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# Quantum Mechanics on Discrete Phase Space: a Group Theoretical Approach


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# Contents

# Introduction

Quantum mechanics is the physical theory which describes microscopic phenomena, developed in the first half of the 20th century thanks to the work of outstanding physicists such as Heisenberg and Schrödinger. It became immediately extremely popular on one hand thanks to the excellent accord between experimental data and theoretical results and, on the other hand, for its wide range of applications fields, among which we find nuclear physics, condensed matter, quantum optics and, in more recent years, cryptography and quantum computing. In its standard formulation, quantum states are represented as trace class, semi-positive definite, trace one operators on a proper separable complex Hilbert space, which represents the quantum system, while observables are described as self-adjoint operators on the same Hilbert space. Nevertheless, this approach proved to be strongly counter intuitive and very far from the standard mathematical language in which classical mechanics is usually described. Bearing in mind this fact, the work of Wigner, Weyl, Moyal and Grönewold gave birth to an alternative description of quantum mechanics, which tries to mimic classical mechanics as best as possible. This *phase space quantization* is undoubtedly founded on the work of Wigner of 1932 [50], who was trying to find the quantum corrections to the Boltzmann distribution at low temperatures. For this purpose, a suitable function on phase space associated with a quantum state $\psi$ was introduced, nowadays called the *Wigner function* (or quasidistribution),

$$\mathscr{W}_\psi(q,p) = \int_{\mathbb{R}^n \times \mathbb{R}^n} dx\ e^{-i2\pi p\cdot x} \overline{\psi\left(q - \frac{x}{2}\right)} \psi\left(q + \frac{x}{2}\right).$$

This function, although it could not be interpreted as a classical state since it is not positive definite, has been the milestone of the description of quantum mechanics on phase space [41, 42].

On the other hand, strictly related with Wigner's work, Weyl was looking for a standard representation of classical observables in quantum mechanics, a well known problem in the quantization of classical quantities, which is strictly related with the transition from a commutative theory (classical mechanics) to a non-commutative one (quantum mechanics). For this purpose, Weyl introduced the following map

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} dqdp\ e^{i\frac{\pi}{\hbar}(p\hat{q}-q\hat{p})}f(q,p), \quad f\in L^2(\mathbb{R}^n\times\mathbb{R}^n),$$

often referred at as the *Weyl transform*, which give a non ambiguous quantization rule. The link with the Wigner function is very strict, indeed it turned out that they are, in a certain way, one the adjoint map of the other [38, 52]. However, it comes without surprise that this is not the unique possible association between classical and quantum objects (respectively, we may find other dequantization maps in place of the Wigner function); some other popular are also found for example in quantum optics and has been applied profitably [41]. Nevertheless, the scheme developed from the works of Weyl and Wigner (the Weyl-Wigner correspondence) is the only one which reproduce faithfully the probability distributions in positions and in momenta [41, 47].
The picture was completed in the following decade, when the works of Grönewold [23] and Moyal [36] put things together introducing a suitable product of functions, the $\star$-*product*, that allows the study of the dynamical evolution of a quantum state from the point of view of the Wigner function. In light of these facts, the phase space approach to quantum mechanics has been applied in various fields of research, for example in quantum transport processes, and it can be fruitfully used to study the quantum-classical transition, which is governed by decoherence [47]. Anyhow, the Wigner function founds its use in many areas other than quantum mechanics, such as classical optics [47] and signal analysis, where the Wigner function may be used to represent time varying signals. Indeed, it is strictly linked with the most fundamental tool of time-frequency analysis, the short time Fourier transform, which allows a simultaneous analysis of a signal both in time (positions) and in frequency (momenta) [22]. Moreover, the Wigner function (and, in general, the phase space approach) offers some insights in wavelet theory too [2, 7].

In recent years, the interest in the Weyl-Wigner scheme has increased drastically thanks to the development of quantum tomography, a technique that allows a direct reconstruction of the Wigner function from experimental measurements [47].
Another important reason behind the popularity of the Wigner function may be found in the birth of quantum information theory and quantum computing, because finite quantum systems can be described in terms of a generalized phase space approach. For instance, in quantum information

theory, quantum teleportation of finite quantum states [37] can be studied in terms of the Weyl-Wigner correspondence only. On the other hand, the phase space approach offers an alternative point of view in quantum computing too, where quantum algorithms, such as Grover's algorithm, can be regarded as quantum maps and can be fruitfully studied in the classical limit, thanks to the common playground between classical and quantum offered by the Weyl-Wigner correspondence [11, 34]. Still this is not the only remarkable field of research which has taken advantage of the Wigner function: some problems in many-body physics can be described using a suitable phase space approach [1, 25].
In order to successfully apply the Weyl-Wigner correspondence to such a vast plethora of applications, this scheme needs to be suitably generalized so that groups different from $\mathbb{R}^n \times \mathbb{R}^n$ can be taken in consideration. In particular, there are two turning points. The first step is to recognize that harmonic analysis on Abelian groups offers a reliable mathematical background where the standard Wigner functions is naturally entailed. Indeed, the whole framework is founded on the irreducible representations of $\mathbb{R}^n \times \mathbb{R}^n$, precisely the projective ones since its unitary representations are physically trivial, because one-dimensional [18, 45]. Projective representations, more specifically, arise in a natural way in quantum mechanics due to Wigner's theorem [35], which fixes simmetry groups to be represented by unitary (or anti-unitary) operators up to phase factors.
Thence, the Wigner function can be introduced as a Fourier transform (the symplectic Fourier transform [5, 7, 9]) of

$$\mathcal{B}_2(L^2(\mathbb{R}^n)) \ni \rho \mapsto \mathrm{tr}(S(\cdot,\cdot)^*\rho) \in L^2(\mathbb{R}^n \times \mathbb{R}^n),$$

where $S$ is an irreducible (infinite dimensional) projective representation of $\mathbb{R}^n \times \mathbb{R}^n$ which acts on $L^2(\mathbb{R}^n)$. The latter map is usually called the characteristic function for the analogies with probability theory [5], or the dequantization map, since it associates a square integrable function to a Hilbert-Schmdit operator. Then, the Weyl transform is recovered considering the adjoint of the Wigner function and the $\star$-product of functions can be introduced as the dequantizer (Wigner function) of the product of the quantized functions (by the Weyl map) [7]. The second fundamental step to generalize the Weyl-Wigner correspondence is given by the notion of *square integrable* representation. Indeed, it turns out that we can suitably define a dequantization map everytime the considered group admits a square integrable (projective) unitary representation [2, 7, 9]. In particular, this map is an isometry defined on the space of Hilbert-Schmidt operators on the space of the representation, and it takes value in the space of square integrable functions of the group considered. Moreover, it admits a pseudo-inverse map, a generalized Weyl transform, which can be regarded as a quantization map. Square integrable representations, on the other hand, also allow an extended study on coherent states, which can be generalized fruitfully to various groups [2].

In this thesis work we will study the Weyl-Wigner scheme from the group theoretical point of view, choosing to give a particular emphasis on discrete phase space, introducing the Weyl-Wigner correspondence in analogy with the standard correspondence that holds for $\mathbb{R}^n \times \mathbb{R}^n$. Representation theory of locally compact second countable groups and harmonic analysis will be introduced so that we can give, as far as possible, a comprehensive introduction to the subject. In particular, we will always discuss the continuous phase space first; next, thanks to the tools developed with the help of representation theory, a similar analysis shall be done for the discrete phase space.
We will start noticing that the symplectic structure of the standard phase space arises from the classification of its projective representations, which can be performed considering a suitable extended group whose unitary representations can be completely analyzed [6, 45]. By an inversion procedure, under suitable hypothesis, we disregard the central character (namely, the irreducible representation of the centre of the extended group) and deduce the projective representations seeked. In particular, the central extension of $\mathbb{R}^n \times \mathbb{R}^n$ via $\mathbb{R}$ is the Heisenberg-Weyl group $\mathbb{H}_n(\mathbb{R})$, whose irreducible unitary representations can be classified by an application of the Mackey machine - a standard technique to classify unitary representations of semi-direct product groups - which also guarantee us that Stone-von Neumann's theorem holds true [20]. Moreover, a rigorous formulation of the canonical commutation relations (CCRs) is also available in terms of Weyl systems, families of jointly irreducible representations which satisfy the CCRs in their exponentiated version; these maps can also be regarded as projective representations of $\mathbb{R}^n \times \mathbb{R}^n$ [4].
To generalize to the case of discrete phase space, the first step is to recognize that $\mathbb{R}^n \times \mathbb{R}^n$ is a group given by the direct product of an Abelian group for its unitary dual, namely $G \times \hat{G}$, where $\hat{G}$ is the group of all equivalence classes of its irreducible unitary representations [18, 40]. Thus, one shall demand that this group structure is preserved in order to have a suitable phase space. Indeed, an interesting counterxample is given by the simplest candidate to be a discretized phase space, namely $\mathbb{Z} \times \mathbb{Z}$. Studying the representations of its central extension via $\mathbb{Z}$, which is given by $\mathbb{H}(\mathbb{Z})$, one realizes that positions and momenta do not keep their dual role, while this can be achieved with a group of the form $G \times \hat{G}$ (with $G$ Abelian). Hence,

the same analysis will be performed for a discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$, which this time satisfy the condition $G \times \hat{G}$. However, some differences arise. Firstly, the dimension of the irreducible unitary representations of the central extension of $\mathbb{Z}_N \times \mathbb{Z}_N$ via $\mathbb{Z}_N$ - namely the discrete Heisenberg-Weyl group $\mathbb{H}(\mathbb{Z}_N)$ - depends on the choice of the representation; in many cases (when such dimension does not match the order of the configurations space $\mathbb{Z}_N$) a rescaling of the quantum system considered is entailed in such a choice.
Another important difference with the standard phase space arises in the study of the discrete Wigner function. Indeed, in the standard case, it is known that the Wigner function can be written in terms of phase-point operators (the quantum counterpart of the phase space points) as [2]

$$\mathscr{W}_S^\rho(q,p) = \mathrm{tr}(A(q,p)\rho), \quad A(q,p) \propto S(q,p)\Pi S(q,p)^*,$$

where $\Pi$ is the parity operator, $S$ is a projective representation of the phase space (continuous or finite) and $\rho$ is a quantum state on the space of the representation. On the other hand, in the discrete case, this alternative formulation of the Wigner function fails when the discrete phase space considered is of even order [34, 46, 53], since the phase-point operators form a basis in the vector space of matrices on a $N$-dimensional Hilbert space only if $N$ is an odd number. In this sense, the discrete Wigner function introduced by means of a square integrable projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ is slightly more general than the one defined in terms of phase-point operators.
The description of finite quantum systems in terms of the Weyl-Wigner correspondence find a natural application in the separability problem [29, 31]. This is a well known problem in quantum information theory, where various important criteria - such as the PPT criterion - provide some conditions which establish if a given state is separable. In particular, we will see that the Weyl-Wigner correspondence offers an interesting alternative point of view, founded on functions of (quantum) positive type.

We summarize these facts in the following chapters:

1. In the first chapter, we will review the most important facts concerning representation theory of locally compact (second countable) groups. We will firstly introduce unitary representations (section 1.2), together with the most important results of the theory, such as Schur's lemma (theorem 1.2.7). We will also investigate the case of compact groups (section 1.2.1), whose irreducible representations are finite dimensional (theorem 1.2.11), while the reducible ones enjoy the complete reducibility property (theorem 1.2.12). A general decomposition

for arbitrary unitary representations will be discussed too, introducing the direct integral of representations (section 1.2.2). Next, the Mackey machine will be discussed (section 1.3.1) in order to characterize the unitary representations of the Heisenberg-Weyl group, both continuous and finite. Nevertheless, the link between the Heisenberg-Weyl group and the phase space (theorem 1.4.8) will also require a more comprehensive discussion on the notion of projective representations (section 1.4).

2. The second chapter is devoted at studying continuous and finite phase space. At first, we will introduce some elementary facts concerning the Heisenberg-Weyl groups, pointing out its semi-direct product structure. Secondly, we will classify the multipliers of $\mathbb{R}^n \times \mathbb{R}^n$ (section 2.2.1); the corresponding projective representations will be obtained classifying the unitary ones of its group extension, $\mathbb{H}_n(\mathbb{R})$, by an application of the Mackey machine (section 2.2.2). Then, an abstract formulation of the CCRs will be discussed introducing Weyl systems (section 2.2.3) and Stone-von Neumann's theorem (theorem 2.2.3).
Next, we will focus on discrete phase space. Studying the representations of $\mathbb{H}(\mathbb{Z})$, we will observe that a general phase space shall be of the form $G \times \hat{G}$, with $G$ Abelian (section 2.3). Hence, we will focus on the case of $\mathbb{Z}_N \times \mathbb{Z}_N$ and we will retrace the same classification done for $\mathbb{R}^n \times \mathbb{R}^n$ up to Stone-von Neumann's theorem, which holds in the finite case too (section 2.4).

3. In the third chapter we study harmonic analysis on locally compact (second countable) Abelian groups. In the first part the Fourier transform will be introduced by virtue of functions of positive type and group algebra (section 3.1); then, we will face the most important theorems such as the inversion formulas (3.3). Finally, the Fourier-Plancherel operator (theorem 3.3.9) and Pontrjagin's duality theorem (theorem 3.3.12) will be discussed (section 3.3.1).
In the second part of the chapter we define the most important tools to deal with the Weyl-Wigner correspondence. Hence, the symplectic Fourier transform (section 3.4.1), the twisted convolution (section 3.4.2) and time-frequency analysis (section 3.4.3) will be introduced. At last, we will deal with square integrable representations and the wavelet transform will be defined (section 3.5), and we will discuss Duflo-Moore's theorem (theorem 3.5.2).

4. In the last chapter we study the Weyl-Wigner correspondence. The first paragraph (section 4.1) is dedicated to the general formulation of

the Wigner (and the Weyl) transform on groups which admit square integrable projective representations. Then, restricting our attention to unimodular groups, we will study the $\star$-product of functions induced by the Weyl-Wigner correspondence (section 4.1.1).
In the second part of the chapter the general framework will be applied to continuous and discrete phase space. Firstly, the standard Wigner function will be defined as the symplectic Fourier transform of the Wigner transform. Some of its most important properties will be examinated too. Then, the discrete phase space case will be studied (section 4.3), pointing out analogies and differences with respect to the standard phase space. Lastly, functions of quantum positive type will be analyzed, with a particular attention to the discrete case, so that the separability problem can be discussed from the point of view of the Weyl-Wigner correspondence (section 4.4).

# Chapter 1

# Basic facts on representation theory

In this chapter we give a brief introduction to representation theory of locally compact second countable (l.c.s.c. in brief) groups, in order to analyze the irreducible representations of the Heisenberg-Weyl groups and the phase space in the future. Naturally, this is an extremely vast subject, thus we will recall just the most elementary facts of our interest.
The chapter is structured as follows. Firstly, we will define the strong and weak operator topologies so that we can introduce the concept of unitary representations of a l.c.s.c. group and the most important facts, such as Schur's lemma. We also briefly study the case of compact groups and the celebrated Peter-Weyl's theorem, which concerns the decomposition into direct sum of irreducible representations of an arbitrary representation of a compact group. Then we will introduce the direct integral decomposition of a unitary representation, which is the unique decomposition into irreducible representations that holds in the most general case.
Next, we will focus on the induced (unitary) representations of locally compact groups, introducing the Mackey machine, by means of which we can analyze the irreducible representations of semi-direct product groups.
Finally, we will spend some time with the theory of projective representations of groups, which are fundamental in quantum mechanics and will be extremely useful when we will deal with Weyl systems.

## 1.1 Unitary operators

To introduce unitary representations, we recall some basic facts concerning unitary operators, in particular about two topologies which will play an

important role soon.
Let $\mathcal{H}$ be a Hilbert space of arbitrary dimension and denote with $\|\cdot\|$ the usual norm on $\mathcal{H}$. In the following we will denote with $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$.

**Definition 1.1.1.** A bounded linear operator $U \in \mathcal{B}(\mathcal{H})$ is *unitary* if it is an isometry, i.e. $\|\psi\| = \|U\psi\|\ \forall\psi \in \mathcal{H}$, and $\operatorname{Ran} U = \mathcal{H}$.

$\mathcal{U}(\mathcal{H}) \equiv \{U \in \mathcal{B}(\mathcal{H}) \mid U \text{ unitary}\}$ will denote the group of all unitary operators on $\mathcal{H}$. For the applications is worth recalling the fact that the following assertions are equivalent [39]:

1. $U$ is a unitary operator.
2. $\operatorname{Ran} U = \mathcal{H}$ and $\langle U\phi, U\psi\rangle = \langle\phi,\psi\rangle\ \forall\phi,\psi \in \mathcal{H}$.
3. $U^*U = UU^* = \mathrm{Id}$, where Id denotes the identity operator in $\mathcal{H}$.
4. $U^*$ is a unitary operator.

As an immediate consequence we have that $U$ is bijective and $U^{-1} = U^*$.
We observe that $\mathcal{U}(\mathcal{H})$ can be equipped with the relative topology with respect to $\mathcal{B}(\mathcal{H})$, i.e. the operator norm topology, but this choice is too strict for physical application. Indeed, if we consider for example a one-parameter group of unitary operators $\mathbb{R} \ni t \mapsto e^{-itH} \in \mathcal{U}(\mathcal{H})$, this group is continuous in the norm topology if and only if $H$ is a bounded operator [39]. Hence, if we require our operators to be continuous in the norm topology, we cannot consider unitary transformations associated with unbounded operators. Therefore it is useful to define the following topologies:

1. The *strong topology*: it is the initial topology induced by the following maps:
$$E_\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{H},\ E_\phi(T) := T\phi,\ \phi \in \mathcal{H}. \tag{1.1}$$
We have the following equivalence for strong convergence [39]:
$$T_\alpha \xrightarrow{s} T \iff \|T_\alpha\psi - T\psi\| \to 0 \quad \forall\psi \in \mathcal{H}, \tag{1.2}$$
where $\{T_\alpha\}_\alpha$ is a net in $\mathcal{B}(\mathcal{H})$ and the 's' "means "strong convergence".
2. The *weak topology*: it is the initial topology induced by the following maps:
$$E_{\psi,\phi} : \mathcal{B}(\mathcal{H}) \to \mathbb{C},\ \psi,\phi \in \mathcal{H},\ E_{\psi,\phi}(T) := \langle\phi, T\psi\rangle. \tag{1.3}$$
Weak convergence is equivalent to require that [39]
$$|\langle\phi, T_\alpha\psi\rangle - \langle\phi, T\psi\rangle| \to 0 \quad \forall\phi,\psi \in \mathcal{H}. \tag{1.4}$$

We observe that these topologies are weaker than the norm topology, indeed the map $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \ni (A, B) \mapsto AB \in \mathcal{B}(\mathcal{H})$ is continuous in the norm topology, but it is not in the weak and in the strong topology [39].

If we restrict our attention to $\mathcal{U}(\mathcal{H})$, it is worth mentioning that, in such a case, the weak and the strong topologies coincide [39]. Lastly, recall that $\mathcal{U}(\mathcal{H})$ equipped with strong topology is a completely metrizable space.

## 1.2 Unitary representations

**Definition 1.2.1.** Let $G$ be a l.c.s.c. group. The map $\pi : G \to \mathcal{U}(\mathcal{H})$ is a *unitary representation* if it is a strongly continuous homomorphism.

In order to have proper representations, we demand $\mathcal{H}$ to be a nonzero space; the latter is usually called the *representation or carrier space* and its dimension is called the *dimension of the representation.* Observe that, due to unitarity, $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$.
It is also worth mentioning that a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ is strongly continuous iff it is weakly Borel, namely the map $G \ni g \mapsto \langle \phi, \pi(g)\psi \rangle \in \mathbb{C}$ is a Borel map $\forall \phi, \psi \in \mathcal{H}$ [45].

**Definition 1.2.2.** Let $\pi_1 : G \to \mathcal{U}(\mathcal{H}_{\pi_1}), \pi_2 : G \to \mathcal{U}(\mathcal{H}_{\pi_1})$ be two unitary representations. We will say that $\pi_1, \pi_2$ are *intertwined* if there exists a bounded linear operator $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $T\pi_1(g) = \pi_2(g)T \; \forall g \in G$, operator that will be called the *intertwining operator.*

We will denote with $\mathscr{C}(\pi_1, \pi_2)$ the set of all the intertwining operators between $\pi_1$ and $\pi_2$; we also observe that if $T \in \mathscr{C}(\pi_1, \pi_2)$, then $T^* \in \mathscr{C}(\pi_2, \pi_1)$, because

$$T^*\pi_2(g) = (\pi_2(g^{-1})T)^* = (T\pi_1(g^{-1}))^* = \pi_1(g)T^*.$$

**Definition 1.2.3.** We will say that the representations are *unitarily equivalent* if $\mathscr{C}(\pi_1, \pi_2)$ contains a unitary operator $U$, so that $\pi_2 = U\pi_1 U^*$. The set $\mathscr{C}(\pi) \equiv \mathscr{C}(\pi, \pi)$ will be called the *commutant or centralizer* of the representation $\pi$.

**Example 1.2.4.** Let us consider the left and right translations $(L_g f)(h) := f(g^{-1}h)$, $(R_g f)(h) := f(hg)$, where $f : G \to \mathbb{C}$.
Let $\mathcal{H} = L^2(G, \lambda, \mathbb{C})$, where $\lambda$ is the left Haar measure on $G$. The map

$$\pi_L : G \to \mathcal{U}(L^2(G, \lambda, \mathbb{C})), \quad (\pi_L(g)f)(h) := f(g^{-1}h) \tag{1.5}$$

is a unitary representation, called the *left regular representation.* Indeed we have:

$$(\pi_L(g_1g_2)f)(h) = f(g_2^{-1}g_1^{-1}h) = \pi_L(g_2)f(g_1^{-1}h) = \pi_L(g_1)\pi_L(g_2)f(h).$$

Moreover, it is unitary, because $L_g$ is a surjective map and $\pi_L$ is an isometry due to the left-invariance of the Haar measure.
In a similar way, $(\pi_R(g)f)(h) := f(hg)$, where $f \in L^2(G, \rho, \mathbb{C})$ and $\rho$ is the right Haar measure on $G$, is a representation, called the *right regular representation.*
We can also define a right regular representation on $L^2(G, \lambda, \mathbb{C})$ thanks to the modular function, as

$$\tilde{\pi}_R : G \to \mathcal{U}(L^2(G, \lambda, \mathbb{C})), \quad (\tilde{\pi}_R(g))f(h) := \Delta(g)^{1/2}f(hg).$$

Obviously, we can do the analogous on $L^2(G, \rho, \mathbb{C})$.
The right regular representations $\pi_R$ and $\tilde{\pi}_R$ are unitarily equivalent. Indeed, recalling that the map $M_2 : L^2(G, \lambda, \mathbb{C}) \to L^2(G, \rho, \mathbb{C})$ such that $(M_2f)(g) := \Delta(g)^{1/2}f(g)$ is an isomorphism between Banach spaces [18], by direct calculation we have that

$$\left(M_2^{-1}\pi_R(g)M_2f\right)(h) = \Delta(g)^{1/2}f(hg) = \left(\tilde{\pi}_R(g)f\right)(h).$$

A similar proof will show that $\tilde{\pi}_L$ and $\pi_L$ are unitarily equivalent.

We can now discuss of the notion of irreducibility of unitary representations.

**Definition 1.2.5.** If $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a unitary representation, a closed subspace $\mathscr{I} \leq \mathcal{H}_\pi$ is *invariant* with respect to $\pi$ if $\pi(g)\mathscr{I} \subseteq \mathscr{I}$ $\forall g \in G$.

We notice that, if $\mathscr{I} \neq 0$, the strongly continuous homomorphism

$$\pi_{\mathscr{I}} : G \to \mathcal{U}(\mathcal{H}_\pi) \mid \pi_{\mathscr{I}}(g)\psi := \pi(g)\psi \ \forall \psi \in \mathcal{H} \tag{1.6}$$

is a unitary representation, called the *subrepresentation* of $\pi$.

**Definition 1.2.6.** We will say that the unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is *reducible* if admits a non-trivial invariant subspace, i.e. there exists an invariant subspace $\mathscr{I} \leq \mathcal{H}_\pi$ different from $\mathcal{H}$ and $\{0\}$; otherwise the representation is called *irreducible.*

An easy way to obtain reducible representations is via a direct sum of representations: let $\{\pi_i\}_{i\in I}$ be a sequence of unitary representation with carrier space $\mathcal{H}_{\pi_i}$. The map

$$\pi := \bigoplus_{i\in I} \pi_i \;\mid\; \pi(g)(\sum_{i\in I} \psi_i) := \sum_{i\in I} \pi_i(g)\psi_i, \tag{1.7}$$

where $\psi_i \in \mathcal{H}_{\pi_i}$ is called the *direct sum representation.* This representation is reducible, since each $\mathcal{H}_{\pi_i}$ is an invariant subspace for $\pi$; therefore each $\pi_i$ is a subrepresentation of $\pi$. Another useful fact worth mentioning is a way to decompose the representation space. Indeed it is easy to prove that if $\mathscr{I}$ is an invariant subspace of $\mathcal{H}$, then $\mathscr{I}^\perp$ is invariant too [18]. Therefore, if $\pi$ is a reducible representation with invariant subspace $\mathscr{I}$, we can write $\pi = \pi_{\mathscr{I}} \oplus \pi_{\mathscr{I}^\perp}$, where $\pi_{\mathscr{I}}$ and $\pi_{\mathscr{I}^\perp}$ denote the respective subrepresentations on $\mathscr{I}$ and $\mathscr{I}^\perp$.
We can now introduce Schur's lemma, one of the most important results in unitary representation theory [18]:

**Theorem 1.2.7** (Schur's lemma)**.** *Let $\pi_1, \pi_2, \pi$ be unitary representations of a l.c.s.c. group $G$. Then the following sentences hold:*

- *$\pi$ is irreducible iff $\mathscr{C}(\pi) = \{c\mathrm{Id}\}_{c\in\mathbb{C}}$.*
- *Suppose $\pi_1, \pi_2$ are irreducible. If they are also unitary equivalent, then $\mathscr{C}(\pi_1, \pi_2)$ is one-dimensional. Otherwise, the commutant is empty.*

*Proof.* Suppose firstly that $\mathcal{M}$ is a closed subspace of $\mathcal{H}_\pi$ and let $P$ be the orthogonal projection onto $\mathcal{M}$. If $P \in \mathscr{C}(\pi)$ and $v \in \mathcal{M}$, then we have $\pi(g)v = \pi(g)Pv = P\pi(g)v \in \mathcal{M}$, hence $\mathcal{M}$ is invariant. Conversely, if $\mathcal{M}$ is invariant, we have $\pi(g)Pv = \pi(g)v = P\pi(g)v$ if $v \in \mathcal{M}$ and $\pi(g)Pv = 0 = P\pi(g)v$ if $v \in \mathcal{M}^\perp$. Hence $P \in \mathscr{C}(\pi)$. Therefore we have proven that $\mathcal{M}$ is invariant iff $P \in \mathscr{C}(\pi)$.
Now let us suppose $\pi$ is a reducible representation. Then $\mathscr{C}(\pi)$ contains a nontrivial projections. Conversely, if $T \in \mathscr{C}(\pi)$ where $T \neq cId$, then the self-adjoint operators $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$ are elements of $\mathscr{C}(\pi)$ and at least one of them is different from a multiple of the identity operator. Suppose for example $A \neq cId$. Thus $\mathscr{C}(\pi)$ is reducible, because it contains nontrivial projections. Indeed, spectral theory guarantees us that every operator that commutes with $A$, commutes with all the projections $\chi_E$, $E \subset \mathbb{R}$ [18, 39], where $\chi_E$ is the characteristic function of $E$.
Suppose now $T \in \mathscr{C}(\pi_1, \pi_2)$, hence $T^* \in \mathscr{C}(\pi_2, \pi_1)$. Then we have $T^*T \in \mathscr{C}(\pi_1)$, because

$$TT^*\pi_2 = T\pi_1 T^* = \pi_2 TT^*.$$

A similar argument shows that $T^*T \in \mathscr{C}(\pi_1)$, so we have $T^*T = c\mathrm{Id}$ and $TT^* = c'\mathrm{Id}$, hence we must have $T = 0$ or $c^{-1/2}T$ is a unitary operator. Thus if $\pi_1$ and $\pi_2$ are inequivalent, $\mathscr{C}(\pi_1, \pi_2) = \{0\}$ and it consists of scalar multiples of unitary operators. Therefore, if $T_1, T_2 \in \mathscr{C}(\pi_1, \pi_2)$, then $T_2^{-1}T_1 = T_2^*T_1 \in \mathscr{C}(\pi_1)$, so we have $T_2^{-1}T_1 = c\mathrm{Id}$, so $\dim \mathscr{C}(\pi_1, \pi_2) = 1$. □

The following corollary is fundamental in physics:

**Corollary 1.2.8.** *If $G$ is an Abelian group, its irreducible unitary representations are one-dimensional.*

*Proof.* If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary irreducible representation, then $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$, which means $\pi(G) \subseteq \mathscr{C}(\pi)$. Thus, we have $\pi(g) = c_g\mathrm{Id}\ \forall g \in G$, therefore every invariant subspace is one-dimensional. □

As a last topic of this section, we present a first way to decompose a unitary representation into a direct sum of cyclic representations.

**Definition 1.2.9.** Let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation and let $\psi \in \mathcal{H}_\pi$. Then

$$\mathscr{I}_\psi := \mathrm{clos}(\mathrm{span}\{\pi(g)\psi \mid g \in G\}) \tag{1.8}$$

is a closed subspace of $\mathcal{H}_\pi$, called the *cyclic subspace of* $\mathcal{H}_\pi$ generated by $\psi$.

We notice that $\mathscr{I}_\psi$ is an invariant subspace. Indeed let us consider $\sum_{n\in\mathbb{N}} c_n\pi(g_n)\psi,\ \sum_{n\in\mathbb{N}} c_n\pi(gg_n)\psi \in \mathscr{I}_\psi$. Then we have

$$\pi(g)(\sum_{n\in\mathbb{N}}^{\|\cdot\|} c_n\pi(g_n)\psi) = \sum_{n\in\mathbb{N}}^{\|\cdot\|} c_n\pi(gg_n)\psi \in \mathscr{I}_\psi,$$

where the apex means the sum converges in the norm sense. If $\mathscr{I}_\psi$ is a cyclic subspace such that $\mathscr{I}_\psi = \mathcal{H}_\pi$, then we will say that $\psi$ is a *cyclic vector* and the corresponding representation will be called the *cyclic representation* of $\psi$.

**Theorem 1.2.10.** *A unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ can be written as a direct sum of cyclic representations.*

*Proof.* Let us consider the following poset of cyclic subspaces:

$$\{\mathscr{I}'_\beta\}_{\beta\in B} \leq \{\mathscr{I}_\alpha\}_{\alpha\in A} \iff \bigoplus_{\beta\in B}\mathscr{I}'_\beta \subset \bigoplus_{\alpha\in A}\mathscr{I}_\alpha. \tag{1.9}$$

Now let $\{\{\mathscr{I}^{(i)}_{\alpha_i}\}_{\alpha\in A}\}_{i\in\mathcal{I}}$ be a chain. Then it is possible to show that $\bigcup_{i\in\mathcal{I}}\bigoplus_{\alpha_i\in A_i}\mathscr{I}^{(i)}_{\alpha_i} = \mathcal{H}_0$ is an upper bound for that chain. By Zorn's lemma[1], there will exist a maximal family of mutually orthogonal subspaces, denoted by $\{\mathscr{I}_\alpha\}_{\alpha\in A}$. Now suppose there exists $\psi \in \mathcal{H}, \psi \neq 0$ such that $\psi \perp \mathscr{I}_\alpha \ \forall\alpha \in A$. Hence $\mathscr{I}_\psi := \operatorname{span}\{\pi(g)\psi \mid g \in G\}$ is orthogonal to $\mathscr{I}_\alpha \ \forall\alpha \in A$. Therefore we have $\psi \in \left(\bigoplus_{\alpha\in A}\mathscr{I}_\alpha\right)^\perp$ and, by invariance of each $\mathscr{I}_\alpha$, $\mathscr{I}_\psi \subset \left(\bigoplus_{\alpha\in A}\mathscr{I}_\alpha\right)^\perp$, but this is in contradiction with the maximality of $\{\mathscr{I}_\alpha\}_{\alpha\in A}$. Hence we must have

$$\mathcal{H}_\pi = \bigoplus_{\alpha\in A}\mathscr{I}_\alpha \quad \pi = \bigoplus_{\alpha\in A}\pi_{\mathscr{I}_\alpha}. \tag{1.10}$$

□

### 1.2.1 Unitary representations of compact groups

We now discuss the case of compact groups, whose representations possess the "complete reducibility" property, namely an arbitrary representation can be decomposed into a direct sum of irreducible ones.
Let $G$ be a compact group with Haar measure $dg$ normalized in such a way that $\int_G dg = 1$. In the following $\hat{G}$ will denote the set of equivalence classes of irreducible representations of $G$, and will be called the *dual* of $G$. We can observe that, in the compact case, the only important representations are the unitary ones. Indeed, if $V : G \to GL(\mathbb{V})$ is a representation, where $\mathbb{V}$ is a finite dimension vector space and $\langle\cdot,\cdot\rangle_0$ is a scalar product in $\mathbb{V}$, we can define the inner product

$$\langle\phi,\psi\rangle := \int_G dg\langle V(g)\phi, V(g)\psi\rangle_0 \tag{1.11}$$

with respect to $V$ is unitary. Indeed,

$$\begin{aligned}\langle V(h)\phi, V(h)\psi\rangle &= \int dg\langle V(hg)\phi, V(hg)\psi\rangle_0\\ &= \int dg\langle V(g)\phi, V(g)\psi\rangle_0 = \langle\phi,\psi\rangle,\end{aligned}$$

where we have used the left invariance of the Haar measure $g \mapsto h^{-1}g$. Lastly, with respect to the topology induced by 1.11, $V$ is still continuous [10].

[1] Suppose a partially ordered set $X$, i.e. a set endowed with a reflexive, antisymmetric and transitive binary relation, has the property that every chain in $X$, that is a totally ordered subset, has an upper bound in $X$. Then $X$ contains at least one maximal element [15].

For each unitary representation $U : G \to \mathcal{U}(\mathcal{H})$, let us consider a vector $\psi \in \mathcal{H} : \|\psi\| = 1$ and define the following operator:

$$T_U := \int_G dg \langle U(g)\psi, \cdot \rangle U(g)\psi \equiv \int_G dg U(g) \left|\psi\right\rangle \left\langle\psi\right| U(g)^*. \tag{1.12}$$

Observe that $T_U$ is a positive linear operator:

$$\langle T_U \phi, \phi \rangle = \int \langle \phi, U(g)\psi \rangle \langle U(g)\psi, \phi \rangle dx = \int |\langle \phi, U(g)\psi \rangle|^2 dx \geq 0,$$

Hence, $T_U$ is self-adjoint, since it is positive [39]. Moreover, it belongs to $\mathscr{C}(U)$. Indeed, let $g' \in G$ and $\psi \in \mathscr{H}$. Then we have

$$\begin{aligned} U(g')T_U\phi &= \int_G dg \langle U(g)\psi, \phi \rangle U(g')U(g)\psi \\ &= \int_G dg \langle U(g')U(g)\psi, U(g')\phi \rangle U(g'g)\psi \\ &= \int_G \langle U(g)\psi, U(g')\phi \rangle U(g)\psi = T_U U(g')\phi, \end{aligned} \tag{1.13}$$

where we have used the unitarity of $U(g)$ and the left invariance of the Haar measure with the substitution $g'g \mapsto g$. Lastly, we notice that $T_U$ is a compact operator [18, 10].
As a consequence, if $\pi$ is an irreducible unitary representation, then $T_\pi = cId$, $c > 0$. Thus, we deduce the representation space $\mathcal{H}_\pi$ must be finite dimensional, because the identity operator is not a compact operator in infinite dimension. Therefore, we have proven that the following fact holds true:

**Theorem 1.2.11.** *If $G$ is a compact group, every irreducible unitary representation $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is finite dimensional.*

By the properties of the operator 1.12 and an application of Zorn's lemma as in the proof of 1.2.10, we also have that the following fundamental fact holds [18]:

**Theorem 1.2.12.** *If $G$ is a compact group, its unitary representations are fully reducible. Namely, if $U : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, we have that*

$$U = \bigoplus_{\alpha \in A} \pi_\alpha, \quad \mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{H}_{\pi_\alpha}, \tag{1.14}$$

*where $\pi_\alpha$ is an irreducible representation of $G$ on $\mathcal{H}_{\pi_\alpha}$.*

Such decomposition is far from being unique. Indeed, if we consider the trivial representation $\iota : G \ni g \mapsto \mathrm{Id} \in \mathcal{U}(\mathcal{H})$, $\dim \mathcal{H} > 1$, any orthonormal basis of $\mathcal{H}$ gives a decomposition of $\mathcal{H}$ into irreducible invariant subspaces. Anyway, there is a way to make the decomposition unique in the following sense. We will denote with irrsp$(U)$ the set of all the invariant subspaces $\mathscr{I}$ of $U$ such that $U|_{\mathscr{I}}$ are irreducible. Moreover, let

$$M_\pi \equiv M_{[\pi]}(U) := \mathrm{span}_{\mathbb{C}}\{\mathscr{I} \in \mathrm{irrsp}(U) \mid U|_{\mathscr{I}} \in [\pi]\} \tag{1.15}$$

be the set of complex linear combination of $\mathscr{I}$ such that the subrepresentation of $U$ is still equivalent to the irreducible representation $\pi$. Then, it can be shown that $M_\pi \perp M_{\tilde{\pi}}$ if $[\pi] \neq [\tilde{\pi}]$ and

$$\mathcal{H} = \bigoplus_{[\pi]\in\hat{G}} M_\pi, \;\; M_\pi = \bigoplus_{\alpha\in A} \mathcal{L}_\alpha, \tag{1.16}$$

where $\mathcal{L}_\alpha$ is an irreducible subspace of $M_\pi$ (the decomposition of $M_\pi$ is never unique, unless we have only one addend in the direct sum) [18]. With these notations, it is possible to prove that the cardinality of $A$ is the same for all the decompositions 1.16; moreover, it is also true that $\sharp(A) \equiv \mathrm{mult}(\pi, U) = \dim(\mathscr{C}(\pi, U))$ [18].

We now sketch the contents of Peter-Weyl's theorem, which concerns the link between the aforementioned decomposition and the left regular representation $\pi_L$. Let $U : G \to \mathcal{U}(\mathcal{H})$ a unitary representation. The maps

$$c^U_{\psi,\phi} : G \ni g \mapsto \langle U(g)\psi, \phi\rangle \in \mathbb{C}, \;\; \phi, \psi \in \mathcal{H} \tag{1.17}$$

are called *matrix elements or coefficient functions* of $U$. Then, the set

$$\mathscr{E}_U := \mathrm{span}(\{c^U_{\psi,\phi} \mid \phi, \psi \in \mathcal{H}\}) \tag{1.18}$$

is a subspace of $L^p(G)$ for each $p$ [18, 10] and depends only on the unitary equivalence class of $U$. Indeed,

$$U'(g) = TU(g)T^{-1} \implies \langle U(g)\psi, \phi\rangle = \langle U'(g)T\psi, T\phi\rangle.$$

The latter set is also an invariant subspace for the left (and right) regular representation [18]. The coefficients 1.17 will give us an orthonormal basis in $L^2(G)$. Indeed, the following orthogonality relations discovered by Schur hold true [18]

**Theorem 1.2.13.** *Let $\pi, \pi'$ two irreducible representations of a compact group $G$ and let us consider the associated subspaces $\mathscr{E}_\pi, \mathscr{E}_{\pi'}$ of $L^2(G)$. Then*

- *If* $[\pi] \neq [\pi']$*, then* $\mathscr{E}_\pi \perp \mathscr{E}_{\pi'}$*.*
- *If* $\{\psi_j\}$ *is an orthonormal basis for* $\mathcal{H}_\pi$*, then* $\{\sqrt{d_\pi}\pi_{i,j} \mid i,j = 1,\ldots,d_\pi\}$ *is an orthonormal basis for* $\mathscr{E}_\pi$*, where* $d_\pi = \dim(\mathcal{H}_\pi)$ *and* $\pi_{i,j} = \langle \pi(g)\psi_i, \psi_j\rangle$*.*

*Moreover, if* $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{H}_\pi$*, the following* orthogonality relations *hold:*

$$\int dg \langle \phi_1, \pi(g)\psi_1\rangle\langle \pi(g)\psi_2, \phi_2\rangle = \langle c^\pi_{\psi_1,\phi_1}, c^\pi_{\psi_2,\phi_2}\rangle_{L^2(G)} = d_\pi^{-1}\langle \phi_1, \phi_2\rangle\langle \psi_2, \psi_1\rangle. \tag{1.19}$$

In section 3.5 we will introduce square integrable representations, which satisfy some orthogonality relations similar to 1.19; these representations will play a fundamental role in the construction of quantum mechanics on phase space.

Let now $\psi \in \mathcal{H}$ be a nonzero normalized vector: $\|\psi\| = 1$. The isometry

$$\mathcal{W}^\pi_\psi : \mathcal{H} \ni \phi \mapsto d_\pi^{1/2} c^\pi_{\psi,\phi} \in L^2(G) \tag{1.20}$$

intertwines $\pi$ with the left regular representation of $G$ [18]. Indeed, let us consider the following equivalence:

$$c^\pi_{\psi,\pi_L(g)\phi} = \langle \pi(\cdot)\psi, \pi_L(g)\phi\rangle = \langle \pi_L(g^{-1})\pi(\cdot)\psi, \phi\rangle.$$

Then we have

$$c^\pi_{\psi,\pi(g)\phi} = \langle \pi(g^{-1}(\cdot))\psi, \phi\rangle = \pi_L(g) c^\pi_{\psi,\phi}.$$

We now observe the following facts [18]:

- $\operatorname{Ran}\mathcal{W}^\pi_\psi \leq \mathscr{E}_\pi$ is an invariant subspace of $\pi_L$. The restriction $\pi_L|_{\mathscr{E}_\pi}$ is unitarily equivalent to $\pi$.
- If $\psi_1, \psi_2 \in \mathcal{H}$ are such that $\psi_1 \perp \psi_2$, then $\operatorname{Ran}\mathcal{W}^\pi_{\psi_1} \perp \operatorname{Ran}\mathcal{W}^\pi_{\psi_2}$.

