the gravitational red-shift by comparing time intervals measured by identical clocks, which are placed at different points of a gravitational field. Any anomalies would show the violation of LPI.

Improving the sensibility of these tests, not only has the goal of finding an actual violation, but also, it allows to discriminate against unified theories which carry a EEP violations at some level. In addition, spacial probes could detect quantum effects associated to small frequencies having a cosmological origin. Even if we cannot overcome a certain energy range, extreme high-energy cosmological phenomena (*e.g.* black holes collisions and gamma ray bursts) could produce very small frequency signals. Therefore, the detection of such data could show non-metric (quantum) effects, giving validity tests for high-energies theories.

[2] SME is the most general effective (quantum) field theory that describes Lorentz violations for elementary particles, by including all Lorentz-violating operators coming from Standard Model fields [112, 113]. Moreover, SME can be extended in order to study Lorentz-violating and CPT-violating gravitational interactions [114], taking into account both curvature and torsion effects.

[3] We refer to atomic clocks, atom interferometers, high-performance time and frequency links and classical accelerometers.

[4] As seen the Chapter 1, EEP can be decomposed in three sub-principles: WEP, LLI, LPI. They cannot be considered as independent because it possible to "show" that if one of the them is violated, then so are the other two. This statement is known as *Schiff's conjecture*, formulated around 1960 [115].

[5] Usually, Eöstöv ratio is commonly denoted by η_{AB}, for two test objects A and B, and it is defined as $\eta_{AB} = 2\frac{(m_i/m_g)_A-(m_i/m_g)_B}{(m_i/m_g)_A+(m_i/m_g)_B} = 2\frac{a_A-a_B}{a_A+a_B} = \beta_A - \beta_B$, where a_i ($i = A$, B) is the i-th acceleration and β_i is the WEP-violating parameter.

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Appendix

Perplexity About Teleparallel Field Strength

Here personal considerations about the teleparallel gravity as gauge theory of the translations group follow.
The perplexity is caused by the "intimate" relation between the tangent space and the internal gauge space. This implies two points of view, the tetrad formalism and the gauge theory one. They should be equivalent to define with consistency a gauge theory of gravity.
According to who writes, the ambiguities arise from the fact that it is necessary to consider an holonomic frame. This is because the gauge theory is on the group of translations whose generators are $P_a \equiv \partial_a$ the ordinary partial derivatives and $[P_a,\, P_b] = 0$. Moreover, the structure of the gauge theory presents substantial differences from the general framework of gauge theories [61].
In the literature there is another formulation of gravity as a gauge theory on the group of translations: "*Einstein Lagrangian as the translational Yang-Mills Lagrangian*" by Y. M. Cho [42]. The latter theory differs from [41] for the holonomicity of the frame and the interpretation of translations. In [42], the gauge potential are identified as the non-trivial part of tetrad and the gauge field strength is given in terms of the commutator coefficients (i.e, the anholonomicity) of the local orthonormal basis one starts with. Thus, it is stressed out that although the gauge group $\mathcal{T}^{(1,3)}$ is Abelian, it is not an internal-symmetry group and acts on spacetime itself; therefore, the "abelianicity" is preserved and one can consider a generic (anholonomic) internal frame.

Returning to the initial point of view, used in Chapter 3, *i.e.* $\mathcal{T}^{(1,3)}$ as symmetry of the internal space, the following problem arises:

- Usually, when one introduces tetrad, one starts from a coordinate basis $\{\partial_\mu\}$ which commutes. The definition (3.109) goes in the opposite direction, starting from the holonomic tetrad $\{\partial_a\}$ and going to $\{h_\mu\}$. Therefore, the usual Cartan's formulas for tetrad should be used in reverse.

- The commutator $[h_\mu,\, h_\nu]$ of eq. (3.113) can be calculated from the tetrad point of view and from the gauge potential point of view. The latter corresponds to the equation (3.113) which means $[\partial_a,\, \partial_\mu] = [P_a,\, \partial_\mu] = 0$ because they belong to different spaces, in contrast with tetrad point of view where there exists a relation between internal and external space [6].