Therefore if we consider an orthonormal basis $\{\psi_j\}_{j=1}^{d_\pi}$ in $\mathcal{H}$, we have the following decomposition:

$$\mathscr{E}_\pi = \bigoplus_{j=1}^{d_\pi} \operatorname{Ran}(\mathcal{W}^\pi_{\psi_j}), \tag{1.21}$$

where $\dim \operatorname{Ran}(\mathcal{W}^\pi_{\psi_j}) = d_\pi \ \forall j$.
With these ingredients, we can enounce the Peter-Weyl's theorem [18]:

**Theorem 1.2.14** (Peter-Weyl)**.** *Let $G$ be a compact group and let $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ an irreducible representation. Let us consider an orthonormal basis $\{\psi_j\}_{j=1}^{d_\pi}$ in $\mathcal{H}_\pi$. Then we have*

$$L^2(G) = \bigoplus_{[\pi]\in\hat{G}} \mathscr{E}_\pi = \bigoplus_{[\pi]\in\hat{G}} \left( \bigoplus_{j=1}^{d_\pi} \mathrm{Ran}(\mathcal{W}^\pi_{\psi_j}) \right). \tag{1.22}$$

*Moreover,*

$$\{d_\pi^{1/2}\pi_{jk} \equiv c^\pi_{\psi^\pi_j,\psi^\pi_k} \mid j,k = 1,\dots,d_\pi,\ [\pi]\in\hat{G}\}, \tag{1.23}$$

*is an orthonormal basis in $L^2(G)$ and $Ran(\mathcal{W}^\pi_{\psi_j})$ is invariant under the action of the left regular representation. Lastly, defining the restriction $\pi^{(j)} := \pi|_{Ran(\mathcal{W}^\pi_{\psi_j})}$, we have the following definition for the left regular representation:*

$$\pi_L = \bigoplus_{[\pi]\in\hat{G}} \left( \bigoplus_{j=1}^{d_\pi} \pi^{(j)} \right), \tag{1.24}$$

*where $\pi^{(j)}$ is unitarily equivalent to $\pi$ for each $j$. Therefore, each $[\pi]$ occurs with multiplicity $d_\pi$ in the decomposition 1.24.*

**Example 1.2.15.** We want to sketch the representation theory of $SU(2)$, i.e. the group of unitary matrices whose determinant is 1. Recall that the most general element in $SU(2)$ is of the form

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a,b\in\mathbb{C},\ |a|^2+|b|^2 = 1. \tag{1.25}$$

Let $\mathcal{P}$ be the space of the complex polynomials in two variables $P(z,w) = \sum_{j,k} c_{jk} z^j w^k$ and let $\mathcal{P}_m$ be the space of homogeneous polynomials of degree m, i.e. the space of the polynomials which can be written as $P = \sum_{j=1}^m c_j z^j w^{m-j}$. If we look at $\mathcal{P}$ as a subset of $L^2(\sigma)$, where $\sigma$ is the surface measure of the unit sphere $S^3$ such that $\sigma(S^3) = 1$, we can endow $\mathcal{P}$ with the following inner product:

$$\langle P, Q\rangle = \int_{S^3} Q\bar{P} d\sigma, \tag{1.26}$$

with respect to $\mathcal{P}$ is not complete, but so it is each $\mathcal{P}_m$, because they are finite dimensional. Observe that a suitable choice of orthonormal basis of $\mathcal{P}_m$ is $\{A(m,j)z^j w^{m-j}\}$ [18], where $A$ is a normalization constant depending on $m$ and $j$. We observe that

$$U\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ -\bar{b}z+\bar{a}w \end{pmatrix},$$

where $(z, w) \in \mathbb{C}^2$ and $U$ is as in 1.25. Then we can define the representation $\pi$ of $SU(2)$ onto $\mathcal{P}$ as follows:

$$(\pi(U)P)(z, w) := P(U^{-1}(z, w)) = P(\bar{a}z - bw, \bar{b}z + aw). \tag{1.27}$$

$\mathcal{P}_m$ is invariant under $\pi$ by definition, hence we can consider the subrepresentation $\pi_m \equiv \pi|_{\mathcal{P}_m} : SU(2) \to \mathcal{P}_m$, which is unitary because we are considering a rotational invariant inner product. For each $m \geq 0$, $\pi_m$ is an irreducible representation and $\{\pi_m\}_m$ form a complete list of irreducible representations of $SU(2)$. Lastly, the matrix coefficients of these representations $\{\pi_m^{j,0} \mid 0 \leq j \leq m\}$ span $\mathcal{P}_m$ [18]. If we compute the matrix elements of $\pi_m$ with respect to the basis $\{A(m, j)z^j w^{m-j}\}$, we see that the functions

$$\pi_m^{j0}(a, b) = A(m, j)b^j a^{m-j}, \quad 0 \leq j \leq m \tag{1.28}$$

span $\mathcal{P}_m$ [18].

### 1.2.2 Direct integral of representations

We now briefly review how the decomposition of a generic unitary representation of a l.c.s.c. group into a direct integral of irreducible representations is performed. The construction requires several ingredients concerning measure theory which are beyond our aims, so we will sketch only the most basic facts.

Let $(A, \mathcal{M})$ a measurable space. The family of nonzero separable Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ will be called *field of Hilbert spaces* over $A$ and the map $f : A \to \prod_{\alpha \in A} \mathcal{H}_\alpha$ will be called *vector field* over $A$. Due to separability, we can consider the countable set $\{e_j(\alpha)\}_{j=1}^{\infty}$ in $\mathcal{H}$; the couple $(\{\mathcal{H}_\alpha\}, \{e_j(\alpha)\})$ will be called *measurable field of Hilbert spaces* if:

1. The maps $A \ni \alpha \mapsto \langle e_j(\alpha), e_k(\alpha)\rangle_\alpha \in \mathbb{C}$, where the subscript means the scalar product is in the $\alpha$'s Hilbert space, are measurable for all $j, k \in \mathbb{N}$.

2. $\text{span}\{e_j(\alpha)\}_{j=1}^{\infty}$ is dense in $\mathcal{H}_\alpha$ for each $\alpha$.

We will say the vector field $f$ on $A$ is *measurable* if, given a measurable field of Hilbert spaces $(\{\mathcal{H}_\alpha\}, \{e_j(\alpha)\})$ on $A$, the map $f \mapsto \langle f(\alpha), e_j(\alpha)\rangle_\alpha$ is a measurable function on $A$ for each $j$. We also notice that, if $f$ and $g$ are two measurable vector fields, then $\langle f(\alpha), g(\alpha)\rangle_\alpha$ is a measurable function [18]. This fact allows us to give the following

**Definition 1.2.16.** Suppose $(\{\mathcal{H}_\alpha\}, \{e_j(\alpha)\})$ is a measurable field of Hilbert spaces over $A$ and suppose $\mu$ is a measure on $A$. The *direct integral* of the $\mathcal{H}_\alpha$ with respect to the measure $\mu$, denoted as $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$, is the space of measurable vector fields $f$ on $A$ such that

$$\|f\|^2 = \int \|f(\alpha)\|_\alpha^2 d\mu(\alpha) < \infty, \tag{1.29}$$

where $\|\cdot\|_\alpha$ denotes the $\alpha$'s Hilbert space norm.

In particular, such space is complete under the following inner product [18], [19]:

$$\langle f, g\rangle := \int \langle f(\alpha), g(\alpha)\rangle_\alpha d\mu(\alpha), \tag{1.30}$$

hence it is an Hilbert space[2].

**Example 1.2.17.** Let $\mathcal{H}$ be a separable Hilbert spaces and let $\{e_j\}$ be an orthonormal basis in $\mathcal{H}$. The *constant field* is a measurable field of Hilbert spaces over $A$, where we have set $\mathcal{H}_\alpha = \mathcal{H}$ and $e_j(\alpha) = e_j$ $\forall \alpha \in A$. Hence $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$ is the space of measurable square integral functions from $A$ to $\mathcal{H}$ with respect to $\mu$, sometimes denoted by $L^2(A, \mu, \mathcal{H})$.

**Example 1.2.18.** Now we can see that the direct integral is a correct generalization of the direct sum of Hilbert spaces. Indeed, suppose $A$ is a discrete set so that its $\sigma$-algebra is simply the power set $\mathcal{P}(A)$ and let $\{\mathcal{H}_\alpha\}$ be an arbitrary field of Hilbert spaces over $A$. For each $\alpha \in A$, $\{e_j(\alpha)\}_{j=1}^{d(\alpha)}$ is an orthonormal basis for $\mathcal{H}_\alpha$. $\{\mathcal{H}_\alpha\}$ is a measurable field if we set $e_j(\alpha) = 0$ for each $j > d(\alpha)$. Now, because $A$ is discrete, we can choose $\mu$ as the counting measure on $A$, hence $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$ is nothing but $\bigoplus_{\alpha\in A} \mathcal{H}_\alpha$.

Let now $(\{\mathcal{H}_\alpha\}, \{e_j(\alpha)\})$ be a measurable field of Hilbert spaces over the parameter space $A$. The element $T : A \to \Pi_{\alpha\in A}\mathcal{B}(\mathcal{H}_\alpha)$ is called *field of operators* over $A$. We will say $T$ is *measurable* if the maps $\alpha \in A \mapsto T(\alpha)f(\alpha) \in \Pi_{\alpha\in A}\mathcal{H}_\alpha$ are measurable.
Let now $\mu$ be a measure on $A$ with respect to the measurable field of operators $T$ is such that $\|T\|_\infty < \infty$. Hence $\|T(\alpha)f(\alpha)\|_\alpha \leq \|T\|_\infty \|f(\alpha)\|_\alpha$, which means $T$ defines a bounded operator on $\int_A^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$, denoted by $\int^{\oplus} T(\alpha) d\mu(\alpha)$, and called the *direct integral of the field* $T$.

[2] This definition depends on the equivalence class of the measure $\mu$, indeed, if $\mu'$ is another measure on $A$ such that $\mu, \mu'$ are mutually absolutely continuous, the map $f \mapsto \sqrt{d\mu/d\mu'}$ is a unitary isomorphism between $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$ and $\int^{\oplus} \mathcal{H}_\alpha d\mu'(\alpha)$ [18].

**Definition 1.2.19.** Suppose $G$ is a l.c.s.c. group and let $\pi_\alpha$ be a unitary representation of $G$ on $\mathcal{H}_\alpha$, where $\alpha \in A$. Suppose the map $\alpha \mapsto \pi_\alpha(g)$ is a measurable field of operators for each $g \in G$. The set $\{\pi_\alpha\}_{\alpha \in A}$ is called *measurable field of representations* of $G$.

Since the $\pi_\alpha$ are unitary representations, we can form the direct integral

$$\pi(g) := \int^{\oplus} \pi_\alpha(g) d\mu(\alpha), \tag{1.31}$$

which is still a unitary representation of $G$ on $\int^{\oplus} \mathcal{H}_\alpha d\mu(\alpha)$ [18]. 1.31 is called the *direct integral* of the representations $\pi_\alpha$. Then, the following fact holds:

**Theorem 1.2.20.** *If $G$ is a l.c.s.c. Abelian group and if $U : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then $U$ is unitarily equivalent to a direct integral of irreducible unitary representations.*

When $G$ is non Abelian, we must fulfill stricter conditions. In particular, if $\pi$ is a unitary representation of a l.c.s.c. group $G$, we will say that $\pi$ is *primary* if $\mathscr{C}(\pi)$ coincides with scalar multiples of the identity (hence every irreducible representation is primary thanks to Schur's lemma). If every primary representation of $G$ is a direct sum of copies of some irreducible representation, then $G$ is a *type I* group. In this setting, it is possible to prove that, in order to have a unique direct integral decomposition (up to unitarily equivalence) of a unitary representation of a group $G$, $G$ shall be a type I group [18], otherwise terrible things happen. Indeed, in such a case, a primary representation $\sigma$ may be such that

$$\sigma \sim \int^{\oplus} \pi_\alpha d\mu(\alpha) \sim \int^{\oplus} \tilde{\pi}_\beta d\nu(\beta), \tag{1.32}$$

where all the $\pi_\alpha$'s and the $\tilde{\pi}_\beta$'s are irreducible and no $\pi_\alpha$ is equivalent to any $\tilde{\pi}_\beta$ [18]. Thus, the direct integral decomposition would not be unique anymore.

## 1.3 Induced representations

In this section we show a way to build up unitary representations of a locally compact (second countable) group starting from unitary representations of its closed subgroups.

Let $G$ be a locally compact group, $H$ a closed subgroup and $\sigma : H \to \mathcal{U}(\mathcal{H}_\sigma)$ a unitary representation of $H$. We denote with $q$ the canonical quotient map of $G$ on $G/H$. We will suppose $G/H$ admits a Radon measure $\mu$, which is invariant under the following left-action of $G$ on $G/H$ [18, 28]:

$$g[\tilde{g}H] := (g\tilde{g})H, \quad g \in G, \ \tilde{g}H \in G/H. \tag{1.33}$$

The inducing construction starts considering the following space of function:

$$\mathcal{F}_0 := \{f \in C(G, \mathcal{H}_\sigma) \mid q(\operatorname{supp} f) \text{ compact}, \ f(g\xi) = \sigma(\xi^{-1})f(g), \quad g \in G, \ \xi \in H\}, \tag{1.34}$$

where $C(G, \mathcal{H}_\sigma)$ denotes the space of continuous functions from $G$ to $\mathcal{H}_\sigma$. Observe that all the functions of the form

$$f_\alpha(g) := \int_H \sigma(h)\alpha(gh)dh, \tag{1.35}$$

where $\alpha : G \to \mathcal{H}_\sigma$ is continuous with compact support, are in $\mathcal{F}_0$. Indeed, because $q(\operatorname{supp} f_\alpha) \subset q(\operatorname{supp} \alpha)$, we have $q(\operatorname{supp} f_\alpha)$ is compact. Moreover, by definition,

$$f_\alpha(g\xi) = \int_H \sigma(h)\alpha(g\xi h)dh = \int_H \sigma(\xi^{-1}h)\alpha(gh)dh = \sigma(\xi^{-1})f_\alpha(g),$$

where we have used the left-invariance of $dh$ and the substitution $h \mapsto \xi^{-1}h$, thus $f \in \mathcal{F}_0$. We can also prove the converse, but we have to recall the following fact: if $f \in C_c(G)$, there exists a unique function $\hat{f} \in C_c(G/H)$ such that $\hat{f}(gH) = \int_H f(gh)dh \ \forall g \in G$ [28].
Therefore, if $\phi \in C_c(G/H)$, there exists $f \in C_c(G)$ such that $\hat{f} = \phi$ [28] (the correspondence is still true if we replace $C_c(G/H)$ with $C_c^+(G/H)$ and $C_c(G)$ with $C_c^+(G)$, where $C_c^+(G)$ denotes the set of strictly positive function with compact support [18]).
Now let us suppose $f \in \mathcal{F}_0$. Then there exists $\phi \in C_c(G)$ such that $\int_H \phi(gh)dh = 1$ for $g \in \operatorname{supp} f$. If we set $\alpha = \phi f$, we have

$$f_\alpha(g) = \int_H \phi(gh)\sigma(h)f(gh)dh = \int_H \phi(gh)f(g)dh = f(g),$$

thus $f = f_\alpha$. We also remark these functions are left uniformly continuous [28].

Now we can naturally consider an action of $G$ on $\mathcal{F}_0$ by left translation $f(h) \mapsto L_g f := f(g^{-1}h)$. Indeed, $L_h f \in \mathcal{F}_0$, because

$$L_h f(g\xi) = f(h^{-1}g\xi) = \sigma(\xi^{-1})f(h^{-1}g) = \sigma(\xi^{-1})L_h f(g).$$

Moreover, recall these maps are surjections, hence we have to find an inner product with respect to such translations are isometries in order to obtain unitary operators. If $f, g \in \mathcal{F}_0$, we observe that $\langle f(g), h(g)\rangle_\sigma$, where $\sigma$ denotes the inner product in $\mathcal{H}_\sigma$, depends only on the coset $q(g)$ of $g$ since $\sigma$ is unitary, thus it defines a function in $C_c(G/H)$. Recalling that if $\mu$ is a left Haar measure then $\int f d\mu > 0 \ \forall f \in C_c^+(G)$ [18], we can define the following inner product in $\mathcal{F}_0$:

$$(f, g) := \int_{G/H} \langle f(g), h(g)\rangle_\sigma d\mu(gH), \tag{1.36}$$

which is preserved under left translation thanks to left invariance of the Haar measure [28]. Finally, denoting with $\mathcal{F}$ the Hilbert space completion of $\mathcal{F}_0$, the translation operators $L_g$ extend to unitary operators on $\mathcal{F}$. We also note that $g \mapsto L_g f$ is continuous from $G$ to $\mathcal{F}$ $\forall f \in \mathcal{F}_0$. Moreover, the maps $L_g$ are strongly continuous on $\mathcal{F}$ [28]. Consequently, we have a unitary representation on $G$, called the *representation induced by* $\sigma$ which will be denoted by $\mathrm{ind}_H^G(\sigma)$.

### 1.3.1 The Mackey machine

We now give a brief introduction to the *Mackey machine* [45], a technique that is very useful to analyze the irreducible representations of a semi-direct product group [45, 18]. Because most of the proofs require the notion of systems of imprimitivity, which are beyond our interest, we only expose the main results of the theory without demonstrations (we refer to [18, 28, 45] for them).

Let $G$ be a l.c.s.c. group and suppose $N \neq \{e\}$ is a closed Abelian normal subgroup of $G$. We can consider the action of $G$ on $N$ by conjugation $n \mapsto gng^{-1}$, which induces an action of $G$ on the dual group $\hat{N}$ as follows:

$$\langle n, g\nu\rangle := \langle g^{-1}ng, \nu\rangle, \quad g \in G, \nu \in \hat{N}, n \in N. \tag{1.37}$$

where $\langle n, \nu\rangle \equiv \nu(n)$. Let $G_\nu := \{g \in G \mid g\nu = \nu\}$ be the stabilizer of $\nu$ for each $\nu \in \hat{N}$ and let $\mathcal{O}_\nu = \{g\nu \mid g \in G\}$ the orbit of $\nu$ for each $\nu \in \hat{N}$. Nevertheless, the action of $G$ on $\hat{N}$ is never transitive and the structure of the set of all the orbits could be very hard to analyze, since, for example, we have that $\mathcal{O}_e = \{e\}$ [18]. Thus we shall introduce a less strict condition for such action.

**Definition 1.3.1.** We will say the action of $G$ on $\hat{N}$ is *regular* if:

- There exists a countable family of $G$-invariant Borel set $\{E_j\}$ in $\hat{N}$ such that each orbit in $\hat{N}$ is the intersection of all the $E_j$'s that contain it.
- $\forall \nu \in \hat{N}$ the map $G/G_\nu \ni gG_\nu \mapsto g\nu \in \mathcal{O}_\nu$ is a homeomorphism.

We remark that, for second countable groups, these two conditions are both implied by requiring that there exists a Borel set in $\hat{N}$ which intersects each orbit in exactly one point [18].
With these notations, if $\pi : G \to \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation, there exists $\nu \in \hat{N}$ and an irreducible representation $\sigma$ of $G_\nu$, with $\sigma(n) = \langle n, \nu \rangle \mathrm{Id}\ \forall n \in N$, such that $\pi$ is unitarily equivalent to $\mathrm{ind}_{G_\nu}^G(\sigma)$ [18, 28, 45]. Observe that the choice of $\nu$ in the orbit $\mathcal{O}_\nu$ is arbitrary, although $\mathcal{O}_\nu$ is uniquely determined. Indeed, if $\nu' \in \mathcal{O}_\nu$, namely $\nu' = g\nu$, we have that $G_\nu \cong G_{\nu'}$ since $G_{\nu'} = gG_\nu g^{-1}$ [18]. Moreover, if $\sigma, \sigma'$ are representations respectively of $G_\nu$ and $G_{\nu'}$, where $\sigma'(h) = \sigma(g^{-1}hg)$, then there exists a bijection between these representations and the induced representations $\mathrm{ind}_{G_\nu}^G(\sigma), \mathrm{ind}_{G_{\nu'}}^G(\sigma')$ are unitarily equivalent [18].
The converse is also true. Namely, if $\nu \in \hat{N}$ and $\sigma$ is such that $\sigma(n) = \langle n, \nu \rangle \mathrm{Id},\ \forall n \in N$, then the representation $\pi = \mathrm{ind}_{G_\nu}^G(\sigma)$ is irreducible. Lastly, if $\sigma'$ is another such representation of $G_\nu$ such that $\mathrm{ind}_{G_\nu}^G(\sigma) = \mathrm{ind}_{G_\nu}^G(\sigma')$, then $\sigma$ and $\sigma'$ are unitarily equivalent [18, 28, 45].

We have to remark that these results are not satisfactory when the representations of $G_\nu$ are not easy to analyze. In such cases, however, we can restrict the analysis to $G/N$. Indeed, if $\nu \in \hat{N}$, we can extend $\nu$ to a representation of $G_\nu$, which will be denoted by $\tilde{\nu} : G_\nu \to \mathbb{T}$, such that $\tilde{\nu}|_N = \nu$. If $\rho$ is an irreducible representation of $G_\nu/N$, then $\sigma : G_\nu \to \mathcal{U}(\mathcal{H}_\rho)$ with $\sigma(y) = \tilde{\nu}(y)\rho(yN)$ and $\sigma(n) = \langle n, \nu \rangle \mathrm{Id}$ for $n \in N$, is an irreducible representation, because $\rho$ is irreducible [18].
Observe that every representation of $G_\nu$ is of this kind. Indeed, if $\sigma$ is an irreducible representation of $G_\nu$ with $\sigma(n) = \langle n, \nu \rangle \mathrm{Id},\ n \in N$, we have that the representation $\sigma'$, which is such that $\sigma'(y) = \tilde{\nu}(y)^{-1}\sigma(y)$, is irreducible and it is trivial on $N$ by construction, thus $\sigma'(y) = \rho(yN)$ and $\sigma(y) = \tilde{\nu}(y)\rho(yN)$.

This phenomenon occurs every time we analyze semi-direct product groups. Let $G = N \ltimes H$ where $N$ is a closed normal subgroup and $H$ is a closed subgroup of $G$ and suppose $G$ acts regularly on $\hat{N}$. Recall that the composition law in $G$ is given by the conjugation, i.e.

$$(n_1h_1)(n_2h_2) := \left(n_1(h_1n_2h_1^{-1})\right)(h_1h_2),$$

where $n_1, n_2 \in N,\ h_1, h_2 \in H$. Now suppose $N$ is an Abelian subgroup and let us call the irreducible representations of $N$ (which are one dimensional due to Schur's lemma) *characters.* For each $\nu \in \hat{N}$ we define the little group $H_\nu := G_\nu \cap H$. Hence we have $H_\nu \cong G_\nu/N$, because $G_\nu \supset N$ and $G_\nu = N \ltimes H_\nu$ [18]. Moreover, by definition of left action of $G$ on $\hat{N}$, every character $\nu$ can be extended to a homomorphism $\tilde{\nu} : G_\nu \to \mathbb{T}$ with $\tilde{\nu}(nh) = \nu(n) \equiv \langle n, \nu\rangle$. Therefore, if $\nu \in \hat{N}$ and $\rho : H_\nu \to \mathcal{U}(\mathcal{H}_\rho)$ is a unitary irreducible representation of $H_\nu$, then

$$(\nu\rho)(nh) = \langle n, \nu\rangle \rho(h), \tag{1.38}$$

is an irreducible representation of $G_\nu$ and every irreducible representation of $G_\nu$ is of this form [18] . Moreover, observe that $\nu\rho$ is equivalent to $\nu\rho'$, where $\rho'$ is an irreducible representation of $H_{\nu'}$, iff $\rho$ is equivalent to $\rho'$ [18]. Hence we can sum up our results in a complete classification of the irreducible representations of $G = N \ltimes H$, where $N$ is Abelian, in terms of the characters $\nu$ of $N$ and the irreducible representations of their little groups $H_\nu$.

Lastly, observe that, since $N$ acts trivially on $\hat{N}$, the $G$-orbit of the character $\nu \in \hat{N}$ is the same as its $H$-orbit. Moreover, if $\nu' = g\nu$, where $g \in H$, then the little groups of $\nu$ and $\nu'$ are such that $H_{\nu'} = gH_\nu g^{-1}$, then we have $H_{\nu'} \cong H_\nu$ [18].

**Example 1.3.2.** We sketch the classification of the irreducible representations of the Poincaré group, which was firstly done by Wigner in his remarkable paper [49]; more details can also be found in [10].

Recall that the Lorentz group $SO(3,1)$ is the group of matrices in $\Lambda \in M_4(\mathbb{R})$ such that

$$\Lambda\eta\Lambda^T = \eta, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

or, equivalently, the group of transformations that leave the Lorentz pseudo-inner product invariant. $SO(3,1)$ admits two non trivial homomorphism in $\{\pm 1\}$, the determinant $\Lambda \mapsto \det\Lambda$ and the sign map $\Lambda \mapsto \operatorname{sgn}\Lambda$. We usually prefer to use the *proper orthochronus Lorentz subgroup*, denoted by $SO_0^\uparrow(3,1)$, whose matrices are such that $\det\Lambda = 1,\ \operatorname{sgn}\Lambda = 1,\ \Lambda \in SO_0^\uparrow(3,1)$. In literature we often see the $SL(2,\mathbb{C})$ group instead of $SO_0^\uparrow(3,1)$, because it is its universal covering group [10]. The Poincaré group is therefore given by the semi-direct product between the proper orthochronus Lorentz group and the group of translations $\mathcal{T}(4)$ in $\mathbb{R}^4$ which is

Abelian and therefore is normal; hence, we will write $P = \mathcal{T}(4) \ltimes SO(3,1)$. The group composition law is the natural composition given as follows:

$$(a, \Lambda)(a', \Lambda') := (a + \Lambda a', \Lambda\Lambda'). \tag{1.39}$$

As in the case of the Lorentz group, we can also consider the universal covering of the Poincarè group $\Pi := \mathcal{T}(4) \ltimes SL(2, \mathbb{C})$. Observe that this group is simply connected, because $\mathcal{T}(4)$ and $SL(2,\mathbb{C})$ are, therefore $\Pi$ is properly the universal covering group of $P$ [10].
We can now sketch how representation theory works for $\Pi$. Observe that we can identify $\hat{\mathcal{T}}(4)$ with $\mathcal{T}(4)$. The characters are given by the pairing $\langle n, \hat{n}\rangle = e^{2\pi i n^{\mu}\hat{n}_{\mu}}$, $\mu = 0, 1, 2, 3$, where $n \in \mathcal{T}(4)$, $\hat{n} \in \hat{\mathcal{T}}(4)$. We can then write the action of $\Pi$ on $\hat{\mathcal{T}}(4)$:

$$\langle \Lambda n, \hat{n}\rangle = e^{2\pi i \Lambda n^{\mu}\hat{n}_{\mu}} = e^{2\pi i n^{\mu}\Lambda^{T}\hat{n}_{\mu}} = \langle n, \Lambda^{-1}\hat{n}\rangle, \tag{1.40}$$

therefore the action of $\Lambda$ is such that $\hat{n} \mapsto \Lambda^{-1}\hat{n}$. Hence we have that every orbit is contained in the following hyperboloids:

$$\hat{n}_0^2 - \hat{n}_1^2 - \hat{n}_2^2 - \hat{n}_3^2 = m^2, \quad m^2 \in \mathbb{R}. \tag{1.41}$$

We can now distinguish three cases, which will require to analyze the irreducible representations of different stabilizer groups:

- Case $m^2 > 0$. In this case 1.41 describes a two-sheet hyperboloid. The stabilizer group is $SU(2)$ ($SO(3)$ for $P$) and the irreducible representations of $\Pi$ are labelled by a integer or semi-integer numbers.

- Case $m^2 < 0$. In this case 1.41 describes a one-sheet hyperboloid and the stabilizer group is $SL(2, \mathbb{R})$.

- Case $m^2 = 0$. In this case 1.41 describes a cone which will consist of three possible orbits: the origin, the upper cone and the lower cone. These orbits are stabilized by the Euclidean group $E(2)$, whose representations can be studied again with the theory of induced representation [10].

## 1.4 Projective representations

We now introduce the theory of projective (unitary) representations, which are fundamental for the phase space description of quantum mechanics. Roughly speaking, projective representations arise in physics because unitary representations are not always the most suitable ones to describe physical simmetries. Indeed, due to Wigner's theorem, we know that simmetries in quantum mechanics are represented by unitary (or anti-unitary) transformations, which are uniquely determined up to phase factors[3] [35]. We will analyze such representations using a standard procedure: starting with a group, we will "lift" such group via the *central extension* and we study the irreducible unitary representations of the latter. Then, we restrict such representations, obtaining the projective ones.

Firstly we need to define the projective group. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{U}(\mathcal{H})$ be the group of unitary operators on $\mathcal{H}$ and let $\mathcal{Z}(\mathcal{U}(\mathcal{H})) := \{z\mathrm{Id} \mid z \in \mathbb{T}\} \equiv \mathcal{T}(\mathcal{H})$ be its center.

**Definition 1.4.1.** The quotient group $\mathscr{P}(\mathcal{H}) := \mathcal{U}(\mathcal{H})/\mathcal{T}(\mathcal{H})$ is called the *(unitary) projective group.*

It can be shown that such group is a polish second countable group [45]. In the following, $\mathbf{p} : \mathcal{U}(\mathcal{H}) \to \mathscr{P}(\mathcal{H})$ will denote the canonical projection epimorphism. Now let $\psi \in \mathcal{H}$ be a normalized vector, then set $\hat{\psi} \equiv |\psi\rangle\langle\psi|$ and define the following maps:

$$\tau_{\hat{\phi},\hat{\psi}}(\mathbf{p}(V)) := \mathrm{Tr}\Big(\hat{\phi} V \hat{\psi} V^*\Big) = |\langle \phi, V\psi\rangle|^2 \quad \forall \hat{\phi}, \hat{\psi} \in \mathcal{P}_1(\mathcal{H}),\ \forall \mathbf{p}(V) \in \mathscr{P}(\mathcal{H}) \tag{1.42}$$

(here we denote with $\mathcal{P}_1(\mathcal{H})$ the set of rank 1 projectors). The quotient topology is equivalent to the initial topology given by the maps 1.42 [45]

$$\{\tau_{\hat{\phi},\hat{\psi}} : \mathscr{P}(\mathcal{H}) \to \mathbb{R}\}_{\hat{\phi},\hat{\psi} \in \mathcal{P}_1(\mathcal{H})}. \tag{1.43}$$

We can now introduce the notion of projective representation.

**Definition 1.4.2.** Let $G$ be a group and $K$ be an Abelian group. The map $\mu : G \times G \to K$ such that $\mu$ is Borel and

$$\begin{aligned} \mu(g_1, g_2 g_3)\mu(g_2, g_3) &= \mu(g_1 g_2, g_3)\mu(g_1, g_2), \\ \mu(g, e) &= \mu(e, g) = 1 \quad \forall g \in G. \end{aligned} \tag{1.44}$$

is called *K-multiplier*. If $K \equiv \mathbb{T}$, we just say $\mu$ is a multiplier.

We denote with $\tilde{M}_K(G)$ the group of the $K$-multipliers, where the composition law is given by the pointwise product. If $\mu_1, \mu_2$ are two $K$-multipliers and there exists a Borel map $B : G \to K$ such that

$$\mu_2(g, \tilde{g}) = \beta(g\tilde{g})\beta(g)^{-1}\beta(\tilde{g})^{-1}\mu_1(g, \tilde{g}) \quad \forall g, \tilde{g} \in G, \tag{1.45}$$

[3] To be fair, it is possible to prove that we can also consider unitary transformations and transpositions instead of unitary and anti-unitary transformations [35].

then $\mu_1, \mu_2$ are said to be *equivalent* and we will write $\mu_1 \sim \mu_2$; if $\mu \sim 1$, then $\mu$ is *exact*. Exact $K$-multipliers form an Abelian group with the pointwise product, denoted by $E_K(G)$, which is a normal subgroup of $\tilde{M}_K(G)$. Therefore we can consider the group $M_K(G) := \tilde{M}_K(G)/E_K(G)$, called the *$K$-multipliers group.*

**Definition 1.4.3.** Let $G$ be a l.c.s.c. group and let $\mu : G \times G \to \mathbb{T}$ a multiplier. The map $U : G \to \mathcal{U}(\mathcal{H})$ is a *$\mu$-representation* if:

1. $g \mapsto U(g)$ is weakly Borel.
2. $U(e) = \mathrm{Id}$.
3. $U(gh) = \mu(g,h)U(g)U(h), \ \forall g, h \in G$.

We will say that $U$ is a *projective representation* if there exists a $\mu$-multiplier such that $U$ is a $\mu$-representation.

Observe that if $\mathbf{p}$ is the projection epimorphism, then $\breve{U} \equiv \mathbf{p} \circ U : G \to \mathscr{P}(\mathcal{H})$ is a homomorphism, because

$$\breve{U}(gh) = \mathbf{p}(\mu(g,h)U(g)U(h)) = \mathbf{p}(U(g))\mathbf{p}(U(h)) = \breve{U}(g)\breve{U}(h).$$

Moreover, $\breve{U}$ is continuous [45]. Viceversa, if $\breve{U}$ is a Borel homomorphism, $\breve{U}$ is continuous and there exists a projective representation $U$ such that $\breve{U} = \mathbf{p}(U)$. The following fact also holds [45]:

**Proposition 1.4.4.** *If $U, V$ are projective representations with multipliers $\mu, \nu$ respectively, such that $\breve{V}(g) = \breve{U}(g)\ \forall g \in G$, then $\mu \sim \nu$.*
*Conversely, if $\nu \sim \mu$, there exists a $\nu$-representation $V$ such that $\breve{U}(g) = \breve{V}(g)\ \forall g \in G$.*

Another important fact is the existence of a $\mu$-representation:

**Proposition 1.4.5.** *If $\mu : G \times G \to \mathbb{T}$ is a multiplier, then there exists a $\mu$-representation $U$.*

*Proof.* Let us consider $\mathcal{H} = L^2(G, \lambda, \mathbb{C})$, where $\lambda$ is the left Haar measure, and consider $U : G \to \mathcal{U}(\mathcal{H})$ defined as follows:

$$(U(g)f)(h) := \mu(h^{-1}, g)^{-1} f(g^{-1}h), \quad f \in L^2(G). \tag{1.46}$$

Hence we have

$$(U(g\tilde{g})f)(h) = \mu(h^{-1}, g\tilde{g})^{-1} f(\tilde{g}^{-1}g^{-1}h). \tag{1.47}$$

and

$$U(g)U(\tilde{g})f(h) = \mu(h^{-1}, g)^{-1}\mu(h^{-1}g, \tilde{g})^{-1}f(\tilde{g}^{-1}g^{-1}h). \qquad (1.48)$$

By 1.47 and 1.48 we have

$$\begin{aligned}
U(g\tilde{g}) &= \mu(h^{-1}, g\tilde{g})^{-1}\mu(h^{-1}, g)\mu(h^{-1}g, \tilde{g})U(g)U(\tilde{g}) \\
&= \mu(h^{-1}, g\tilde{g})^{-1}\mu(g, \tilde{g})\mu(h^{-1}, g\tilde{g})U(g)U(\tilde{g}) \\
&= \mu(g, \tilde{g})U(g)U(\tilde{g}).
\end{aligned}$$

□

The previous results lead us to the following important corollary [45]:

**Corollary 1.4.6.** *If $U$ is a $\mu$-representation, $\mu$ is an exact multiplier iff there exists a unitary representation $V : G \to \mathcal{U}(\mathcal{H})$ such that the projections $\mathbf{p}\circ V$, $\mathbf{p}\circ U$ coincide. Moreover, if $V, \tilde{V}$ are two unitary representations of $G$ on $\mathcal{H}$ such that $\mathbf{p} \circ V = \mathbf{p} \circ \tilde{V}$, then there exists a continuous homomorphism $\chi : G \to \mathbb{T}$ (which we will call a character in the future) such that $\tilde{V}(g) = \chi(g)V(g)$ and viceversa.*

Therefore, if we have an exact multiplier, we can always find a unitary representation projectively equivalent to the projective one.

We now introduce the notion of group extension. Let $G, K$ be two l.c.s.c. groups, $K$ Abelian, let $M_K(G)$ be the group of $K$-multipliers and let us consider a topological group $H$.

**Definition 1.4.7.** If $i : K \to H$ is a monomorphism which restrict to an isomorphism from $K$ to $\tilde{K} \equiv i(K) \lhd H$ and if $j : H \to G$ is a continuous epimorphism such that $\tilde{K} = \ker(j)$, the triad $(H, i, j)$ is called the *group extension* of $G$ via $K$.
If $im(K) = \tilde{K} \subseteq \mathcal{Z}(H)$, the extension is *central.*

Notice that, in general, we may consider equivalence classes of group extension defined as follows. If $(H, i, j)$, $(H', i', j')$ are group extensions of $G$ via $K$, they are *equivalent* if there exists an isomorphism $\phi : H \to H'$ such that $\phi(i(k)) = i'(k)\ \forall k \in K$ and $j'(\phi(h)) = j(h)\ \forall h \in H$.
A fundamental result due to Mackey plays a central role in the analysis of projective representations; roughly speaking, its statement gives the standard form of the central extension of a group [45]:

**Theorem 1.4.8** (Mackey)**.** *Let $\mu : G \times G \to K$ be a $K$-multiplier and let us consider the group $K \times_\mu G$ of pairs in $K \times G$ such that*

$$(k, g)(\tilde{k}, \tilde{g}) = (k\tilde{k}\mu(g, \tilde{g}), g\tilde{g}), \tag{1.49}$$

$$(k, g)^{-1} = (k^{-1}\mu(g, g^{-1}), g^{-1}), \tag{1.50}$$

*where $(1, e)$ is the identity element. Then there exists a unique topology, called the* Weil topology, *with respect to $K \times_\mu G$ is a l.c.s.c. group and the Borel structure induced by the Weil topology coincides with the Borel structure of the product topology.*
*Furthermore, if $i_0 : K \to K \times_\mu G$, $j_0 : K \times_\mu G \to G$ are such that $i_0(k) := (k, e)$ and $j_0(k, g) := g$, then $(K \times_\mu G, i_0, j_0)$ is a central extension of $G$. Each central extension of $G$ is of this kind and the central extensions are equivalent iff the respective $K$-multipliers are.*

It is easy to show that the Haar measure on $K \times_\mu G$ is given by the product $\kappa \otimes \rho$, where $\kappa$ is the Haar measure on $H$ and $\rho$ is the left Haar measure on $G$. Indeed, for each $f \in L^1(K \times_\mu G, \kappa \otimes \lambda)$ we have

$$\int_{K\times G} d\kappa \otimes \lambda(k, g) f((\tilde{k}, \tilde{g})(k, g)) = \int_G d\lambda(g) \int_K d\kappa(k) f(\tilde{k}k\mu(\tilde{g}, g), \tilde{g}g) =$$
$$= \int_K d\kappa(k) \int_G d\lambda(g) f(k, g) = \int_{K\times G} d\kappa \otimes \lambda(k, g) f(k, g),$$

where we have used Fubini-Tonelli's theorem and the left invariance of the Haar measures $\kappa, \rho$. Moreover, it is still a Radon measure because the product of two Radon measures is still Radon.

Finally, let us briefly discuss the link between projective representations of the l.c.s.c. group $G$ and unitary representations of its central extensions via the Abelian group $K$.

**Proposition 1.4.9.** *Let $\nu : G \times G \to K$ be a $K$-multiplier and let $\mathbf{h} : K \to \mathbb{T}$ be a group homomorphism. Then,*

$$\mu \equiv \nu_{\mathbf{h}} := \mathbf{h} \circ \nu : G \times G \to \mathbb{T} \tag{1.51}$$

*is a multiplier.*

*Proof.* Observe that $\mathbf{h}$ is a Borel map, because it is a homomorphism between polish spaces [45]. Then, since $\nu$ is a multiplier, 1.44 holds, hence

$$\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mathbf{h}(\nu(g_1, g_2g_3)\nu(g_2, g_3)) =$$
$$= \mathbf{h}(\nu(g_1g_2, g_3)\nu(g_1, g_2)) = \mu(g_1g_2, g_3)\mu(g_1, g_2).$$

Similarly,

$$\mu(g,e) = \mathbf{h}(\nu(g,e)) = \mathbf{h}(e) = 1 = \mathbf{h}(\nu(e,g)) = \mu(e,g).$$

Moreover, $\mu$ is a Borel map, because it is the composition of the Borel maps $\nu$ and $\mathbf{h}$ [45]. □

Similarly, we also have the following fact:

**Proposition 1.4.10.** *If $U : G \to \mathcal{U}(\mathcal{H})$ is a $\mu$-representation, where $\mu \equiv \nu_{\mathbf{h}}$, the map*

$$V : K \times_\nu G \ni (k,g) \mapsto \mathbf{h}(k)^{-1} U(g) \in \mathcal{U}(\mathcal{H}) \tag{1.52}$$

*is a unitary representation. Conversely, if $U : G \to \mathcal{U}(\mathcal{H})$ is such that $U(g) := V(1,g)$ and if $V(k,e) = \mathbf{h}(k)^{-1}\mathrm{Id}$, then $U$ is a $\mu$-representation.*

*Proof.* By a direct computation we have:

$$\begin{aligned} V((k,g)(\tilde{k},\tilde{g})) &= \mathbf{h}(k\tilde{k}\nu(g,\tilde{g}))^{-1}U(g\tilde{g}) \\ &= \mathbf{h}(k)^{-1}\mathbf{h}(\tilde{k})^{-1}\mathbf{h}(\nu(g,\tilde{g}))^{-1}\mu(g,\tilde{g})U(g)U(\tilde{g}) \\ &= \mathbf{h}(k)^{-1}\mathbf{h}(\tilde{k})^{-1}U(g)U(\tilde{g}) = V(k,g)V(\tilde{k},\tilde{g}) \end{aligned}$$

Moreover, $V$ is a weakly Borel homomorphism from $K \times_\mu G$ to $\mathcal{U}(\mathcal{H})$, hence it is a strongly continuous map.
Suppose now $U : G \to \mathcal{U}(\mathcal{H})$ is such that $U(g) := V(1,g)$ and suppose $V(k,e) = \mathbf{h}(k)^{-1}\mathrm{Id}$. Then $U$ is a $\mu$-representation, because

$$\begin{aligned} U(g\tilde{g}) &= V(1,g\tilde{g}) = V((\nu(g,\tilde{g})^{-1},g)(1,\tilde{g})) = V((\nu(g,\tilde{g})^{-1},e)(1,g)(1,\tilde{g})) \\ &= \mathbf{h}(\nu(g,\tilde{g})^{-1})^{-1}V(1,g)V(1,\tilde{g}) = \mu(g,\tilde{g})U(g)U(\tilde{g}). \end{aligned}$$

Such a map is continuous iff the multiplier $\mu$ is continuous [45]. □

# Chapter 2

# The Heisenberg-Weyl group and Weyl systems

In this chapter we will study the Heisenberg-Weyl group, a fundamental tool in the description of quantum mechanics on phase space, which helps us in the classification of the irreducible projective representations of the phase space translations group. In particular, in order to study the discrete Heisenberg-Weyl group, it is convenient to consider the standard one at first, because the classification of their irreducible unitary representations can be done in a similar way thanks to the Mackey machine.

The chapter is structured as follows. In the first part we introduce the standard Heisenberg-Weyl group, both in its unpolarized and polarized form, the last one which is useful for generalizations. We will also review the latter from a symplectic point of view.

Then, we will study the projective representations of $\mathbb{R}^{2n}$, which, by central extension with $\mathbb{R}$, will lead to the irreducible unitary representations of the Heisenberg-Weyl group. Next we will deal with Weyl systems and the canonical commutation relations and we will discuss the celebrated Stone-von Neumann's theorem.

In the second part we will consider the discrete version of the Heisenberg-Weyl groups. In particular, by mean of an interesting example, we will see that it is convenient to consider a general phase space in the form $G \times \hat{G}$, where $G$ is a l.c.s.c. Abelian group (these phase spaces are very interesting for many physical application, for example in many body systems and in quantum information [1]).

Finally, we will concentrate our attention on the discrete case $\mathbb{Z}_N \times \mathbb{Z}_N$, studying the irreducible unitary representations of the discrete Heisenberg-Weyl group. In this context we will also point out an interesting difference with respect the standard Heisenberg-Weyl group, which is due to the finiteness of the group considered. As a last topic, we will discuss of the discrete analogue of Weyl systems and Stone-von Neuamnn's theorem.

## 2.1 Some fundamentals on the Heisenberg-Weyl group

In this section we define the Heisenberg-Weyl group, which play a crucial role in the classification of the projective representations of the phase space, since it is its central extension (see section 2.2.1). We will illustrate different *realizations* of such a group, which are equivalent to each other.

**Definition 2.1.1.** The set of upper triangular matrices

$$M(\tau, q, p) = \begin{pmatrix} 1 & p & \tau \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_{(n+2)}(\mathbb{R}), \tag{2.1}$$

where $q, p \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$, is a group with respect to the standard matrix product, called the *polarized Heisenberg-Weyl group* and it is usually denoted as $\mathbb{H}_n^{POL}(\mathbb{R})$.

We will also write the composition law in the form

$$M(\tau, q, p)M(\tau', q', p') = M(\tau + \tau' + pq', q + q', p + p'), \tag{2.2}$$

hence the inverse element of $(\tau, q, p)$ is given by

$$M(qp - \tau, -q, -p). \tag{2.3}$$

Observe that $\mathbb{H}_n^{POL}(\mathbb{R})$ is decomposed as $N^{POL} \times'_\alpha H^{POL}$, where $\alpha_p(\tau, q) = (\tau + qp, q)$ is the semi-direct product action and $N^{POL} = \{(\tau, q, 0) \mid \tau, \in \mathbb{R}, q \in \mathbb{R}^n\}$ and $H^{POL} = \{(0, 0, p) \mid p \in \mathbb{R}^n\}$.
The polarized Heisenberg-Weyl group admits another equivalent realization:

**Definition 2.1.2.** The *(unpolarized) Heisenberg-Weyl group* $\mathbb{H}_n(\mathbb{R})$ is the group of triples $(\tau, q, p)$, $\tau \in \mathbb{R}$, $q, p \in \mathbb{R}^n$ with respect to the composition law

$$(\tau, q, p)(\tau', q', p') := \left(\tau + \tau' + \frac{1}{2}(qp' - pq'), q + q', p + p'\right), \tag{2.4}$$

and where the inverse element of $(\tau, q, p)$ is given by

$$(\tau, q, p)^{-1} = (-\tau, -q, -p). \tag{2.5}$$

Indeed, it is straightforward to prove the following fact:

**Proposition 2.1.3.**

$$j : \mathbb{H}_n^{POL}(\mathbb{R}) \ni (\tau, q, p) \mapsto \left(\frac{1}{2}qp - \tau, q, p\right) \in \mathbb{H}_n(\mathbb{R}), \tag{2.6}$$

*is a group isomorphism, whose inverse is*

$$j^{-1} : \mathbb{H}_n(\mathbb{R}) \ni (\tau, q, p) \mapsto \left(\frac{1}{2}qp - \tau, q, p\right) \in \mathbb{H}_n^{POL}(\mathbb{R}).$$

We remark that

$$\mathcal{Z}(\mathbb{H}_n(\mathbb{R})) = \{(\tau, 0, 0) \in \mathbb{H}_n(\mathbb{R}) \mid \tau \in \mathbb{R}\}$$

is the center of the Heisenberg-Weyl group. Indeed, $(\tau, q, p) \in \mathcal{Z}(\mathbb{H}_n(\mathbb{R}))$ iff

$$(\tau, q, p) = (\tau', q', p')(\tau, q, p)(\tau', q', p')^{-1} =$$
$$= \left(\tau' + \tau - \tau' + \frac{1}{2}(pq' - qp') + \frac{1}{2}[q'(p - p') - p'(q - q')], q, p\right),$$

hence we must have $q = p = 0$.

We can now highlight the semi-direct product structure of $\mathbb{H}_n(\mathbb{R})$. In particular, let us consider the subgroups

$$N = \{(\tau, q, 0) \in \mathbb{H}_n(\mathbb{R}) \mid \tau \in \mathbb{R},\ q \in \mathbb{R}^n\}, \quad H = \{(0, 0, p) \in \mathbb{H}_n(\mathbb{R}) \mid p \in \mathbb{R}^n\}. \tag{2.7}$$

Of course $N$ is a normal subgroup of $G$ since it is Abelian. Moreover, observe that the following decomposition holds:

$$(\tau, q, p) = \left(\tau - \frac{1}{2}qp, q, 0\right)(0, 0, p),$$

thus

$$(\tau, q, p)(\tau', q', p') = \left(\tau - \frac{1}{2}qp, q, 0\right)\left(\tau' - \frac{1}{2}q'p' - pq', q', 0\right)(0, 0, p + p').$$

Hence, $\tilde{\alpha}_p(\tau, q) = (\tau - qp, q)$ is a semi-direct product action, therefore $\mathbb{H}_n(\mathbb{R}) = N \times'_{\tilde{\alpha}_p} H$.
Lastly, observe that the map

$$i : \mathbb{H}_n(\mathbb{R}) = N \ltimes H \ni nh \mapsto (n, h) \in N^{POL} \times'_{\alpha} H^{POL} =: \hat{\mathbb{H}}_n^{POL}(\mathbb{R}) \tag{2.8}$$

such that

$$(\tau, q, p)(\tau', q', p') = (\tau + \tau' - pq', q + q', p + p')$$

is a group isomorphism. We notice that $\hat{\mathbb{H}}_n^{POL}(\mathbb{R})$ is another polarization of the Heisenberg-Weyl group, which is isomorphic to $\mathbb{H}_n^{POL}(\mathbb{R})$ via the map

$$i \circ j : \mathbb{H}_n^{POL}(\mathbb{R}) \ni (\tau, q, p) \mapsto (-\tau, q, p) \in \hat{\mathbb{H}}_n^{POL}(\mathbb{R}). \tag{2.9}$$

We also remark that the semi-direct product decompositions given above are not the unique ones possible for the Heisenberg-Weyl group. Indeed, we can also choose as a normal factor the Abelian subgroup $\tilde{N} = \{(\tau, 0, p) \mid \tau \in \mathbb{R},\ p \in \mathbb{R}^n\}$ and as a homogeneous factor the subgroup $\tilde{H} = \{(0, q, 0) \mid q \in \mathbb{R}^n\}$; the decomposition still holds (the same holds for the polarized realization).

### 2.1.1 The symplectic point of view

Now we clarify some facts concerning the polarizations of the Heisenberg-Weyl group which are related with the standard symplectic form defined on the phase space.

Let us consider a symplectic space $(\mathbb{V}, \omega)$, where $\mathbb{V}$ is a vector space and $\omega : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is a symplectic form, namely an antisymmetric, non-degenerate bilinear form. Given a set $S \subset \mathbb{V}$, we will consider

$$S^\omega := \{v \in \mathbb{V} \mid \omega(v, w) = 0\ \forall w \in S\}, \tag{2.10}$$

which is the set of "symplectic orthogonal" vectors of $S$. We will say that $S$ is a *lagrangian set* if $S^\omega = S$. Moreover, we recall that a symplectic space must be of even dimension [44].

**Definition 2.1.4.** If $(\mathbb{V}, \omega)$ is a symplectic space ($\dim \mathbb{V} = 2n$), the set $\{e_1, \cdots, e_n\} \cup \{e'_1, \ldots, e'_n\}$ is a *symplectic basis* if

$$\omega(e_j, e_k) = \omega(e'_j, e'_k) = 0, \quad \omega(e_j, e'_k) = \delta_{j,k} \quad \forall j, k \tag{2.11}$$

We notice that, if $(\mathbb{V}, \omega)$ is a symplectic space, there exists a symplectic basis in $\mathbb{V}$ [44]. Now, if $\{e_1, \ldots, e_n, e'_1, \ldots, e'_n\}$ is a symplectic basis of $\mathbb{V}$ with $\dim \mathbb{V} = 2n$, then the subspaces $\mathbb{L} = \operatorname{span}\{e_1, \ldots, e_n\}, \mathbb{L}' = \operatorname{span}\{e'_1, \ldots, e'_n\}$ are lagrangian and they are such that $\mathbb{L} \cap \mathbb{L}' = \emptyset$, hence $\mathbb{V} = \mathbb{L} + \mathbb{L}'$, with $\dim \mathbb{L} = \dim \mathbb{L}' = n$. Moreover, these two lagrangian subspaces are isomorphic, since

$$\mathbb{L} \ni w \mapsto \omega(\cdot, w) \equiv w^* \in \mathbb{L}'$$

is a linear isomorphism [44].

If we now consider a symplectic space $(\mathbb{V}, \omega)$, $\dim \mathbb{V} = 2n$, we have that $\mathbb{H}_\omega \equiv \mathbb{R} \times \mathbb{V}$ is a group with respect to the composition law

$$(\tau, v)(\tau', v') := \left(\tau + \tau' + \frac{1}{2}\omega(v, v'), v + v'\right), \tag{2.12}$$

where the inverse element is $(-\tau, -v)$ [44].
Hence, we can define a polarization by the choice of a pair of lagrangian subspaces $(\mathbb{L}, \mathbb{L}')$. Indeed, observe that $\mathbb{V} \ni v = Q+P,\ Q \in \mathbb{L},\ P \in \mathbb{L}' \cong \mathbb{L}^*$. Then let us define $P(Q) \equiv P \cdot Q \equiv Q \cdot P := \omega(Q, P) = -\omega(P, Q)$. Thus, the composition law 2.12 can be written as

$$\begin{aligned}
&(\tau, Q + P)(\tau', Q' + P') = \\
&\quad = \left(\tau + \tau' + \frac{1}{2}\omega(Q + P, Q' + P'), (Q + Q') + (P + P')\right) \\
&\quad = \left(\tau + \tau' + \frac{1}{2}(Q \cdot P' - P \cdot Q'), (Q + Q') + (P + P')\right).
\end{aligned}$$

Moreover, since $Q \in \mathbb{L},\ P \in \mathbb{L}'$, we have $Q = \sum_{i=1}^n q_i e_i$ and $P = \sum_{i=1}^n p_i e_i'$.

**Proposition 2.1.5.** *The map*

$$\eta_\omega : \mathbb{H}_\omega \ni (\tau, Q + P) \mapsto (\tau, q, p) \in \mathbb{H}_n(\mathbb{R}) \tag{2.13}$$

*is a group isomorphism.*

*Proof.* By direct calculation and setting $q \cdot p \equiv \sum_{i=1}^n q_i p_i$, we have that

$$\begin{aligned}
&\eta_\omega[(\tau, Q + P)(\tau', Q' + P')] = \\
&\quad = \eta_\omega\left(\tau + \tau' + \frac{1}{2}(Q \cdot P' - P \cdot Q'), (Q + Q') + (P + P')\right) \\
&\quad = \left(\tau + \tau' + \frac{1}{2}(q \cdot p' - p \cdot q'), q + q', p + p'\right) \\
&\quad = \eta_\omega[(\tau, Q + P)]\eta_\omega[(\tau', Q' + P')].
\end{aligned}$$

□

Thus, $\mathbb{H}_\omega$ is a generalized form of the Heisenberg-Weyl group.
Therefore, we have two possible polarized forms of the Heisenberg-Weyl group, denoted with $\mathbb{H}_\omega^{POL^+}, \mathbb{H}_\omega^{POL^-}$, with the following respective product laws:

$$(\tau, Q + P)(\tau', Q' + P') = (\tau + \tau' - P \cdot Q', (Q + Q') + (P + P')), \tag{2.14}$$
$$(\tau, Q + P)(\tau', Q' + P') = (\tau + \tau' + P \cdot Q', (Q + Q') + (P + P')). \tag{2.15}$$

## 2.2 Weyl systems

In this section we will discuss the irreducible representations of the additive group of phase space translations $\mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^{2n}$. Since this is an Abelian group, by Schur's lemma 1.2.7, its irreducible unitary representations are all one-dimensional, hence physycally trivial, thus we will concentrate on the projective ones.
In particular, we will study the multipliers of the vector group $\mathbb{R}^{2n}$ and its central extension via $\mathbb{R}$, which will lead us to the (generalized) Heisenberg-Weyl group $\mathbb{H}_\omega$, whose irreducible unitary representations will be discussed by means of the Mackey machine. Finally, we will deal with Weyl systems and Stone-Von Neumann's theorem.

### 2.2.1 The projective representations of the phase space translations group

Let $\nu : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a $\mathbb{R}$-multiplier. Observe that, if $\nu$ is exact, then $\nu$ is symmetric. Indeed, since $\mathbb{R}^{2n}$ is an Abelian additive group,

$$\nu(x, y) = \beta(x + y) - \beta(x) - \beta(y) = \nu(y, x) \quad \forall x, y \in \mathbb{R}^{2n}.$$

Furthermore, if $\gamma : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is a bilinear form, then $\gamma$ is a $\mathbb{R}$-multiplier. Indeed, if $x, y, z \in \mathbb{R}^{2n}$,

$$\gamma(x, y + z) + \gamma(y, z) = \gamma(x, y) + \gamma(x, z) + \gamma(y, z) = \gamma(x + y, z) + \gamma(x, y),$$
$$\gamma(x, 0) = \gamma(0, x) = 0.$$

Besides, we also have that a bilinear symmetric form $\sigma : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is an exact multiplier. Indeed, if we set $\beta = \frac{1}{2}\sigma(x, x)$, we have that

$$\beta(x + y) - \beta(x) - \beta(y) = \frac{1}{2} \left(\sigma(x + y, x + y) - \sigma(x, x) - \sigma(y, y)\right) = \sigma(x, y).$$

Therefore, we can analyze $\gamma$ by observing the behaviour of its antisymmetric component, defined as

$$\alpha(x, y) = \frac{1}{2}(\gamma(x, y) - \gamma(y, x));$$

if $\alpha$ is not null, then the multiplier is not exact.

Let us now consider the covering homomorphism $\mathbf{h} : \mathbb{R} \ni t \mapsto e^{it} \in \mathbb{T}$, which is a Borel map because $\mathbb{R}$ and $\mathbb{T}$ are polish spaces [19, 45]. Therefore, following the general construction seen in section 1.4, we have that

$$\mathbf{h} \circ \nu : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{T}$$

is a multiplier. In particular, by the previous discussion, the latter is non-exact if $\nu$ is non-exact, hence if it is an antisymmetric bilinear form. Thus, we can consider the group of multipliers

$$M(\mathbb{R}^{2n}) = \{[\mathbf{h} \circ \alpha] \mid \alpha \text{ is a bilinear antisymmetric form}\} \tag{2.16}$$

(in particular, $E(\mathbb{R}^{2n}) := [1]$ is the set of the exact multipliers of $\mathbb{R}^{2n}$). Hence, the most generic non-exact multiplier $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{T}$ is given by

$$\mu(x, y) = \beta(xy)\beta(x)^{-1}\beta(y)^{-1}e^{i\alpha(x,y)}, \tag{2.17}$$

where $\alpha$ is a bilinear antisymmetric form. Therefore, if $\alpha \equiv 0$, we have an exact multiplier.

**Definition 2.2.1.** We will say that the multiplier $\mu : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{T}$ is *non-degenerate* if the corresponding bilinear form is a symplectic form, i.e. it is an antisymmetric non-degenerate bilinear form.

Observe that $\mu$ is a non-degenerate multiplier of $\mathbb{R}^{2n}$ if and only if

$$\forall x \in \mathbb{R}^{2n}, x \neq 0,\ \exists y \in \mathbb{R}^{2n} \mid \mu(x, y)\mu(y, x)^* \neq 1. \tag{2.18}$$

Indeed,

$$\mu(x, y)\mu(y, x)^* = e^{i(\alpha(x,y)-\alpha(y,x))} = e^{i2\alpha(x,y)} \in [(2\alpha)_{\mathbf{h}}].$$

Let now $\mu$ be a multiplier related to the antisymmetric form $\alpha$ as in 2.17 and let us consider the left radicals

$$\mathrm{rad}_L(\alpha) := \{x \in \mathbb{R}^{2n} \mid \alpha(x, y) = 0\}. \tag{2.19}$$

Due to antisymmetry, we have that the set of right radicals

$$\mathrm{rad}_R(\alpha) := \{y \in \mathbb{R}^{2n} \mid \alpha(x, y) = 0\}$$

will coincide with $\mathrm{rad}_L(\alpha)$, thus we set $\mathbb{V}_0 \equiv \mathrm{rad}_L(\alpha) = \mathrm{rad}_R(\alpha)$. Let now $W : \mathbb{R}^{2n} \to \mathcal{U}(\mathcal{H})$ be an irreducible projective representation with multiplier $\mu$ defined as in 2.17. The projective representation $V$ defined as

$$V(x) := \beta(x)^* W(x) \quad \forall x \in \mathbb{R}^{2n}$$

is still irreducible, with multiplier $\alpha_{\mathbf{h}} = e^{i\alpha(\cdot,\cdot)}$. Let us now consider $v \in \mathbb{V}$, $v_0 \in \mathbb{V}_0$, then we have

$$V(v_0 + v) = e^{i\alpha(v_0,v)}V(v_0)V(v) = V(v)V(v_0),$$

because $v_0 \in \mathrm{rad}(\alpha)$. Conversely, if $v_1, v_2 \in \mathbb{V}$, then

$$V(v_1+v_2) = e^{i\omega(v_1,v_2)}V(v_1)V(v_2) = e^{-i\omega(v_2,v_1)}V(v_2)V(v_1),$$

where $\omega := \alpha|_{\mathbb{V}\times\mathbb{V}}$ is a symplectic form. Hence, we have that

$$V|_{\mathbb{V}_0} : \mathbb{V}_0 \to \mathcal{U}(\mathcal{H})$$

is a unitary representation such that $V_0(\mathbb{V}_0) \subset \mathscr{C}(V) = \{z\mathrm{Id}_{\mathcal{H}} \mid z \in \mathbb{T}\}$ due to Schur's lemma 1.2.7, so $V_0(v_0) = e^{ik\cdot v_0}\mathrm{Id}_{\mathcal{H}}$. Conversely, the representation $U \equiv V|_{\mathbb{V}} : \mathbb{V} \to \mathcal{U}(\mathcal{H})$ is projective, with multiplier $\omega_{\mathbf{h}} = e^{i\omega(\cdot,\cdot)}$, hence

$$V(v_0+v) = e^{ik\cdot v_0}U(v),\ \forall v_0 \in \mathbb{V}_0, \forall v \in \mathbb{V}.$$

In other words, $V$ is projectively equivalent to the irreducible projective representation with multiplier $\omega_{\mathbf{h}}$. In summary, we can always disregard the phase factor given by $\mathrm{rad}(\alpha)$ and we can focus our attention onto multipliers related with symplectic forms.
Now, following Mackey's theorem 1.4.8, we can consider the group $\mathbb{H}_{\omega/2} = \mathbb{R} \times_{\omega/2} \mathbb{V}$, where $\omega/2$ is the symplectic form associated with the multiplier $e^{i\omega(\cdot,\cdot)/2}$. The composition law of $\mathbb{H}_{\omega/2}$ is given by

$$(\tau, v)(\tilde{\tau}, \tilde{v}) = \left(\tau + \tilde{\tau} + \frac{1}{2}\omega(v, \tilde{v}), v + \tilde{v}\right), \tag{2.20}$$

which is exactly the product law of the generalized Heisenberg-Weyl group associated to a symplectic form. Therefore, the projective representations of $\mathbb{R}^{2n}$ can be analyzed via the irreducible unitary representations of $\mathbb{H}_{\omega/2}$, which is isomorphic to $\mathbb{H}_n(\mathbb{R})$. Just for completeness, we explicitly display those representations that will be studied in the next paragraph:

$$\{S_h : \mathbb{H}_n(\mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^n))\}_{h\in\mathbb{R}_*} \cup \{R_{u,v} : \mathbb{H}_n(\mathbb{R}) \to \mathcal{U}(\mathbb{C})\}_{u,v\in\mathbb{R}^n}, \tag{2.21}$$

where

$$R_{u,v}(\tau, q, p)z = \exp(i(u\cdot q + v\cdot p))z,\ \forall z \in \mathbb{C} \tag{2.22}$$

and

$$(S_h(\tau, q, p)f)(x) := e^{-i\frac{2\pi}{h}(\tau+q\cdot p/2)}e^{i\frac{2\pi}{h}p\cdot x}f(x-q), \tag{2.23}$$

with $f \in L^2(\mathbb{R}^n)$ and where $q\cdot p = \sum_{i=1}^n q_i p_i$.
Hence, it follows that the map

$$\mathbb{R}^n \times \mathbb{R}^n \ni (q,p) \mapsto S_h(0,q,p) \equiv S_h(q,p) \in \mathcal{U}(L^2(\mathbb{R}^n)) \tag{2.24}$$

is an irreducible projective representation of $\mathbb{R}^{2n}$ [4, 45]. In particular, by direct calculation,

$$\begin{aligned} S_h(q+q', p+p') &= e^{i\frac{\pi}{h}(q\cdot p' - p\cdot q')} S_h(q,p) S_h(q',p') \\ &= e^{i\frac{\pi}{h}\omega((q,p),(q',p'))} S_h(q,p) S_h(q',p'), \end{aligned} \tag{2.25}$$

thus the multiplier $\mu$ associated with the representation 2.24 is

$$\mu_h\left((q,p),(q',p')\right) = e^{i\frac{\pi}{h}(q\cdot p' - p\cdot q')} = e^{i\frac{\pi}{h}\omega((q,p),(q',p'))}. \tag{2.26}$$

Lastly, we remark that the representation 2.24 can also be expressed in the form [4]

$$S_h(q,p) = e^{i\frac{2\pi}{h}(p\cdot\hat{q} - q\cdot\hat{p})}, \tag{2.27}$$

where $\hat{q}$ and $\hat{p}$ are the canonical position and momentum operators, which satisfy the canonical commutation relations $[\hat{q},\hat{p}] = ih$.

### 2.2.2 The irreducible representations of the Heisenberg-Weyl group

We now reconsider the classification of the irreducible unitary representations of the Heisenberg-Weyl group $\mathbb{H}_n(\mathbb{R})$ from the point of view of the Mackey machine. Recall that $\mathbb{H}_n(\mathbb{R})$ can be written as $N \ltimes H$, where $N = \{(\tau, q, 0) \mid \tau \in \mathbb{R},\ p \in \mathbb{R}^n\} \cong \mathbb{R} \times \mathbb{R}^n$ and $H = \{(0,0,p) \mid p \in \mathbb{R}^n\} \cong \mathbb{R}^n$ and the action is $\alpha_p(\tau, q) = (\tau - q\cdot p, q)$.
Observe that, since $N$ is an Abelian group, the maps

$$\chi_{\lambda,\xi} : N \ni (\tau, q, 0) \mapsto e^{i2\pi(\lambda\tau + q\cdot\xi)} \in \mathbb{C}, \quad \xi \in \mathbb{R}^n,\ \lambda \in \mathbb{R}, \tag{2.28}$$

are the only irreducible unitary representations, called the (unitary) characters of $N$. $\hat{N}$ will denote the group of characters of $N$. Hence, acting with $H$ on $\hat{N}$, we have

$$(0,0,p)[\chi_{\lambda,\xi}(\tau,q)] = \chi_{\lambda,\xi}(\tau + q\cdot p, q) = \chi_{\lambda,\xi+\lambda p}(\tau, q), \quad (\tau,q) \in N. \tag{2.29}$$

Thus we have two cases:

- If $\lambda = 0$, we have that $(0,0,p)[\chi_{0,\xi}] = \chi_{0,\xi}$, hence we have a singleton orbit [6, 28] and the stabilizer is all $H$, which leads us to the family of one-dimensional representations of $\mathbb{H}_n(\mathbb{R})$ 2.22.

- If $\lambda \neq 0$, the orbits $\mathcal{O}_\chi = \{\chi_{\lambda,\xi} \mid \xi \in \mathbb{R}^n\}$ are homeomorphic to $H$ and the action is regular [28]. Moreover, observe that the latter group-action is free. Thus, for each orbit $\mathcal{O}_\chi$ there is only one irreducible unitary representation, which is induced by the character $\chi_{\lambda,0}$. Therefore, we can consider the induced representation $S^\lambda \equiv \mathrm{ind}_N^{\mathbb{H}_n(\mathbb{R})}(\chi_{\lambda,0})$ [28], which can be realized on $L^2(\mathbb{R}^n)$ as [42]

$$S^\lambda(\tau,q,p)f(\tilde{p}) = e^{i2\pi\lambda(\tau-\frac{q\cdot p}{2})}e^{i2\pi\lambda q\cdot\tilde{p}}f(\tilde{p}-p), \quad f\in L^2(\mathbb{R}^n). \tag{2.30}$$

Therefore, we have completely characterized the irreducible representations of $\mathbb{H}_n(\mathbb{R})$, hence its dual group:

$$\widehat{\mathbb{H}_n(\mathbb{R})} = \{R_{u,v} \mid u,v\in\mathbb{R}^n\} \cup \{S^\lambda \mid \lambda\in\mathbb{R}_* \equiv \mathbb{R}\setminus\{0\}\}. \tag{2.31}$$

As we already pointed out, the first class of irreducible representations of the Heisenberg-Weyl group are physically trivial, since they are one-dimensional, hence we will focus only on the infinite-dimensional ones.
We also observe the following fact: the representations built by the Mackey machine act on the momentum space instead of the configuration space. However, this will not bring any trouble. Indeed, we can construct the Schrödinger representations acting on the configuration space by swapping $q$ and $p$ (namely, considering the decomposition of $\mathbb{H}_n(\mathbb{R})$ in $\{(\tau,0,p) \mid \tau\in\mathbb{R},\ p\in\mathbb{R}^n\}$ and $\{(0,q,0) \mid q\in\mathbb{R}^n\}$), and via the Mackey machine (and setting $\lambda = 1/h$) we obtain the representations 2.23, which act on the configurations space. Moreover, such representations are unitarily equivalent to 2.30, since the Mackey machine gives us a complete set of unitarily inequivalent irreducible representations. In particular, the intertwining unitary operator is the Fourier-Plancherel operator on $L^2(\mathbb{R}^n)$ [6, 42], which we will discuss in section 3.3.1.

We can also investigate the intertwining properties of the representations $S_h, S_{-h}$. By the previous discussion, we know that they are unitarily inequivalent. Equivalently, we can say that they are anti-unitarily equivalent. Indeed, if $J : L^2(\mathbb{R}^n)\ni f\mapsto \bar{f}\in L^2(\mathbb{R}^n)$ is a complex conjugation, we have that

$$\begin{aligned}(JS_h(\tau,q,p)J^*f)(x) &= e^{i\frac{2\pi}{h}(\tau+q\cdot p/2)}e^{-i\frac{2\pi}{h}p\cdot x}(J^*\bar{f})(x-q)\\ &= e^{i\frac{2\pi}{h}(\tau+q\cdot p/2)}e^{-i\frac{2\pi}{h}p\cdot x}f(x-q) = (S_{-h}(\tau,q,p)f)(x).\end{aligned}$$

hence

$$S_{-h}(\tau,q,p) = JS_h(\tau,q,p)J^*, \tag{2.32}$$

Lastly, we remark that the equivalence classes of irreducible representations of $\mathbb{H}_n(\mathbb{R})$ are identified by the *central characters*, namely by the maps of the form

$$\mathcal{Z}(\mathbb{H}_N(\mathbb{R})) \ni (\tau,0,0)\mapsto V(\tau,0,0) = e^{-i\frac{2\pi}{h}\tau}\mathrm{Id}, \tag{2.33}$$

which are all phase factors due to Schur's lemma 1.2.7.

### 2.2.3 Canonical commutation relations (CCRs)

The Schrödinger representation plays an important role in the abstract formulation of the CCRs

$$[\hat{q}, \hat{p}] = i\hbar \mathrm{Id}, \tag{2.34}$$

which entails non trivial mathematical issues in this "infinitesimal form".
In order to illustrate this point from a very abstract point of view, let us consider a unital Banach algebra $\mathcal{A}$ and suppose $A, B \in \mathcal{A}$ are such that $[A, B] = c\mathrm{Id}$, for some $c \in \mathbb{C}$. Then we have that [18]

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\},$$

where $\sigma(A)$ denotes the spectrum of $A \in \mathcal{A}$. Hence,

$$\sigma(AB) = \sigma(c\mathrm{Id} + BA) = c + \sigma(BA),$$

and

$$\sigma(BA) \cup \{0\} = (c + \sigma(BA)) \cup \{0\},$$

which implies $c = 0$.
From this general argument, since the set of bounded operators forms a unital Banach algebra, it follows that a pair of bounded operators in an infinite-dimensional Hilbert space cannot satisfy the CCRs, since $\hbar > 0$.
Besides, even if $\hat{q}$ or $\hat{p}$ is an unbounded operator, it is still troublesome to give a formal infinitesimal formulation of the CCRs, since not all the operators which satisfy 2.34 are canonical pairs of position and momentum operators.
For example, let us consider the operators

$$Q : L^2([0,1]) \ni f(x) \mapsto xf(x) \in L^2([0,1]),$$
$$P : \mathcal{D} \ni f \mapsto -i\hbar \frac{\mathrm{d}f}{\mathrm{d}x} \in L^2([0,1]),$$

where $\mathcal{D} := \{f \in C^1([0,1]) \mid f(0) = f(1) = 0\}$ is a dense domain. Then we have that $\mathcal{D}$ is invariant under the action of $Q$ and $[Q, P] = i\hbar\, \mathrm{Id}|_{\mathcal{D}}$, but they are not "true" position and momentum operators (we can also find more abstract formulations of this kind of operators, see [4, 39]).
Therefore, we are led to an alternative definition of the CCRs, an "exponentiated version", which relies on the definition of Weyl systems:

**Definition 2.2.2.** For each $h \in \mathbb{R}$, a *$h$-Weyl system* is a family of $2n$ unitary representations

$$\mathbb{R} \ni p_j \mapsto {}^h M_j(p_j) \in \mathcal{U}(\mathcal{H}), \tag{2.35}$$
$$\mathbb{R} \ni q_j \mapsto {}^h T_j(q_j) \in \mathcal{U}(\mathcal{H}), \tag{2.36}$$

where $j = 1, \ldots, n$, such that

- The representations $M_j$ and $T_j$ are jointly irreducible, namely, if $\mathcal{H}_0 \leq \mathcal{H}$ is a closed subspace with ${}^hM_j\mathcal{H}_0 \subset \mathcal{H}_0$ and ${}^hT_j\mathcal{H}_0 \subset \mathcal{H}_0$ for each $j$, then $\mathcal{H}_0 = \{0\}$ or $\mathcal{H}_0 = \mathcal{H}$.

- The *integrated form* of the CCRs holds:

$$ {}^hM_j(p_j)\,{}^hT_k(q_k) = e^{i\frac{2\pi}{h}\delta_{jk}p_jq_k}\,{}^hT_k(q_k)\,{}^hM_j(p_j), \tag{2.37}$$

$$ [{}^hM_j(p_j), {}^hM_k(p_k)] = 0 = [{}^hT_j(q_j), {}^hT_k(q_k)] \tag{2.38}$$

Note that $n$ denotes the number of degrees of freedom of the system. Moreover, the operators

$$\{\hat{q}_1, \dots, \hat{q}_n, {}^h\hat{p}_1, \dots {}^h\hat{p}_n\} \tag{2.39}$$

denotes a *canonical system of position and momentum observables in the Schrödinger representation*, where

$$\hat{q}_j : L^2(\mathbb{R}^n) \ni f(q_1, \dots, q_n) \mapsto q_jf(q_1, \dots, q_n) \in L^2(\mathbb{R}^n),$$
$${}^h\hat{p}_j := h\mathcal{F}^*q_j\mathcal{F},$$

where $\mathcal{F}$ is the Fourier-Plancherel operator defined as

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} d^nxf(x)e^{-2\pi i\xi\cdot x}$$

(in section 3.3.1 we define such operator for a generic l.c.s.c. Abelian group).

Observe now that, since the Schrödinger representations 2.23 are constructed ultimately via the Mackey machine, these are the only irreducible non-unidimensional representations of the phase space. This is nothing but the contents of the celebrated *Stone-von Neumann*'s theorem [18, 20, 35], which we report below:

**Theorem 2.2.3** (Stone-von Neumann)**.** *For any $h > 0$ let us consider a $h$-Weyl system as in 2.2.2. Then there exists a unique unitary operator $U : \mathcal{H} \to L^2(\mathbb{R}^n)$ - up to phase factors - such that*

$$U\,{}^hM_j(p_j)U^{-1} = \exp\left(\frac{i}{h}p_j\hat{q}_j\right), \tag{2.40}$$

$$U\,{}^hT_j(q_j)U^{-1} = \exp\left(-\frac{i}{h}q_j\,{}^h\hat{p}_j\right). \tag{2.41}$$

Hence, we notice that the operators ${}^hT(q)$ and ${}^hM(p)$, which define the $h$-Weyl system, are respectively the displacement operators in position and momentum coordinates (or, equivalently, in the language of time-frequency analysis, they are the translation and modulation operators) [4, 22]. Therefore, we can also define a Weyl system as an irreducible projective representation of $\mathbb{R}^n \times \mathbb{R}^n$. We also remark that the projective representation 2.24 can be expressed in terms of the translation and modulation operators as [4, 22]

$$S_h(q,p) = e^{-i\frac{\pi}{h}q\cdot p}\,{}^hM(p)\,{}^hT(q). \tag{2.42}$$

Observe that Stone-von Neumann's theorem holds in the case $h < 0$ too. Indeed, let us consider the map $(\tau, 0, 0) \mapsto e^{i\frac{2\pi}{h}}\mathrm{Id}$, which is the central character of $S_{-h}$ and recall that $S_{-h}$ and $S_h$ are anti-unitarily equivalent. Then, by 2.32 and theorem 2.2.3, we will have that the unique operator which intertwines the representations will be unitary in the case $h > 0$, and anti-unitary in the case $h < 0$, since $W = UJ$, where $U$ is a unitary operator and $J$ is a complex conjugation, is an anti-unitary operator [39, 35].
We also observe that if we drop the joint irreducibility in definition 2.2.2, everything still works, but the unitary (or anti-unitary) operator which appears in theorem 2.2.3 will be such that

$$U\,{}^hM_j(p_j)U^{-1} = \bigoplus_{\alpha\in A}\exp\left(\frac{i}{h}p_j\hat{q}_j^\alpha\right), \tag{2.43}$$

$$U\,{}^hT_j(q_j)U^{-1} = \bigoplus_{\alpha\in A}\exp\left(-\frac{i}{h}q_j\,{}^h\hat{p}_j^\alpha\right), \tag{2.44}$$

where $A$ is a denumerable set and, for each $\alpha$, $\{\hat{q}_1^\alpha, \ldots, \hat{q}_n^\alpha, {}^h\hat{p}_1^\alpha, \ldots, {}^h\hat{p}_n^\alpha\}$ is a canonical system of Schrödinger operators in $L^2(\mathbb{R}^n)_\alpha$ [4].

Finally, we remark that it is possible to define a Weyl system without refering to a particular choice of position and momentum variables (namely, without the choice of a polarization).
Indeed, let us consider a real vector group $\mathbb{V}$, $\dim\mathbb{V} = 2n$, $n \in \mathbb{N}$, and let $\tilde{U} : \mathbb{V} \to \mathcal{U}(\mathcal{H})$ be a projective representation with non-degenerate multiplier $\tilde{\mu}$ (i.e. it is similar to the multiplier $\mu^\circ : \mathbb{V}\times\mathbb{V} \ni (v_1, v_2) \mapsto e^{i\pi\omega(v_1,v_2)} \in \mathbb{T}$ where $\omega$ is a symplectic form on $\mathbb{V}$). Hence, we can consider a representation $U^\circ : \mathbb{V} \to \mathcal{U}(\mathcal{H})$ with multiplier $\mu^\circ$ which is projectively equivalent to $\tilde{U}$ [4]. Then the choice a polarization of $\mathbb{V}$ allows us to recover the standard symplectic form on the phase space and, consequently, the Schrödinger representations 2.24 [4].

Therefore, we can also identify a Weyl system with an irreducible projective representation of a real vector group, whose multiplier is nondegenerate [4].

## 2.3 From continuous to discrete phase space

In this section we want to discuss why it is useful to consider a general phase space of the form $G \times \hat{G}$, where $G$ is a l.c.s.c. Abelian group. In particular, we will sketch with an interesting example how a straightforward discretization of the Heisenberg-Weyl group will not be the best choice, since its irreducible representations, which are formally similar to the Schrödinger representations 2.23, have a very different behaviour (the latter discretization would also implie some issues in the definition of a discrete Wigner function, see [25, 37]).

Recall that $(\mathbb{Z}, +)$ is an Abelian group which becomes a ring when equipped with the multiplication. We will consider the group $\mathbb{H}(\mathbb{Z})$, i.e. the discrete Heisenberg-Weyl group of matrices

$$\begin{pmatrix} 1 & j & l \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}, \quad j, k, l \in \mathbb{Z}$$

with the composition law 2.2, namely

$$(l, j, k)(l', j', k') = (l + l' + jk', j + j', k + k').$$

We remark that $\mathbb{H}(\mathbb{Z})$ is a proper subgroup of the polarized Heisenberg-Weyl group $\mathbb{H}^{POL}(\mathbb{R}) \equiv \mathbb{H}_1^{POL}(\mathbb{R}) \cong \mathbb{H}_1(\mathbb{R}) \equiv \mathbb{H}(\mathbb{R})$ and its center is given by the group $\mathcal{Z}(\mathbb{H}(\mathbb{Z})) := \{(l, 0, 0) \mid l \in \mathbb{Z}\}$.
Since $\mathbb{H}(\mathbb{R})$ is the central extension of the phase space $\mathbb{R} \times \mathbb{R}$ via $\mathbb{R}$, by analogy, we could also say that we are essentially considering a "discrete phase space" $\mathbb{Z} \times \mathbb{Z}$ (thus, positions and momenta are discrete variables) and $\mathbb{H}(\mathbb{Z})$ is its central extension via $\mathbb{Z}$.
Let us consider the following representation of $\mathbb{H}(\mathbb{Z})$ on $L^2(\mathbb{R})$, defined in such a way that

$$(\rho_\omega(l, j, k)f)(t) = e^{i2\pi\omega l} e^{i2\pi\omega kt} f(t + j). \tag{2.45}$$

We remark that 2.45 is the discrete analogous of the Schrödinger representation 2.23 (in the "momenta coordinates") and can be considered as the restriction of the latter to $\mathbb{H}(\mathbb{Z})$ [18, 21, 20]. This representation is reducible and will reveal a pathological behaviour, as we are going to sketch.

Due to Stone-Von Neumann theorem 2.2.3, the analysis of such representations will follow the central characters $\chi_\omega(l) = e^{i2\pi\omega l}\mathrm{Id}$, where $\omega \in \mathbb{R}$, which are the irreducible representations of $\mathcal{Z}(\mathbb{H}(\mathbb{Z}))$ (since it is an Abelian group, they are the only irreducible ones by Schur's lemma 1.2.7). In particular, we will see how the representation 2.45 acts in different ways with respect to the Schrödinger representation 2.23, depending on the nature of the label $\omega$.

Let us firstly consider the case of trivial central characters, namely $\omega = p,\ p \in \mathbb{Z}$. In such a case, let us consider the one-dimensional representations of $\mathbb{Z}^2$

$$R_{u,v}(l,j,k) \equiv R_{u,v}(k,j) = e^{i2\pi(ju+vk)}, \quad u,v \in \mathbb{R}. \tag{2.46}$$

Hence, the representation $\rho_{\omega=p}$ is unitarily equivalent to the direct integral representation

$$\int_{[0,1]\times[0,1]}^{\oplus} R_{u,pv} du dv,$$

which acts on $L^2([0,1]^2) \cong \int_{[0,1]\times[0,1]}^{\oplus} \mathbb{C} du dv$ [21].

A second decomposition arises when $\omega$ is of the form $p/q$, where $\gcd(p,q) = 1$ and $q > 1$ (gcd means here 'greatest common divisor'). In such a case, we have $\rho_{\omega=p/q}(l,0,0) = \mathrm{Id}$ if $q$ divides $l$, hence it is convenient to analyze the group $\mathbb{H}_q(\mathbb{Z}) := \{(l,j,k) \mid j,k \in \mathbb{Z},\ l \in \mathbb{Z}_q\}$, i.e. a discrete Heisenberg-Weyl group where the variable $l$ is an integer modulo $q$ (the composition law is still like 2.2).
A complete list of irreducible representations of such a group is given by an application of the Mackey machine [18], and these are of the form

$$\pi_{u,v}(l,j,k) f(m) = e^{i2\pi lp/q} e^{i2\pi k(v-(p/q)m)} f(m-j), \tag{2.47}$$

where $f$ is an element of the $q$-dimensional Hilbert space

$$\mathcal{H}_u := \{f : \mathbb{Z} \to \mathbb{C} \mid f(m+nq) = e^{-i2\pi unq} f(m)\ \forall m,n \in \mathbb{Z}\}.$$

We can again intertwine the representation 2.45, with $\omega = p/q$ with the 2.47. In particular, $\rho_{p/q}$ is unitarily equivalent to [21]

$$\int_{[0,1/q)}^{\oplus} du dv\ \pi_{u,v} \underbrace{\oplus \cdots \oplus}_{p \text{ times}} \int_{[0,1/q)}^{\oplus} du dv\ \pi_{u,v}.$$

The last case is when $\omega$ is a fixed positive irrational number. Here things are very different, since it cannot be possible to fully use the Mackey machine, and the irreducible representations have a very different behaviour. Indeed, if we decompose $\mathbb{H}(\mathbb{Z})$ as a semi-direct product $N \ltimes H$, where $N := \{(l,0,k) \mid l,k \in \mathbb{Z}\}$ and $H := \{(0,j,0) \mid j \in Z\}$, we will have that the action

of $H$ on $\hat{N}$, whose element are of the form $\nu_{\beta,\gamma}(l,k) = e^{i2\pi(\beta k+\gamma l)}$, $\beta,\gamma \in \mathbb{R}\backslash\mathbb{Z}$, is not regular [18]. We remark that this is a consequence of the fact that $\mathbb{H}(\mathbb{Z})$ is not a type I group [21].
In such a case, the maps $\sigma_\nu : \mathbb{H}(\mathbb{Z}) \to \mathcal{U}(L^2(\mathbb{Z})) \equiv \mathcal{U}(l^2(\mathbb{Z})),\ \nu \in \mathbb{R}$ such that

$$[\sigma_\nu(l,j,k)f](m) = e^{2\pi i\omega l}e^{2\pi ik(v-\omega m)}f(m-j) \tag{2.48}$$

are irreducible unitary representations of $\mathbb{H}(\mathbb{Z})$ [18] and $\rho_\omega$ is unitarily equivalent to [21]

$$\sigma := \int_{[0,\omega)}^{\oplus} \sigma_\nu d\nu.$$

The point here is that the representations $\sigma_\nu$ are not all inequivalent, hence the direct integral decomposition is not unique. We can illustrate this fact by an application of the Stone-Von Neumann theorem 2.2.3. Firstly, recall that we are considering $\rho_\omega$ as a restriction of an irreducible representation of $\mathbb{H}_N(\mathbb{R})$, which we now denote with $S_\omega$. Moreover, let us consider an automorphism $\phi$ of $\mathbb{H}(\mathbb{R})$ that leaves the center fixed pointwise and suppose that $\phi$ is an automorphism of $\mathbb{H}(\mathbb{Z})$ too. Then we have that $S_\omega \circ \phi$ is another irreducible representation with the same central character of $S_\omega$, thus, by the Stone-Von Neumann theorem 2.2.3, they are unitarily equivalent. Therefore,

$$\int_{[0,\omega)}^{\oplus} \sigma_\nu d\nu \sim \rho_\omega \sim \rho_\omega \circ \phi \sim \int_{[0,\omega)}^{\oplus} \sigma_\nu \circ \phi \; d\nu$$

($\sim$ means here "unitarily equivalent"), but each $\sigma_\nu \circ \phi$ could be inequivalent to all the $\sigma_\nu,\ \nu \in \mathbb{R}$ [21] (this is a consequence of the fact that $\mathbb{H}(\mathbb{Z})$ is not a type I group [18]).

In conclusion, we cannot make a "straightforward discretization" of the Heisenberg-Weyl group $\mathbb{H}(\mathbb{R})$ in order to obtain a description of quantum mechanics on discrete phase space, because the "discrete Schrödinger representation" 2.45 behaves in a deeply different way from the continuous one. The turning point here is the group-theoretical structure, since $\mathbb{Z} \times \mathbb{Z}$ is not of the form $G \times \hat{G}$ (indeed, $\hat{Z} \cong \mathbb{T}$, see section 3.2), which is, on the other hand, what we expect, due to the dual nature of position and momentum coordinates in the description of physical systems [1]. In the next sections we will see that this choice will lead us to representations which have the same behaviour of the Schrödinger representations 2.23.

### 2.3.1 The discrete Heisenberg-Weyl group

The previous discussion leads us to consider a general phase space in the form $G \times \hat{G}$, where $G$ is a l.c.s.c. Abelian group; here $G$ may be regarded as the space of "positions", whereas $\hat{G}$ is the space of "momenta". We now choice to focus our attention on the discrete phase space, which has numerous applications in physics, e.g. in quantum information theory and many-body systems (see [25, 1] for a discussion on the phase space $\mathbb{Z} \times \mathbb{T}$ too).
We now observe the following fact: since we suppose that $G \times \hat{G}$ is a discrete group (i.e. it is endowed with the discrete topology), we have that $G$ and $\hat{G}$ shall be discrete groups too. Then, let us now focus on the discrete group $G$ and recall that $\hat{G}$ denotes the group of (equivalence classes of) irreducible unitary representations of $G$. We also recall that a discrete group $G$ is compact if and only if it is finite. Then, we have that $\hat{G}$ is a compact Abelian group, since $G$ is a discrete Abelian group (see proposition 3.2.3). However, $\hat{G}$ is also discrete by hypothesis, hence it is finite.
Conversely, we can consider the dual group of $\hat{G}$, namely $\hat{\hat{G}}$, which can be identified with $G$ itself (see Pontrjagin's theorem 3.3.12). Therefore, since $\hat{G}$ is discrete, we also have that $G$ is compact (and discrete by hypothesis), hence it is a finite group. As a consequence, the discrete group $G \times \hat{G}$, where $G$ is Abelian, is a finite group.