- Performing a local Lorentz transformation the tetrad $h^a{}_\mu$ results different from the eq. (3.114). Moreover, the explicit computation of the commutator of the covariants derivative does not return the expression (3.117).

Let us start with the computation of commutator $[h_\mu,\, h_\nu]$ in absence of inertial effect, explaining internal degrees of freedom. It results:

$$
\begin{aligned}
[h_\mu,\, h_\nu]\psi &= (h^d{}_\mu \partial_d h^a{}_\nu - h^d{}_\nu \partial_d h^a{}_\mu)\partial_a \equiv T^a{}_{\mu\nu}\partial_a \psi && (37)\\
&= \big[(\partial_\mu x^d + B^d{}_\mu)\partial_d(\partial_\nu x^a + B^a{}_\nu) - (\mu \leftrightarrow \nu)\big]\partial_a\psi = \\
&= (\partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu)\partial_a\psi + \\
&\quad + \big[(B^d{}_\mu \partial_d B^a{}_\nu - B^d{}_\nu \partial_d B^a{}_\mu) + (B^d{}_\mu \partial_d \partial_\nu x^a - B^d{}_\nu \partial_d \partial_\mu x^a)\big]\partial_a\psi. && (38)
\end{aligned}
$$

Here $T^a{}_{\mu\nu}\partial_a = (\partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu)\partial_a$ is obtained iff $(B^d{}_\mu \partial_d B^a{}_\nu - B^d{}_\nu \partial_d B^a{}_\mu) + (B^d{}_\mu \partial_d \partial_\nu x^a - B^d{}_\nu \partial_d \partial_\mu x^a) = 0$. This means $B^a{}_\mu$ and $\partial_\mu x^a$ are independent of internal coordinates and it corresponds to choose the origin of tangent space coincident with the tetrad one, while the anholonomy is given by the gauge potential: $h^a{}_\mu = \delta^a_\mu + B^a{}_\mu$.
Therefore, this is the link between the gauge viewpoint and the tetrad one. It is extremely delicate because of the "intimate" relation between the tangent space and the internal gauge space.

Now, let us perform a local Lorentz transformation inside the internal space [7] :

$$
\begin{aligned}
h_\mu &= h^a{}_\mu \partial_a \to \tilde{h}^a{}_\mu \Lambda_a{}^b \partial_b && (39)\\
B_\mu &\to \tilde{B}_\mu = \tilde{B}^a{}_\mu \Lambda_a{}^b \partial_b && (40)\\
\partial_\mu &= \partial_\mu x^a \partial_a \to \partial_\mu(\Lambda^a{}_b x^b)\Lambda_a{}^c \partial_c = \partial_\mu x^a \partial_a + \Lambda_a{}^c \partial_\mu(\Lambda^a{}_b) x^b \partial_c = \\
&\qquad\qquad = (\partial_\mu x^a - \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b)\partial_a. && (41)
\end{aligned}
$$

Therefore, if $B^a{}_\mu \to \Lambda^a{}_\mu B^a{}_\mu$ then the new tetrad is

$$
h^a{}_\mu = \partial_\mu x^a - \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b + B^a{}_\mu = \overset{\bullet}{\mathscr{D}}_\mu x^a + B^a{}_\mu, \tag{42}
$$

that differs from (3.114) by the sign of the inertial connection.
To obtain the tetrad (3.114), taking into account that $B^a{}_\mu \to B^a{}_\mu B^a{}_\mu$, one has to manually replace ∂_a with $\partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b$ that is the anholonomic tetrad (3.89) iff $\{x^a\}$ is referred

[6] Tetrads by definitions are the set of coefficients $\{e^a{}_\mu\} \in GL(4, \mathbb{R})$, namely they are invertible.