As a next step, we can observe that we can focus on the finite group $\mathbb{Z}/N\mathbb{Z} \equiv \mathbb{Z}_N$ only. Firstly, recall that $\mathbb{Z}_N$ is the Abelian group of equivalence classes of remainders modulo $N$ with respect to the addition law. In particular, we will say that $a$ is *congruent* $a'$ modulo $N$ if $a - a'$ is an integer multiple of $N$, and we will write $a \cong a' \mod N$. We also observe that $\hat{\mathbb{Z}}_N \cong \mathbb{Z}_N$ (see section 3.2). Moreover, $\mathbb{Z}_N$ is a ring when endowed with the product law defined as the product of the representatives $[a][b] := [ab]$. The addition and multiplication are well-defined, since they do not depend on the choice of representatives [15]. We also remark that $(\mathbb{Z}_N, +, \cdot)$ is in general a ring and it is a field if and only if $N$ is a prime number (or it can be extended to a field if and only if $N$ is a power of a prime) [15]. In the future we will drop the square bracket, and we will identify the equivalence classes with their representatives.
We can now notice that, when dealing with a discrete phase space, we can restrict our attention to $\mathbb{Z}_N \times \mathbb{Z}_N$ only. Indeed, by the structure theorem for finite Abelian groups, we have that, if $G$ is a finite Abelian group, the following decomposition holds [15]:

$$G = \mathbb{Z}_{{p_1}^{n_1}} \times \cdots \times \mathbb{Z}_{{p_k}^{n_k}},$$

where $p_1, \dots, p_k$ are prime numbers (not necessary distinct between each other).

It is also important to point out the fact that we will work with a $N$-dimensional Hilbert space, which can be identified with $\mathbb{C}^N$ with a periodicity property (namely, the space $L^2(\mathbb{Z}_N)$ [17]). In particular, the "configuration" space (this is not effectively a positions space, because we cannot give a direct physical interpretation [1, 34]) will be spanned by the standard orthonormal basis $\{\psi_n\}_{n\in\mathbb{Z}_N} \subset \mathbb{C}^N$, where each $\psi_n$ enjoys the periodicity property, i.e. $\psi_n(j+lN) = \psi_n(j)\ \forall l \in \mathbb{Z},\ j = 0, \dots, N-1$. In Dirac notation, this is the basis of vectors $\{|n\rangle \mid n = 0, \dots, N-1\}$ with the periodicity property $|n+lN\rangle = |n\rangle$, which is often called the *computational basis* [1, 34].
The "momenta" space can be introduced via the discrete Fourier transform, which is defined as

$$(\mathcal{F}f)(k) = \frac{1}{\sqrt{N}} \sum_{j\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}jk} f(j), \quad f \in L^2(\mathbb{Z}_N)$$

(we will discuss the latter in section 3.3 example 3.3.8). In particular, the basis vectors can be written in terms of the positions basis vectors as

$$\hat{\psi}_n(k) = \frac{1}{\sqrt{N}} \sum_{j\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}jk} \psi_n(j). \tag{2.49}$$

We also remark that in this finite space it is not possible to define position and momentum operators, since they will not satisfy the CCRs, as we have seen in section 2.2.3, hence we must introduce the notion of (finite) Weyl system.

An important remark: since we have to consider the irreducible projective representations of $\mathbb{Z}_N \times \mathbb{Z}_N$ (because the unitary ones are physically trivial), we have to find its group extension and classify its irreducible unitary representations. However, we will obtain the right projective representations restricting our attention to the ones associated with the antisymmetric, non-exact multipliers, as we have seen for $\mathbb{R}^n \times \mathbb{R}^n$. Thus, if we extend $\mathbb{Z}_N \times \mathbb{Z}_N$ via $\mathbb{Z}_N$, the extended group will be the discrete Heisenberg-Weyl group $\mathbb{H}(\mathbb{Z}_N)$, which is the group of triples such that

$$(\tau, j, k)(\tau', j', k') := (\tau + \tau' + jk', j + j', k + k'), \tag{2.50}$$

$$(\tau, j, k)^{-1} = (jk - \tau, -j, -k), \tag{2.51}$$

$(\tau, j, k), (\tau', j', k') \in \mathbb{H}(\mathbb{Z}_N)$. Here, the addition and multiplication shall be intended modulo $N$. Notice that the order of $\mathbb{H}(\mathbb{Z}_N)$ is $N^3$. Here $j$ and $k$ play respectively the role of labels in positions and momenta space.
In order to analyze its irreducible representations, we notice that - as in the continuous case - $\mathbb{H}(\mathbb{Z}_N)$ may be written as a semi-direct product group $A \ltimes H$, where

$$A := \{(\tau, 0, k) \mid \tau, k \in \mathbb{Z}_N\}, \ H := \{(0, j, 0) \mid j \in \mathbb{Z}_N\}$$

and the semi-direct product action is defined as $(0, j, 0)[(\tau, 0, k)] = (\tau + jk, 0, k)$ (the second decomposition with $j$ and $k$ swapped obviously holds too).
In the next section, analyzing the irreducible representations of $\mathbb{H}(\mathbb{Z}_N)$, we will see that $\mathbb{Z}_N \times \mathbb{Z}_N$ provides a good description of a discrete quantum system.

We conclude this brief presentation mentioning an interesting fact concerning the discrete phase space: it turns out that the order $N$ of $\mathbb{Z}_N$ plays a crucial role in $\mathbb{Z}_N \times \mathbb{Z}_N$, which has some remarkable consequences in physics. For example, in the study of the mutually unbiased bases, it is known that there are exactly $N + 1$ bases of this kind if we are dealing with a field, while in the general case of a ring we can only say that there are *at most* $N + 1$ mutually unbiased bases [46]. However, we will not investigate these facts any further, since they are beyond our aims.

## 2.4 Representation theory of the discrete phase space translations group

In this section we will discuss the irreducible representations of the discrete phase space translations group $\mathbb{Z}_N \times \mathbb{Z}_N$ - obviously the projective ones since it is an Abelian group - and discrete (or finite) Weyl systems. Thence, as in the continuous case, we will firstly analyze the irreducible representations of its central extension via $\mathbb{Z}_N$, namely the discrete Heisenberg-Weyl group $\mathbb{H}(\mathbb{Z}_N)$, next we will introduce finite Weyl systems.

### 2.4.1 Irreducible representations of the discrete Heisenberg-Weyl group

The classification of the irreducible unitary representations of $\mathbb{H}(\mathbb{Z}_N)$ is very similar to the case of the continuous group described in section 2.2.2, but there are some slight differences, since we are considering a finite group.

We will consider $\mathbb{H}(\mathbb{Z}_N)$ decomposed as $A \ltimes H$, as we have seen in the previous section.
In the finite case we have $N^2$ characters for $A$, since $A$ is of order $N^2$, which are given by

$$\chi_{\lambda,\xi} : A \ni (\tau, 0, k) \equiv (\tau, k) \mapsto e^{i\frac{2\pi}{N}(\lambda\tau+\xi k)} \in \mathbb{T}, \quad \lambda, \xi \in \mathbb{Z}_N. \tag{2.52}$$

Hence, the action of $H$ on $\hat{A}$, which is formally analogous to the action 2.29, is given by

$$(0, j, 0)[\chi_{\lambda,\xi}(\tau, 0, k)] = \chi_{\lambda,\xi-\lambda j}(\tau, 0, k), \tag{2.53}$$

and the following fact holds:

**Proposition 2.4.1.** *The orbits of the action of $H$ on $\hat{A}$ are given by*

$$\mathcal{O}_{\lambda,\kappa} = \{\chi_{\lambda,\xi} \mid \xi \equiv \kappa \mod \gcd(\lambda, N)\} \tag{2.54}$$

*Proof.* Recall that, if $a, b \in \mathbb{Z}$ and $n \geq 1$, the congruence $ma = b \mod n$ has a solution $m \geq 1$ if and only if $b = 0 \mod \gcd(a, n)$ [12]. Therefore, since $\chi_{\lambda_1,\xi_1}$ and $\chi_{\lambda_2,\xi_2}$ belong to the same orbit if and only if $\lambda_1 = \lambda_2 \equiv \lambda$ and there exists $\alpha \in \mathbb{Z}_N$ such that $\xi_1 - \alpha\lambda = \xi_2 \mod N$; the latter as a solution iff $\xi_1 = \xi_2 \mod \gcd(\lambda, N)$. □

We notice that, for a fixed character in $\hat{A}$, the stabilizer of its orbit is the following set [12]:

$$G_{\chi_{\lambda,\xi}} = \left\{(0, j, 0) \in H \mid j \equiv 0 \mod \frac{N}{\gcd(\lambda, N)}\right\} \cong \mathbb{Z}_{\gcd(\lambda,N)} \tag{2.55}$$

Thus, we have the following cases [43, 12]:

- $\lambda = 0\,(\iff \gcd(\lambda, N) = N)$. In such a case, we have $N$ orbits of order 1, which are stabilized by $\mathbb{Z}_N$.
- $\gcd(\lambda, N) = 1$. Then we have one orbit of order $N$, stabilized by the trivial subgroup $\{(0, 0, 0)\}$.
- $1 < \gcd(\lambda, N) < N$. In such a case, for each $\lambda$, there are $\gcd(\lambda, N)$ orbits of order $N/\gcd(\lambda, N)$, whose stabilizer group is isomorphic to $\mathbb{Z}_{\gcd(\lambda,N)}$.

Observe that the $\gcd(\lambda, N)$ characters of $G_{\chi_{\lambda,\xi}}$ are given by

$$\sigma_{\lambda,\alpha} : G_{\chi_{\lambda,\xi}} \ni \left(0, j\frac{N}{\gcd(\lambda, N)}, 0\right) \mapsto e^{i\frac{2\pi}{\gcd(\lambda,N)}\alpha j} \in \mathbb{T}. \tag{2.56}$$

We can now define the little groups $H_{\chi_{\lambda,\xi}} := A \ltimes G_{\chi_{\lambda,\xi}}$ as

$$H_{\chi_{\lambda,\xi}} = \left\{ \left( \tau, j\frac{N}{\gcd(\lambda, N)}, k \right) \mid 0 \leq j \leq \gcd(\lambda, N) - 1, \tau, k \in \mathbb{Z}_N \right\} \quad (2.57)$$

and we extend $\chi_{\lambda,\xi}$ and $\sigma_{\lambda,\alpha}$ to $H_{\chi_{\lambda,\xi}}$. Therefore, the family

$$\left\{ \rho_{\lambda,\alpha,\xi} = \mathrm{ind}_{H_{\chi_{\lambda,\xi}}}^{\mathbb{H}(\mathbb{Z}_N)} (\chi_{\lambda,\xi}\sigma_{\lambda,\alpha}) \mid \lambda \in \mathbb{Z}_N,\ \alpha, \xi \in \{0, \dots, \gcd(\lambda, N) - 1\} \right\} \quad (2.58)$$

is a complete set of irreducible, inequivalent, unitary representations of $\mathbb{H}(\mathbb{Z}_N)$ [12, 43]. Each representation $\rho_{\lambda,\alpha,\xi}$ has dimension $\frac{N}{\gcd(\lambda,N)}$.
Now recall that the *Euler totient function*, which counts the numbers which are coprime with $N$, is defined as [46]

$$\varphi(N) = N \prod_{p|N} \left( 1 - \frac{1}{p} \right), \quad (2.59)$$

where the product is over the set of prime numbers which divide $N$. Then the following fact holds [12, 43]:

**Proposition 2.4.2.** *For each divisor $d$ of $N$, $\mathbb{H}(\mathbb{Z}_N)$ has $(\frac{N}{d})^2\varphi(d)$ irreducible inequivalent representations of dimension $N/d$.*
*In particular, there are $\varphi(N)$ $N$-dimensional representations and $N^2$ one-dimensional representations.*

As in the continuous case, we will focus our attention on the representations $\rho_{\lambda,0,0}$ induced by the characters $\chi_{\lambda,0}$, which will be denoted with $S_\lambda$ from now on. In this way we will be able to obtain the irreducible projective representations of $\mathbb{Z}_N \times \mathbb{Z}_N$ associated with antisymmetric multipliers from the unitary ones of the extended group $\mathbb{H}(\mathbb{Z}_N)$. In order to give an explicit formula for those representations, let us consider the following operators on $L^2(\mathbb{Z}_{d_\lambda})$, $d_\lambda \equiv \frac{N}{\gcd(\lambda,N)}$:

- ${}^\lambda T_j$, defined by

$$ {}^\lambda T_j \psi(n) := \psi([n] - [j]), \quad \lambda, j \in \mathbb{Z}_N,\ \psi \in L^2(\mathbb{Z}_{d_\lambda}). \quad (2.60)$$

A matrix representation of these operators is given by

$$ {}^\lambda T_j = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \vdots & \ddots & \ddots & 0 \\ 0 & \vdots & \vdots & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 \end{pmatrix}^j \in \mathcal{M}_{d_\lambda}(\mathbb{C}). \quad (2.61)$$

- ${}^{\lambda}M_k$, defined by

$$ {}^{\lambda}M_k\psi(n) = e^{i2\pi\frac{\lambda}{N}[k]\cdot[n]}\psi(n), \quad \lambda, k \in \mathbb{Z}_N,\ \psi \in L^2(\mathbb{Z}_{d_\lambda}), \tag{2.62} $$

whose matrix representation is

$$ {}^{\lambda}M_k = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & e^{i2\pi\frac{\lambda}{N}1\cdot k} & 0 & \ddots & \vdots \\ 0 & 0 & e^{i2\pi\frac{\lambda}{N}2\cdot k} & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & e^{i2\pi\frac{\lambda}{N}(d_\lambda-1)k} \end{pmatrix} \in \mathcal{M}_{d_\lambda}(\mathbb{C}). \tag{2.63} $$

We remark that such operators - which are the position and momentum displacement operators - are often called, in the context of signal analysis, the *translation* and *modulation* operators [22, 17].

**Example 2.4.3.** In the case $N = 2$ and $\lambda = 1$, the matrix representation for the operators 2.60, 2.62 is

$$ T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}, \tag{2.64} $$

hence, if $\{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli matrices, we have that $\sigma_1 = T_1$ and $\sigma_3 = M_1$.

We can realize the representation $S_\lambda$ on $L^2(\mathbb{Z}_{d_\lambda})$ as [12, 43]

$$ [S_\lambda(\tau, j, k)\psi](n) := e^{-i2\pi\frac{\lambda}{N}\tau}\, {}^{\lambda}M_k\, {}^{\lambda}T_j\psi(n) = e^{-i2\pi\frac{\lambda}{N}\tau}e^{i2\pi\frac{\lambda}{N}[k]\cdot[n]}\psi([n]-[j]); \tag{2.65} $$

in the future we will drop the square brackets that specify the modular arithmetic operations (which are on $\mathbb{Z}_{d_\lambda}$) if there are no ambiguities. Notice that 2.65 is a proper unitary representation of $\mathbb{H}(\mathbb{Z}_N)$, indeed:

$$ \begin{aligned} (S((\tau, j, k), (\tau', j', k'))\psi)(n) &= (S(\tau + \tau' + jk', j + j', k + k')\psi)(n) \\ &= e^{-i2\pi\frac{\lambda}{N}(\tau+\tau'+jk')}e^{i2\pi\frac{\lambda}{N}(k+k')n}\psi(n - j - j'), \end{aligned} \tag{2.66} $$

$$ (S(\tau, j, k)S(\tau', j', k')\psi)(n) = e^{-i2\pi\frac{\lambda}{N}(\tau+\tau')}e^{i2\pi\frac{\lambda}{N}kn}e^{i2\pi\frac{\lambda}{N}k'(n-j)}\psi(n - j - j'). \tag{2.67} $$

Here we have to highlight an important fact: every $S_\lambda$ can of course be realized on $L^2(\mathbb{Z}_N)$, but they will not be all irreducible, because the only

$N$-dimensional irreducible representations of $\mathbb{H}(\mathbb{Z}_N)$ are the ones where $\lambda$ is coprime with $N$, If $\gcd(\lambda, N) \neq 1$, we have that the dimension of $S_\lambda$ is strictly less than $N$, in particular it is equal to $d_\lambda$. Thus, in such a case, we obtain a $N$-dimensional reducible representation - which we denote with $\tilde{S}_\lambda$ - if we consider the direct sum of $\gcd(\lambda, N)$ copies of $S_\lambda$, namely

$$\tilde{S}_\lambda = S_\lambda \underbrace{\oplus \cdots \oplus}_{\gcd(\lambda,N)\ \text{times}} S_\lambda. \tag{2.68}$$

Therefore, we have that the value of $\lambda$ (which is the finite analogous of $\hbar$) fixes the "scale" of the system. More specifically, the fact the $\lambda$ is not coprime with $N$ means that the system must be rescaled in order to recover the irreducibility.

**Example 2.4.4.** Let us consider the simple case of $\mathbb{H}(\mathbb{Z}_4)$ and recall that we denote with $\{\psi_n\}$, $n = 0, \ldots, 3$ the standard basis in $L^2(\mathbb{Z}_4)$. Here we have that the representations with $\lambda = 1, 3$ are irreducible on $L^2(\mathbb{Z}_4)$, while for $\lambda = 2$ we have that $S_2$ acts on $L^2(\mathbb{Z}_2)$ as

$$[S_2(\tau, j, k)\psi](n) = e^{-i\pi\tau} e^{i\pi kn} \psi(n - j), \quad \tau, j, k \in \mathbb{Z}_4,\ \psi \in L^2(\mathbb{Z}_2).$$

Therefore, the 4-dimensional representations will be given by $\tilde{S}_2 = S_2 \oplus S_2$ and $L^2(\mathbb{Z}_4)$ is decomposed as $L^2(\mathbb{Z}_2) \oplus L^2(\mathbb{Z}_2)$. In particular, observe that $\operatorname{span}\{\psi_+, \psi_-\}$, where

$$\psi_+ = \frac{1}{\sqrt{2}}(1, 0, 1, 0) = \frac{1}{\sqrt{2}}(\psi_0 + \psi_2),\ \psi_- = \frac{1}{\sqrt{2}}(0, 1, 0, 1) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_3),$$

is an invariant space under the action of $\tilde{S}_2$. Indeed, for $j, k \in \mathbb{Z}_4$, we have that

$$M_k\psi_+ = \psi_+, \quad T_j\psi_+ = \begin{cases} \psi_+, & j \text{ even}, \\ \psi_-, & j \text{ odd} \end{cases},$$
$$M_k\psi_- = -\psi_-, \quad T_j\psi_- = \begin{cases} \psi_-, & j \text{ even}, \\ \psi_+, & j \text{ odd} \end{cases}.$$

In a similar way, $\operatorname{span}\{\phi_+, \phi_-\}$, where

$$\phi_+ = \frac{1}{\sqrt{2}}(1, 0, -1, 0) = \frac{1}{\sqrt{2}}(\psi_0 - \psi_2),$$
$$\phi_- = \frac{1}{\sqrt{2}}(0, 1, 0, -1) = \frac{1}{\sqrt{2}}(\psi_1 - \psi_3),$$

is an invariant subspace for $\tilde{S}_2$.
At the same time, we can also consider the representation

$$(S^{\circ}_{\lambda}(\tau, j, k)\psi)(n) = e^{-i\frac{2\pi}{4}\tau} e^{i2\pi\frac{\lambda}{4\gcd(\lambda,4)}kn}\psi(n - \gcd(\lambda, 4)j), \quad \tau, j, k, \lambda \in \mathbb{Z}_4, \tag{2.69}$$

which acts on $\psi \in L^2(\mathbb{Z}_4)$. In the case $\lambda = 2$ the latter is a reducible unitary representation which is unitarily equivalent to $\tilde{S}_2$ (the unitarily equivalence is realized by the dilation $j \mapsto \gcd(\lambda, N)j$, $k \mapsto \frac{1}{\gcd(\lambda,N)}k$). In particular, we have that $\mathrm{span}\{\psi_0, \psi_2\}$ and $\mathrm{span}\{\psi_1, \psi_3\}$ are invariant subspaces for $S^{\circ}_2$.

Lastly, we have to remark two facts. Firstly, we have analyzed the irreducible unitary representations of $\mathbb{H}(\mathbb{Z}_N)$ on the space of "positions". However, as in the continuous case, we can also construct these representations on the dual space of momenta; the link between the two point of views is given by the discrete Fourier transform acting on the $\frac{N}{\gcd(\lambda,N)}$-dimensional space of the representation $S_\lambda$ [43].
Secondly, we remark that these representations are identified uniquely by the central characters of $\mathbb{H}(\mathbb{Z}_N)$, i.e. by the maps

$$\mathcal{Z}(\mathbb{H}(\mathbb{Z}_N)) \ni (\tau, 0, 0) \mapsto e^{-i2\pi\frac{\lambda}{N}\tau}\mathrm{Id}. \tag{2.70}$$

### 2.4.2 Finite Weyl systems

We can now introduce finite Weyl systems, namely, following the discussion in section 2.2, the irreducible projective representations of the discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$. We will mostly focus on the case $\gcd(\lambda, N) = 1$, where $\lambda \in \mathbb{Z}_N$, in order to consider irreducible representations which are $N$-dimensional only. Later on, we will briefly sketch what occurs when $1 < \gcd(\lambda, N) \leq N$.

As in the continuous case, the CCRs (in their exponentiated version) hold true. Indeed,

$$^{\lambda}T_j\, {}^{\lambda}T_{j'} = {}^{\lambda}T_{j'}\, {}^{\lambda}T_j, \quad {}^{\lambda}M_k\, {}^{\lambda}M_{k'} = {}^{\lambda}M_{k'}\, {}^{\lambda}M_k, \tag{2.71}$$

are trivially satisfied, while, for each $\psi \in L^2(\mathbb{Z}_N)$ and $j, k, \lambda, n \in \mathbb{Z}_N$,

$$[{}^{\lambda}M_k\, {}^{\lambda}T_j\psi](n) = e^{i2\pi\frac{\lambda}{N}kn}\psi(n - j), \quad [{}^{\lambda}T_j\, {}^{\lambda}M_k\psi](n) = e^{i2\pi\frac{\lambda}{N}k(n-j)}\psi(n - j),$$

thus

$$^{\lambda}M_k\, {}^{\lambda}T_j = e^{i2\pi\frac{\lambda}{N}jk}\, {}^{\lambda}T_j\, {}^{\lambda}M_k. \tag{2.72}$$

Moreover, if $\mathcal{F}$ is the discrete Fourier operator such that

$$\hat{\psi}(k) := \sum_{k=0}^{N} \psi(j) e^{-i\frac{2\pi}{N}jk}, \quad \psi \in L^2(\mathbb{Z}_N)$$

(see section 3.3 example 3.3.8), the translation and modulation operators satisfy the following intertwining properties:

**Proposition 2.4.5.** *The discrete Fourier transform intertwines the translation and modulation operators, namely*

$$\mathcal{F}\,{}^{\lambda}T_{\lambda j} = {}^{\lambda}M_{-j}\mathcal{F}, \quad \mathcal{F}\,{}^{\lambda}M_j = {}^{\lambda}T_{\lambda j}\mathcal{F},$$
$$\mathcal{F}^{-1}\,{}^{\lambda}T_{\lambda j} = {}^{\lambda}M_j\mathcal{F}^{-1}, \quad \mathcal{F}^{-1}\,{}^{\lambda}M_j = {}^{\lambda}T_{-\lambda j}\mathcal{F}^{-1}.$$

*Proof.* Indeed, for $\psi \in L^2(\mathbb{Z}_N)$, we have that

$$\begin{aligned}
\left({}^{\lambda}M_j\mathcal{F}\psi\right)(n) &= e^{i2\pi\frac{\lambda}{N}jn} \sum_{m\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}mn}\psi(m) \\
&= e^{i2\pi\frac{\lambda}{N}jn} \sum_{m\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}(m+\lambda j)n}\psi(m+\lambda j) \\
&= \sum_{m\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}mn}\left({}^{\lambda}T_{-\lambda j}\psi\right)(m) = \left(\mathcal{F}\,{}^{\lambda}T_{-\lambda j}f\right)(n).
\end{aligned}$$

In the same way,

$$\begin{aligned}
\left({}^{\lambda}T_{\lambda j}\mathcal{F}\psi\right)(n) = \sum_{m\in\mathbb{Z}_N} e^{-i2\pi\frac{\lambda}{N}m(n-\lambda j)}\psi(m) &= \sum_{m\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}mn}\left({}^{\lambda}M_j\psi\right)(m) \\
&= \left(\mathcal{F}\,{}^{\lambda}M_j\psi\right)(n).
\end{aligned}$$

The proof of the other intertwining relations is straightforward. □

We remark that this is exactly what happens in the continuous case too, where the operator $\mathcal{F}$ is the Fourier-Plancherel operator in $L^2(\mathbb{R}^n)$ [22].

Let us consider again the irreducible $N$-dimensional unitary representation 2.65 (recall that we are considering the case $\lambda$ coprime with $N$). Then we have that

$$S_\lambda(j,k) \equiv S_\lambda(0,j,k) = {}^{\lambda}M_k\,{}^{\lambda}T_j \tag{2.73}$$

is an irreducible projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ such that

$$S_\lambda(j+j',k+k') = e^{i2\pi\frac{\lambda}{N}jk'}S_\lambda(j,k)S_\lambda(j',k'), \tag{2.74}$$

hence its multiplier is given by

$$\mu_{S_\lambda}((j,k),(j',k')) = e^{i2\pi \frac{\lambda}{N} jk'}, \quad (j,k),(j',k') \in \mathbb{Z}_N \times \mathbb{Z}_N. \tag{2.75}$$

Observe that the latter is similar to the (formal) symplectic multiplier, which is given by

$$\mu_\lambda((j,k),(j',k')) := e^{i\pi \frac{\lambda}{N}(jk'-kj')} \tag{2.76}$$

Indeed, it is sufficient to consider the Borel map $\beta : \mathbb{Z}_N \times \mathbb{Z}_N \ni (j,k) \mapsto e^{-i\pi \frac{\lambda}{N} jk} \in \mathbb{T}$ and relation 1.45 is trivially satisfied:

$$\frac{e^{-i\pi \frac{\lambda}{N}[jk+jk'+j'k+j'k']}}{e^{-i\pi \frac{\lambda}{N} jk} e^{-i\pi \frac{\lambda}{N} j'k'}} e^{i2\pi \frac{\lambda}{N} jk'} = e^{i\pi \frac{\lambda}{N}(jk'-kj')}.$$

Therefore, by proposition 1.4.4, there exists a projective representation projectively equivalent to $S_\lambda$ (namely, the projections defined via the canonical projection epimorphism coincide). In particular, we notice that

$$D_\lambda(j,k) = e^{-i\pi \frac{\lambda}{N}[j]\cdot[k]} \, {}^\lambda M_k \, {}^\lambda T_j, \quad j,k \in \mathbb{Z}_N \tag{2.77}$$

is the aforementioned representation (as already discussed in the previous section, we will drop the square bracket and identify the equivalence classes with their representatives). Indeed we have that

$$(D_\lambda(j+j',k+k')\psi)(n) = e^{-i\pi \frac{\lambda}{N}(jk+jk'+j'k+j'k')} e^{i2\pi \frac{\lambda}{N}(k+k')n} \psi(n-j-j'), \tag{2.78}$$

$$(D_\lambda(j,k)D_\lambda(j',k')\psi)(n) = e^{-i\pi \frac{\lambda}{N} jk} e^{-i\pi \frac{\lambda}{N} j'k'} e^{i2\pi \frac{\lambda}{N} kn} e^{i2\pi \frac{\lambda}{N} k'(n-j)} \psi(n-j-j') \tag{2.79}$$

implies

$$D_\lambda(j+j',k+k') = e^{i\pi \frac{\lambda}{N}(jk'-kj')} D_\lambda(j,k) D_\lambda(j',k'). \tag{2.80}$$

Also observe that this fact can be understood by looking at the commutation relations of the representations 2.73, 2.77 [17], for which we have:

$$S_\lambda(j,k)S_\lambda(j',k') = e^{i2\pi \frac{\lambda}{N}(jk'-kj')} S_\lambda(j',k') S_\lambda(j,k),$$
$$D_\lambda(j,k)D_\lambda(j',k') = e^{i2\pi \frac{\lambda}{N}(jk'-kj')} D_\lambda(j',k') D_\lambda(j,k).$$

Because the (formal) symplectic structure is more evident in its multiplier, we will mostly consider the representation $D_\lambda$ instead of $S_\lambda$ hereinafter. Furthermore, when we deal with the discrete Wigner function in chapter 4, we will see that the choice of $D_\lambda$ is more suitable for some physical applications. However, until then, choosing $D_\lambda$ instead of $S_\lambda$ will not cause any relevant difference in light of Stone-Von Neumann's theorem too, which we are going to review in the discrete case soon.

We are now ready to define discrete Weyl systems. Infact, since ${}^\lambda M_k$ and ${}^\lambda T_j$ are jointly irreducible representations on $L^2(\mathbb{Z}_N)$ [17] and satisfy the CCRs, we can give the following

**Definition 2.4.6.** If $\lambda$ is such that $\gcd(\lambda, N) = 1$, then a *discrete (or finite) $\lambda$-Weyl system* is a pair of jointly irreducible representations

$$\mathbb{Z}_N \ni j \mapsto {}^\lambda T_j \in \mathcal{U}(L^2(\mathbb{Z}_N)), \quad \mathbb{Z}_N \ni k \mapsto {}^\lambda M_k \in \mathcal{U}(L^2(\mathbb{Z}_N)) \tag{2.81}$$

which satisfy the relations

$$[M_k, M_{k'}] = [T_j, T_{j'}] = 0, \quad {}^\lambda M_k \, {}^\lambda T_j = e^{i2\pi \frac{\lambda}{N} jk} \, {}^\lambda T_j \, {}^\lambda M_k. \tag{2.82}$$

Equivalently, a finite $\lambda$-Weyl system is an irreducible $N$-dimensional projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$.

Stone-von Neumann's theorem holds true in the discrete case too, and it can be formulated in the following way [17]:

**Theorem 2.4.7** (Stone-von Neumann)**.** *Let us consider a $\lambda$-Weyl system $D_\lambda$ for $\mathbb{Z}_N \times \mathbb{Z}_N$ defined as in 2.73, where $\lambda$ is such that* $\gcd(\lambda, N) = 1$*. If $\rho$ is another projective irreducible representation of $\mathbb{Z}_N \times \mathbb{Z}_N$, then $\rho$ is unitarily equivalent to $D_\lambda$.*
*Equivalently, if $U$ and $V$ are jointly irreducible representations on $L^2(\mathbb{Z}_N)$ which satisfy the CCRs, then there exists a unitary operator $W$ - unique up to a phase factor - such that $W^*UW = {}^\lambda T_j$ and $W^*VW = {}^\lambda M_k$.*

Lastly, let us consider the case $1 < \gcd(\lambda, N) < N$, which corresponds to drop the hypothesis of joint irreducibility of the translation and modulation operators. If $\Gamma$ is a denumerable index set, then $\tilde{D}_\lambda = \bigoplus_{l \in \Gamma} D^l_\lambda$ is a reducible representation on a complex separable Hilbert space $\mathcal{H}$, where each $D^l_\lambda(j,k) = e^{-i\frac{\pi}{\lambda} jk} \, {}^\lambda M^l_k \, {}^\lambda T_j{}^l$ acts on a different $L^2(\mathbb{Z}_{d_\lambda})$, $d_\lambda \equiv \frac{N}{\gcd(\lambda, N)}$. Observe that, since each $L^2(\mathbb{Z}_{d_\lambda})$ is disjoint from the others, we have the following slightly generalized form of the CCRs:

$${}^\lambda M^l_k \, {}^\lambda T^m_j = e^{i2\pi \frac{\lambda}{N} \delta_{l,m} jk} \; {}^\lambda T^m_j \, {}^\lambda M^l_k. \tag{2.83}$$

In such a case, by Stone-von Neumann's theorem, if $U$ and $V$ are unitary representations which satisfy the above CCRs, there exists a unique unitary operator $W$ such that

$$WUW^* = \bigoplus_{l\in\Gamma}{}^{\lambda} T_j^l, \quad WVW^* = \bigoplus_{l\in\Gamma}{}^{\lambda} M_k^l. \tag{2.84}$$

# Chapter 3

# Harmonic analysis on Abelian groups

This chapter aims at introducing some fundamental tools of harmonic analysis on l.c.s.c. Abelian groups (and, in particular, on phase space), an important topic in representation theory (the theory still works if we drop the second countability, but proofs become trickier [18, 40]). Since we will mostly deal with Abelian groups, we recall that such groups are unimodular, thus the left and right Haar measures will coincide [18]. The chapter is structured as follows. We will briefly recall the notion of group algebra and function of positive type at first. In this way, we will finally be able to introduce rigorously the group $\hat{G}$ of (equivalence classes of) irreducible unitary representations of the Abelian group $G$, called the characters of $G$. Next, we will define the Fourier transform and the Fourier-Stieltjes transform and we will study the most important theorems, such as Fourier's inversion formulas and Plancherel's theorem.
In the second part of this chapter we will briefly deal with harmonic analysis on phase space, introducing some tools which will play a relevant role in the discussion of the Wigner function. We also remark that this is a topic of great interest in signal analysis too, since it corresponds to the time-frequency analysis [22]. In particular, we will firstly define the symplectic Fourier transform on phase space. Next we will introduce the twisted group algebra, an "alternative" Banach $*$-algebra, which can be defined for every group which admits non-exact multiplier and it is connected with the notion of $\star$-product of functions. Then we will define the Gabor transform in the continuous case of $\mathbb{R}^n$ and in the finite case of $\mathbb{Z}_N$.
As a last topic, we recall the notions of square integrable representation of a l.c.s.c. group (dropping the Abelian hypothesis) and wavelet transform, two fundamental tools in harmonic analysis on phase space and, in general,

in the discussion of the Weyl-Wigner correspondence.

## 3.1 Some basic facts

In this section we introduce some preliminary notions before we discuss Fourier analysis. At first, we will not assume the group to be Abelian, since the concepts we will recall are quite general.
In particular, in the first part of this section we will recall some facts concerning the Banach space of complex Radon measures [19], then we introduce an important subspace, i.e. the group algebra $L^1(G, \lambda, \mathbb{C})$ ($\lambda$ is a left Haar measure on $G$). In the second part we will briefly recall the link between functions of positive type and cyclic representations.

### 3.1.1 The group algebra

Let us denote with $\mathcal{M}(G) \equiv \mathcal{M}(G, \mathbb{C})$ the space of complex Radon measures on a l.c.s.c. group $G$. Recall that, by Riesz's theorem [19], if

$$F_{\mu,\nu}(\phi) := \int d\mu(g) \int d\nu(h)\phi(gh), \quad \mu, \nu \in \mathcal{M}(G),\ \phi \in C_0(G, \mathbb{C}), \quad (3.1)$$

is a bounded linear functional, we can find a unique complex Radon measure $\mu * \nu$ such that

$$\int d(\mu * \nu)(g)\phi(g) := \int d\mu(g) \int d\nu(h)\phi(gh) = F_{\mu,\nu}(\phi), \qquad (3.2)$$

which will be called the *convolution measure of* $\mu$ *and* $\nu$. This operation is associative and it is commutative if and only if $G$ is an Abelian group [18]. Moreover, the *Dirac measure* defined as

$$\delta_g(\mathcal{E}) := \begin{cases} 1, & g \in \mathcal{E}, \\ 0, & g \notin \mathcal{E}, \end{cases} \qquad (3.3)$$

where $\mathcal{E}$ is a Borel set of $G$, is the identity element with respect to the measure convolution.
Lastly, we can endow $\mathcal{M}(G)$ with an involution defined as

$$\int d\mu^*(\mathcal{E}) := \overline{\mu(\mathcal{E}^{-1})}. \qquad (3.4)$$

Hence we have that $\mathcal{M}(G)$ is a unital $*$-algebra with respect to the convolution 3.2 and the involution 3.4 [19].

We now recall that $L^1(G)$ can be embedded in $\mathcal{M}(G)$ via the following map:

$$L^1(G, \lambda, \mathbb{C}) \ni f \mapsto f(g)dg \in \mathcal{M}(G), \tag{3.5}$$

where $dg \equiv d\lambda(g)$, $g \in G$ is a left Haar measure on $G$. Hence the following fact holds:

**Proposition 3.1.1.** *The space $L^1(G, \lambda, \mathbb{C})$ of $\mathbb{C}$-valued integrable functions on $G$ is a Banach $*$-algebra when equipped with the convolution*

$$(f_1 * f_2)(h) := \int_G d\lambda(g) f_1(h) f_2(h^{-1}g), \quad f_1, f_2 \in L^1(G, \lambda, \mathbb{C}) \tag{3.6}$$

*and the involution $\mathcal{I}$ which maps $f \in L^1(G, \lambda, \mathbb{C})$ in*

$$f^*(g) := \Delta(g^{-1})\overline{f(g^{-1})} \tag{3.7}$$

*($\Delta$ denotes the modular function on $G$).*

**Definition 3.1.2.** The Banach $*$-algebra $(L^1(G, \lambda, \mathbb{C}), *, \mathcal{I})$ is called the *group algebra* of $G$.

We notice that the convolution 3.6 does not admit a unit element unless $G$ is a discrete or compact group [20]. Moreover, observe that the convolution between functions coincides with the convolution measure (the correspondence is given by 3.5) [18].
We also remark that $f \in L^1(G, \lambda, \mathbb{C})$ is positive if it can be written in the form $f' * f'^*$ for some $f' \in L^1(G, \lambda, \mathbb{C})$ [18].

Lastly, we observe that there is a correspondence between the unitary representation of $G$ and the representations of $L^1(G)$ by virtue of the following fact [18]:

**Proposition 3.1.3.** *If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, we have that*

$$\tilde{\pi}(f) := \int_G d\lambda(g)\ f(g)\pi(g), \quad f \in L^1(G), \tag{3.8}$$

*is a $*$-representation of the group algebra into the $*$-algebra $\mathcal{B}(\mathcal{H})$, i.e. it is an algebra homomorphism and $\tilde{\pi}(f^*) = \tilde{\pi}(f)^*$.*
*Conversely, if $\tilde{\pi}$ is a non-degenerate $*$-representation of $L^1(G)$, it arises from a unique unitary representation of $G$ as in 3.8.*

We remark that the operator $\tilde{\pi}$ is defined in the weak sense; namely if $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}$, then 3.8 is to be intended as

$$\langle \phi, \tilde{\pi}(f)\psi \rangle = \int_G d\lambda(g) f(g) \langle \phi, \pi(g)\psi \rangle.$$

**Example 3.1.4.** If $\pi_L$ is the left regular representation, we have that the convolution is the corresponding $*$-representation, indeed

$$((\tilde{\pi}_L(f_1))f_2)(g) = \int_G d\lambda(y) f_1(h) f_2(h^{-1}g) = (f_1 * f_2)(g), \quad f_1, f_2 \in L^1(G).$$

We remark that 3.8 is essentially the prototype of the Weyl quantization map. Indeed, if we consider a group which admits square integrable representations (see section 3.5), then 3.8 is well-defined even for $L^2(G)$. In particular, in the latter case, the image of the $*$-representation $\tilde{\pi}$ will be a Hilbert-Schmidt operator [7], while in general it is only a bounded linear operator [18].

### 3.1.2 Functions of positive type

Suppose $G$ is a l.c.s.c. group and $d\lambda(g) \equiv dg$ is a left Haar measure on $G$. Then $\lambda$ is $\sigma$-finite, since $\lambda$ is regular on compacts. Hence we have that $L^\infty(G) = L^1(G)^*$[1] [19].

**Definition 3.1.5.** A *function of positive type* is a positive linear functional $\chi$ on $L^1(G, \lambda, \mathbb{C})$, i.e. it is a function $\chi \in L^\infty(G)$ such that

$$\int d\lambda(g)\ \chi(g)(f^* * f)(g) \geq 0 \quad \forall f \in L^1(G). \tag{3.9}$$

**Proposition 3.1.6.** *If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, and if $\psi \in \mathcal{H}$, then the map $c_\phi : G \ni g \mapsto \langle \phi, \pi(g)\phi \rangle$ is a function of positive type.*

*Proof.* Observe that $c_\phi$ is a continuous map. Then, if $f \in L^1(G)$, we have

$$\begin{aligned}\int d\lambda(g)d\lambda(h)\ f(g)\overline{f(h)}c_\phi(h^{-1}g) &= \int d\lambda(g)d\lambda(h)\ \langle f(h)\pi(h)\phi, f(g)\pi(g)\phi \rangle \\ &= \|\pi(f)\phi\|^2 \geq 0,\end{aligned}$$

where we have used the definition 3.8 and the unitarity of $\pi$. □

These functions are important for our aims since it is possible to associate a cyclic representation to a function of positive type in the following way [18, 40]:

**Theorem 3.1.7.** *If $\chi$ is a function of positive type, and if $\pi_\chi : G \to \mathcal{U}(\mathcal{H}_\chi)$ is a unitary representation, there exists a cyclic vector $\psi$ of $\pi_\chi$ such that $\chi(g) = \langle \psi, \pi_\chi(g)\psi \rangle$ locally a.e..*

[1] Recall that $L^\infty(G, \mu, \mathbb{C}) \equiv L^\infty(G)$ is a Banach space with the norm $\|f\|_\infty := \inf\{\alpha \in \mathbb{R} \mid \mu(\{g \in G \mid |f(g)| < \alpha\}) = 0\}$.

This fact leads us to two interesting properties of functions of positive type, which we summarize in the following [18]

**Proposition 3.1.8.** *Every function of positive type agrees locally a.e. with a (bounded) continuous function in* $L^\infty(G)$.
*Moreover, if* $\chi$ *is a function of positive type, then we have*

$$\|\chi\|_\infty = \chi(e), \quad \chi(g^{-1}) = \overline{\chi(g)}. \tag{3.10}$$

Therefore, we can always consider our functions of positive type to be continuous. In particular, we will denote with $P(G)$ the set of all continuous functions of positive type on $G$.
A last remark on the link between functions of positive type and cyclic representations; with the same notations of theorem 3.1.7, the representation $\pi_\chi$ is canonical in the following sense [18, 40]:

**Theorem 3.1.9.** *If* $\pi : G \to \mathcal{U}(\mathcal{H})$ *is a ciclyc representation with cyclic vector* $\psi$ *such that* $\chi(g) = \langle\psi, \pi(g)\psi\rangle$, *then* $\pi$ *and* $\pi_\chi$ *are unitarily equivalent.*

Observe now that, thanks to proposition 3.1.8, we can consider the set of normalized functions of positive type

$$P_1(G) := \{\chi \in P(G) \mid \|\chi\|_\infty = 1\} \, (= \{\chi \in P(G) \mid \chi(e) = 1\}), \tag{3.11}$$

which is a convex set, as well as $P(G)$ [18]. Thus we can consider the set of its extremal points. The following result holds [18, 40]:

**Theorem 3.1.10.** *If* $\chi \in P_1(G)$ *and* $\pi_\chi : G \to \mathcal{U}(\mathcal{H}_\chi)$ *is the canonical cyclic representation associated with* $\chi$, *then* $\chi$ *is an extremal point in* $P_1(G)$ *if and only if* $\pi_\chi$ *is irreducible.*

Lastly, recall that $L^1(G) \subset L^\infty(G)^*$ via the embedding

$$L^1(G) \ni f \mapsto F_f(\phi) := \phi(f) = \int_G d\lambda(g)\phi(g)f(g) \in L^\infty(G)^* \tag{3.12}$$

for each $\phi \in L^\infty(G)$. Hence we can define the *weak* topology* on $L^\infty(G)$. This is the initial topology given by the maps

$$F_f : L^\infty(G) \to \mathbb{C}, \quad f \in L^1(G). \tag{3.13}$$

Thus we can induce this topology on $P(G)$ and $P_1(G)$ [18]. Observe that in the case of $P_1(G)$ the weak* topology is equivalent to the compact convergence topology, whose neighbourhood basis is given by the open sets of the form $\mathcal{N}_{\xi_0}(\epsilon, K) := \{\xi \in \hat{G} \mid |\xi(g) - \xi_0(g)| < \epsilon \ \forall x \in K\}$ where $K$ is a compact subset of $\hat{G}$ [18].

## 3.2 The dual group $\hat{G}$

Let $G$ be a l.c.s.c. Abelian group. By Schur's lemma 1.2.7, if $\pi : G \to \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation, then $\mathcal{H} = \mathbb{C}$, thus $\pi(g)z = \xi(g)z,\ \forall g \in G,\ \forall z \in \mathbb{C},\ \xi(g) \in \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.

**Definition 3.2.1.** With the above notations, the continuous homomorphism $\xi : G \to \mathbb{T}$ is called the *(unitary) character* of the group $G$.

The set of all characters of $G$ will be denoted with $\hat{G}$. We notice that this is an Abelian group with respect to the pointwise product, because the Torus group is Abelian. We can also observe that, if $\xi \in \hat{G}$, $\xi$ is a function of positive type normalized at 1, since we can write $\xi(g) = \langle 1, \pi(g)1\rangle$ and of course $\xi(e) = 1$. Thus we have $\hat{G} \subset P_1(G)$; in particular, since each character is an irreducible representation of $G$, $\hat{G}$ coincides with the set of extremal points of $P_1(G)$. Therefore we can endow $\hat{G}$ with the compact convergence topology, with respect to it is a l.c.s.c. Abelian group [18].

**Definition 3.2.2.** The group $\hat{G}$ of all characters of $G$, endowed with the compact convergence topology, is called the *(unitary) dual group* of $G$.

We can now observe that, if $G$ is a compact group with a normalized Haar measure $\lambda$ ($\lambda(G) = 1$), then $\hat{G}$ is an orthonormal set in $L^2(G)$.
Indeed, if $\xi \in \hat{G}$, then $|\xi|^2 = 1$, hence $\|\xi\|_2 = \lambda(G)^{1/2} = 1$, where $\lambda$ is the Haar measure on $G$ such that $\lambda(G) = 1$. Now let us consider $\xi_1, \xi_2 \in \hat{G},\ \xi_1 \neq \xi_2$. Then there exists $g_0 \in G$ such that $\langle g_0, \xi_1^{-1}\xi_2\rangle \neq 1$. Therefore we have

$$\begin{aligned}\int_G d\lambda(g)\ \overline{\xi_1(g)}\xi_2(g) &= \int d\lambda(g)\ \langle g, \xi_1^{-1}\xi_2\rangle \\ &= \langle g_0, \xi_1^{-1}\xi_2\rangle \int d\lambda(g)\ \langle g_0^{-1}g, \xi_1^{-1}\xi_2\rangle.\end{aligned}$$

If we now apply the substitution $g \mapsto g_0 g$ we have

$$\int_G d\lambda(g)\ \overline{\xi_1(g)}\xi_2(g) = \langle g_0, \xi_1^{-1}\xi_2\rangle \int d\lambda(g)\ \langle g, \xi_1^{-1}\xi_2\rangle,$$

for $\langle \xi_1, \xi_2\rangle_{L^2} \neq 0$. Hence, if $\xi_1 \neq \xi_2$, then $\langle \xi_1, \xi_2\rangle_{L^2} = 0$.

We will say the group $G$ is *discrete* if it is endowed with the discrete topology. Then the following fact holds [18]:

**Proposition 3.2.3.** *Let $G$ be a l.c.s.c. Abelian group. If $G$ is discrete, $\hat{G}$ is compact.*
*If $G$ is compact, $\hat{G}$ is discrete.*

In the following examples, we list some dual groups which we have already encountered in chapter 2.

**Example 3.2.4.** $\hat{\mathbb{R}} \cong \mathbb{R}$ with $\langle x, \xi \rangle = e^{i2\pi x\xi}$ [18]. Indeed, let us consider $f \in \hat{\mathbb{R}}$. Hence, there exists $\alpha > 0$ such that

$$A = \int_0^\alpha dx\ f(x) \neq 0.$$

Moreover, since $f$ is a continuous homomorphism and $f(0) = 1$, we have that the expression

$$Af(t) = \int_0^\alpha dx\ f(x+t) = \int_t^{\alpha+t} dx\ f(x)$$

is differentiable, thus $f$ is differentiable and we have

$$f'(x) = \frac{f(x+t) - f(x)}{A}.$$

Therefore, setting $a = \frac{f(t)-1}{A}$, we have that $f(x) = e^{at}$. Moreover, since $|f| = 1$, we must have $a = 2\pi i\xi$ for some $\xi \in \mathbb{R}$ and so the assertion is proven.

**Example 3.2.5.** Let us consider the torus group, i.e. the group of elements $z \in \mathbb{C}$ such that $|z| = 1$, and the set of integer numbers $\mathbb{Z}$, which is a group with respect to the addition.
Hence, $\hat{\mathbb{T}} \cong \mathbb{Z}$ with the pairing $\langle \alpha, n \rangle = \alpha^n,\ \alpha = e^{i2\pi x} \in \mathbb{T},\ n \in \mathbb{Z}$. Indeed, everything works the same as in the case $G = \mathbb{R}$, but the character $f$ must satisfy the condition $f(x + 2\pi) = f(x)$ too, since $\mathbb{T}$ can be identified with $\mathbb{R}/\mathbb{Z}$ [18, 40].
Conversely, we have that $\hat{\mathbb{Z}} \cong \mathbb{T}$ via the pairing $\langle n, \alpha \rangle = \alpha^n$, because, given $\phi \in \hat{\mathbb{Z}}$, we have $\alpha = \phi(1) \in \mathbb{T}$ and $\phi(n) = \phi(1)^n = \alpha^n$.

**Example 3.2.6.** Let us consider the group $\mathbb{Z}/N\mathbb{Z} \equiv \mathbb{Z}_N$, i.e. the group of equivalence classes of numbers which are congruent modulo $N$ with respect to the addition $[a + b] := [a] + [b]$.
Hence, $\hat{\mathbb{Z}}_N \cong \mathbb{Z}_N$ via the pairing $\langle m, n \rangle = e^{i\frac{2\pi}{N}mn}$, because the characters of $\mathbb{Z}_N$ are the characters of $\mathbb{Z}$ which are trivial on $N\mathbb{Z}$ [18, 40], therefore they are of the form $\phi(n) = \alpha^n$, where $\alpha$ is the $N$th root of 1.

Lastly, if $G_1, \dots, G_n$ are l.c.s.c. Abelian groups, we can consider the dual group of $G_1 \times \cdots \times G_n$, which is isomorphic to $\hat{G}_1 \times \cdots \times \hat{G}_n$. Indeed, $\xi = (\xi_1, \dots, \xi_n) \in \hat{G}_1 \times, \dots, \times \hat{G}_n$ is a character of $G_1 \times \cdots \times G_n$ via the natural pairing

$$\langle (g_1, \dots, g_n), (\xi_1, \dots, \xi_n) \rangle = \langle g_1, \xi_1 \rangle \dots \langle g_n, \xi_n \rangle \tag{3.14}$$

and each character $\tilde{\xi}$ of $G_1 \times \cdots \times G_n$ is of this form, since each $\xi_i$ can be defined as

$$\langle g_i, \xi_i \rangle = \langle (e, \dots, e, g_i, e, \dots, e), \tilde{\xi} \rangle.$$

Therefore, we have that every finite Abelian group is self-dual, i.e. $\hat{G} \cong G$. Indeed, since by the structure theorem for finite Abelian group

$$G = \mathbb{Z}_{{p_1}^{n_1}} \times \cdots \times \mathbb{Z}_{{p_k}^{n_k}},$$

and since $\mathbb{Z}_n$ is self-dual for each $n \in \mathbb{N}$, the assertion follow from 3.14. Moreover, we will also have that $\hat{\mathbb{R}}^n \cong \mathbb{R}^n$, $\hat{\mathbb{T}}^n \cong \mathbb{Z}^n$ and $\hat{\mathbb{Z}}^n \cong \mathbb{T}^n$.

## 3.3 The Fourier transform on l.c.s.c. Abelian groups

**Definition 3.3.1.** Let $G$ be a l.c.s.c. Abelian group and let $dg \equiv d\lambda(g)$ be its Haar measure. If $f$ is a function in $L^1(G)$ the *Fourier transform* of $f$ is the map $\mathcal{F} : L^1(G) \to C_0(\hat{G})$ such that

$$(\mathcal{F}f)(\xi) \equiv \hat{f}(\xi) := \int_G dg\ \overline{\langle g, \xi \rangle} f(g), \tag{3.15}$$

Observe that $\hat{f}$ is a continuous function. Moreover, $\mathcal{F}$ is a norm-decreasing linear map and the Riemann-Lebesgue lemma holds, i.e. $\operatorname{Ran}\mathcal{F}$ is a dense subspace of $C_0(\hat{G})$ [18]. We can also observe that $\mathcal{F}$ is a $*$-homomorphism, since $C_0(\hat{G})$ is a $*$-algebra, where the convolution is the pointwise product and the involution is $f^*(g) = \overline{f(g^{-1})}$. Indeed, if $f, f_1, f_2 \in L^1(G)$, we have $\mathcal{F}(f_1 * f_2) = (\mathcal{F}f_1)(\mathcal{F}f_2)$ and $\mathcal{F}(f^*) = \overline{\mathcal{F}f}$, since

$$\begin{aligned} \int_G dg\ \overline{\langle g, \xi \rangle} (f_1 * f_2)(g) &= \int_G dg\ \overline{\langle g, \xi \rangle} \int_G dh f_1(h) f_2(h^{-1}g) \\ &= \int_G dgdh\ \overline{\langle hg, \xi \rangle} f_1(h) f_2(g) = \mathcal{F}(f_1)\mathcal{F}(f_2), \end{aligned}$$

$$\int_G dg\ \overline{\langle g,\xi\rangle} f^*(g) = \int_G dg\ \overline{\langle g,\xi\rangle f(g^{-1})} = \int_G dg\ \overline{\langle g^{-1},\xi\rangle f(g)}$$
$$= \int_G dg\ \langle g,\xi\rangle \overline{f(g)} = \overline{\mathcal{F}f}.$$

It is useful to extend the Fourier transform on $L^1(G)$ to $\mathcal{M}(G)$:

**Definition 3.3.2.** The *Fourier-Stieltjes* transform of a Radon measure $\mu$ is the map $\mathcal{F} : \mathcal{M}(G) \to \mathrm{BC}(G)$, where $\mathrm{BC}(G)$ is the set of continuous bounded function on $G$, such that

$$\hat{\mu}(\xi) := \int_G d\mu(g)\ \overline{\langle g,\xi\rangle}. \tag{3.16}$$

It is still true that $\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}$, where $\mu, \nu \in \mathcal{M}(G)$, since

$$\widehat{\mu * \nu}(\xi) = \int d\mu(g) d\nu(h)\ \overline{\langle gh,\xi\rangle} = \int d\mu(g) d\nu(h)\ \overline{\langle g,\xi\rangle\langle h,\xi\rangle} = \hat{\mu}(\xi)\hat{\nu}(\xi).$$

We can also define a Fourier transform on $\hat{G}$ instead of $G$ (this is possible since we can identify $\hat{\hat{G}}$ with $G$, as we will see in theorem 3.3.12): this will be a map $\mathcal{M}(\hat{G}) \to \mathrm{BC}(\hat{G})$ such that

$$\phi_\mu(g) \equiv \check{\mu}(g) := \int_{\hat{G}} d\mu(\xi)\ \langle g,\xi\rangle \in \mathrm{BC}(\hat{G}). \tag{3.17}$$

Observe that Bochner's theorem holds [18]:

**Theorem 3.3.3.** *$\mu \in \mathcal{M}^+(\hat{G}) \iff \check{\mu} \equiv \phi_\mu \in P(G)$.*

We remark that, if $\mu$ is a probability measure and $\chi \in P_1(G)$, then $\check{\mu} = \chi$ for some $\mu \in \mathcal{M}^+(\hat{G})$ such that $\mu(\hat{G}) = 1$. Hence, functions of positive type play an important role in physics, since they are nothing but the Fourier-Stieltjes transform of probability measures, which are the classical states in classical statistical mechanics (we will return on this fact in section 4.4).
Let us now consider $\mathcal{B}(G) := \{\check{\mu} \in \mathrm{BC}(G) \mid \mu \in \mathcal{M}^+(\hat{G})\}$ (we remark that $\mathcal{B}(G)$ contains all the functions of the form $f_1 * f_2$, with $f_1, f_2 \in C_c(G)$ [18]) and define

$$\mathcal{B}^p(G) := \mathcal{B}(G) \cap L^p(G), \quad p < \infty, \tag{3.18}$$

which is a dense set in $L^p(G)$ [18]. Hence the following Fourier's inversion formula holds [18]:

**Theorem 3.3.4.** *If $f \in \mathcal{B}^1(G)$, then $\hat{f} \in L^1(\hat{G})$. Moreover, if the Haar measures $dg \in \mathcal{M}(G)$, $d\mu(\xi) \in \mathcal{M}(\hat{G})$ are suitably normalized, we have that*

$$d\mu(\xi) = \hat{f}(\xi)d\xi \iff \check{\mu} = f. \tag{3.19}$$

*Hence*

$$f(g) = \int_{\hat{G}} d\mu(\xi)\ \langle g, \xi\rangle = \int_{\hat{G}} d\xi\ \langle g, \xi\rangle \int_G dh\overline{\langle h, \xi\rangle} f(h) \quad \forall f \in \mathcal{B}^1(G). \tag{3.20}$$

If we now fix $dg$ to be an Haar measure on $G$, there will exist a unique normalization of the measure $d\xi$ on $\hat{G}$ such that 3.20 holds. In such a case, we will say that the Haar measures $(dg, d\xi)$ are *conjugate*.
We also point out the following interesting result [18, 40]:

**Proposition 3.3.5.** *If $G$ is a compact group and its Haar measure is normalized in such a way that $|G| = 1$, then the dual measure on $\hat{G}$ is the counting measure. Conversely, if $G$ is discrete and the Haar measure is the counting measure, then the dual measure on $\hat{G}$ is normalized to the unit.*

**Example 3.3.6.** The Lebesgue measure on $\mathbb{R}$ is self-dual [18] if $\hat{\mathbb{R}} \cong \mathbb{R}$ with the pairing $\langle x, \xi\rangle = e^{2\pi i x\xi}$; in such a case we have

$$\hat{f}(\xi) = \int dx\ e^{-2\pi i x\xi} f(x), \quad f(x) = \int d\xi\ e^{2\pi i x\xi}\hat{f}(\xi),$$

then we have

$$f(x) = \int_{\mathbb{R}} d\xi\ e^{2\pi i x\xi} \int_{\mathbb{R}} dx'\ e^{-2\pi i x'\xi} f(x').$$

Observe that, if we identify $\hat{R}$ with $\mathbb{R}$ via the pairing $\langle x, \xi\rangle = e^{ix\xi}$, the dual measure of $dx$ will be $(2\pi)^{-1}d\xi$, hence

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int dx\ e^{-ix\xi} f(x), \quad f(x) = \frac{1}{\sqrt{2\pi}} \int d\xi\ e^{ix\xi}\hat{f}(\xi).$$

**Example 3.3.7.** Let us consider $G \equiv \mathbb{T}$, hence $\hat{T} \cong \mathbb{Z}$. If $\lambda = (2\pi)^{-1}d\theta$ is the normalized Haar measure on $\mathbb{T}$, then the dual measure on $\mathbb{Z}$ is the counting measure. Therefore, by Fourier's first inversion theorem 3.3.4, we have

$$\hat{f}(k) = \int_0^{2\pi} f(\theta)e^{-i\theta k}\frac{d\theta}{2\pi}, \tag{3.21}$$

$$f(\theta) = \sum_{k=-\infty}^{+\infty} \hat{f}(k)e^{i\theta k}.$$

**Example 3.3.8.** We now consider $G \equiv \mathbb{Z}_N$, thus $\hat{\mathbb{Z}_N} \cong \mathbb{Z}_N$. Since $\mathbb{Z}_N$ is a finite compact group the dual of the counting measure is still a counting measure up to a rescaling factor that preserves the normalization. Hence we will have

$$\hat{f}(k) = \sum_{k=0}^{N} f(j) e^{-2\pi i jk/N}, \tag{3.22}$$
$$f(j) = \frac{1}{N} \sum_{k=0}^{N} \hat{f}(k) e^{2\pi i jk/N}.$$

Of course, the normalizing factor can be placed in the Fourier transform, it is a matter of choice (as well as the choice $1/\sqrt{N}$ both for the transform and the anti-transform). In such a case, the first Fourier's inversion formula is given by

$$f(j) = \frac{1}{N} \sum_{k \in \mathbb{Z}_N} e^{\frac{2\pi i}{N} jk} \sum_{l \in \mathbb{Z}_N} e^{-\frac{2\pi i}{N} lk} f(l).$$

### 3.3.1 Other useful theorems

Let us consider $f \in L^2(G)$. Then $f * f^* \in P(G)$, since

$$\langle f, \pi_L(g) f \rangle = \int dh \, \overline{f(h)} f(g^{-1}h) = \int dh \, \overline{f(h) f^*(h^{-1}g)} = \overline{f * f^*}.$$

Thus, if $f \in L^1(G) \cap L^2(G)$, we have that $f * f^* \in L^1(G) \cap P(G) \subset \mathcal{B}^1(G)$. Moreover, since the Fourier transform on $L^1(G)$ is a $*$-homomorphism, we have that $\widehat{f * f^*} = |\hat{f}|^2$. Now, if $f \in L^1(G) \cap L^2(G)$ and if $dg$ and $d\xi$ are conjugate measures, we have

$$\int_G dg |f(g)|^2 = \int_G dg \, f(g) \overline{f(e^{-1}g)} = (f * f^*)(e) =$$
$$= \int_{\hat{G}} d\xi \, \langle e, \xi \rangle \widehat{f * f^*}(\xi) = \int_{\hat{G}} d\xi \, \widehat{f * f^*}(\xi) = \int_{\hat{G}} d\xi \, |\hat{f}(\xi)|^2.$$

Therefore

$$\|f\|_{L^2(G)} \equiv \int_G dg \, |f(g)|^2 = \int_{\hat{G}} d\xi \, |\hat{f}(\xi)|^2 \equiv \|\hat{f}\|_{L^2(\hat{G})}. \tag{3.23}$$

Hence we have proven that $f \mapsto \hat{f}$ is an isometry in the $L^2$ norm. The latter extends uniquely to an isometry from $L^2(G)$ to $L^2(\hat{G})$, which is also surjective, hence it is a unitary map [18]. Thus, we have that the Plancherel's theorem holds:

**Theorem 3.3.9** (Plancherel). *If $f \in L^1(G) \cap L^2(G)$, then $\mathcal{F}(f) \equiv \hat{f} \in L^2(\hat{G})$ and this is true $\forall f \in L^1(G) \cap L^2(G)$ if $dx$ and $d\xi$ are conjugate Haar measures. Moreover, the map*

$$L^1(G) \cap L^2(G) \ni f \mapsto \hat{f} \in L^2(\hat{G}) \tag{3.24}$$

*extends uniquely to a unitary operator $\mathcal{F} : L^2(G) \to L^2(\hat{G})$, which is called the Fourier-Plancherel transform.*

**Corollary 3.3.10.** *If $G$ is a compact Abelian group and $\lambda$ is its Haar measure such that $\lambda(G) = 1$, then $\hat{G}$ is an orthonormal basis in $L^2(G)$*

*Proof.* We have seen in the previous section that $\hat{G}$ is an orthonormal set in $L^2(G)$. Observe that, if $f \in L^2(G) \perp \xi \ \forall \xi \in \hat{G}$, we have that

$$0 = \int d\lambda(g)\ \overline{\xi(g)} f(g) = \int d\lambda(g)\ \overline{\langle g, \xi \rangle} f(g) = \hat{f}(\xi) \quad \forall \xi \in \hat{G},$$

hence, by Plancherel's theorem, we have $f \equiv 0$ and $\hat{G}$ is an orthonormal basis in $L^2(G)$. □

As a last topic of this section, we consider the dual group of $\hat{G}$, namely $\hat{\hat{G}}$.

**Theorem 3.3.11** (Gelfand-Raikov). *Let $G$ be a locally compact group. The irreducible representations of $G$ separate points, i.e. if $g, h \in G, g \neq h$, hence there exists an irreducible unitary representation $\pi$ such that $\pi(g) \neq \pi(h)$.*

Suppose now $g \in G$ and let us consider the map $\langle g, \cdot \rangle : \hat{G} \ni \xi \mapsto \langle g, \xi \rangle \equiv \xi(g) \in \mathbb{T}$ and observe that $\langle g, \xi_1 \xi_2 \rangle = \langle g, \xi_1 \rangle \langle g, \xi_2 \rangle$. Hence we have that $g$ is a character of $\hat{G}$. If we now consider the map $\Phi : G \ni g \mapsto \langle g, \cdot \rangle \in \hat{\hat{G}}$, we have that

$$\Phi(g_1 g_2) = \langle g_1 g_2, \cdot \rangle = \langle g_1, \cdot \rangle \langle g_2, \cdot \rangle = \Phi(g_1)\Phi(g_2) \quad \forall g_1, g_2 \in G,$$

hence $\Phi$ is a group homomorphism from $G$ to $\hat{\hat{G}}$ and it is injective due to the Gelfand-Raikov's theorem 3.3.11. It is also possible to prove that $\Phi$ is a continuous surjection , hence we have the following duality theorem, due to Pontrjagin [18]:

**Theorem 3.3.12** (Pontrjagin). *The map $\Phi : G \ni g \mapsto \langle g, \cdot \rangle \in \hat{\hat{G}}$ is an isomorphism of topological groups.*

Therefore we will identify $G$ with $\hat{\hat{G}}$ from now on and we will not pay attention to the pairings $\langle g, \xi\rangle$ and $\langle \xi, g\rangle$.

We have various interesting corollaries and here we present two of them. The first is a second inversion formula:

**Theorem 3.3.13.** *If $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$, then $f(g) = \hat{\hat{f}}(g^{-1})$ a.e., i.e.*

$$f(g) = \int_{\hat{G}} d\xi\ \langle g, \xi\rangle \hat{f}(\xi) = \int_{\hat{G}} d\xi\ \langle g, \xi\rangle \int_G dh\ \overline{\langle h, \xi\rangle} f(h). \tag{3.25}$$

*If $f$ is also a continuous function, the equivalence is true pointwise.*

*Proof.* Observe that, by Bochner's theorem 3.3.3, for each $f \in L^1(G)$ we have that $\hat{f} \in \mathcal{B}(\hat{G})$, because

$$\hat{f}(\xi) = \int_G dg\ \overline{\langle g, \xi\rangle} f(g) = \int_G dg\ \langle g^{-1}, \xi\rangle f(g) = \int_G dg\ \langle g, \xi\rangle f(g^{-1}).$$

Hence we have that $\breve{\mu} = \hat{f} = \widehat{f((\cdot)^{-1})}$, where $d\mu(g) = f(g^{-1})dg$ and $f \in L^1(G)$, then $\hat{f} \in \mathcal{B}(\hat{G})$. Suppose now $\hat{f} \in L^1(\hat{G})$ (and $\hat{f} \in \mathcal{B}^1(G)$). Then, by the first Fourier's inversion theorem 3.3.4, we have that $\breve{\mu} = \hat{f}$ implies $f(g^{-1})dg = d\mu(g) = \hat{\hat{f}}(g)dg$, hence $f(g^{-1}) = \hat{\hat{f}}(g)$ a.e..
If $f$ is a continuous function, $\hat{\hat{f}}$ is also continuous, therefore $f(g^{-1}) = \hat{\hat{f}}(g)\ \forall g \in G$. □

The second one is the so called Fourier uniqueness theorem [18]:

**Theorem 3.3.14.** *If $\mu, \nu$ are two complex Radon measure such that $\hat{\mu} = \hat{\nu}$, then $\mu = \nu$. In particuar, if $f_1, f_2 \in L^1(G)$ are such that $\hat{f}_1 = \hat{f}_2$, then $f_1 = f_2$.*

## 3.4 Harmonic analysis on phase space

Harmonic analysis on phase space is at foundation of the phase space description of quantum mechanics. In this section we will introduce the most important facts regarding this topic, which are related with the symplectic structure of the phase space, and will be used in the next chapter to define the Wigner function.
In particular, after we have defined the symplectic Fourier transform, we will briefly recall the notion of twisted group algebra, which is strictly related with the $\star$-product of functions [7]. Then, we will introduce the Gabor transform, which is at foundation of time-frequency analysis and it is an interesting tool due to its link with wavelets and coherent states [2].

### 3.4.1 The symplectic Fourier transform

The *symplectic Fourier transform*, as the name suggests, is strictly intertwined with the symplectic structure of the phase space (the link will be explicit in examples 3.4.2 and 3.4.3). Recall that the symplectic structure arises from the underlying group structure, as we have seen in chapter 2.

Let $G$ be a l.c.s.c. Abelian group. If we consider the (abstract) phase space $G\times\hat{G}$, then, by Pontrjagin's duality theorem 3.3.12, $\widehat{G\times\hat{G}} = \hat{G}\times G \cong G\times\hat{G}$. Hence we can give the following definition:

**Definition 3.4.1.** The *symplectic Fourier transform* is the unitary operator $\mathcal{F}_{\mathrm{Sp}} : L^2(G\times\hat{G}) \to L^2(G\times\hat{G})$ such that

$$(\mathcal{F}_{\mathrm{Sp}}f)(t,\omega) := \int_{G\times\hat{G}} dx d\xi \; \overline{\langle x,\omega\rangle}\langle t,\xi\rangle f(x,\xi), \quad f\in L^2(G\times\hat{G}). \tag{3.26}$$

The quantity

$$\langle (x,\xi),(t,\omega)\rangle = \overline{\langle x,\omega\rangle}\langle t,\xi\rangle = \overline{\langle (t,\omega),(x,\xi)\rangle} \tag{3.27}$$

is called the *symplectic character* of $G\times\hat{G}$.

We will also denote $\mathcal{F}_{\mathrm{Sp}}f$ as $\hat{f}$.
We remark that it is also possible to give an alternative definition of the symplectic Fourier transform, given by the complex conjugate of the symplectic character as $\langle x,\omega\rangle\overline{\langle t,\xi\rangle} = \overline{\langle (x,\xi),(t,\omega)\rangle}$. The alternative definition of $\mathcal{F}_{\mathrm{Sp}}$ is analogous to 3.26.
We observe that, due to Pontrjagin's duality theorem 3.3.12 and due to the second inversion formula 3.25, we have that $\mathcal{F}_{\mathrm{Sp}}^2 = \mathrm{Id}$. Indeed,

$$\begin{aligned}(\mathcal{F}_{\mathrm{Sp}}\hat{f})(x,\xi) &= \int_{G\times\hat{G}} dt d\omega \; \overline{\langle t,\xi\rangle}\langle x,\omega\rangle \hat{f}(t,\omega)\\ &= \int_{G\times\hat{G}} dt d\omega \; \overline{\langle t,\xi\rangle}\langle x,\omega\rangle \int_{G\times\hat{G}} dx' d\xi' \; \overline{\langle x',\omega\rangle}\langle t,\xi'\rangle f(x',\xi')\\ &= f(x,\xi),\end{aligned}$$

Therefore, since it is a unitary and self-inverse operator, the symplectic Fourier transform is a self-adjoint operator. We remark that the Fourier transform 3.15 does not enjoy this property, because it is such that $\mathcal{F}^4 = \mathrm{Id}$, while $\mathcal{F}^2$ is the parity operator [19].

**Example 3.4.2.** Let us consider the continuous phase space $\mathbb{R}^n \times \mathbb{R}^n$. Hence, since the characters on $\mathbb{R}^n$ are given by $\langle q, p\rangle = e^{i2\pi q\cdot p}$, the symplectic character will be

$$\overline{\langle q, p'\rangle}\langle p, q'\rangle = e^{-i2\pi(q\cdot p' - p\cdot q')} = e^{-i2\pi\omega((q,p),(q',p'))}, \tag{3.28}$$

where $\omega$ is the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$ and $q \cdot p = \sum_{i=1}^n q_i p_i$. Hence, the symplectic Fourier transform 3.26 is defined as

$$(\mathcal{F}_{\mathrm{Sp}} f)(q,p) := \int_{\mathbb{R}^n\times\mathbb{R}^n} dq'dp'\ e^{i2\pi\omega((q,p),(q',p'))} f(q',p'), \quad f \in L^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{3.29}$$

**Example 3.4.3.** For the finite phase space $\mathbb{Z}_N \times \mathbb{Z}_N$, since the characters on $\mathbb{Z}_N$ are of the form $\langle j, k\rangle = e^{i\frac{2\pi}{N}jk}$, the symplectic characters are defined as

$$\overline{\langle j, k'\rangle}\langle k, j'\rangle = e^{-i\frac{2\pi}{N}(jk'-kj')}, \tag{3.30}$$

and the discrete symplectic Fourier transform is given by

$$(\mathcal{F}_{\mathrm{Sp}} f)(j,k) = \frac{1}{N}\sum_{j',k'}^{N} e^{i\frac{2\pi}{N}(jk'-kj')} f(j',k'), \quad f \in L^2(\mathbb{Z}_N \times \mathbb{Z}_N). \tag{3.31}$$

### 3.4.2 The twisted group algebra

We can now see how it is possible to endow $L^1(G)$ with an algebraic structure alternative to the group algebra (namely, the twisted group algebra) in the case of groups which admit projective representations. In particular, we see that the twisted convolution plays an important role in the phase space description of quantum mechanics, since it is strictly related with the $\star$-product of functions [5, 7, 42, 52].

**Definition 3.4.4.** Let $G$ be a l.c.s.c. group and let $\mu : G \times G \to \mathbb{T}$ be a multiplier. If $\psi_1, \psi_2 \in (L^1(G), d\lambda(g) \equiv dg, \mathbb{C})$, then the function defined by

$$(\psi_1 \circledast_\mu \psi_2)(g) := \int_G dh\ \psi_1(h)\psi_2(h^{-1}g)\overline{\mu(h, h^{-1}g)} \tag{3.32}$$

is called the *twisted convolution* of $\psi_1$ and $\psi_2$.

We remark that the integral is well-defined since $\mu$ is a bounded, weakly Borel function [16].

**Proposition 3.4.5.** *$(L^1(G), \circledast_\mu)$ is a Banach algebra.*

*Proof.* for $\psi_1, \psi_2, \psi_3 \in L^1(G)$ we have

$$((\psi_1 \circledast_\mu \psi_2) \circledast_\mu \psi_3)(g) = \int dhdk\ \psi_1(k)\psi_2(k^{-1}h)\psi_3(h^{-1}g) \cdot \\ \cdot \overline{\mu(h, h^{-1}g)\mu(k, k^{-1}h)}$$

and

$$(\psi_1 \circledast_\mu (\psi_2 \circledast_\mu \psi_3))(g) = \int dhdk\ \psi_1(k)\psi_2(h)\psi_3(h^{-1}k^{-1}g) \cdot \\ \cdot \overline{\mu(h, h^{-1}k^{-1}g)\mu(k, k^{-1}g)}.$$

Hence, by the substitution $h \mapsto kh$ in the first expression, we have

$$((\psi_1 \circledast_\mu \psi_2) \circledast_\mu \psi_3)(g) = \int dhdk\ \psi_1(k)\psi_2(h)\psi_3(h^{-1}k^{-1}g) \cdot \\ \cdot \overline{\mu(kh, h^{-1}k^{-1}g)\mu(k, h)}.$$

Then, since $\mu$ is a multiplier, we have

$$\mu(kh, h^{-1}k^{-1}g)\mu(k, h) = \mu(k, k^{-1}g)\mu(h, h^{-1}k^{-1}g)$$

and the equivalence is proven.
Moreover,

$$\left| \int dgdh\ \psi_1(h)\psi_2(h^{-1}g)\overline{\mu(h, h^{-1}g)} \right| \leq \int dgdh\ |\psi_1(h)\psi_2(h^{-1}g)\overline{\mu(h, h^{-1}g)}| \\ = \int dgdh\ |\psi_1(h)\psi_2(g)| = \|\psi_1\|_{L^1}\|\psi_2\|_{L^1}.$$

□

We can also define the map $I_\mu : L^1(G) \ni \psi \mapsto \psi^{\circledast_\mu} \in L^1(G)$, where

$$\psi^{\circledast_\mu}(g) := \Delta(g^{-1})\overline{\mu(g, g^{-1})\psi(g^{-1})}, \tag{3.33}$$

which is an involution, because we have [7, 16]

$$\|\psi^{\circledast_\mu}\|_1 = \|\psi\|, \quad (\psi^{\circledast_\mu})^{\circledast_\mu} = \psi, \quad \forall \psi \in L^1(G). \tag{3.34}$$

Moreover [16],

$$(\psi_1 \circledast_\mu \psi_2)^{\circledast_\mu} = (\psi_2)^{\circledast_\mu} \circledast_\mu (\psi_1)^{\circledast_\mu},$$

hence we can give the following definition:

**Definition 3.4.6.** The Banach $*$-algebra $(L^1(G), \circledast_\mu, I_\mu)$ is called the *twisted group algebra* of $G$.

Observe that in the case of exact multipliers, the twisted convolution reduces to the ordinary one, as well as the twisted involution, hence the twisted group algebra coincides with the group algebra of $G$.
We also note incidentally that the twisted group algebra is a unital algebra if and only if $G$ is a discrete group [16]. Moreover, it is a commutative algebra if and only if $\mu \sim 1$ and $G$ is an Abelian group.
Hence, we can observe that twisted convolution is a good candidate to "mimic" the product of operators on the space of functions on the phase space. In particular, it turns out that twisted convolution is involved in the Weyl-Wigner scheme, which we will discuss in the next chapter, because it arises when we consider the $\star$-product induced by a square integrable projective representation of the phase space [7]. In such a case, if $\psi, \phi \in L^2(G \times \hat{G})$, then $\psi \circledast_\mu \phi \in L^2(G \times \hat{G})$ [5, 7, 20, 22]. Moreover, $L^2(G \times \hat{G})$ becomes a non-commutative $*$-algebra when equipped with the twisted convolution and the twisted involution [5, 7].

**Example 3.4.7.** Let us consider the continuous phase space $\mathbb{R}^n \times \mathbb{R}^n$ and an irreducible projective representation with multiplier $\mu((q,p),(q',p')) = e^{i\pi(q\cdot p' - p\cdot q')}$ $(h = 1)$. The twisted convolution 3.32 is given by

$$(f \circledast_\mu g)(q,p) := \int_{\mathbb{R}^n \times \mathbb{R}^n} dq'dp' \; f(q',p')g(q-q',p-p')e^{i\pi(q\cdot p' - p\cdot q')}, \quad (3.35)$$

where $f, g \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. We again remark that it is the "standard" $\star$-product of functions, which is well-defined in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ too.

**Example 3.4.8.** We now consider the discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$, and a finite Weyl system with symplectic multiplier $\mu\left((j,k),(j',k')\right) \equiv \mu_1\left((j,k),(j',k')\right) = e^{i\frac{\pi}{N}(jk' - kj')}$ $(\lambda = 1)$. Then,

$$(f \circledast_\mu g)(j,k) = \sum_{j',k' \in \mathbb{Z}_N} f(j',k')g(j-j',k-k')e^{i\frac{\pi}{N}(jk'-kj')}, \qquad (3.36)$$

where $f, g \in L^1(\mathbb{Z}_N \times \mathbb{Z}_N)(\cong \mathbb{C}^{N\times N} \cong \mathcal{M}_N(\mathbb{C}))$. We also observe that, by construction, the twisted group algebra on $\mathbb{Z}_N \times \mathbb{Z}_N$ coincides with the matrix algebra $\mathcal{M}_N(\mathbb{C})$, where the latter is a Banach $*$-algebra with respect to the trace norm $\|A\|_{HS}^2 := \mathrm{tr}(AA^*)$ (in the finite case all the operator norms are equivalent [39]) and the involution is given by the adjoint map [17].
Finally, by compactness of $\mathbb{Z}_N \times \mathbb{Z}_N$, we also have that the irreducible

projective representations of $\mathbb{Z}_N \times \mathbb{Z}_N$ $D_\lambda(j,k)$ defined in 2.77 form an orthonormal basis for the twisted group algebra $L^1(\mathbb{Z}_N \times \mathbb{Z}_N)$ [17].

A last remark on twisted group algebra: it turns out that a projective representation of a l.c.s.c. group G gives rise to a $*$-representation of its twisted group algebra, in the same way as we have seen for unitary representations and $*$-representations of the group algebra in section 3.1.1 [16, 17].

### 3.4.3 Gabor analysis on finite Abelian groups

In the context of signal analysis it turns out that the Fourier transform is not the best tool. Indeed, let us consider for example the signal $t \mapsto f(t)$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We are often interested in both the time and frequency information of such signal, but we cannot "select" the spectral information for a fixed frequency $\omega$, since, by $\mathcal{F}$, it is related to all times. In other terms, in order to evaluate $\hat{f}(\omega)$, we need to know $f(t)$ $\forall t \in \mathbb{R}$.
Moreover, we also find out that the uncertainty principle holds, hence the concept of "istantaneous frequency" is also troublesome [22].
To accomplish this task it is necessary to introduce time-frequency analysis and the Gabor transform, which is, roughly speaking, the Fourier transform of $f$ restricted to an interval by a smooth cut-off $g$, often called window [13, 22]. We remark that the time-frequency plane is nothing but the phase space, since it can also be identified with the direct product of an Abelian group for its unitary dual [17, 22].
Hence, we now define the Gabor transform on discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$; we will briefly discuss the case $\mathbb{R}^n \times \mathbb{R}^n$ too, bearing in mind that, in such a case, things can be trickier because of convergence issues (in particular, some facts require the notion of square integrable representations, see section 3.5). For convenience, in the following we will set the label of the projective representations of the phase space (both discrete and continuous) to 1.

**Definition 3.4.9.** Let $D(j,k) \equiv D_1(j,k) = e^{-i\frac{\pi}{N}jk} M_k T_j$ be the projective irreducible representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ defined in 2.77. The *short-time Fourier transform or Gabor transform* of $f \in L^2(\mathbb{Z}_N)$ with respect to a fixed window $0 \neq w \in L^2(\mathbb{Z}_N)$ is the map $\mathcal{G}_w : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ such that

$$\mathcal{G}_w f(j,k) := \langle D(j,k)w, f\rangle_{L^2(\mathbb{Z}_N)} = \sum_{l \in \mathbb{Z}_N} e^{i\frac{\pi}{N}jk} e^{-i\frac{2\pi}{N}kl} \overline{w(l-j)} f(l). \tag{3.37}$$

We notice incidentally that the discrete Gabor transform can be expressed as a convolution product as follows:

$$\begin{aligned} e^{-i\frac{\pi}{N}jk}\left(f * M_k w^*\right)(j) &= e^{-i\frac{\pi}{N}jk}\sum_{l\in Zn} f(l)\left(M_k w^*\right)(j-l) = \\ &= e^{-i\frac{\pi}{N}jk}\sum_{l\in Zn} f(l) e^{i\frac{2\pi}{N}k(j-l)}\overline{w(l-j)} = \\ &= \sum_{l\in\mathbb{Z}_N} e^{i\frac{\pi}{N}jk}e^{-i\frac{2\pi}{N}kl} f(l)\overline{w(l-j)} = \mathcal{G}_w f(j,k). \end{aligned}$$

We also observe that Schur's orthogonality relations 1.19 hold, namely:

$$\langle \mathcal{G}_{w_1} f_1, \mathcal{G}_{w_2} f_2\rangle_{L^2(\mathbb{Z}_N\times\mathbb{Z}_N)} = \langle f_1, f_2\rangle_{L^2(\mathbb{Z}_N)}\overline{\langle w_1, w_2\rangle}_{L^2(\mathbb{Z}_N)}. \tag{3.38}$$

In the context of analysis on phase space, the latter identity is often called the *Moyal's identity*.

In the case of $G = \mathbb{R}$ things are not much different, but proofs become trickier since we have to deal with the domain of the Gabor operator (see e.g. [22]). For future reference, the Gabor transform with respect to the window $w$ is usually defined as

$$\mathcal{G}_w f(q,p) := \int_{-\infty}^{+\infty} dx\ f(x)\overline{w(x-q)}e^{-i2\pi x\cdot p} \tag{3.39}$$

(the generalization to $\mathbb{R}^n$ is straightforward). Obviously, if we consider the projective representation $S(q,p)$ of $\mathbb{R}\times\mathbb{R}$ given in 2.24, we have that $\mathcal{G}_w f(q,p) = \langle S(q,p)w, f\rangle$ (up to a phase factor). We can also give another simple, but interesting, equivalent expression for 3.39. Indeed, by the substitution $x \mapsto x + \frac{q}{2}$, we will have

$$\mathcal{G}_w f(q,p) := e^{-i\pi qp}\int_{-\infty}^{+\infty} dt\ f\left(x+\frac{q}{2}\right)\overline{w\left(x-\frac{q}{2}\right)}e^{-2\pi i x p}. \tag{3.40}$$

This expression is also known as the *Fourier Wigner transform* [20] (or, in time-frequency analysis, as the cross-ambiguity function [22]) and will return (as its discrete counterpart) in the definition of the Wigner map on the continuous (respectively, the discrete) phase space. Indeed, we will see that the Wigner map is given by the symplectic Fourier transform of the Fourier Wigner transform.

We also remark that the Moyal's identity 3.38 holds in the continuous case as well [22, 20] (see next section).

## 3.5 Square integrable representations and wavelets

In the discussion of harmonic analysis on phase space we have understood that the notion of square integrable representation is an invaluable tool in the phase space description of quantum mechanics. Moreover, in the next chapter, we will see that the Weyl-Wigner correspondence solely relies on the existence of square integrable representations. Thus, as a last topic of this chapter, we will give some basic results regarding the latter concept and we will introduce the wavelet transform. We will also sketch how the (standard) wavelet transform, which is an alternative tool to time-frequency analysis [13, 22], arises from group theoretical considerations.

Let $G$ be a l.c.s.c. group with left Haar measure $d\lambda(g)$.

**Definition 3.5.1.** Let $U : G \to \mathcal{U}(\mathcal{H})$ be a unitary (projective) representation and let $c_{\psi,\phi} : G \ni g \mapsto \langle U(g)\psi, \phi\rangle \in \mathbb{C}$ be its matrix coefficients. The set

$$\mathcal{A}(U) := \{\psi \in \mathcal{H} \mid \exists\phi \in \mathcal{H},\ \phi \neq 0 : c_{\psi,\phi} \in L^2(G, \lambda, \mathbb{C})\} \tag{3.41}$$

is called the *set of admissible vectors* for the representation $U$. In other words, $\psi \in \mathcal{H}$ is an admissible vector if

$$\int_G d\lambda(g)\ |\langle U(g)\psi, \psi\rangle|^2 < \infty. \tag{3.42}$$

If $\mathcal{A}(U) \neq \{0\}$, the representation $U$ is said to be *square integrable* (or, equivalently, $U$ is *in the discrete series*).

Observe that $\mathcal{A}(U)$ is always nonempty, since $0 \in \mathcal{A}(U)$. The elements $\phi \in \mathcal{A}(U)$ such that $\|\phi\| = 1$ are often called *wavelets*. We notice that everything works the same if we consider right invariant measures. Indeed, recall that if $d\lambda(g)$ is a left Haar measure, then $d\lambda(g^{-1})$ is a right Haar measure, [18]. Hence, from the admissibility condition 3.42, we have

$$\begin{aligned}\int_G d\lambda(g)\ |\langle U(g)\phi, \phi\rangle|^2 &= \int_G d\lambda(g)\ \left|\langle \phi, U(g^{-1})\phi\rangle\right|^2 \\ &= \int_G d\lambda(g^{-1})\ |\langle \phi, U(g)\phi\rangle|^2.\end{aligned}$$

Thus we can restrict our attention to left Haar measures only.
Moreover, $\mathcal{A}(U)$ is also a stable under the action of $U(g)$. Indeed, if $\phi \in$

$\mathcal{A}(U)$, then

$$\begin{aligned}\int_G d\lambda(g')\ |\langle U(g')U(g)\phi, U(g)\phi\rangle|^2 &= \int_G d\lambda(g')\ \left|\langle U(g^{-1}g'g)\phi,\phi\rangle\right|^2 \\ &= \int_G d\lambda(g')\ |\langle U(g'g)\phi,\phi\rangle|^2 \\ &= \frac{1}{\Delta(g)}\int_G d\lambda(g')\ |\langle U(g')\phi,\phi\rangle|^2,\end{aligned}$$

where we have used the left invariance of $\lambda$, $\Delta$ is the modular function on $G$, and $d\lambda(g^{-1}) = \Delta(g^{-1})d\lambda(g)$ [18]. It is also remarkable that $\mathcal{A}(U)$ is a dense linear span in $\mathcal{H}$ [2].
The next theorem, due to Duflo and Moore [14] (see also [6]), will provide us some orthogonality relations that are analogous to Schur's orthogonality relations 1.19 which holds for compact groups:

**Theorem 3.5.2** (Duflo-Moore)**.** *Let $U : G \to \mathcal{U}(\mathcal{H})$ be a unitary (projective) representation. Then for any pair of vectors $\phi \in \mathcal{H}$ and $\psi \in \mathcal{A}(U)$, the coefficient $c_{\phi,\psi}$ is an element of $L^2(G,\lambda,\mathbb{C})$. Moreover, there exists a unique positive selfadjoint, injective linear operator $D_U$ such that its domain corresponds with $\mathcal{A}(U)$ and the following orthogonality relations hold:*

$$\langle c_{\psi_1,\phi_1}, c_{\psi_2,\phi_2}\rangle_{L^2} = \langle\phi_1,\phi_2\rangle\langle D_U\psi_2, D_U\psi_1\rangle \quad \forall\phi_1,\phi_2\in\mathcal{H},\ \forall\psi_1,\psi_2\in\mathcal{A}(U). \tag{3.43}$$

*Lastly, $D_U$ is a bounded operator if and only if $G$ is a unimodular group; in such a case we also have that $D_U = d_U\mathrm{Id}$, where $d_U$ is a positive constant.*

Notice that the Duflo-Moore operato $D_U$ is linked to the normalization of the Haar measure. In particular, recall that a Haar measure on a locally compact group $G$ is unique up to positive factors [18]. Hence, if we rescale the Haar measure by a positive constant $k > 0$, $D_U$ will be rescaled correspondly by the square root of $k$ [2, 7].