[7] In tetrad formalism, one has eq.s (3.86 - 3.89): $\hat{e}_a = e_a{}^\mu \partial_\mu$, $\hat{e}'_a = e'_a{}^\mu \partial_\mu$ and $e_a{}^\mu = \Lambda_a{}^b e'_b{}^\mu$. Therefore, the changing of the internal basis causes the changing of the tetrad and ∂_μ is unchanged with respect to this transformation. If one follows this procedure, considering h_μ unchanged, the constraint to express the covariant derivative with respect to the holonomic bases $\partial_a \equiv P_a$ makes sure there is no change. Therefore the only thing one can do is to transform the inner coordinates into tetrad.

to the anholonomic coordinate frame.
However, this ambiguity is the consequence to require that the internal space is described by the holonomic bases $\{\partial_a\} \equiv \{P_a\}$.
Therefore, it would be more correct to redefine $h^a{}_\mu$ with a generic spin connection, such that the tetrad is invariant with respect to local Lorentz transformation.

According to who writes, what deceives is the following procedure: one defines

$$[h_\mu,\, h_\nu] = T^a{}_{\mu\nu} P_a, \tag{43}$$

requiring that

$$T^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \dot{\omega}^a{}_{b\mu} B^b{}_\nu - \dot{\omega}^a{}_{b\nu} B^b{}_\mu = \dot{\mathscr{D}}_\mu B^a{}_\nu - \dot{\mathscr{D}}_\nu B^a{}_\mu, \tag{44}$$

therefore, since $[\dot{\mathscr{D}}_\mu,\, \dot{\mathscr{D}}_\nu] x^a = 0$, it result

$$T^a{}_{\mu\nu} = \dot{\mathscr{D}}_\mu h^a{}_\nu - \dot{\mathscr{D}}_\nu h^a{}_\mu = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu + \dot{\omega}^a{}_{b\mu} h^b{}_\nu - \dot{\omega}^a{}_{b\nu} h^b{}_\mu. \tag{45}$$

However, it is not enough to say that

$$h_\mu = h^a{}_\mu P_a, \tag{46}$$

because (here) one could not use the Cartan's equation in the usual way and then, the relation between h_μ and $h^a{}_\mu$ is not the usual for tetrads. The Mismatch can be observed by explicitly calculating the commutator, assuming that $h^a{}_\mu = \partial_\mu x^a + \dot{\omega}^a{}_{b\mu} x^b + B^a{}_\mu$:

$$\begin{aligned}
[h_\mu,\, h_\nu] f &= h^c{}_\mu \partial_c (h^a{}_\nu \partial_a \psi) - (\mu \leftrightarrow \nu) = \\
&= h^c{}_\mu \partial_c (h^a{}_\nu) \partial_a \psi + h^c{}_\mu h^a{}_\nu \partial_c (\partial_a \psi) - (\mu \leftrightarrow \nu) = \\
&= (\partial_\mu x^c + \dot{\omega}^c{}_{d\mu} x^d + B^c{}_\mu) \partial_c (\partial_\nu x^a + \dot{\omega}^a{}_{b\nu} x^b + B^a{}_\nu) \partial_a \psi + \\
&+ (\partial_\mu x^c + \dot{\omega}^c{}_{d\mu} x^d + B^c{}_\mu)(\partial_\nu x^a + \dot{\omega}^a{}_{b\nu} x^b + B^a{}_\nu) \partial_d \partial_a \psi - (\mu \leftrightarrow \nu) = \\
&= (\partial_\mu x^c + \dot{\omega}^c{}_{d\mu} x^d + B^c{}_\mu) \partial_c (\partial_\nu x^a + \dot{\omega}^a{}_{b\nu} x^b + B^a{}_\nu) \partial_a \psi - (\mu \leftrightarrow \nu) = \\
&= \big[\partial_\mu \dot{\omega}^a{}_{b\nu} x^b + \dot{\omega}^a{}_{b\nu} \partial_\mu x^b + \partial_\mu B^a{}_\nu + \\
&+ \dot{\omega}^c{}_{d\mu} x^d \partial_c \partial_\nu x^a + \dot{\omega}^c{}_{d\mu} x^d \partial_c \dot{\omega}^a{}_{b\nu} x^b + \dot{\omega}^c{}_{d\mu} x^d \dot{\omega}^a{}_{c\nu} + \dot{\omega}^c{}_{d\mu} x^d \partial_c B^a{}_\nu + \\
&+ B^c{}_\mu \partial_c \partial_\nu x^a + B^c{}_\mu \partial_c \dot{\omega}^a{}_{b\nu} x^b + B^c{}_\mu \dot{\omega}^a{}_{c\nu} + B^c{}_\mu \partial_c B^a{}_\nu - (\mu \leftrightarrow \nu) \big] \partial_a \psi,
\end{aligned} \tag{47}$$