As a consequence of the Duflo-Moore's theorem, we have that, for $0 \neq \psi \in \mathcal{A}(U)$ and $\phi_1,\phi_2 \in \mathcal{H}$,

$$\langle\phi_1,\phi_2\rangle = \|D_U\psi\|^2\int_G d\lambda(g)\ \langle\phi_1, U(g)\psi\rangle\langle U(g)\psi,\phi_2\rangle \tag{3.44}$$

or analogously, in the Dirac notation, the following *resolution of the identity* holds:

$$\mathrm{Id} = \|D_U\psi\|^2\int_G d\lambda(g)\,|U(G)\psi\rangle\langle U(g)\psi|\,. \tag{3.45}$$

Moreover, theorem 3.5.2 allows us to introduce the following concept:

**Definition 3.5.3.** If $U : G \to \mathcal{U}(\mathcal{H})$ is a square integrable representation of $G$ and if $\psi \in \mathcal{A}(U)$, then the linear operator

$$\mathcal{W}_U^\psi : \mathcal{H} \ni \phi \mapsto \frac{1}{\|D_U\psi\|} c_{\psi,\phi} \in L^2(G, \lambda, \mathbb{C}) \tag{3.46}$$

is called the *(generalized) wavelet transform* generated by $U$ with *fiducial (or analyzing) vector* $\psi$.

Observe that 3.46 is an isometry. We can also easily compute the adjoint operator, which will be a map from $L^2(G, \lambda, \mathbb{C})$ to $\mathcal{H}$; if $f \in L^2(G)$ and $\phi \in \mathcal{H}$, we have

$$\begin{aligned}
\langle \phi, (\mathcal{W}_U^\psi)^* f\rangle_{\mathcal{H}} = \langle \mathcal{W}_U^\psi \phi, f\rangle_{L^2} &= \frac{1}{\|D_U\psi\|}\langle c_{\psi,\phi}, f\rangle_{L^2} \\
&= \frac{1}{\|D_U\psi\|}\int_G d\lambda(g)\ \overline{\langle U(g)\psi, \phi\rangle_{\mathcal{H}}} f(g) \\
&= \frac{1}{\|D_U\psi\|}\int_G d\lambda(g)\ \langle \phi, U(g)\psi\rangle_{\mathcal{H}} f(g).
\end{aligned}$$

Hence we have the following *reconstruction formula*:

$$(\mathcal{W}_U^\psi)^* f = \frac{1}{\|D_U\psi\|}\int_G d\lambda(g) f(g) U(g)\psi, \quad \forall f \in L^2(G, \lambda, \mathbb{C}). \tag{3.47}$$

We can now stress the strong analogy with some results that hold for compact groups. Firstly, observe that, since the irreducible representation of a compact group $G$ are finite dimensional, we have that they are also square integrable, since the Haar measure on $G$ is finite and every coefficient of such representations is a bounded function. Moreover, if $\lambda$ is normalized in such a way that $\lambda(G) = 1$, then, from theorem 3.5.2,

$$\sqrt{d_U}\int_G d\lambda(g)\ \langle \phi_1, U(g)\psi_1\rangle\langle U(g)\psi_2, \phi_2\rangle = \langle \phi_1, \phi_2\rangle\langle \psi_2, \psi_1\rangle,$$

where $\phi_1, \phi_2 \in \mathcal{H}$, $\psi_1, \psi_2 \in \mathcal{A}(U)$, and $d_U \equiv \dim \mathcal{H}$. Hence we have

$$D_U = \frac{1}{\sqrt{d_U}}\mathrm{Id}. \tag{3.48}$$

As a consequence, we have that the wavelet transform 3.46 and the isometry 1.20 (which we will refer as the prototype wavelet transform) coincide.

**Example 3.5.4.** We now define the standard wavelet transform which is often encountered in time-frequency analysis [13].
Recall that $\mathbb{R} \ltimes \mathbb{R}_+^* \equiv G$ is the *affine group* with composition law

$$(b, a)(\tilde{b}, \tilde{a}) := (b + a\tilde{b}, a\tilde{a}); \tag{3.49}$$

the inverse element is given by $(b, a)^{-1} = (-a^{-1}b, a^{-1})$ [2]. This group is not unimodular, since

$$d\lambda(b, a) = \frac{1}{a^2} dadb, \quad d\rho(b, a) = \frac{1}{a} dbda$$

are the left and right Haar measures [2], hence the Duflo-Moore operator is unbounded. The maps

$$U^{(+)} : G \to \mathcal{U}(L^2(R_+^*)) \mid (U^{(+)}(b, a)f)(\xi) := a^{1/2} e^{ib\xi} f(a\xi), \tag{3.50}$$

$$U^{(-)} : G \to \mathcal{U}(L^2(R_-^*)) \mid (U^{(-)}(b, a)f)(\xi) := a^{1/2} e^{ib\xi} f(a\xi), \tag{3.51}$$

are irreducible, inequivalent square integrable representations of $R \ltimes R_+^*$ [2]. The function $f \in L^2(\mathbb{R}_+^*)$ is an admissible element of $U^{(+)}$ if and only if $\mathbb{R}_+^* \ni \xi \mapsto |\xi|^{-1}|f(\xi)|^2 \in L^2(\mathbb{R}_+^*)$ [2]. The same holds for $\mathcal{A}(U^{(-)})$.
If we now consider the representation $U : G \to \mathcal{U}(L^2(\mathbb{R}))$ such that $U := U^{(-)} \oplus U^{(+)}$ and if $\mathcal{F}$ is the Fourier-Plancherel transform, we have that

$$(\tilde{U}(b, a)\psi)(x) := (\mathcal{F}U(b, a)\mathcal{F}^*\psi)(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x - b}{a}\right), \quad \psi \in L^2(\mathbb{R}). \tag{3.52}$$

The quantity

$$\psi_{b,a}(x) \equiv \frac{1}{\sqrt{a}} \psi\left(\frac{x - b}{a}\right) \tag{3.53}$$

is often called the *core of the wavelet transform.* The isometry $\mathcal{W}_\psi : L^2(\mathbb{R}) \to L^2(\mathbb{R} \ltimes R_+^*, \lambda = a^{-2}dbda)$ defined in such a way that

$$(\mathcal{W}_\psi \phi)(b, a) := \int_{\mathbb{R}} dx \overline{\psi_{b,a}(x)} \phi(x), \quad \phi \in L^2(\mathbb{R}) \tag{3.54}$$

is called the *(standard) wavelet transform with analyzing vector* $\psi$. In particular, $\psi$ is a *mother wavelet* for the representation $U$ if the following admissibility condition hold:

$$\mathbb{R}^{\pm} \ni \xi \mapsto |\xi|^{-1} \left|(\hat{\mathcal{F}}\psi)(\xi)\right|^2 \in L^1(\mathbb{R}_*^{\pm}),$$

$$2\pi \int_{\mathbb{R}_*^-} |\xi|^{-1} \left|(\hat{\mathcal{F}}\psi)(\xi)\right|^2 d\xi = 2\pi \int_{\mathbb{R}_*^+} |\xi|^{-1} \left|(\hat{\mathcal{F}}\psi)(\xi)\right|^2 d\xi = 1.$$

Lastly, we briefly return to the projective representations of $\mathbb{R}^n \times \mathbb{R}^n$. In particular, it turns out that the projective representations 2.24 are square integrable [2, 4, 6, 20], hence orthogonality relations 3.43 hold (we remark that this is true only for the projective representations of the continuous phase space, since the irreducible representations of $\mathbb{H}_n(\mathbb{R})$ are not square integrable [2, 6]; in such a case we say that the Schrödinger representation is square integrable *modulo* its center). Hence, the Moyal's identity 3.38 holds even for $G = \mathbb{R}^n \times \mathbb{R}^n$, since it is nothing but a restatement of the orthogonality relations for square integrable representations. Moreover, we also have that the Gabor transform is the (generalized) wavelet transform with respect to the Schödinger representation, since it is defined as the coefficient of the representation (as we have seen in section 3.4.3).
Lastly, we remark that the square-integrability of representations is the key property to define coherent states [2]. For instance, as it is well known, the (canonical) coherent states are generated by the displacement operators acting on the ground state of the Hamiltonian operator of the harmonic oscillator as [2, 41]

$$|\alpha\rangle = D(q,p)\,|0\rangle\,, \tag{3.55}$$

and satisfy the resolution of the identity 3.45, which is a consequence of Duflo-Moore's theorem. This scheme is suitable for the study of coherent states over groups different from the phase space $\mathbb{R}^n \times \mathbb{R}^n$ (see [2]); however, this is beyond our aims and we will not investigate this facts hereafter.

# Chapter 4

# Quantum mechanics on phase space

We are now ready to introduce quantum mechanics on phase space exploiting the tools developed in the previous chapter; here we will review the most important facts of quantum mechanics, paying more attention to quantum states, both in continuous and in finite cases.
The chapter is structured as follows. At first, we will describe the general quantization-dequantization (or Weyl-Wigner) scheme: given a square integrable projective representation of a l.c.s.c. group $G$, we will define an isometry - called the (generalized) Wigner transform or dequantization map - from the space of Hilbert-Schmidt operators on the space of the representation to the space of square integrable functions defined on $G$. Then, the Weyl transform (or quantization map) will simply be the adjoint of the latter. In such a scheme we have that the Wigner transform is in general only an isometry, thus its adjoint is a partial isometry, while in the original Weyl-Wigner scheme of $G = \mathbb{R}^n \times \mathbb{R}^n$ they are unitary operators [20, 38].
Since we are mostly interested in the continuous and the discrete phase space, we will focus our attention on the case of unimodular groups. Thereby, we can define the $\star$-product of functions by means of the Weyl-Wigner scheme, and we will be able to clarify the link with the twisted group algebra (in the general case of non-unimodular groups the $\star$-product is not equivalent to the twisted convolution, see [7] for further details).

Next, we will describe the Wigner function on $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{Z}_N \times \mathbb{Z}_N$. In particular, we will see that the standard Wigner function associated with the Hilbert-Schmidt operator $\rho$, which appeared for the first time in [50], is recovered if we consider the symplectic Fourier transform of the dequantization map; we will also review some interesting properties of the Wigner function.

The discrete Wigner function will be defined in analogy with the standard one as the discrete symplectic Fourier transform of the discrete Wigner transform. Furthermore, we will study some fundamental properties and we will compare it with the Wigner function defined by means of phase-point operators [2, 34], which, roughly speaking, can be regarded as the quantum counterpart of the phase space points. In particular, we will observe that the latter is not well defined for phase space of even order, unlike the discrete Wigner function defined by means of the Weyl-Wigner correspondence.
Eventually, thanks to Weyl-Wigner correspondence, we will be able to study quantum states on discrete phase space by means of functions of quantum positive type, a generalization of functions of positive type encountered in section 3.1.2. In this way, we will be able to discuss of some criteria which establish if a state is separable.

## 4.1 The Weyl-Wigner correspondence

In this section we will introduce the generalized Wigner transform (or dequantization map), the basic tool in the phase space approach to quantum mechanics. By means of a given projective representation of a l.c.s.c. group $G$, the dequantization map associates a square integrable function on $G$ to each Hilbert-Schmidt operator on the space of the representation.

Thus, let us briefly recall that a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is a *Hilbert-Schmidt* operator if $\operatorname{tr} T^*T < \infty$; we will denote with $\mathcal{B}_2(\mathcal{H})$ the set of such operators. We also need the following facts [39]:

- $\mathcal{B}_2(\mathcal{H})$ is a $*$-ideal in $\mathcal{B}(\mathcal{H})$, namely $\mathcal{B}_2(\mathcal{H})$ is such that if $T \in \mathcal{B}_2(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, then $TS$, $ST$ and $T^*$ are still Hilbert-Schmidt operators.

- $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space, where the inner product is defined as

$$\langle A, B\rangle := \operatorname{tr}(A^*B), \quad A, B \in \mathcal{B}_2(\mathcal{H}). \tag{4.1}$$

- The norm $\|T\|_2 := \sqrt{\operatorname{tr} T^*T}$, $T \in \mathcal{B}_2(\mathcal{H})$, is such that $\|T\| \leq \|T\|_2$. Moreover, the finite rank operators are dense in $\mathcal{B}_2(\mathcal{H})$ (with respect to $\|\cdot\|_2$).

Let us consider a l.c.s.c. group $G$ (which it is not necessary unimodular yet), a left Haar measure $d\lambda(g)$ on $G$ and a square integrable projective representation $U : G \to \mathcal{U}(\mathcal{H})$ with multiplier $\mu$. In order to give a proper definition of the Wigner map, let us consider the rank one operators in $\mathcal{H}$ of the type

$$|\psi\rangle\langle\phi|\,, \quad \psi \in \mathcal{H},\ \phi \in \mathrm{Dom}(D_U^{-1}),$$

where $D_U$ is the Duflo-Moore operator associated with the representation $U$ normalized according to the left Haar measure $d\lambda$ of the group. The linear span of such operators corresponds to the set of finite rank operators in $\mathcal{H}$ [39], hence it is dense in $\mathcal{B}_2(\mathcal{H})$. Therefore, a finite rank operator $F$ in $\mathcal{H}$ admits a canonical decomposition of the form $F = \sum_{k=1}^{N} |\psi_k\rangle\langle\phi_k|$, where $N$ is a natural number, $\{\phi_k\}_{k=1}^{N}$ and $\{\psi_k\}_{k=1}^{N}$ are linearly indipendent systems in $\mathcal{H}$ and each $\phi_k$ belongs to $\mathrm{Dom}(D_U^{-1})$.
Let us now consider the quantity

$$(\mathscr{D}_U\,|\psi\rangle\langle\phi|)(g) := \mathrm{tr}\big(U(g)^*\,|\psi\rangle\langle D_U^{-1}\phi|\big) = \langle U(g)D_U^{-1}\phi,\psi\rangle, \quad |\psi\rangle\langle\phi| \in \mathcal{B}_2(\mathcal{H}). \tag{4.2}$$

Then, if $|\psi_1\rangle\langle\phi_1|$ and $|\psi_2\rangle\langle\phi_2|$ are rank one operators, by the orthogonality relations 3.43, we have that

$$\begin{aligned}\int_G d\lambda(g)\overline{(\mathscr{D}_U\,|\psi_1\rangle\langle\phi_1|)\,(g)}\,(\mathscr{D}_U\,|\psi_2\rangle\langle\phi_2|)\,(g) &= \\ &= \int_G d\lambda(g)\langle\psi_1, U(g)D_U^{-1}\phi_1\rangle\langle U(g)D_U^{-1}\phi_2,\psi_2\rangle \\ &= \langle\psi_1,\psi_2\rangle\langle\phi_2,\phi_1\rangle = \langle|\psi_1\rangle\langle\phi_1|\,,|\psi_2\rangle\langle\phi_2|\rangle_{\mathcal{B}_2(\mathcal{H})}.\end{aligned}$$

Therefore, we can extend $\mathscr{D}_U$ to finite rank operators by linearity and to the whole $\mathcal{B}_2(\mathcal{H})$ by continuity, so that we are allowed to give the following

**Definition 4.1.1.** Let $G$ be a l.c.s.c. group, let $U : G \to \mathcal{U}(\mathcal{H})$ be a square integrable projective representation with multiplier $\mu$, and let $D_U$ be the associated Duflo-Moore operator.
Then, the map $\mathscr{D}_U : \mathcal{B}_2(\mathcal{H}) \to L^2(G)$ such that

$$(\mathscr{D}_U\rho)(g) := \mathrm{tr}\big(U(g)^*\rho D_U^{-1}\big), \quad \rho \in \mathcal{B}_2(\mathcal{H}), \tag{4.3}$$

is called the *generalized Wigner transform or the dequantization map* induced by the square integrable representation $U$.

If $G$ is a unimodular group, since $D_U = d_U\mathrm{Id}$, $d_U > 0$, the Wigner transform becomes

$$(\mathscr{D}_U\rho)\,(g) = \frac{1}{d_U}\,\mathrm{tr}(U(g)^*\rho), \quad \rho \in \mathcal{B}_2(\mathcal{H}). \tag{4.4}$$

We also remark that, due to orthogonality relations 3.43, the dequantization map is an isometry [7].

In the following, we will denote with $\mathcal{R}_U$ the range of the Wigner transform, which depends on the unitary equivalence class of $U$ only [7]. We also observe that, if $\mathcal{W}_U^\psi$ is the generalized wavelet transform with analyzing vector $\psi \in \mathcal{H}$ defined in 3.46, the following relation holds [7]:

$$\mathcal{R}_U = \overline{\operatorname{span}\{\psi \in \operatorname{Ran} \mathcal{W}_U^\psi \mid \psi \in \mathcal{A}(U),\ \psi \neq 0\}}. \tag{4.5}$$

**Proposition 4.1.2.** *If $U : G \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{H}')$ are two square integrable unitary representations, then the ranges $\mathcal{R}_U, \mathcal{R}_V$ are orthogonal.*

*Proof.* Let us consider the generalized wavelet transform $\mathcal{W}_U^\psi, \mathcal{W}_V^\eta$ with analyzing vectors $\psi \in \mathcal{H}$ and $\eta \in \mathcal{H}'$. Recall that such transforms intertwine the representation $U$ and $V$ with the left regular representation $\pi_L$. Thus, we have that ${\mathcal{W}_V^\eta}^* \mathcal{W}_U^\psi : \mathcal{H} \to \mathcal{H}'$ intertwines $U$ and $V$. However, $U$ and $V$ are inequivalent, hence, by Schur's lemma 1.2.7, ${\mathcal{W}_V^\eta}^* \mathcal{W}_U^\psi$ must be identically zero. Therefore,

$$0 = \langle {\mathcal{W}_V^\eta}^* \mathcal{W}_U^\psi \phi, \xi \rangle = \langle \mathcal{W}_U^\psi \phi, \mathcal{W}_V^\eta \xi \rangle \quad \forall \phi \in \mathcal{H},\ \forall \xi \in \mathcal{H}'$$

and thanks to 4.5 the proof is complete. □

**Corollary 4.1.3.** *If $G$ is a compact group, then $L^2(G) = \bigoplus_{U \in \hat{G}} \mathcal{R}_U$.*

*Proof.* Recall that, by Peter-Weyl's theorem 1.2.14,

$$L^2(G) = \bigoplus_{U \in \hat{G}} \bigoplus_{j=1}^{d_U} \operatorname{Ran}(\mathcal{W}_U^{\psi_j}),$$

where $d_U$ is the dimension of the representation. By the previous consideration, we also have that

$$\bigoplus_{j=1}^{d_U} \operatorname{Ran} \mathcal{W}_U^{\psi_j} = \operatorname{span}\{c_{\psi,\phi} \mid \psi, \phi \in \mathcal{H}\} = \mathcal{R}_U,$$

where $c_{\psi,\phi}$ are the coefficients of the representation 1.17, and the proof is complete. □

We can now study the intertwining properties of the Wigner transform; from now on, we will focus on the case of unimodular groups only in order to simplify some proofs, hence $D_U = d_U \mathrm{Id},\ d_U > 0$.
Let us consider the unitary representation

$$U \vee U : G \to \mathcal{U}(L^2(G)), \quad U \vee U(g) A := U(g) A U(g)^*, \quad \forall g \in G,\ A \in \mathcal{B}_2(\mathcal{H}), \tag{4.6}$$

which can be regarded as the symmetry action of the group $G$ onto quantum-mechanical operators.
We also consider the map

$$\mathcal{T}_\mu : G \to \mathcal{U}(L^2(G)) \mid (\mathcal{T}_\mu(g)f)(g') := \mu^{\leftrightarrows}(g,g')f(g^{-1}g'g), \qquad (4.7)$$

where $\mu^{\leftrightarrows}(g,g') := \overline{\mu(g,g^{-1}g')}\mu(g^{-1}g',g)$, which is a unitary representation too.

**Proposition 4.1.4.** *Let $G$ be a unimodular l.c.s.c. group and let $U$ be a projective square integrable representation of $G$ with multiplier $\mu$. Then,*

$$\mathscr{D}_U U \vee U(g) = \mathcal{T}_\mu(g)\mathscr{D}_U, \ \forall g \in G \qquad (4.8)$$

*(namely, the dequantization map intertwines $U \vee U$ and $\mathcal{T}_\mu$). As a consequence, $\mathcal{R}_U$ is an invariant subspace for $\mathcal{T}_\mu$ and $U \vee U$ is unitarily equivalent to the subrepresentation $\mathcal{T}_\mu|_{\mathcal{R}_U}$.*

*Proof.* Observe that

$$\begin{aligned}(\mathscr{D}_U U \vee U(g)A)(g') = (\mathscr{D}_U U(g)AU(g)^*)(g') &= \frac{1}{d_U}\operatorname{tr}(U(g')^*U(g)AU(g)^*) \\ &= \frac{1}{d_U}\operatorname{tr}(U(g)^*U(g')^*U(g)A).\end{aligned}$$

We notice now that, since $U$ is a projective representation, we have

$$U(g^{-1}g'g) = \mu(g^{-1},g'g)\mu(g',g)U(g^{-1})U(g')U(g).$$

Thus,

$$\begin{aligned}&(\mathcal{T}_\mu(g)\mathscr{D}_U A)(g') = \mathcal{T}_\mu(g)\frac{1}{d_U}\operatorname{tr}(U(g')^*A) = \\ &= \frac{1}{d_U}\overline{\mu(g,g^{-1}g')}\mu(g^{-1}g',g)\overline{\mu(g^{-1},g'g)\mu(g',g)}\operatorname{tr}\big((U(g^{-1})U(g')U(g))^*A\big) \\ &= \frac{1}{d_U}\overline{\mu(g,g^{-1}g')}\mu(g^{-1}g',g)\overline{\mu(g^{-1},g'g)\mu(g',g)}\operatorname{tr}\big(U(g)^*U(g')^*U(g^{-1})^*A\big)\end{aligned}$$

By the defining property of the multipliers, we have that

$$\begin{aligned}&\overline{\mu(g,g^{-1}g')}\mu(g^{-1}g',g)\overline{\mu(g^{-1},g'g)\mu(g',g)} = \\ &\qquad = \overline{\mu(g,g^{-1}g')}\mu(g^{-1}g',g)\overline{\mu(g^{-1}g',g)\mu(g^{-1},g')} \\ &\qquad = \overline{\mu(g,g^{-1}g')\mu(g^{-1},g')} \\ &\qquad = \overline{\mu(gg^{-1},g')\mu(g,g^{-1})} = \overline{\mu(g,g^{-1})}.\end{aligned}$$

Now, recalling that $\mathrm{Id} = U(gg^{-1}) = \overline{\mu(g,g^{-1})}U(g)U(g^{-1})$, we see that

$$(\mathcal{T}_\mu(g)\mathscr{D}_U A)(g') = \frac{1}{d_U}\,\mathrm{tr}\left(U(g)^* U(g')^* U(g) A\right).$$

□

Let us now consider the map $J_\mu : L^2(G) \to L^2(G)$ such that

$$(J_\mu f)(g) := \mu(g,g^{-1})\overline{f(g^{-1})}, \quad \forall f \in L^2(G) \tag{4.9}$$

(recall that we are considering unimodular groups only), which is a well-defined involution in $L^2(G)$. We also remark that the latter is a selfadjoint anti-unitary map.

**Proposition 4.1.5.** *Let $G$ be a l.c.s.c. unimodular group and let $U$ be a square integrable projective representation of $G$. If $\mathcal{J}$ is a standard complex conjugation in the Hilbert space $\mathcal{B}_2(\mathcal{H})$, namely $\mathcal{J}$ maps $A \in \mathcal{B}_2(\mathcal{H})$ to $A^* \in \mathcal{B}_2(\mathcal{H})$, then the dequantization map $\mathscr{D}_U$ induced by the representation $U$ intertwines the involution 4.9 with $\mathcal{J}$, i.e.*

$$\mathscr{D}_U \mathcal{J} = J_\mu \mathscr{D}_U. \tag{4.10}$$

*Proof.* Observe that

$$(\mathscr{D}_U \mathcal{J} A)(g) = (\mathscr{D}_U A^*)(g) = \frac{1}{d_U}\mathrm{tr}(U(g)^* A^*) = \frac{1}{d_U}\overline{\mathrm{tr}(U(g)A)}.$$

Moreover, we have

$$(J_\mu \mathscr{D}_U A)(g) = J_\mu \frac{1}{d_U}\mathrm{tr}(U(g)^* A) = \frac{1}{d_U}\mu(g,g^{-1})\overline{\mathrm{tr}(U(g^{-1})^* A)}.$$

Again, since $U(g^{-1})^* = \overline{\mu(g^{-1},g)}U(g)^*$, we have that

$$(J_\mu \mathscr{D}_U A)(g) = \frac{1}{d_U}\overline{\mathrm{tr}(U(g)A)}.$$

□

We can now introduce the adjoint map of the Wigner transform.

**Definition 4.1.6.** Let $U$ be a square integrable projective representation of a l.c.s.c. unimodular group $G$ and let $\mathscr{D}_U$ be the corresponding Wigner transform. The adjoint map

$$\mathscr{Q}_U \equiv \mathscr{D}_U{}^* : L^2(G) \to \mathcal{B}_2(\mathcal{H}) \tag{4.11}$$

such that

$$\mathscr{Q}_U \mathscr{D}_U = \mathrm{Id}, \quad \mathscr{D}_U \mathscr{Q}_U = P_{\mathcal{R}_U}, \tag{4.12}$$

where $P_{\mathcal{R}_U}$ is the orthogonal projection on the range of the Wigner map, is called the *Weyl transform (or quantization map).*

We remark that, since the Wigner transform is an isometry, the Weyl transform, which is the pseudo-inverse of $\mathscr{D}_U$, is a partial isometry such that $\mathcal{R}_U = \ker \mathscr{Q}_U$ [7, 39]. As we briefly mentioned at the end of section 3.1.1, we can observe that the following fact holds [7]:

**Proposition 4.1.7.** *If $G$ is a l.c.s.c. unimodular group, $U$ is a square integrable projective representation of $G$ and $f = \mathscr{D}_U |\psi\rangle\langle\phi|$, then the integral*

$$\frac{1}{d_U} \int_G d\lambda(g) f(g) U(g) \tag{4.13}$$

*converges weakly to $|\psi\rangle\langle\phi|$. Therefore, if $f \in \mathcal{R}_U(G)$, we have*

$$\mathscr{Q}_U f = \frac{1}{d_U} \int_G d\lambda(g) f(g) U(g). \tag{4.14}$$

*Proof.* If $\psi', \phi' \in \mathcal{H}$ and $L^2(G) \ni f = \mathscr{D}_U |\phi'\rangle\langle\psi'|$, $|\phi'\rangle\langle\psi'| \in \mathcal{B}_2(\mathcal{H})$, by orthogonality relations 3.43, we have that

$$\begin{aligned}
\frac{1}{d_U} \int_G d\lambda(g) \langle \phi, U(g)\psi \rangle f(g) &= \frac{1}{d_U^2} \int_G d\lambda(g) \langle \phi, U(g)\psi \rangle \operatorname{tr}(U(g)^* |\phi'\rangle\langle\psi'|) \\
&= \frac{1}{d_U^2} \int_G d\lambda(g) \langle \phi, U(g)\psi \rangle \langle U(g)\psi', \phi' \rangle \\
&= \langle \phi, \phi' \rangle \langle \psi', \psi \rangle.
\end{aligned}$$

Moreover, we have

$$\left| \int_G d\lambda(g) \frac{1}{d_U} \langle \phi, U(g)\psi \rangle f(g) \right| \le \|\psi\| \|\psi'\| \|\phi\| \|\phi'\|. \tag{4.15}$$

Therefore the integral

$$\frac{1}{d_U} \int_G d\lambda(g) U(g) f(g)$$

can be interpreted in the weak sense. □

### 4.1.1 The star-product

We can now introduce the $\star$-product of functions $f_1, f_2 \in L^2(G)$ induced by the Weyl-Wigner scheme, which, roughly speaking, will be given by the dequantization map of the product of the operators associated to $f_1$ and $f_2$ via the quantization map.
We will focus our attention on the case of unimodular groups only, because we are interested in the case of the Weyl-Wigner correspondence in the phase space. Besides, in the general case of non unimodular groups, it is much more difficult to give explicit formulas (see [7] section 5).

**Definition 4.1.8.** Let $G$ be a l.c.s.c. unimodular group and let $U : G \to \mathcal{U}(\mathcal{H})$ be a square integrable projective representation. If $\mathscr{D}_U$ ($\mathscr{Q}_U$) is the associated dequantization (quantization) map, then the bilinear map

$$(\cdot) \star_U (\cdot) : L^2(G) \times L^2(G) \ni (f_1, f_2) \mapsto \mathscr{D}_U\big( (\mathscr{Q}_U f_1)\, (\mathscr{Q}_U f_2) \big) \in L^2(G) \tag{4.16}$$

is the $\star$-*product* induced by $U$.

We can easily prove that the $\star$-product is associative. Indeed, let us consider $f_1, f_2, f_3 \in L^2(G)$ and recall that $\mathscr{Q}_U \mathscr{D}_U = \mathrm{Id}$. Hence we have

$$\begin{aligned}
(f_1 \star_U f_2) \star_U f_3 &= \mathscr{D}_U\Big( \big[\mathscr{Q}_U \mathscr{D}_U\big( (\mathscr{Q}_U f_1)\, (\mathscr{Q}_U f_2) \big)\big]\, (\mathscr{Q}_U f_3) \Big) \\
&= \mathscr{D}_U\big( (\mathscr{Q}_U f_1)\, (\mathscr{Q}_U f_2)\, (\mathscr{Q}_U f_3) \big) \\
&= \mathscr{D}_U\Big( (\mathscr{Q}_U f_1)\, \big[\mathscr{Q}_U \mathscr{D}_U\big( (\mathscr{Q}_U f_2)\, (\mathscr{Q}_U f_3) \big)\big] \Big) \\
&= \mathscr{D}_U\big( (\mathscr{Q}_U f_1)\, (\mathscr{Q}_U f_2)\, (\mathscr{Q}_U f_3) \big) = f_1 \star_U (f_2 \star_U f_2)\,.
\end{aligned}$$

Next, we can observe that

$$\|f_1 \star_U f_2\|_{L^2} \le \|f_1\|_{L^2} \|f_2\|_{L^2} \quad \forall f_1, f_2 \in L^2(G). \tag{4.17}$$

Indeed, since $\mathscr{D}_U$ is an isometry and $\mathscr{Q}_U$ is a partial isometry, we have

$$\begin{aligned}
\|\mathscr{D}_U\left((\mathscr{Q}_U f_1)(\mathscr{Q}_U f_2)\right)\|_{L^2} &= \|(\mathscr{Q}_U f_1)(\mathscr{Q}_U f_2)\|_{\mathcal{B}_2(\mathcal{H})} \\
&\le \|\mathscr{Q}_U f_1\|_{\mathcal{B}_2(\mathcal{H})} \|\mathscr{Q}_U f_2\|_{\mathcal{B}_2(\mathcal{H})} \le \|f_1\|_{L^2} \|f_2\|_{L^2}.
\end{aligned}$$

Let us now consider a multiplier $\mu$ of the representation $U$ and recall that the involution $J_\mu$ defined in 4.9 is such that $\mathscr{D}_U \mathcal{J} = J_\mu \mathscr{D}_U$, where $\mathcal{J}$ is the complex conjugation in $\mathcal{B}_2(\mathcal{H})$. As a consequence, we have that [7]

$$\mathcal{J} \mathscr{Q}_U = \mathscr{Q}_U J_\mu. \tag{4.18}$$

Thus, for each $f_1, f_2 \in L^2(G)$ we have that

$$\begin{aligned} J_\mu\left(f_1 \star_U f_2\right) &= J_\mu \mathscr{D}_U\left((\mathscr{Q}_U f_1)(\mathscr{Q}_U f_2)\right) = \mathscr{D}_U \mathcal{J}\left((\mathscr{Q}_U f_1)(\mathscr{Q}_U f_2)\right) \\ &= \mathscr{D}_U\left((\mathcal{J}\mathscr{Q}_U f_2)(\mathcal{J}\mathscr{Q}_U f_1)\right) = \mathscr{D}_U\left((\mathscr{Q}_U J_\mu f_2)(\mathscr{Q}_U J_\mu f_1)\right) \\ &= (J_\mu f_2) \ \star_U \ (J_\mu f_1). \end{aligned}$$

Therefore, since $L^2(G)$ is an algebra with respect to $(\cdot) \star_U (\cdot)$ (due to bilinearity), we have proved the following

**Proposition 4.1.9.** *The space $(L^2(G), \star_U, J_\mu)$ is a Banach $*$-algebra.*

We also notice that $\mathcal{R}_U$ is a closed two-sided ideal in such algebra, namely $f_1 \star_U f_2 \in \mathcal{R}_U$ [7].

The $\star$-product of functions admits an explicit formula in terms of integral kernels. However, we will not investigate this property (see [7] section 5 for further details), but we will focus on the case of unimodular groups, where we can recognize the formal expression of the twisted convolution 3.32. Indeed, the following fact holds true [7]:

**Theorem 4.1.10.** *Let $G$ be a l.c.s.c. unimodular group and let $U$ be a square integrable projective representation. Then, for any $f_1, f_2 \in \mathcal{R}_U$ and for almost all $g \in G$, we have that*

$$(f_1 \star_U f_2)(g) = \frac{1}{d_U} \int_G d\lambda(g) f_1(h) f_2(h^{-1}g) \overline{\mu(h, h^{-1}g)}, \quad d_U > 0. \tag{4.19}$$

For compact groups the following fact also holds [7]:

**Corollary 4.1.11.** *Let $G$ be a l.c.s.c. compact group and let $\lambda$ be an Haar measure normalized in such a way that $\lambda(G) = 1$. Let $U$ be a square integrable representation of $G$ with dimension $d_U$. Then, $\forall f_1, f_2 \in L^2(G)$, we have that*

$$G \ni g \mapsto \int_G d\lambda(h) f_1(h) f_2(h^{-1}g) = \sum_{U \in \hat{G}} d_U (f_1 \star_U f_2), \tag{4.20}$$

*where the sum on the right hand side converges in the norm sense, is a map in $L^2(G)$.*

## 4.2 Wigner function on continuous phase space

We now apply the general formalism developed in section 4.1 to $\mathbb{R}^n \times \mathbb{R}^n$. We will focus on the case of a Weyl system $S \equiv S_1$ defined in 2.24 whose symplectic multiplier $\mu$ is given in 2.26.

Firstly, let us briefly review the intertwining properties, so let $\rho$ be a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$. Then, the generalized Wigner transform (also called the *Fourier-Wigner transform* or the *characteristic function* in this context) $\mathscr{D}_S : \mathcal{B}_2(L^2(\mathbb{R}^n)) \to L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is given by

$$(\mathscr{D}_S\rho)(q,p) := \mathrm{tr}(S(q,p)^*\rho) \tag{4.21}$$

(the Haar measure is normalized in such a way that the Duflo-Moore operator is $D_S = \mathrm{Id}$). Let us consider the unitary representation $S \vee S : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{U}(L^2(\mathbb{R}^n \times \mathbb{R}^n))$ such that $(S \vee S)(q,p)\rho = S(q,p)\rho S(q,p)^*$ and observe that

$$\begin{aligned}
&\overline{\mu((q,p),(q'-q,p'-p))}\mu((q'-q,p'-p),(q,p)) = \\
&= \exp\{-i\pi\,(q\cdot(p'-p) - p\cdot(q'-q))\}\exp\{i\pi\,((q'-q)\cdot p - (p'-p)\cdot q)\} \\
&= \exp\{-i2\pi(q\cdot p' - p\cdot q')\}.
\end{aligned}$$

Thus, $S \vee S$ is intertwined by $\mathscr{D}_S$ with the representation 4.7, which turns out to be defined in such a way that

$$(T_\mu(q,p)f)\,(q',p') = e^{-i2\pi(q\cdot p' - p\cdot q')}f(q',p'), \quad \forall f \in L^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{4.22}$$

Similarly, bearing in mind that

$$\mu((q,p),(q,p)^{-1}) = e^{i\pi(q\cdot(-p) - p\cdot(-q))} = 1, \tag{4.23}$$

where $\mu$ is the symplectic multiplier 2.26, the generalized Wigner transform $\mathscr{D}_S$ intertwines the involution

$$\mathcal{J} : \mathcal{B}_2(L^2(\mathbb{R}^n)) \ni A \mapsto A^* \in \mathcal{B}_2(L^2(\mathbb{R}^n)) \tag{4.24}$$

with (see proposition 4.1.5)

$$(Jf)(q,p) \equiv (J_\mu f)(q,p) = \overline{f(-q,-p)}, \quad \forall f \in L^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{4.25}$$

We now observe that, in order to recover the *standard* Wigner function [50]

$$\mathscr{W}_S^\psi = \int_{\mathbb{R}^n} dx\; e^{-i2\pi p\cdot x}\overline{\psi\left(q - \frac{x}{2}\right)}\psi\left(q + \frac{x}{2}\right), \quad \psi \in L^2(\mathbb{R}^n), \tag{4.26}$$

it is necessary to consider the symplectic Fourier transform of the characteristic function 4.21. Indeed, let us consider the finite rank operator $|\psi\rangle\langle\phi|$, $\psi, \phi \in L^2(\mathbb{R}^n)$ at first. Then, we see that

$$\begin{aligned}
(\mathscr{D}_S\,|\psi\rangle\langle\phi|)(q,p) &= \mathrm{tr}(S(q,p)^*\,|\psi\rangle\langle\phi|) = \langle S(q,p)\phi, \psi\rangle_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} dx\psi(x)e^{i\pi q\cdot p}e^{-i2\pi p\cdot x}\overline{\phi(x-q)}.
\end{aligned} \tag{4.27}$$

Hence, mapping $x \mapsto x + \frac{q}{2}$, we obtain

$$(\mathscr{D}_S\,|\psi\rangle\langle\phi|)(q,p) = \int_{\mathbb{R}^n} dx \psi\left(x+\frac{q}{2}\right)\overline{\phi\left(x-\frac{q}{2}\right)} e^{-i2\pi p\cdot x} \tag{4.28}$$

(notice that the latter is essentially the Gabor transform of $\psi$ with respect to the window $\phi$ [20, 22]).
Thus, by an application of the symplectic Fourier transform we recover the "right" Wigner function. Indeed, observe that the first Fourier's inversion formula 3.20 can be realized as

$$\psi(q) = \int_{\mathbb{R}^n} dp'\; e^{i2\pi q\cdot p'} \int_{\mathbb{R}^n} dx\; e^{-i2\pi p'\cdot x}\psi(x) \quad \psi \in L^2(\mathbb{R}^n).$$

Then,

$$\begin{aligned}
&(\mathcal{F}_{\mathrm{Sp}}(\mathscr{D}_S\,|\psi\rangle\langle\phi|))\,(q,p) = \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} dq' dp'\; e^{-i2\pi(q'\cdot p - p'\cdot q)} \int_{\mathbb{R}^n} dx\; \overline{\phi\left(x-\frac{q'}{2}\right)}\psi\left(x+\frac{q'}{2}\right) e^{-i2\pi p'\cdot x} \\
&= \int dq'\; \overline{\phi\left(q-\frac{q'}{2}\right)}\psi\left(q+\frac{q'}{2}\right) e^{-i2\pi p'\cdot q'}.
\end{aligned}$$

Therefore, we can give the following definition:

**Definition 4.2.1.** The *standard Wigner function* is a map from $\mathcal{B}_2(L^2(\mathbb{R}^n))$ to $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\mathscr{W}_S^{\rho} = \mathcal{F}_{\mathrm{Sp}}\mathscr{D}_S\rho, \quad \rho \in \mathcal{B}_2(L^2(\mathbb{R}^n)). \tag{4.29}$$

We remark that definition 4.29 can be written in Dirac notation as

$$\mathscr{W}_S^{\rho}(q,p) = \int_{\mathbb{R}^n} dx\; e^{-i2\pi p\cdot x}\left\langle q-\frac{x}{2}\middle|\rho\middle|q+\frac{x}{2}\right\rangle.$$

We also notice that, since $\mathscr{D}_S$ is a unitary operator on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ [38], $\mathscr{W}_S$ is unitary too.
Moreover, orthogonality relations 3.43 hold true, because $S(q,p)$ is a square integrable projective representation. Thus, by the unitarity of the operator $\mathcal{F}_{\mathrm{Sp}}$, we have

**Theorem 4.2.2.** *If $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^n)$, the Wigner function satisfy the* Moyal identity*, namely*

$$\langle \mathscr{W}_S^{|\psi_1\rangle\langle\phi_1|}, \mathscr{W}_S^{|\psi_2\rangle\langle\phi_2|}\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = \langle\psi_1,\psi_2\rangle_{L^2(\mathbb{R}^n)}\langle\phi_2,\phi_1\rangle_{L^2(\mathbb{R}^n)}. \tag{4.30}$$

Another brief remark concerning the intertwining properties: let us consider the unitary representation

$$\mathcal{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{U}(L^2(\mathbb{R}^n \times \mathbb{R}^n)), \quad \mathcal{V}(q,p) := \mathcal{F}_{\mathrm{Sp}} \mathcal{T}_\mu \mathcal{F}_{\mathrm{Sp}}, \tag{4.31}$$

which is intertwined with $S \vee S$, because

$$\mathscr{W}_S = \mathcal{F}_{\mathrm{Sp}} \mathscr{D}_S \implies \mathscr{W}_S S \vee S = \mathcal{F}_{\mathrm{Sp}} \mathcal{T}_\mu \mathcal{F}_{\mathrm{Sp}} \mathscr{W}_S = \mathcal{V} \mathscr{W}_S.$$

Observe that $\mathcal{V}$ is the translation operator on the phase space, because

$$\begin{aligned}
&(V(q,p)f)(q',p') = (\mathcal{F}_{\mathrm{Sp}} \mathcal{T}_\mu \mathcal{F}_{\mathrm{Sp}} f)(q',p') = \\
&= \int dq_1 dp_1 \; e^{i2\pi(q'\cdot p_1 - p'\cdot q_1)} (\mathcal{T}_\mu(q,p) \mathcal{F}_{\mathrm{Sp}} f)(q_1,p_1) = \\
&= \int dq_1 dp_1 \; e^{i2\pi(q'\cdot p_1 - p'\cdot q_1)} e^{-i2\pi(q\cdot p_1 - p\cdot q_1)} (\mathcal{F}_{\mathrm{Sp}} f)(q_1,p_1) = \\
&= \int dq_1 dp_1 \; e^{i2\pi(q'\cdot p_1 - p'\cdot q_1)} e^{-i2\pi(q\cdot p_1 - p\cdot q_1)} \int dq_2 dp_2 \; e^{i2\pi(q_1\cdot p_2 - p_1\cdot q_2)} f(q_2,p_2) = \\
&= f(q'-q, p'-p).
\end{aligned}$$

Thus, the simmetry action on Hilbert-Schmidt operators is intertwined with translations on phase space. In other terms, the latter unitary equivalence means that the Wigner function behaves “well” under position and momentum translations, namely [9]

$$\begin{aligned}
\psi(q) \mapsto \psi(q-q') &\implies \mathscr{W}_S^\psi(q,p) \mapsto \mathscr{W}_S^\psi(q-q',p), \\
\psi(q) \mapsto e^{2\pi i p' q} \psi(q) &\implies \mathscr{W}_S^\psi(q,p) \mapsto \mathscr{W}_S^\psi(q,p-p'),
\end{aligned}$$

where $\psi$ is a function on the configuration space.

Before we go any further, let us briefly review an alternative expression of the Wigner function 4.29, that relies on the definition of the following self-adjoint operators, often called the *phase-point operators*,

$$A(q,p) := 2S(q,p)\Pi S(q,p)^*, \quad (q,p) \in \mathbb{R}^n \times \mathbb{R}^n \tag{4.32}$$

where $\Pi \equiv \mathcal{F}^2$ is the parity operator such that $f(x) \mapsto f(-x),\ f \in L^2(\mathbb{R}^n)$ ($\mathcal{F}$ is the Fourier-Plancherel operator on $L^2(\mathbb{R}^n)$); the factor 2 will be necessary in order to retrieve exactly $\mathscr{W}_S$. Observe that we have an operator for each point $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$; in this sense, these operators are the quantum mechanical counterparts of the phase space points. Moreover, each phase-point operator $A(q,p)$ can be interpreted as a *displaced* parity operator, namely as a parity operator around the phase space point $(q, p)$ [46]. Since the translation and modulation operators are intertwined by the Fourier transform on $L^2(\mathbb{R}^n)$ as in proposition 2.4.5 [22], we also have the following equivalent expressions, which highlight a dilation involved in the definition of the phase-point operator:

$$\begin{aligned} A(q,p) &= 2S(q,p)\Pi S(q,p)^* = 2e^{-i2\pi qp}M(p)T(q)\Pi M(-p)T(-q) \\ &= 2e^{-i2\pi qp}M(p)T(q)M(p)T(q)\Pi = 2e^{-i4\pi qp}M(2p)T(2q)\Pi \\ &= 2S(2q,2p)\Pi, \end{aligned}$$

or, in a similar way [34],

$$A(q,p) = 2e^{i4\pi qp}T(2q)\Pi M(-2p).$$

Hence, the Wigner function 4.29 can be written as [2]

$$\mathscr{W}_S^\rho(q,p) = \operatorname{tr}(A(q,p)\rho), \quad \rho \in \mathcal{B}_2(L^2(\mathbb{R}^n)). \tag{4.33}$$

For instance, if $\psi \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} \operatorname{tr}(A(q,p)\,|\psi\rangle\langle\psi|) &= 2\int_{\mathbb{R}^n} dx\ e^{i\pi q\cdot p}e^{i2\pi p\cdot x}\overline{\psi(q+x)}e^{-i\pi q\cdot p}e^{i2\pi p\cdot x}\psi(q-x) \\ \left(x \mapsto -\frac{x}{2}\right) &= \int_{\mathbb{R}^n} dx\ e^{-i2\pi p\cdot x}\overline{\psi\left(q-\frac{x}{2}\right)}\psi\left(q+\frac{x}{2}\right) = \mathscr{W}_S^\psi(q,p). \end{aligned}$$

Finally, we review some well-known facts concerning quantum states. Recall that an operator $\rho$ is a quantum state if

$$\rho \in \mathcal{B}_1(\mathcal{H}), \quad \operatorname{tr}\rho = 1, \quad \rho \geq 0, \tag{4.34}$$

where $\mathcal{B}_1(\mathcal{H})$ denotes the set of trace class operators on the Hilbert space $\mathcal{H}$ (namely, $A \in \mathcal{B}_1(\mathcal{H})$ if $\operatorname{tr}|A| < \infty$, where $|A| = \sqrt{A^*A}$). We will denote with $\mathcal{S}(\mathcal{H})$ the convex set [35] of quantum states on $\mathcal{H}$. Furthermore, we recall that every quantum state $\rho$ admits the decomposition

$$\rho = \sum_{i\in\Gamma} p_i P_i, \quad \sum_{i\in\Gamma} p_i = 1,$$

where $\Gamma$ is denumerable index set and each $P_i$ is a rank one projector, hence it can be written as $|\psi_i\rangle\langle\psi_i|$, $\psi_i \in \mathcal{H}$ [35].

Firstly, as it was already noted by Wigner [50], the Wigner function 4.29 is not positive definite[1]. Indeed, as a trivial example, let us consider an odd non-null function $\psi \in L^2(\mathbb{R}^n)$. Then we have

$$\mathscr{W}_S^\psi(0,0) = \int_{\mathbb{R}^n} dx\ \overline{\psi\left(-\frac{x}{2}\right)}\psi\left(\frac{x}{2}\right) = -\int_{\mathbb{R}^n} dx\ \overline{\psi\left(\frac{x}{2}\right)}\psi\left(\frac{x}{2}\right) = -2\|\psi\|_{L^2}^2,$$

[1] Rigorously, it is remarkable that the only (pure) states whose Wigner functions are non-negative are the gaussian states [20].

which is a negative quantity, since the norm is positive definite. Hence it cannot be interpreted as a probability density and the correspondence with quantum states is not “genuine”.

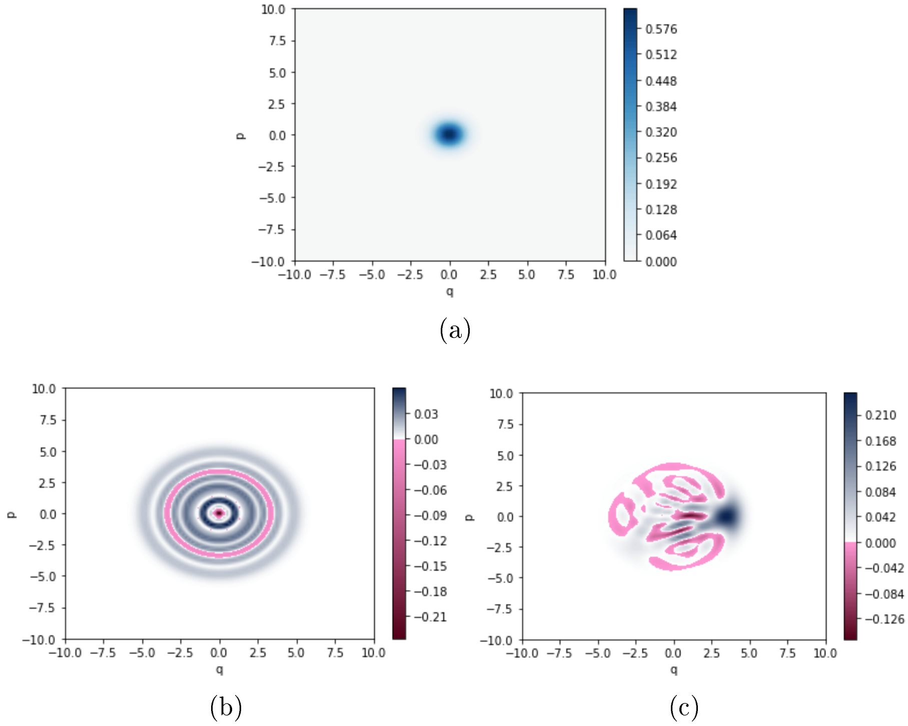


Figure 4.1: Some example of Wigner functions, generated with the QuTiP module of Python [26, 27]. (a) is the ground state of the harmonic oscillator; (b) is a superposition of the following eigenstates of the harmonic oscillator: $|1\rangle, |3\rangle, |9\rangle, |13\rangle$; (c) is a superposition of the coherent states $|\alpha_1\rangle, |\alpha_2\rangle, |\alpha_3\rangle$, where $\alpha_1 = 17.45 + i20.01, \alpha_2 = 14.24 + i14, \alpha_3 = 16.52$.

However, it enjoys some other remarkable properties:

**Proposition 4.2.3.** *Let* $|\psi\rangle\langle\psi| \in \mathcal{S}(L^2(\mathbb{R})n))$ *be a pure state. Then,* $\mathscr{W}_S^\psi$ *is a real function. Moreover, if* $\psi, \mathcal{F}\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*, where* $\mathcal{F}$ *is the Fourier-Plancherel operator in* $L^2(\mathbb{R}^n)$*, the marginal distributions*

$$\int_{\mathbb{R}^n} dp\, \mathscr{W}_S^\psi(q,p) = |\psi(q)|^2, \tag{4.35}$$

$$\int_{\mathbb{R}^n} dq\, \mathscr{W}_S^\psi(q,p) = |(\mathcal{F}\psi)(p)|^2 \tag{4.36}$$

*hold true.*
*Lastly, given two pure states* $|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| \in \mathcal{S}(L^2(\mathbb{R}^n))$, *we have that*

$$|\langle\psi_1,\psi_2\rangle_{L^2}|^2 = \int_{\mathbb{R}^n\times\mathbb{R}^n} dqdp\ \mathscr{W}_S^{\psi_1}(q,p)\mathscr{W}_S^{\psi_2}(q,p). \tag{4.37}$$

*Proof.* We firstly observe that the Wigner function is real. Indeed, since

$$\begin{aligned}\overline{\mathscr{W}_S^{|\phi\rangle\langle\psi|}(q,p)} &= \int_{\mathbb{R}^n} dx\ e^{2\pi ip\cdot x}\overline{\phi\left(q+\frac{x}{2}\right)}\psi\left(q-\frac{x}{2}\right)\\ (x\mapsto -x)\ &= \int_{\mathbb{R}^n} dx\ e^{-2\pi ip\cdot x}\overline{\phi\left(q-\frac{x}{2}\right)}\psi\left(q+\frac{x}{2}\right) = \mathscr{W}_S^{|\psi\rangle\langle\phi|}(q,p)\end{aligned}$$

where $\phi,\psi \in L^2(\mathbb{R}^n)$, we have $\mathscr{W}_S^{\psi}(q,p) = \overline{\mathscr{W}_S^{\psi}(q,p)}$.
Now we can prove the marginal properties. Indeed, observe that

$$\begin{aligned}\int_{\mathbb{R}^n} dq\ \mathscr{W}_S^{\psi}(q,p) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} dqdx\ e^{-i2\pi p\cdot x}\overline{\psi\left(q-\frac{x}{2}\right)}\psi\left(q+\frac{x}{2}\right)\\ \left(u=q+\frac{x}{2},\ v=q-\frac{x}{2}\right)\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} dudv\ e^{-i2\pi p\cdot u}\psi(u)\overline{e^{-i2\pi p\cdot v}\psi(v)}\\ &= |(\mathcal{F}\psi)(p)|^2.\end{aligned}$$

The marginal distribution

$$\int_{\mathbb{R}^n} dp\ \mathscr{W}_S^{\psi}(q,p) = |\psi(x)|^2$$

follows from a direct application of the first Fourier's inversion formula 3.20 [18].
Similarly, we have

$$\begin{aligned}&\int dqdp\ \mathscr{W}_S^{\psi_1}(q,p)\mathscr{W}_S^{\psi_2}(q,p) =\\ &\quad= \int dqdpdx_1dx_2\ e^{-i2\pi p\cdot x_1}e^{-i2\pi p\cdot x_2}\overline{\psi_1\left(q-\frac{x_1}{2}\right)}\psi_1\left(q+\frac{x_1}{2}\right)\cdot\\ &\quad\cdot\overline{\psi_2\left(q-\frac{x_2}{2}\right)}\psi_2\left(q+\frac{x_2}{2}\right)\\ &\quad= \int dqdx\ \overline{\psi_1\left(q+\frac{x}{2}\right)}\psi_1\left(q-\frac{x}{2}\right)\overline{\psi_2\left(q-\frac{x}{2}\right)}\psi_2\left(q+\frac{x}{2}\right)\\ &\quad= \int dudv\ \overline{\psi_1(u)}\psi_1(v)\overline{\psi_2(v)}\psi_2(u) = \langle\psi_1,\psi_2\rangle\langle\psi_2,\psi_1\rangle = |\langle\psi_1,\psi_2\rangle|^2,\end{aligned}$$

where $u = q+\frac{x}{2}, v = q-\frac{x}{2}$. □

Moreover, if $\hat{q}_i$ and $\hat{p}_i$ denote respectively the $i$-th position and momentum operators and $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we have that [20]

$$\int_{\mathbb{R}^n} dqdp\ p_i \mathscr{W}_S^{\psi}(q,p) = \langle \psi, \hat{p}_i \psi \rangle, \quad \int_{\mathbb{R}^n} dqdp\ q_i \mathscr{W}_S^{\psi}(q,p) = \langle \psi, \hat{q}_i \psi \rangle. \tag{4.38}$$

We remark that the above results can be suitably extended to any density operator $\rho$ in $\mathcal{S}(L^2(\mathbb{R}^n))$, since each quantum state can be decomposed in a convex combination of pure states [35].

### 4.2.1 The star-product on the continuous phase space

We can now define the $\star$-product of functions on $\mathbb{R}^n \times \mathbb{R}^n$. In particular, from the discussion in section 4.1.1, we will quickly recall that the twisted convolution arises if we consider the characteristic function 4.21. Next, we will consider the $\star$-product induced by the standard Wigner transform, which is linked with the twisted convolution by the symplectic Fourier transform [7].

Recall that, in such a case we have that $\mathcal{R}_S = L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and the formula given in theorem 4.1.10 holds for every couple of functions on the phase space. Therefore, 4.16 corresponds exactly with the twisted convolution 3.35 on the whole $L^2(\mathbb{R}^n \times \mathbb{R}^n)$, which we report below for the sake of completeness:

$$(f_1 \star_S f_2)(q,p) = \int_{\mathbb{R}^n \times \mathbb{R}^n} dq'dp'\ f_1(q',p') f_2(q-q', p-p') e^{i\pi(q\cdot p' - p\cdot q')} \tag{4.39}$$

(as always, $S \equiv S_1$ denotes a Weyl system).

Let us now switch to the case of the $\star$-product induced by the Weyl-Wigner scheme induced by the standard Wigner transform. In particular, the latter will be a map (denoted simply with $\star$) of the form

$$L^2(\mathbb{R}^n \times \mathbb{R}^n) \times L^2(\mathbb{R}^n \times \mathbb{R}^n) \ni (f_1, f_2) \mapsto \mathscr{W}_S \left( (\mathscr{W}_S^* f_1)(\mathscr{W}_S^* f_2) \right) \in L^2(\mathbb{R}^n \times \mathbb{R}^n). \tag{4.40}$$

Hence, since $\mathscr{W}_S = \mathcal{F}_{\mathrm{Sp}} \mathscr{D}_S$,

$$\begin{aligned} f_1 \star f_2 &= \mathscr{W}_S \left( (\mathscr{W}_S^* f_1)(\mathscr{W}_S^* f_2) \right) = \mathcal{F}_{\mathrm{Sp}} \mathscr{D}_S \left( (\mathscr{Q}_S \mathcal{F}_{\mathrm{Sp}} f_1)(\mathscr{Q}_S \mathcal{F}_{\mathrm{Sp}} f_2) \right) \\ &= \mathcal{F}_{\mathrm{Sp}} \left( (\mathcal{F}_{\mathrm{Sp}} f_1) \star_S (\mathcal{F}_{\mathrm{Sp}} f_2) \right). \end{aligned}$$

In particular, we have that

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Sp}}\left((\mathcal{F}_{\mathrm{Sp}}f_1)\star_S(\mathcal{F}_{\mathrm{Sp}}f_2)\right)\\
&\qquad=\int dq'dp'\; e^{i2\pi(q\cdot p'-p\cdot q')}\left((\mathcal{F}_{\mathrm{Sp}}f_1)\star_S(\mathcal{F}_{\mathrm{Sp}}f_2)\right)(q',p')\\
&\qquad=\int dq'dp'\; e^{i2\pi(q\cdot p'-p\cdot q')}\int dq_2dp_2\; e^{i\pi(q'\cdot p''-p'\cdot q'')}(\mathcal{F}_{\mathrm{Sp}}f_1)(q'',p'')\cdot\\
&\qquad\cdot(\mathcal{F}_{\mathrm{Sp}}f_2)(q'-q'',p'-p'')\\
&\qquad=\int dq_1dq_2dp_1dp_2\;\kappa((q,p),(q_1,p_1),(q_2,p_2))f_1(q_1,p_1)f_2(q_2,p_2),
\end{aligned}$$

where

$$\kappa((q,p),(q_1,p_1),(q_2,p_2))=\exp\left(i4\pi\left[q\cdot p_1-p\cdot q_1+q_1\cdot p_2-p_1\cdot q_2+q_2\cdot p-p_2\cdot q\right]\right) \tag{4.41}$$

is the *Grönewold-Moyal kernel* [7, 38]. We remark that the $\star$-product 4.40 appears in the evaluation of the dynamical evolution of the Wigner function as a quantum deformation of the Poisson brackets of classical mechanics [52]. On the other hand, we must observe that the kernel 4.41 makes sense for functions in $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ only [7]. Hence, in this sense, the twisted convolution 4.39 is more useful in practical applications, since it applies for functions in $L^2(\mathbb{R}^n\times\mathbb{R}^n)$.
Nevertheless, we notice that the $\star$-product 4.40, as well as the twisted convolution 4.39, allows us to define an algebra structure on $L^2(\mathbb{R}^n\times\mathbb{R}^n)$ too. In particular, if we consider the standard complex conjugation in $L^2(\mathbb{R}^n\times\mathbb{R}^n)$, we have again a Banach $*$-algebra [7, 38].

## 4.3 Wigner function on discrete phase space

We can finally apply the quantization-dequantization scheme on $\mathbb{Z}_N\times\mathbb{Z}_N$. In analogy with the continuous case, the Wigner function will be defined as the discrete symplectic Fourier transform (defined in 3.31) of a discrete Wigner transform (also called dequantization map, or characteristic function) induced by a square integrable representation of $\mathbb{Z}_N\times\mathbb{Z}_N$. In this section, we will mostly consider the discrete Weyl system $D\equiv D_1$ defined in 2.77, which is a $N$-dimensional irreducible projective representation of $\mathbb{Z}_N\times\mathbb{Z}_N$ acting on $L^2(\mathbb{Z}_N)$ with multiplier $\mu((j,k),(j',k'))=e^{i\frac{\pi}{N}(jk'-kj')}$. We will also observe that the finite analogous of function 4.33, defined by means of the finite phase-point operators, will not always enjoy the properties of a Wigner function.

Let us now consider a discrete Weyl system $D(j,k) \equiv D_1(j,k)$ of $\mathbb{Z}_N \times \mathbb{Z}_N$ with symplectic multiplier 2.76. The Haar measure on $\mathbb{Z}_N \times \mathbb{Z}_N$ (which is the counting measure since the group is discrete) is normalized in agreement with the representation $D$ in such a way that the Duflo-Moore operator is given by $D_D = N\mathrm{Id}$. Hence, the generalized discrete Wigner transform (or the discrete characteristic function) is given by

$$(\mathscr{D}_D\rho)(j,k) := \frac{1}{N}\,\mathrm{tr}(D(j,k)^*\rho), \quad \rho \in \mathcal{B}_2(L^2(\mathbb{Z}_N)); \tag{4.42}$$

we remark that, if $\rho = |\psi\rangle\langle\phi|$, $\psi,\phi \in L^2(\mathbb{Z}_N)$, the latter resembles the (discrete) Gabor transform 3.37.
We notice that the intertwining properties are formally analogous to the continuous ones, namely,

$$\mathscr{D}_D D \vee D = \mathcal{T}_\mu \mathscr{D}_D, \quad (T_\mu(j,k)f)(j',k') = e^{-i\frac{2\pi}{N}(jk'-kj')} f(j',k'), \tag{4.43}$$

$$\mathscr{D}_D \mathcal{J} = J_\mu \mathscr{D}_D, \quad (Jf)(j,k) := \overline{f(-j,-k)}, \tag{4.44}$$

since $\mu((j,k),(-j,-k)) = 1$ ( as well as in the continuous case). Moreover, for future reference, we explicitly display the discrete Weyl transform, which is determined by 4.14:

$$\rho := \frac{1}{N} \sum_{j,k\in\mathbb{Z}_N} (\mathscr{D}_D\rho)(j,k) D(j,k). \tag{4.45}$$

Thence, we define the discrete Wigner function by an application of the discrete symplectic Fourier transform:

**Definition 4.3.1.** If $D \equiv D_1$ is a finite Weyl system on $\mathbb{Z}_N \times \mathbb{Z}_N$, the *discrete (or finite) Wigner function* is a map from $\mathcal{B}_2(L^2(\mathbb{Z}_N))$ to $L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ such that

$$\mathscr{W}_D := \mathcal{F}_{\mathrm{Sp}}(\mathscr{D}_D\rho), \quad \rho \in \mathcal{B}_2(L^2(\mathbb{Z}_N)), \tag{4.46}$$

where $\mathcal{F}_{\mathrm{Sp}}$ is the discrete symplectic Fourier transform defined in 3.31.

For a pure state $|\psi\rangle\langle\psi| \in \mathcal{S}(L^2(\mathbb{Z}_N))$, the discrete Wigner function is given by

$$\begin{aligned}\mathscr{W}_D^{\psi}(j,k) &:= \frac{1}{N^2} \sum_{j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')} \left(\mathscr{D}_D\, |\psi\rangle\langle\psi|\right)(j',k') \\ &= \frac{1}{N^2} \sum_{j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')} \sum_{l\in\mathbb{Z}_N} e^{i\frac{\pi}{N}j'k'} e^{-i\frac{2\pi}{N}k'l}\,\overline{\psi(l-j')}\psi(l).\end{aligned} \tag{4.47}$$

Again, in perfect analogy with the continuous case, the latter intertwines the unitary representation $D \vee D$ with the translation on the discrete phase space, namely $\mathscr{W}_D D \vee D = \mathcal{V}\mathscr{W}_D$, where $\mathcal{V} := \mathcal{F}_{\mathrm{Sp}}\mathcal{T}_\mu\mathcal{F}_{\mathrm{Sp}}$ is such that

$$(\mathcal{V}(j,k)\psi)(j',k') = \psi(j'-j, k'-k), \quad \psi \in L^2(\mathbb{Z}_N). \tag{4.48}$$

We remark that, in the following, we can mostly consider pure states $|\psi\rangle\langle\psi| \in \mathcal{S}(L^2(\mathbb{Z}_N))$, since the extension to mixed states is obvious. Indeed, if $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is a mixed state, we have

$$\begin{aligned}\mathscr{W}_D^\rho(j,k) &= \frac{1}{N}\mathcal{F}_{\mathrm{Sp}}\,\mathrm{tr}(D(j,k)^*\rho) = \frac{1}{N}\sum_i p_i\mathcal{F}_{\mathrm{Sp}}\,\mathrm{tr}(D(j,k)^*|\psi_i\rangle\langle\psi_i|)\\ &= \sum_i p_i \mathscr{W}_D^{\psi_i}(j,k).\end{aligned}$$

We can now observe that the $\mathscr{W}_D$ enjoys the same properties of the standard Wigner function 4.26:

**Proposition 4.3.2.** *Let $|\psi\rangle\langle\psi| \in \mathcal{S}(L^2(\mathbb{Z}_N))$ be a pure state. Then, $\mathscr{W}_D^\psi$ is a real function.*

*Proof.* Firstly, observe that, for each $\phi, \psi \in L^2(\mathbb{Z}_N)$,

$$\begin{aligned}\langle D(j,k)^*\phi, \psi\rangle = \langle \phi, D(j,k)\psi\rangle &= \sum_{l\in\mathbb{Z}_N} \overline{\phi(l)} e^{-i\frac{\pi}{N}jk} e^{i\frac{2\pi}{N}kl}\psi(l-j)\\ &= \sum_{l\in\mathbb{Z}_N} \overline{\phi(l+j)} e^{i\frac{\pi}{N}jk} e^{i\frac{2\pi}{N}kl}\psi(l).\end{aligned}$$

Hence, we have that

$$(D(j,k)^*\phi)(l) = e^{-i\frac{\pi}{N}jk}e^{-i\frac{2\pi}{N}kl}\phi(l+j) = (D(-j,-k)\phi)(l). \tag{4.49}$$

Therefore,

$$\begin{aligned}\overline{\mathscr{W}_D^\psi(j,k)} &= \frac{1}{N^2}\sum_{j',k'\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}(jk'-kj')}\overline{\langle D(j',k')\psi,\psi\rangle}\\ &= \frac{1}{N^2}\sum_{j',k'\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}(jk'-kj')}\langle D(j',k')^*\psi,\psi\rangle\\ ((j',k')\mapsto(-j',-k')) &= \frac{1}{N^2}\sum_{j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')}\langle D(j',k')\psi,\psi\rangle = \mathscr{W}_D^\psi(j,k).\end{aligned}$$

□

**Proposition 4.3.3.** *If* $|\psi\rangle\langle\psi|, |\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| \in \mathcal{S}(L^2(\mathbb{Z}_N))$*, then the following relations hold true:*

$$\sum_{j\in\mathbb{Z}_N} \mathscr{W}_D^{\psi}(j,k) = |(\mathcal{F}\psi)(k)|^2, \qquad \sum_{k\in\mathbb{Z}_N} \mathscr{W}_D^{\psi}(j,k) = |\psi(j)|^2, \tag{4.50}$$

$$\sum_{j,k\in\mathbb{Z}_N} \mathscr{W}_D^{\psi_1}(j,k)\mathscr{W}_D^{\psi_2}(j,k) = \frac{1}{N}|\langle\psi_1,\psi_2\rangle|^2. \tag{4.51}$$

*Proof.* Indeed, adding all over the $j$'s in $\mathbb{Z}_N$ equation 4.47, we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}_N} \mathscr{W}_D^{\psi}(j,k) &= \frac{1}{N^2}\sum_{j,j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')}\sum_{l\in\mathbb{Z}_N} e^{i\frac{\pi}{N}j'k'}e^{-i\frac{2\pi}{N}k'l}\overline{\psi(l-j')}\psi(l)\\
&= \frac{1}{N}\sum_{j',l\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}kj'}\overline{\psi(l-j')}\psi(l)\\
&= \frac{1}{N}\sum_{l\in\mathbb{Z}_N}\psi(l)\sum_{j\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}kj'}\overline{\psi(l-j')}\\
&= \frac{1}{N}\sum_{l\in\mathbb{Z}_N}\psi(l)e^{-i\frac{2\pi}{N}kl}\sum_{j\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}kj'}\overline{\psi(-j')} = |(\mathcal{F}\psi)(k)|^2.
\end{aligned}$$

The second marginal distribution is a direct consequence of the first Fourier's inversion formula, as in the continuous case. Lastly,

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}_N} \mathscr{W}_D^{\psi_1}(j,k)\mathscr{W}_D^{\psi_2}(j,k)\\
&= \frac{1}{N^4}\sum_{\substack{j,j_1,j_2,\\ k,k_1,k_2\in\mathbb{Z}_N}} e^{i\frac{2\pi}{N}(jk_1-kj_1)}e^{i\frac{2\pi}{N}(jk_2-kj_2)}\langle D(j_1,k_1)\psi_1,\psi_1\rangle\langle D(j_2,k_2)\psi_2,\psi_2\rangle\\
&= \frac{1}{N^2}\sum_{j_1,k_1\in\mathbb{Z}_N}\langle D(j_1,k_1)\psi_1,\psi_1\rangle\langle D(-j_1,-k_1)\psi_2,\psi_2\rangle\\
&= \frac{1}{N^2}\sum_{j,k,l_1,l_2\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}jk}e^{-i\frac{2\pi}{N}k(l_1-l_2)}\overline{\psi_1(l_1-j)}\psi_1(l_1)\overline{\psi_2(l_2+j)}\psi_2(l_2)\\
&= \frac{1}{N}\sum_{l\in\mathbb{Z}_N}\psi_1(l)\overline{\psi_2(l)}\sum_{j\in\mathbb{Z}_N}\overline{\psi_1(-j)}\psi_2(-j) = \frac{1}{N}|\langle\psi_1,\psi_2\rangle|^2.
\end{aligned}$$

□

Before we discuss of the phase-point operators approach, let us briefly review why we have choosen the Weyl system $D \equiv D_1$ 2.77 instead of $S \equiv S_1$ defined in 2.73 (recall that by Stone-von Neumann's theorem, since $D$ and

$S$ are projectively equivalent, they describe the same physical system).
For instance, let us consider the pure state $|\psi\rangle\langle\psi| \in \mathcal{S}(L^2(\mathbb{Z}_N))$. Then, the discrete Wigner transform (or characteristic function) induced by $S$ (whose multiplier $\mu_S((j,k),(j',k')) = e^{i\frac{2\pi}{N}jk'}$ is defined in 2.75) is given by

$$(\mathscr{D}_S\,|\psi\rangle\langle\psi|)(j,k) := \frac{1}{N}\operatorname{tr}(S(j,k)^*\,|\psi\rangle\langle\psi|) \tag{4.52}$$

($\mathscr{D}_S$ differs from $\mathscr{D}_D$ for a global phase factor, since $D(j,k) = e^{-i\frac{\pi}{N}jk}M_kT_j = e^{-i\frac{\pi}{N}jk}S(j,k)$). Clearly this is a valid characteristic function and, by propositions 4.1.4, 4.1.5, we have

$$\mathscr{D}_S S \vee S = \mathcal{T}_{\mu_S}\mathscr{D}_S, \quad (\mathcal{T}_{\mu_S}f)(j',k') = e^{-i\frac{2\pi}{N}(jk'-kj')}f(j',k'), \tag{4.53}$$

$$\mathscr{D}_S\mathcal{J} = J_{\mu_S}\mathscr{D}_S, \quad (J_{\mu_S}f)(j,k) = e^{-i\frac{2\pi}{N}jk'}\overline{f(-j,-k)}. \tag{4.54}$$

Hence, bearing in mind the first Fourier's inversion formula 3.20, the discrete Wigner function induced by the Weyl system $S$ is given by

$$\begin{aligned}\mathscr{W}_S^{\psi}(j,k) &= \frac{1}{N^2}\sum_{j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')}\sum_{l\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}k'l}\psi(l)\overline{\psi(l-j')}\\ &= \frac{1}{N}\sum_{j'\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}kj'}\psi(j)\overline{\psi(j-j')}.\end{aligned} \tag{4.55}$$

Moreover, since 4.53 is formally analogous to 4.43, we have that $\mathscr{W}_S$ intertwines $S \vee S$ with the translation on the discrete phase space 4.48 too. However, since

$$S(j,k)^* = e^{-i\frac{2\pi}{N}jk}S(-j,-k),$$

the Wigner function 4.55 is not real:

$$\begin{aligned}\overline{\mathscr{W}_S^{\psi}(j,k)} &= \sum_{j',k'\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}(jk'-kj')}\overline{\langle S(j',k')\psi,\psi\rangle} = \sum_{j'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}kj'}\overline{\psi(j)}\psi(j-j')\\ &\neq \mathscr{W}_S^{\psi}(j,k).\end{aligned}$$

Therefore, from the point of view of the discrete Wigner function, the choice of the Weyl system $D$ is more suitable than $S$. We remark that this different behaviour is a consequence of the explicit expression of the generalized Wigner transform 4.3, which depends on the choice of the representative in the equivalence class of the projective representation considered.

For each finite phase space point $(j,k) \in \mathbb{Z}_N \times \mathbb{Z}_N$, we now define the (finite) phase-point operator as

$$A(j,k) := \frac{1}{N}D(j,k)\Pi D(j,k)^* = \frac{1}{N}D(2j,2k)\Pi = \frac{1}{N}e^{i\frac{4\pi}{N}jk}T_{2j}\Pi M_{-2k}, \tag{4.56}$$

where $\Pi = \mathcal{F}^2$ is the parity operator ($\mathcal{F}$ is the discrete Fourier transform 3.22). Hence, by analogy with the standard Wigner function defined on $\mathbb{R}^n \times \mathbb{R}^n$, we can define a function in terms of the phase-point operator as

$$\tilde{\mathscr{W}}_D^\rho := \mathrm{tr}(A(j,k)\rho), \quad \rho \in \mathcal{S}(L^2(\mathbb{Z}_N)), \tag{4.57}$$

which is real by definition. However, 4.57 cannot be always interpreted as a Wigner function. We investigate this fact splitting our analysis in two different cases, namely $N$ even and odd.

Let us consider the case of $N$ odd at first. Then, everything works as in $\mathbb{R}^n \times \mathbb{R}^n$ and $\tilde{\mathscr{W}}_D$ is a Wigner function:

**Proposition 4.3.4.** *If $N \in \mathbb{N}$ is an odd number and $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N))$, the functions $\mathscr{W}_D^\rho$ and $\tilde{\mathscr{W}}_D^\rho$ will coincide.*

*Proof.* We will work with a pure state $\psi \in \mathcal{S}(L^2(\mathbb{Z}_N))$, since the extension to mixed states is trivial. Observe that, since $N$ is odd, there exists the inverse of $2 \in \mathbb{Z}_N$ with respect to the product rule, namely $2^{-1} = \frac{N+1}{2} \in \mathbb{Z}_N$ (recall that $a \in \mathbb{Z}_N$ admits an inverse element with respect to the product if and only if $\gcd(a,N) = 1$ [15]). Hence, by a simple computation, we see that

$$\begin{aligned}
\mathscr{W}_D^\psi(j,k) &= \frac{1}{N^2} \sum_{j',k' \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')} \sum_{l \in \mathbb{Z}_N} e^{i\frac{\pi}{N}j'k'} e^{-i\frac{2\pi}{N}k'l} \overline{\psi(l-j')}\psi(l) \\
&= \frac{1}{N^2} \sum_{j',k' \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')} \sum_{l \in \mathbb{Z}_N} e^{-i\frac{2\pi}{N}k'l} \overline{\psi\left(l-2^{-1}j'\right)}\psi\left(l+2^{-1}j'\right) \\
&= \frac{1}{N} \sum_{j' \in \mathbb{Z}_N} e^{-i\frac{2\pi}{N}kj'} \overline{\psi\left(j-2^{-1}j'\right)}\psi\left(j+2^{-1}j'\right),
\end{aligned}$$

where in the second step we have used the substitution $l \mapsto l + 2^{-1}j'$. Similarly, we have

$$\begin{aligned}
\tilde{\mathscr{W}}_D^\psi &= \frac{1}{N}\langle D(j,k)^*\psi, \Pi D(j,k)^*\psi\rangle = \frac{1}{N}\sum_{l \in \mathbb{Z}_N} e^{i\frac{4\pi}{N}kl}\overline{\psi(j+l)}\psi(j-l) \\
\left(l \mapsto -2^{-1}l\right) &= \frac{1}{N}\sum_{l \in \mathbb{Z}_N} e^{-i\frac{2\pi}{N}kl}\overline{\psi\left(j-2^{-1}l\right)}\psi\left(j+2^{-1}l\right),
\end{aligned}$$

hence, $\tilde{\mathscr{W}}_D^\psi = \mathscr{W}_D^\psi$. □

When $N$ is an even number things work differently and $\tilde{\mathscr{W}}_D$ does not enjoy the properties of a standard Wigner function anymore. From a heuristic point of view we can understand this fact immediately. Indeed, notice that, for a pure state $|\psi\rangle\langle\psi| \in \mathcal{S}(L^2(\mathbb{Z}_N))$, we have

$$\tilde{\mathscr{W}}_D^{\psi}(j,k) = \frac{1}{N} \sum_{l \in \mathbb{Z}_N} e^{i\frac{4\pi}{N}kl} \overline{\psi(j+l)}\psi(j-l). \tag{4.58}$$

Hence, if $|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| \in \mathcal{S}(L^2(\mathbb{Z}_N))$, $\langle\psi_1,\psi_2\rangle$ cannot be expressed in terms of $\tilde{\mathscr{W}}_D^{\psi_1}$ and $\tilde{\mathscr{W}}_D^{\psi_2}$ as in 4.51, indeed

$$\begin{aligned}
&\sum_{j,k \in \mathbb{Z}_N} \tilde{\mathscr{W}}_D^{\psi_1}(j,k)\tilde{\mathscr{W}}_D^{\psi_2}(j,k) \\
&= \sum_{j,k,l_1,l_2 \in \mathbb{Z}_N} e^{i\frac{4\pi}{N}k(l_1+l_2)} \overline{\psi_1(j+l_1)}\psi_1(j-l_1)\overline{\psi_2(j+l_2)}\psi_2(j-l_2) \\
&= \sum_{j,l} \overline{\psi_1(j-l)}\psi_1(j+l)\overline{\psi_2(j-l)}\psi_2(j-l) \neq |\langle\psi_1,\psi_2\rangle|^2
\end{aligned}$$

(similarly, the first marginal in 4.50 cannot be written in terms of $\tilde{\mathscr{W}}_D$).
More formally, the reason why $\tilde{\mathscr{W}}_D$ is not a Wigner function is inscribed in the definition 4.56 of the finite phase-point operator, which is affected by the finiteness of $\mathbb{Z}_N \times \mathbb{Z}_N$. Indeed, the finite phase-point operators behave in a very different way with respect to the continuous ones (defined in 4.32) because, ultimately, the dilation involved in their definition is not negligible in the finite case (this fact was observed by Zak in [53], who called this property the *doubling feature* of $\tilde{\mathscr{W}}_D$).
To unfold this point, it is useful to focus on the following formula, which highlights the aforementioned dilation of the phase space point involved:

$$A(j,k) = \frac{1}{N} D(2j, 2k)\Pi, \quad \forall (j,k) \in \mathbb{Z}_N \times \mathbb{Z}_N \tag{4.59}$$

(roughly speaking, the point $(j,k) \in \mathbb{Z}_N \times \mathbb{Z}_N$ in which we evaluate the phase-point operator corresponds to the dilated point $(2j, 2k) \in \mathbb{Z}_N \times \mathbb{Z}_N$ in which we evaluate $D$). As already pointed out by Wootters [51], we can define a Wigner function via the phase-point operators if they form a basis of linear operators on $L^2(\mathbb{Z}_N)$. However, the research of such a basis is affected by the parity of $N$. In particular, when $N$ is odd, the set of operators $\{A(j,k) \mid j,k \in \mathbb{Z}_N\}$ is a set of $N^2$ independent operators, hence it is an operator basis on $L^2(\mathbb{Z}_N)$ [53]. On the other hand, when $N$ is even, we have $N^2/4$ independent operators only [34]. The difference between the even and the odd cases follows from the fact that $2 \in \mathbb{Z}_N$, when $N$ is even, does not admit a multiplicative inverse. Indeed, let us consider the homomorphism

$$\mathbf{h} : \mathbb{Z}_N \ni j \mapsto 2j \in \mathbb{Z}_N. \tag{4.60}$$

Then, if $N$ is odd, $\ker(\mathbf{h}) = \{(0,0)\}$ since there exists the multiplicative inverse of $2 \in \mathbb{Z}_N$, namely $2^{-1} = \frac{N+1}{2}$, and $\mathbf{h}$ is a group isomorphism. Conversely, if $N$ is even, we have that $\ker(\mathbf{h}) \neq \{0\}$ and $\mathbf{h}$ is not a group isomorphism[2]. Indeed, for example, $j = N/2 \in \ker(\mathbf{h})$ if $N$ is even.

**Example 4.3.5.** Let us consider the case $N = 4$. We observe that

$$A(0,0) = \frac{1}{4}D(0,0)\Pi, \quad A(1,0) = \frac{1}{4}D(2,0)\Pi, \tag{4.61}$$

$$A(0,1) = \frac{1}{4}D(0,2)\Pi, \quad A(1,1) = \frac{1}{4}D(2,2)\Pi, \tag{4.62}$$

are the only independent operators. Indeed, for example, we have that

$$A(3,0) = \frac{1}{4}D(6,0)\Pi = \frac{1}{4}D(2,0)\Pi, \ A(3,3) = \frac{1}{4}D(6,6)\Pi = \frac{1}{4}D(2,2)\Pi,$$
$$A(2,0) = \frac{1}{4}D(4,0)\Pi = \frac{1}{4}D(0,0)\Pi, \ A(2,2) = \frac{1}{4}D(4,4)\Pi = \frac{1}{4}D(0,0)\Pi,$$

and so on. Therefore, the phase-point operators cannot form an operator basis on $L^2(\mathbb{Z}_4)$.
On the other hand if $N = 3$ we have for example

$$A(0,0) = \frac{1}{3}D(0,0)\Pi, \quad A(1,0) = \frac{1}{3}D(2,0)\Pi, \tag{4.63}$$

$$A(2,0) = \frac{1}{3}D(4,0)\Pi = \frac{1}{3}D(1,0)\Pi. \tag{4.64}$$

Then it is clear that, in such a case, $\{A(j,k) \mid j,k \in \mathbb{Z}_3\}$ is a set of 9 independent operators on $L^2(\mathbb{Z}_3)$.

Therefore, from this point of view, the discrete Wigner function defined following the Weyl-Wigner correspondence is slightly more general than the one defined by means of the phase-point operators, since it does not distinguish the even and the odd cases. However, we remark that a discrete Wigner function defined as in 4.57 can still be defined when $N$ is even, but it requires some ad hoc tweaks [34, 53] which we will not investigate any further.

Lastly, we recall that, as in the continuous case, the $\star$-product induced by the finite Weyl system $D$ corresponds to the twisted convolution on $\mathbb{Z}_N \times \mathbb{Z}_N$, namely

$$(f_1 \star_D f_2)(j,k) = \sum_{j',k' \in \mathbb{Z}_N} f_1(j',k') f_2(j-j',k-k') e^{\frac{i\pi}{N}(jk'-kj')}, \tag{4.65}$$

[2] These facts follow from the first isomorphism theorem: if $h : G \to \tilde{G}$ is a group homomorphism, then $\ker h$ is a normal subgroup of $G$ and $G/\ker h \cong h(G)$ [15].

where $f_1, f_2 \in L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$.

### 4.3.1 Some simple examples

Here we briefly present some simple examples of Wigner function 4.47 induced by $D \equiv D_1$, which - for a generic mixed state $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N))$ - we recall is given by

$$\mathscr{W}_D^{\rho}(j,k) = \frac{1}{N^2} \sum_{j',k' \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')} \operatorname{tr}(D(j',k')^* \rho) \tag{4.66}$$

**Example 4.3.6.** At first, let us consider the case of the position pure state $|\psi_{j_0}\rangle\langle\psi_{j_0}|$. Hence, we have

$$\begin{aligned}
\mathscr{W}_D^{j_0}(j,k) &= \frac{1}{N^2} \sum_{j',k' \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')} \operatorname{tr}(D(j',k')^* \, |\psi_{j_0}\rangle\langle\psi_{j_0}|) \\
&= \frac{1}{N^2} \sum_{j',k' \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')} \langle D(j',k')\psi_{j_0}, \psi_{j_0}\rangle = \frac{1}{N^2} \sum_{k' \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}k'(j-j_0)} \\
&= \frac{1}{N}\delta_N(j-j_0),
\end{aligned}$$

where $\delta_N(j-j_0)$ is such that

$$\delta_N(j-j_0) = \begin{cases} 0, & j \neq j_0 \mod N, \\ 1, & j = j_0 \mod N. \end{cases} \tag{4.67}$$

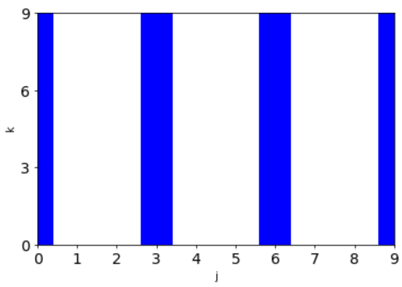


Figure 4.2: Discrete Wigner function of the position basis vector $\psi_0$ for $N = 3$.

**Example 4.3.7.** Let us now consider the case of the pure state $|\hat{\psi}_{k_0}\rangle\langle\hat{\psi}_{k_0}|$, where $\hat{\psi}_{k_0}$ is a vector in the "momenta" basis, whose vectors are defined as

$$\hat{\psi}_n(k) = \frac{1}{\sqrt{N}} \sum_{j\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}jk}\psi_n(j).$$

Hence we have

$$\begin{aligned}\mathscr{W}_D^{\hat{\psi}_{k_0}} &= \frac{1}{N^2} \sum_{j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')}\langle D(j',k')\hat{\psi}_{k_0}, \hat{\psi}_{k_0}\rangle \\ &= \frac{1}{N^2} \sum_{j'\in\mathbb{Z}_N} e^{-i\frac{2\pi}{N}j'(k+k_0)} = \frac{1}{N}\delta_N(k+k_0).\end{aligned}$$

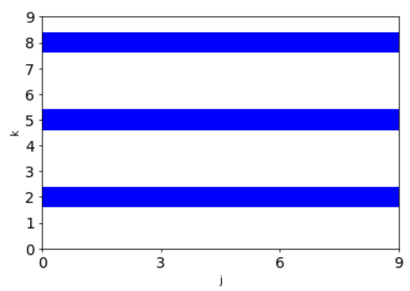


Figure 4.3: Discrete Wigner function of the momentum basis vector $\hat{\psi}_1$ for $N = 3$.