deleting the inertial spin curvature,

$$
\begin{aligned}
[h_\mu,\, h_\nu]\psi = \big[&\overset{\bullet}{\omega}{}^a{}_{b\nu}\partial_\mu x^b + \partial_\mu B^a{}_\nu + \\
&+ \overset{\bullet}{\omega}{}^c{}_{d\mu} x^d \partial_c \partial_\nu x^a + \overset{\bullet}{\omega}{}^c{}_{d\mu} x^d \partial_c \overset{\bullet}{\omega}{}^a{}_{b\nu} x^b + \overset{\bullet}{\omega}{}^c{}_{d\mu} x^d \partial_c B^a{}_\nu + \\
&+ B^c{}_\mu \partial_c \partial_\nu x^a + B^c{}_\mu \partial_c \overset{\bullet}{\omega}{}^a{}_{b\nu} x^b + B^c{}_\mu \overset{\bullet}{\omega}{}^a{}_{c\nu} + B^c{}_\mu \partial_c B^a{}_\nu - (\mu \leftrightarrow \nu)\big]\partial_a \psi = \\
&= \big[\partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu - \overset{\bullet}{\omega}{}^a{}_{c\mu} B^c{}_\nu + \overset{\bullet}{\omega}{}^a{}_{c\nu} B^c{}_\mu\big]\partial_a \psi \\
&+ \big[\overset{\bullet}{\omega}{}^a{}_{b\nu}\partial_\mu x^b + \overset{\bullet}{\omega}{}^c{}_{d\mu} x^d \partial_c \partial_\nu x^a + \overset{\bullet}{\omega}{}^c{}_{d\mu} x^d \partial_c \overset{\bullet}{\omega}{}^a{}_{b\nu} x^b + \\
&+ \overset{\bullet}{\omega}{}^c{}_{d\mu} x^d \partial_c B^a{}_\nu + B^c{}_\mu \partial_c \partial_\nu x^a + B^c{}_\mu \partial_c \overset{\bullet}{\omega}{}^a{}_{b\nu} x^b + B^c{}_\mu \partial_c B^a{}_\nu - (\mu \leftrightarrow \nu)\big]\partial_a \psi = \\
&= \big[\partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu - \overset{\bullet}{\omega}{}^a{}_{c\mu} B^c{}_\nu + \overset{\bullet}{\omega}{}^a{}_{c\nu} B^c{}_\mu\big]\partial_a \psi + \\
&+ \big[\overset{\bullet}{\omega}{}^a{}_{b\nu}\partial_\mu x^b - \overset{\bullet}{\omega}{}^a{}_{b\mu}\partial_\nu x^b\big]\partial_a \psi. \qquad (48)
\end{aligned}
$$

The above expression presents the wrong sign for contractions between $B^a{}_\mu$ and $\omega^a{}_{b\mu}$ with respect to (45) part and, moreover, considering $\overset{\bullet}{\omega}{}^a{}_{b\mu}$, $B^a{}_\mu$ and $\partial_\mu x^a$ independent on that the internal coordinates[8], there is the presence of $\overset{\bullet}{\omega}{}^a{}_{b\nu}\partial_\mu x^b - \overset{\bullet}{\omega}{}^a{}_{b\mu}\partial_\nu x^b$, which is in general non-null.

Finally, a new internal basis is introduced in the eq. (3.124),

$$h^a = dx^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b dx^\mu + B^a{}_\mu dx^\mu. \qquad (49)$$

The new basis $h_a = h^a{}_\mu dx^\mu$ is anholonomic and it is not related with $h_\mu = h^a{}_\mu \partial_a$, except for $h^a{}_\mu$. There is a change of viewpoint, which now is the usual one: $h^a{}_\mu$ is the usual tetrad with non-trivial part as $B^a{}_\mu$, which cannot be interpreted as connection 1-form with value in the translations group.

[8] This guarantees the equivalence between the tetrad formalism and the gauge point of view.