**Example 4.3.8.** We now briefly discuss the finite analogous of position and momentum operators. Indeed, as already pointed out in section 2.3.1, we can consider position and momentum coordinates as the elements of a l.c.s.c. Abelian group and its unitary dual. Hence, let us consider the standard bases in position and momentum coordinates $\{\psi_n\}_{n\in\mathbb{Z}_N}, \{\hat{\psi}_n\}_{n\in\mathbb{Z}_N} \subset L^2(\mathbb{Z}_N)$, where $\hat{\psi}_n = \mathcal{F}\psi_n \ \forall n \in \mathbb{Z}_N$. Then, the operators

$$\hat{j} := \sum_{j\in\mathbb{Z}_N} j\,|\psi_j\rangle\langle\psi_j|\,, \quad \hat{k} := \sum_{k\in\mathbb{Z}_N} k\,|\hat{\psi}_k\rangle\langle\hat{\psi}_k| \tag{4.68}$$

are the finite position and momentum operators (clearly they are defined modulo $N$). Since $\hat{\psi}_k = \mathcal{F}\psi_k \ \forall k \in \mathbb{Z}_N$, we have that the discrete Fourier transform intertwines them, namely $\hat{k} = \mathcal{F}\hat{j}\mathcal{F}^*$ [46]. Of course these are not "true" position and momentum operators, because they cannot satisfy the CCRs (recall by section 2.2.3 that this is possible in infinite-dimensional

Hilbert spaces only), however they have various applications (see [1, 46]). Thanks to examples 4.3.6, 4.3.7, we can easily calculate the Wigner functions associated with the operators in 4.68. In particular, we have

$$
\begin{aligned}
\mathscr{W}_D^{\hat{j}}(j,k) &= \frac{1}{N^2}\sum_{j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')}\operatorname{tr}\left(D(j',k')^*\sum_{j''\in\mathbb{Z}_N} j''\,|\psi_{j''}\rangle\langle\psi_{j''}|\right)\\
&= \frac{1}{N}\sum_{j''\in\mathbb{Z}_N} j''\delta_N(j-j''),\\
\mathscr{W}_D^{\hat{k}}(j,k) &= \frac{1}{N^2}\sum_{j',k'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(jk'-kj')}\operatorname{tr}\left(D(j',k')^*\sum_{k''\in\mathbb{Z}_N} k''\,|\hat{\psi}_{k''}\rangle\langle\hat{\psi}_{k''}|\right)\\
&= \frac{1}{N}\sum_{k''\in\mathbb{Z}_N} k''\delta_N(k+k'').
\end{aligned}
$$

Hence,

$$
\operatorname{tr}\left(\hat{j}\rho\right) = N\sum_{j,k\in\mathbb{Z}_N}\mathscr{W}_D^{\hat{j}}(j,k)\mathscr{W}_D^{\rho}(j,k) = \sum_{j,k\in\mathbb{Z}_N} j\ \mathscr{W}_D^{\rho}(j,k), \tag{4.69}
$$

$$
\operatorname{tr}\left(\hat{k}\rho\right) = N\sum_{j,k\in\mathbb{Z}_N}\mathscr{W}_D^{\hat{k}}(j,k)\mathscr{W}_D^{\rho}(j,k) = \sum_{j,k\in\mathbb{Z}_N} k\ \mathscr{W}_D^{\rho}(j,-k). \tag{4.70}
$$

Next, we want to analyze some simple Wigner functions of states on a composite system. In this regard, the following result concerning the tensor product of unitary (projective) representation is necessary [18]:

**Theorem 4.3.9.** *Let $G_1$ and $G_2$ be two l.c.s.c. group and let $\pi_1 : G_1 \to \mathcal{U}(\mathcal{H}_1)$ and $\pi_2 : G_2 \to \mathcal{U}(\mathcal{H}_2)$ be two unitary representations. Let us consider the unitary representation*

$$
\pi \equiv \pi_1 \otimes \pi_2 : G_1 \times G_2 \to \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2), \quad \pi(g_1, g_2) = \pi_1(g_1) \otimes \pi_2(g_2). \tag{4.71}
$$

*Then, $\pi$ is irreducible if and only if $\pi_1$ and $\pi_2$ are both irreducible.*

Hence, if we consider the direct product group $\mathbb{Z}_N\times\mathbb{Z}_N\times\mathbb{Z}_M\times\mathbb{Z}_M$, where $N, M \in \mathbb{N}$ (namely, a composite system $AB$), the tensor product of the Weyl systems on the two discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$ and $\mathbb{Z}_M \times \mathbb{Z}_M$, denoted respectively with $D_A \equiv D_1^A$ and $D_B \equiv D_1^B$, is an irreducible projective representation.
We now present some examples of separable states and entangled states; in the next section we will introduce some criteria to characterize separability, which relies on the finite characteristic function.

**Example 4.3.10.** Let us consider a separable state $|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| \in \mathcal{S}(L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M))$. Then, by definition of tensor product, we can factorize the characteristic function:

$$\begin{aligned}(\mathscr{D}_D(|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|))(j_1,k_1,j_2,k_2) :=& \\ &:= \frac{1}{NM}\operatorname{tr}\left((D_A(j_1,k_1)^* \otimes D_B(j_2,k_2)^*)(\psi_1\otimes\psi_2)\right) \\ &= \frac{1}{N}\langle D_A(j_1,k_1)^*\psi_1,\psi_1\rangle \frac{1}{M}\langle D_B(j_2,k_2)^*\psi_2,\psi_2\rangle \\ &= \left(\mathscr{D}_{D_A}|\psi_1\rangle\langle\psi_1|\right)(j_1,k_1)\left(\mathscr{D}_{D_B}|\psi_2\rangle\langle\psi_2|\right)(j_2,k_2),\end{aligned}$$

where $(j_1,k_1) \in \mathbb{Z}_N \times \mathbb{Z}_N$ and $(j_2,k_2) \in \mathbb{Z}_M \times \mathbb{Z}_M$. As a consequence, we can consider the standard Wigner function of the bipartite system applying the discrete symplectic Fourier transform, which can be factorized; we will denote with $\mathcal{F}^A_{\mathrm{Sp}}$ and $\mathcal{F}^B_{\mathrm{Sp}}$ the symplectic Fourier transform on $A$ and $B$. Therefore, the discrete Wigner function of the separable state $\psi_1 \otimes \psi_2$ is given by

$$\mathscr{W}^{\psi_1\otimes\psi_2}_{AB}(j_1,k_1,j_2,k_2) = \mathscr{W}^{\psi_1}_A(j_1,k_1)\mathscr{W}^{\psi_2}_B(j_2,k_2) \tag{4.72}$$

(here we prefer to specify in the subscript the quantum system instead of the representation used to calculate the Wigner function, since we will be stuck with the 1-Weyl system $D = D_A \otimes D_B$). Observe that the partial trace of $\psi_1 \otimes \psi_2$ translates in the discrete phase space as the sum all over the points of only one subsystem, e.g. $\operatorname{tr}_A$ "corresponds" to $\sum_{j_1,k_1\in\mathbb{Z}_N}$.

**Example 4.3.11.** At last, let us consider the Wigner function of some remarkable entangled states, namely the *Bell states*. Let $A$ and $B$ two $N$-dimensional quantum states. Then, let us consider at first the state

$$\theta_0(l_1,l_2) := \frac{1}{\sqrt{N}}\sum_{n\in\mathbb{Z}_N}(\psi_n\otimes\psi_n)(l_1,l_2), \tag{4.73}$$

which is an entangled state, because its reduced density matrices are mixed states [37]. Then we have

$$\begin{aligned}\mathscr{W}^{\theta_0}_{AB}(j_1,k_1,j_2,k_2) &= \frac{1}{N^4}\sum_{j_1',k_1'\in\mathbb{Z}_N}\sum_{j_2',k_2'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(j_1k_1'-k_1j_1')}e^{i\frac{2\pi}{N}(j_2k_2'-k_2j_2')}\times \\ &\langle (D_A(j_1',k_1')\otimes D_B(j_2',k_2'))\theta_0,\theta_0\rangle \\ &= \frac{1}{N^5}\sum_{j_1',k_1'\in\mathbb{Z}_N}\sum_{j_2',k_2'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(j_1k_1'-k_1j_1')}e^{i\frac{2\pi}{N}(j_2k_2'-k_2j_2')}\times \\ &\sum_{n,m\in\mathbb{Z}_N}\langle D_A(j_1',k_1')\psi_n,\psi_m\rangle\langle D_B(j_2',k_2')\psi_n,\psi_m\rangle \\ &= \frac{1}{N^5}\sum_{j_1',k_1'\in\mathbb{Z}_N}\sum_{j_2',k_2'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(j_1k_1'-k_1j_1')}e^{i\frac{2\pi}{N}(j_2k_2'-k_2j_2')}\times \\ &\sum_{l\in\mathbb{Z}_N} e^{i\frac{\pi}{N}j_1'(k_1'+k_1')}e^{-i\frac{2\pi}{N}(k_1'+k_2')l} \\ &= \frac{1}{N^4}\sum_{j_1',k_1'\in\mathbb{Z}_N} e^{i\frac{2\pi}{N}(j_1k_1'-k_1j_1')}e^{-i\frac{2\pi}{N}(j_2k_1'+k_2j_1')} \\ &= \frac{1}{N^2}\delta_N(j_1-j_2)\delta_N(k_1+k_2).\end{aligned}$$

In a similar way, for $i=1,\dots N-1$, we can define the other Bell's states as

$$\theta_i(l_1,l_2) := \frac{1}{\sqrt{N}}\sum_{n\in\mathbb{Z}_N}\left(\mathrm{Id}_A\otimes D_B(j_i,k_i)\right)\left(\psi_n(l_1)\otimes\psi_n(l_2)\right). \tag{4.74}$$

Therefore, by a similar computation, we see that

$$\mathscr{W}^{\theta_i}_{AB}(j_1,k_1,j_2,k_2) = \frac{1}{N^2}\delta_N(j_1-j_2-j_i)\delta_N(k_1+k_2+k_i). \tag{4.75}$$

## 4.4 Quantum states as functions of quantum positive type

In this last section we discuss of functions of quantum positive type, an alternative point of view on quantum states which is inspired by the duality - provided by Bochner's theorem 3.3.3 - between function of positive type and probability measures. For standard phase space, functions of (quantum) positive type provide a common playground in which classical limit [30] and classical-quantum interactions [5] can be studied. On the other hand, for discrete phase space, they also offer an alternative take on entanglement, which we are now going to explore. To be fair, most of the results holds for compact groups (see [29, 31]), anyhow if we focus on the discrete phase space we will be able to exhibit the most interesting facts without worrying of convergence issues.

Let us briefly sketch the standard cases at first. Recall from section 3.1.2 that a function of positive type on phase space $\chi \in P(\mathbb{R}^n \times \mathbb{R}^n)$ is an element of $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} dqdp\ \chi(q,p)\,(f^* * f)\,(q,p) \geq 0, \quad \forall f \in L^1(\mathbb{R}^n \times \mathbb{R}^n). \tag{4.76}$$

For a bounded continuous function $\chi \in P(\mathbb{R}^n \times \mathbb{R}^n)$, the latter is equivalent to say that $\chi$ is a *positive definite function* [18], namely

$$\sum_{j,k} \chi(g_k - g_j)\overline{c_j} c_k \geq 0, \quad \forall g_1, \ldots, g_m \subset \mathbb{R}^n \times \mathbb{R}^n \tag{4.77}$$

($g_i \equiv (q_i, p_i)$) for every finite set $\{g_1, \ldots, g_m\} \subset \mathbb{R}^n \times \mathbb{R}^n$ and $c_1, \ldots, c_m \in \mathbb{C}$. We also recall that $\chi$ agrees a.e. with a bounded continuous function and, if it is continuous, we have $\|\chi\|_\infty = \chi(0,0)$.
According to Bochner's theorem 3.3.3, normalized functions of positive type (i.e. $\chi(0,0) = 1$) can be regarded as the Fourier-Stieltjes transform of probability measures $\mu(q,p) \in \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^n)$, which are exactly classical states on the $C^*$-algebra of classical observables $C_0(\mathbb{R}^n \times \mathbb{R}^n)$ of functions that vanish at infinity [5]. In this sense, functions of positive type represent an alternative, more practical, description of classical states. For instance, the mean value of a classical observable $f \in C_0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ in the state $\mu \in \mathcal{M}(\mathbb{R}^n \times \mathbb{R}^n)$, which is given by

$$\langle f \rangle_\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} d\mu(q,p)\ f(q,p), \tag{4.78}$$

is often hard to deal with, since probability measures are in general quite abstract. Thus, it may be useful to consider the symplectic Fourier transform, so that [5]

$$\langle f \rangle_\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} dqdp\ \chi(q,p) \hat{f}(q,p), \tag{4.79}$$

where

$$\chi(q,p) = \breve{\mu}(q,p) = \int_{\mathbb{R}^n \times \mathbb{R}^n} d\mu(q',p')\ e^{-i2\pi(q\cdot p' - p\cdot q')}, \tag{4.80}$$

$$\hat{f}(q,p) = (\mathcal{F}f)(q,p) = \int_{\mathbb{R}^n \times \mathbb{R}^n} dq'dp'\ e^{i2\pi(q\cdot p' - p\cdot q')} f(q',p'). \tag{4.81}$$

To introduce functions of positive type in quantum mechanics, we shall consider the Weyl-Wigner correspondence, thanks to which quantum states can be espressed in terms of Wigner functions defined as in 4.26 (as always, we consider the Weyl system $S \equiv S_1$ 2.24). Indeed, in such a case, we already have a suitable $*$-algebra of functions on which we can define functions of positive type, the twisted group algebra $(L^2(\mathbb{R}^n \times \mathbb{R}^n), \star_S, \mathcal{J}_\mu)$ induced by the Weyl sistem $S$, where $\star_S$ is the twisted convolution 4.39 and $\mathcal{J}_\mu$ is the involution 4.25. Hence, since $L^2(\mathbb{R}^n \times \mathbb{R}^n)^*$ can be identified with $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ [19], we can give the following

**Definition 4.4.1.** A function $\chi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is *of quantum positive type* if

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} dqdp\; \chi(q,p)\, (\psi^* \star_S \psi)\,(q,p) \geq 0, \quad \forall \psi \in L^2(\mathbb{R}^n \times \mathbb{R}^n), \tag{4.82}$$

Unfolding the twisted convolution, if $\chi$ is continuous, the following equivalent condition for $\psi \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$ holds true [5]:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} dqdpdq'dp'\; \overline{\psi(q',p')}\chi(q-q',p-p')\psi(q,p)e^{i\pi(q'\cdot p - p'\cdot q)} \geq 0. \tag{4.83}$$

The link with the characteristic function $\mathscr{D}_S\rho,\ \rho \in \mathcal{B}_2(L^2(\mathbb{R}^n))$ 4.21 is realized by continuity [5]:

**Theorem 4.4.2.** *If $\chi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is a continuous function of quantum positive type, then $\chi$ is bounded and $\|\chi\|_\infty = \chi(0,0)$. Moreover, $\chi$ is a continuous function of positive type if and only if $\chi$ is the characteristic function induced by the Weyl system $S$ of a positive operator $\rho \in \mathcal{B}_1(L^2(\mathbb{R}^n))$, namely $\chi \equiv \chi_\rho = \mathscr{D}_S\rho,\ \rho \geq 0$.*

We notice that if $\chi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is a continuous function of positive type and $\rho \in \mathcal{B}_2(L^2(\mathbb{R}^n))$ is the corresponding operator, the positivity condition 4.82 is equivalent to the positivity of $\rho$, namely [5]

$$\mathrm{tr}(\rho A^* A) \geq 0, \quad \forall A \in \mathcal{B}_2(L^2(\mathbb{R}^n)). \tag{4.84}$$

Lastly, we remark that, since $\mathscr{D}_S$ is a unitary map, a characteristic function $\chi_\rho$, viewed as a functional on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the following condition:

$$\|\chi_\rho\|_2 = \sqrt{((\mathscr{D}_S\rho) \star_S (\mathscr{D}_S\rho))\,(0,0)} = \sqrt{\mathrm{tr}(\rho^2)} \leq 1. \tag{4.85}$$

Therefore $\|\chi_\rho\|_2 = 1$ if and only if $\rho$ is a pure state (the above trace is indeed called the *purity* of the quantum state $\rho$ [35]).

For finite quantum systems everything works the same. We consider the twisted group algebra $(L^2(\mathbb{Z}_N \times \mathbb{Z}_N), \star_D, \mathcal{J}_\mu)$, where $D \equiv D_1$ is a finite Weyl system with (formal) symplectic multiplier $\mu((j,k),(j',k')) = e^{i\frac{\pi}{N}(jk'-kj')}$, $\star_D$ is the twisted convolution 4.65 and $\mathcal{J}_\mu$ is the involution defined in 4.44. In this way, we can give the following

**Definition 4.4.3.** A function $\chi \in L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, is a *function of quantum positive type* if

$$\sum_{j,k\in\mathbb{Z}_N} \chi(j,k)\,(\psi^* \star_D \psi)\,(j,k) \geq 0, \quad \forall \psi \in L^2(\mathbb{Z}_N \times \mathbb{Z}_N). \tag{4.86}$$

$P_Q(\mathbb{Z}_N \times \mathbb{Z}_N)$ will denote the space of continuous functions of quantum positive type on $\mathbb{Z}_N \times \mathbb{Z}_N$.

Clearly, functions of quantum positive type on discrete phase space correspond to positive definite function, since $\mathbb{Z}_N \times \mathbb{Z}_N$ is a finite group. Equivalently, we also have that $\chi \in P_Q(\mathbb{Z}_N \times \mathbb{Z}_N)$ iff for each $\psi \in L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$

$$\sum_{j,j',k,k'} \overline{\psi(-j',-k')}\chi(j-j',k-k')\psi(j,k)\mu((j',k'),(j,k)) \geq 0. \tag{4.87}$$

Following the same steps as before, we have that continuous functions of quantum positive type form a subset of discrete characteristic functions 4.42. In particular, the following result, which relies on the unitarity of the Wigner (and Weyl) transform, holds true:

**Proposition 4.4.4.** *An operator $A$ on $L^2(\mathbb{Z}_N)$ is positive if and only if the corresponding characteristic function $\chi_A \equiv \frac{1}{N}\operatorname{tr}(D^*A)$ is a function of quantum positive type.*

*Proof.* Suppose $\chi \in P_Q(\mathbb{Z}_N \times \mathbb{Z}_N)$ and let us consider a vector $\psi \in L^2(\mathbb{Z}_N)$. For convenience, we introduce the following simplified shorthand notation: we set $g \equiv (j,k) \in \mathbb{Z}_N \times \mathbb{Z}_N$ and $\frac{1}{N}\sum_{j,k\in\mathbb{Z}_N} \equiv \int dg$, so that $\frac{1}{N}\int dg = 1$. Hence, by Weyl transform 4.45, we have that there exists an operator $A^\chi \in \mathcal{B}_2(L^2(\mathbb{Z}_N))$ such that

$$A^\chi = \int dg\ \chi(g)D(g) \tag{4.88}$$

(recall that $\mathcal{B}_2(L^2(\mathbb{Z}_N)) \cong \mathcal{M}_N(\mathbb{C})$). Moreover, if $\{\varphi_i\}_{i\in\mathbb{Z}_N}$ is an orthonormal basis in $L^2(\mathbb{Z}_N)$ then $\mathrm{Id} = \sum_i |\varphi_i\rangle\langle\varphi_i|$; hence, denoting with $\mu$ the symplectic multiplier 2.76, we have

$$\begin{aligned}\langle\psi, A^{\chi}\psi\rangle &= \int dg\ \chi(g)\langle\psi, D(g)\psi\rangle = \frac{1}{N}\int dgdh\ \chi(g)\langle\psi, D(g)\psi\rangle \\ &= \frac{1}{N}\int dgdh\ \chi(g-h)\langle\psi, D(g-h)\psi\rangle \\ &= \frac{1}{N}\int dgdh\ \chi(g-h)\langle\psi, D(g)D(h)^*\psi\rangle\mu(h,g) \\ &= \frac{1}{N}\sum_i\int dgdh\ \chi(g-h)\langle D(g)^*\psi, \varphi_i\rangle\langle\varphi_i, D(h)^*\psi\rangle\mu(h,g) \geq 0.\end{aligned}$$

We also used the fact that

$$D(g-h) = \mu(g,-h)D(g)D(h^{-1}) = \mu(h,g)D(g)D(h)^* \tag{4.89}$$

since $\mu(g,h) \equiv \mu((j,k),(j',k')) = e^{i\frac{\pi}{N}(jk'-kj')}$ is such that $\mu(-h,h) = 1$. Conversely, if $\mathcal{B}_2(L^2(\mathbb{Z}_N)) \ni A \geq 0$, we have that

$$\begin{aligned}&\int dgdh\ \overline{\psi(h)}\psi(g)\mu(h,g)\chi_A(g-h) = \\ &\qquad = \frac{1}{N}\int dgdh\ \overline{\psi(h)}\psi(g)\mu(h,g)\,\mathrm{tr}(D(g-h)^*A) = \\ &\qquad = \frac{1}{N}\int dgdh\ \overline{\psi(h)}\psi(g)\,\mathrm{tr}(D(h)D(g)^*A) \\ &\qquad = \frac{1}{N}\,\mathrm{tr}(FF^*A) \geq 0,\end{aligned}$$

where $F \equiv \int dh\overline{\psi(h)}D(h)$. □

We observe incidentally that if $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N))$ is a quantum state, we have

$$\chi_\rho(0,0) = \frac{1}{N}\,\mathrm{tr}(\rho) = \frac{1}{N}. \tag{4.90}$$

Moreover, the following fact also holds true:

**Proposition 4.4.5.** *If $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N))$, the corresponding function of positive type $\chi_\rho$ is such that $\|\chi_\rho\|_2^2 \leq 1$; the equivalence holds true if and only if $\rho$ is a pure state.*

*Proof.* Suppose $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N))$ so that $\rho = \sum_{l\in\mathbb{Z}_N} p_l |\psi_l\rangle\langle\psi_l|$. Then, by orthogonality relations 3.43 we have that

$$\begin{aligned}
\|\chi_\rho\|_2^2 &= \frac{1}{N^2}\sum_{j,k\in\mathbb{Z}_N} \overline{\operatorname{tr}(D(j,k)^*\rho)}\operatorname{tr}(D(j,k)^*\rho) \\
&= \frac{1}{N^2}\sum_{j,k,l,m\in\mathbb{Z}_N} p_l p_m \langle\psi_l, D(j,k)\psi_l\rangle\langle D(j,k)\psi_m,\psi_m\rangle \\
&= \frac{1}{N^2}\sum_{l,m\in\mathbb{Z}_N} p_l p_m \langle\psi_l,\psi_m\rangle\langle D_D\psi_m, D_D\psi_l\rangle \\
&= \sum_{l,m\in\mathbb{Z}_N} p_l p_m \langle\psi_l,\psi_m\rangle\langle\psi_m,\psi_l\rangle = \sum_l p_l^2 \le 1,
\end{aligned}$$

where $D_D = N\mathrm{Id}$ is the Duflo-Moore operator associated with the projective representation $D$. In particular, it is clear that $\|\chi_\rho\|_2^2 = 1$ if and only if $\rho$ is a pure state, because in such a case we have that $\rho = |\psi\rangle\langle\psi|$ , $\psi \in L^2(\mathbb{Z}_N)$. □

We also notice that $\|\chi_\rho\|_2^2 = (\chi_\rho^* \star_D \chi_\rho)(0,0)$, indeed

$$\begin{aligned}
(\chi_\rho^* \star_D \chi_\rho)(0,0) &= \sum_{j,k}\chi_\rho(j,k)^*\chi_\rho(-j,-k)\overline{\mu((j,k),(0,0))} \\
&= \sum_{j,k}\overline{\chi_\rho(-j,-k)}\chi_\rho(-j,-k) = \sum_{j,k}\overline{\chi_\rho(j,k)}\chi_\rho(j,k).
\end{aligned}$$

We are now ready to discuss of bipartite systems. In particular, we will consider the discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$ and $\mathbb{Z}_M \times \mathbb{Z}_M$ endowed respectively with the Weyl systems $D_A \equiv D_1^A$ and $D_B \equiv D_1^B$ (recall that $D_A$ is a $N$-dimensional irreducible projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$, while $D_B$ is a $M$-dimensional one of $\mathbb{Z}_M \times \mathbb{Z}_M$). In this way we can consider a bipartite system $AB$ whose Hilbert space is given by $L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M)$, namely the space of the representation $D = D_A \otimes D_B$ of $(\mathbb{Z}_N\times\mathbb{Z}_N)\times(\mathbb{Z}_M\times\mathbb{Z}_M)$.
Recall now that a state $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M))$ is separable if

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| , \tag{4.91}$$

where $\psi_i \in L^2(\mathbb{Z}_N)$ and $\phi_i \in L^2(\mathbb{Z}_M)$. Hence, we can now prove the following correspondence already encountered in the previous section:

**Theorem 4.4.6.** *Let $L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M)$ be a bipartite quantum system. Then $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M))$ is a separable state if and only if the corresponding function of quantum positive type $\chi_\rho$ factorizes.*

*Proof.* If $\rho$ is separable, we have

$$\begin{aligned}\chi_\rho \equiv \mathscr{D}_D\rho = \frac{1}{NM}\operatorname{tr}((D_A\otimes D_B)^*\rho) = \sum_i p_i \frac{1}{N}\langle D_A\psi_i, \psi_i\rangle \frac{1}{M}\langle D_B\phi_i, \phi_i\rangle \\ = \sum_i p_i \mathscr{D}_{D_A} |\psi_i\rangle\langle\psi_i| \, \mathscr{D}_{D_B} |\phi_i\rangle\langle\phi_i| \,.\end{aligned}$$

Conversely, let us consider $\chi_\rho = \sum_i p_i \mathscr{D}_{D_A} |\psi_i\rangle\langle\psi_i| \, \mathscr{D}_{D_B} |\phi_i\rangle\langle\phi_i|$. Then fix $g_1 \equiv (j_1, k_1) \in \mathbb{Z}_N \times \mathbb{Z}_N$, $g_2 \equiv (j_2, k_2) \in \mathbb{Z}_M \times \mathbb{Z}_M$ and $\int dg_1 \equiv \frac{1}{N}\sum_{j_1,k_1\in\mathbb{Z}_N}$, $\int dg_2 \equiv \frac{1}{M}\sum_{j_2,k_2\in\mathbb{Z}_M}$, so that by Weyl transform 4.45 we have

$$\begin{aligned}\rho &= \int dg_1 dg_2 \; \chi_\rho(g_1, g_2) D_A(g_1) \otimes D_B(g_2) \\ &= \sum_i p_i \int dg_1 dg_2 \; (\mathscr{D}_{D_A} |\psi_i\rangle\langle\psi_i|)(g_1) D_A(g_1) \otimes (\mathscr{D}_{D_B} |\phi_i\rangle\langle\phi_i|)(g_2) D_B(g_2) \\ &= \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \,.\end{aligned}$$

□

In light of theorem 4.4.6, we can give the following

**Definition 4.4.7.** If $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N) \otimes L^2(\mathbb{Z}_M))$, the corresponding function of quantum positive type $\chi_\rho$ is *separable* if $\rho$ is a separable state.

Another characterization is available for pure states, which relies on orthogonality relations [29]:

**Theorem 4.4.8.** *If $\rho$ is a pure state on $L^2(\mathbb{Z}_N) \otimes L^2(\mathbb{Z}_M)$, $\chi_\rho$ is separable if and only if*

$$\sum_{j_1,k_1\in\mathbb{Z}_N} |\chi_\rho(j_1, k_1, 0, 0)|^2 = \frac{1}{M^2}, \tag{4.92}$$

$$\sum_{j_2,k_2\in\mathbb{Z}_M} |\chi_\rho(0, 0, j_2, k_2)|^2 = \frac{1}{N^2}. \tag{4.93}$$

For instance, if $\rho = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$ , $\psi \in L^2(\mathbb{Z}_N), \phi \in L^2(\mathbb{Z}_M)$ so that $\chi_\rho$ is separable, we have

$$\begin{aligned}
\sum_{j_1,k_1\in\mathbb{Z}_N} |\chi_\rho(j_1,k_1,0,0)|^2 &= \sum_{j_1,k_1\in\mathbb{Z}_N} \overline{\chi_\rho(j_1,k_1,0,0)}\chi_\rho(j_2,k_2,0,0) \\
&= \frac{1}{N^2M^2}\sum_{j_1,k_1\in\mathbb{Z}_N} \overline{\langle (D_A(j_1,k_1)\otimes \mathrm{Id})(\psi\otimes\phi),\psi\otimes\phi\rangle}\cdot \\
&\cdot \langle (D_A(j_1,k_1)\otimes \mathrm{Id})(\psi\otimes\phi),\psi\otimes\phi\rangle \\
&= \frac{1}{M^2}|\langle\psi,\psi\rangle|^2|\langle\phi,\phi\rangle|^2 = \frac{1}{M^2}
\end{aligned}$$

(clearly the proof of the other relation is analogous). On the other hand, suppose for example that $\rho = |\Psi\rangle\langle\Psi|$ , $\Psi \in L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M)$ is a pure state and that 4.92 holds true. Then we have

$$\begin{aligned}
\frac{1}{M^2} &= \sum_{j_1,k_1\in\mathbb{Z}_N} |\chi_\rho(j_1,k_1,0,0)|^2 \\
&= \frac{1}{N^2M^2}\sum_{j_1,k_1\in\mathbb{Z}_N} \langle\Psi,(D_A(j_1,k_1)\otimes\mathrm{Id})\Psi\rangle\langle(D_A(j_1,k_1)\otimes\mathrm{Id})\Psi,\Psi\rangle.
\end{aligned}$$

However, in order to satisfy the latter equivalence, $\Psi$ shall be a separable state; in this way the action of $D_A \otimes \mathrm{Id}$ on $\Psi$ factorizes and orthogonality relations can be applied. The same holds for relation 4.93.
We remark that the above theorem is a group-theoretical equivalent of a standard result of quantum information, which characterize pure states as the one and only states such that [24]

$$\mathrm{tr}_A(\mathrm{tr}_B(\rho))^2 = \mathrm{tr}_B(\mathrm{tr}_A(\rho))^2 = 1, \quad \rho \in \mathcal{S}(L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M)) \tag{4.94}$$

We now sketch an alternative take on Horodecki's theorem - an important characterization of separable states on finite bipartite systems - founded on functions of quantum positive type. This will be achieved with the following steps: firstly, we briefly review the standard formulation in the Hilbert space formalism. Next, we apply some results that holds for compact groups [31] to the case of $\mathbb{H}(\mathbb{Z}_N)$; in particular, we will briefly discuss of characteristic functions defined on $\mathbb{H}(\mathbb{Z}_N)$, which, on one hand, mimics the discrete characteristic functions 4.42 and, on the other hand, corresponds to functions of positive type on $\mathbb{H}(\mathbb{Z}_N)$. Hence, we will enounce an important theorem which holds for compact groups and that can be regarded as a group-theoretical analogous of Horodecki's theorem. Eventually, we exploit the link between $\mathbb{H}(\mathbb{Z}_N)$ and $\mathbb{Z}_N \times \mathbb{Z}_N$ given by the central extension via $\mathbb{Z}_N$. In this way, we can relate functions of quantum positive type on $\mathbb{Z}_N \times \mathbb{Z}_N$ with functions of positive type on $\mathbb{H}(\mathbb{Z}_N)$, so that we will be able to focus on discrete phase space.

Let us recall that a linear map $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is *positive* if it preserves positivity, namely $\rho \geq 0$ implies $\Lambda\rho \geq 0$. Moreover, the positive map $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is *completely positive* if, given an arbitrary Hilbert space $\mathcal{K}$ with $\dim \mathcal{K} = k < \infty$, the map $\mathrm{Id}_{\mathcal{K}} \otimes \Lambda$ is positive for each value of $k$.
Then, Horodecki's theorem holds true [31]:

**Theorem 4.4.9** (Horodecki)**.** *Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a finite Hilbert space and let us consider $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then, $\rho$ is separable iff for all positive linear maps $\Lambda : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ the operator* $(\mathrm{Id} \otimes \Lambda)\rho$ *on $\mathcal{H}_A \otimes \mathcal{H}_A$ is positive.*

For future reference, we recall the following fundamental result due to Choi [24]:

**Theorem 4.4.10** (Choi)**.** *Let $\Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ a positive map,* $\dim \mathcal{H}_A = d < \infty$, $\dim \mathcal{H}_B = d' < \infty$*. Then, $\Lambda$ is completely positive if and only if it is d-positive, namely* $\mathrm{Id}_A \otimes \Lambda$ *is a positive map.*

Let us now consider the discrete Heisenberg-Weyl group $\mathbb{H}(\mathbb{Z}_N)$ and let $S : \mathbb{H}(\mathbb{Z}_N) \to \mathcal{U}(L^2(\mathbb{Z}_N))$ be an irreducible unitary representation defined as in 2.58 (with $\lambda = 1$). Then, we can define a *non-commutative characteristic function* for a given quantum state $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N))$ as a continuous function such that [29]

$$\xi_\rho(\tau, j, k) := \mathrm{tr}(S(\tau, j, k)^* \rho). \tag{4.95}$$

Observe that 4.95 can be interpreted as a function of positive type on the group algebra $(L^1(\mathbb{H}(\mathbb{Z}_N)), *, \mathcal{I})$, where $*$ is the convolution 3.6 and $\mathcal{I}$ is the involution 3.7) [29, 30]. Indeed, for a given quantum state $\rho$, we have

$$\begin{aligned}
\int dgdh\ \overline{\psi(h)}\psi(g)\xi_\rho(h^{-1}g) &= \int dgdh\ \overline{\psi(h)}\psi(g)\,\mathrm{tr}\big(S(h^{-1}g)^*\rho\big) \\
&= \int dgdh\ \overline{\psi(h)}\psi(g)\,\mathrm{tr}(S(g)^* S(h)\rho) \\
&= \mathrm{tr}(FF^*\rho) \geq 0,
\end{aligned}$$

where $F \equiv \int dg\psi(g)S(g)^*$. As always, in such a case we will denote with $P(\mathbb{H}(\mathbb{Z}_N))$ the set of functions of positive type on $\mathbb{H}(\mathbb{Z}_N)$. Next, we consider a bipartite system $AB$ whose Hilbert space is $L^2(\mathbb{Z}_N) \otimes L^2(\mathbb{Z}_M)$, which is

here viewed as the space of the representation $S \equiv S_A \otimes S_B$. In particular, we remark that $S_A : \mathbb{H}(\mathbb{Z}_N) \to \mathcal{U}(L^2(\mathbb{Z}_N))$ and $S_B : \mathbb{H}(\mathbb{Z}_M) \to \mathcal{U}(L^2(\mathbb{Z}_M))$ are two irreducible unitary representations. Hence, we have that a state $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N) \otimes L^2(\mathbb{Z}_M))$ is separable if and only if $\xi_\rho \in P(\mathbb{H}(\mathbb{Z}_N) \times \mathbb{H}(\mathbb{Z}_M))$ is separable, i.e. it factorizes (the proof is analogous to the one of theorem 4.4.6) [29].
In the following, given a compact group $H$, $\iota : L^p(H) \ni \phi \mapsto \phi \in L^p(H)$ will denote the identity map which sends a function in $L^p(H)$ in itself, where $p = 1, 2, \ldots, \infty$. In this way we can give the following

**Definition 4.4.11.** Let $\tilde{\mathscr{L}} : L^\infty(\mathbb{H}(\mathbb{Z}_M)) \to L^\infty(\mathbb{H}(\mathbb{Z}_N))$ be a linear map. Then

- $\tilde{\mathscr{L}}$ is a *map of positive type* if maps functions of positive type in $L^\infty(\mathbb{H}(\mathbb{Z}_M))$ into functions of positive type in $L^\infty(\mathbb{H}(\mathbb{Z}_N))$.
- If $H$ is a compact group, $\tilde{\mathscr{L}}$ is a map of $H$-*positive type* if $\iota \otimes \tilde{\mathscr{L}}$ is a map of positive type from $L^\infty(H \times \mathbb{H}(\mathbb{Z}_M))$ to $L^\infty(H \times \mathbb{H}(\mathbb{Z}_N))$.
- $\tilde{\mathscr{L}}$ is a *map of completely positive type* if it is of $H$-positive type for any compact group $H$.

We notice that the map $\iota \otimes \tilde{\mathscr{L}}$, is rigorously defined on $L^\infty(H) \otimes L^\infty(\mathbb{H}(\mathbb{Z}_M))$; then, by continuity, we extend it to the whole $L^\infty(H \times \mathbb{H}(\mathbb{Z}_M))$ [31].
Thence, the following result holds true [31]:

**Theorem 4.4.12.** *$\xi \in P(\mathbb{H}(\mathbb{Z}_N) \times \mathbb{H}(\mathbb{Z}_M))$ is separable if and only if for each map of positive type $\tilde{\mathscr{L}} : L^\infty(\mathbb{H}(\mathbb{Z}_M)) \to L^\infty(\mathbb{H}(\mathbb{Z}_N))$ we have that $(\iota \otimes \tilde{\mathscr{L}})\xi \in P(\mathbb{H}(\mathbb{Z}_N) \times \mathbb{H}(\mathbb{Z}_N))$.*

Moreover, we also have the following group-theoretical counterpart of Choi's theorem [31]:

**Proposition 4.4.13.** *$\tilde{\mathscr{L}} : L^\infty(\mathbb{H}(\mathbb{Z}_M)) \to L^\infty(\mathbb{H}(\mathbb{Z}_N))$ is a map of $\mathbb{H}(\mathbb{Z}_M)$-positive type if and only if it is a map of completely positive type.*

We can now observe that the above theorems hold true for functions of quantum positive type on discrete phase space too, since $\mathbb{H}(\mathbb{Z}_N)$ is the central extension via $\mathbb{Z}_N$ of $\mathbb{Z}_N \times \mathbb{Z}_N$. In order to better understand this point, recall that the projective representations of $\mathbb{Z}_N \times \mathbb{Z}_N$ are linked with the unitary ones of $\mathbb{H}(\mathbb{Z}_N)$ (see section 2.4.2). We also consider the group algebra $(L^1(\mathbb{H}(\mathbb{Z}_N)), *)$ and the twisted group algebra $(L^2(\mathbb{Z}_N \times \mathbb{Z}_N), \star_S)$, where $\star_S$ is the twisted convolution induced by a projective representation $S$

with multiplier $\mu_S((j,k),(j',k')) = e^{i\frac{2\pi}{N}jk'}$ defined as in 2.75. In particular, $\star_S$ is given by

$$\begin{aligned}(\psi \star_S \phi)(j,k) &:= \sum_{j',k' \in \mathbb{Z}_N} \psi(j',k')\phi(j-j',k-k')\overline{\mu_S((j',k'),(j-j',k-k'))} \\ &= \sum_{j',k' \in \mathbb{Z}_N} \psi(j',k')\phi(j-j',k-k')e^{i\frac{2\pi}{N}j'k'}e^{-i\frac{2\pi}{N}j'k}.\end{aligned} \tag{4.96}$$

Let $\sigma : \mathbb{Z}_N \ni \tau \to e^{i\frac{2\pi}{N}\tau} \in \mathbb{T}$ be a character on $\mathbb{Z}_N$ and let $\psi, \phi \in L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$. Then we have that $\Psi(\tau,j,k) := \sigma(\tau)\psi(j,k)$ and $\Phi(\tau,j,k) := \sigma(\tau)\phi(j,k)$ are in $L^1(\mathbb{H}(\mathbb{Z}_N))$ and the following relation holds [20]

$$\begin{aligned}(\Psi * \Phi)(\tau,j,k) &= \sum_{\tau',j',k'} \Psi(\tau',j',k')\Phi((\tau',j',k')^{-1}(\tau,j,k)) \\ &= \sum_{\tau',j',k'} \Psi(\tau',j',j')\Phi(\tau-\tau'+j'k'-j'k, j-j', k-k') \\ &= Ne^{i\frac{2\pi}{N}\tau}\sum_{j',k'} \psi(j',k')\phi(j-j',k-k')e^{i\frac{2\pi}{N}j'k'}e^{-i\frac{2\pi}{N}j'k} \\ &= N\sigma(\tau)\left(\psi \star_S \phi\right)(j,k).\end{aligned}$$

Thence, by the extension via the character $\sigma$, we can consider functions of positive type on $\mathbb{H}(\mathbb{Z}_N)$ which act on elements of $L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$. In particular, given $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N))$ and the corresponding non-commutative characteristic function $\xi_\rho \in P(\mathbb{H}(\mathbb{Z}_N))$, we have that $\xi_\rho$ induces a function of quantum positive type on $\mathbb{Z}_N \times \mathbb{Z}_N$:

$$\begin{aligned}0 &\leq \sum_{\tau,j,k} \xi_\rho(\tau,j,k)\left(\Psi^* * \Psi\right)(\tau,j,k) \\ &= N\sum_{\tau,j,k} \mathrm{tr}(S(\tau,j,k)^*\rho)\sigma(\tau)\left(\psi^* \star_S \psi\right)(j,k) \\ &= N^2\sum_{j,k} \mathrm{tr}(S(j,k)^*\rho)\left(\psi^* \star_S \psi\right)(j,k)\end{aligned}$$

(recall that $S(\tau,j,k) = e^{-i\frac{2\pi}{N}\tau}M_kT_j = e^{-i\frac{2\pi}{N}\tau}S(j,k)$). We remark two facts: firstly, $\mathrm{tr}(S(j,k)^*\rho)$ is - up to a positive factor - the discrete Wigner function induced by the projective representation $S$. Secondly, $S$ is projectively equivalent to $D$, hence, in the following, we can consider $\mathrm{tr}(D^*\rho)$ in place of $\mathrm{tr}(S^*\rho)$ .

We can now introduce maps of positive type in analogy with definition 4.4.11. In particular, given a linear map $\mathscr{L} : L^2(\mathbb{Z}_M \times \mathbb{Z}_M) \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, we will say that

- $\mathscr{L}$ is a *map of positive type* if maps functions of quantum positive type in $L^2(\mathbb{Z}_M \times \mathbb{Z}_M)$ into functions of quantum positive type in $L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$.
- Given a compact group $H$, $\mathscr{L}$ is a map of *$H$-positive type* if $\iota \otimes \mathscr{L}$ is a map of positive type.
- $\mathscr{L}$ is a *map of completely positive type* if it is of $H$-positive type for any compact group $H$.

Hence, bearing in mind the link between functions of positive type on $\mathbb{H}(\mathbb{Z}_N)$ and functions of quantum positive type on $\mathbb{Z}_N \times \mathbb{Z}_N$, thanks to theorem 4.4.14 we have the following

**Theorem 4.4.14.** *A function of quantum positive type $\chi_\rho$ on $(\mathbb{Z}_N \times \mathbb{Z}_N) \times (\mathbb{Z}_M \times \mathbb{Z}_M)$ is separable if and only if for each map of positive type $\mathscr{L} : L^2(\mathbb{Z}_M \times \mathbb{Z}_M) \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ the function $(\iota \otimes \mathscr{L})\chi_\rho$ is of quantum positive type on $(\mathbb{Z}_N \times \mathbb{Z}_N) \times (\mathbb{Z}_N \times \mathbb{Z}_N)$.*

Similarly, proposition 4.4.13 gives us the following result:

**Proposition 4.4.15.** *$\mathscr{L} : L^2(\mathbb{Z}_M \times \mathbb{Z}_M) \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ is a map of $(\mathbb{Z}_M \times \mathbb{Z}_M)$-positive type if and only if it is a map of completely positive type.*

We now highlight that theorem 4.4.14 is equivalent to Horodecki's theorem 4.4.9. To fulfill this task, let us start with the following result:

**Proposition 4.4.16.** *There is a one-to-one correspondence between maps of positive type and positive maps.*

*Proof.* Suppose $\mathscr{L} : L^2(\mathbb{Z}_M \times \mathbb{Z}_M) \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ is a map of positive type at first. If $\rho_B \in \mathcal{B}_2(L^2(\mathbb{Z}_M))$ is a positive operator, then $\chi_{\rho_B} = \mathscr{D}_{D_B}\rho_B \in P_Q(\mathbb{Z}_M \times \mathbb{Z}_M)$. Hence, $\mathscr{L}\chi_{\rho_B}$ is a function of quantum positive type on $\mathbb{Z}_N \times \mathbb{Z}_N$ which corresponds, via the Weyl transform, to a positive operator $\rho_A \in \mathcal{B}_2(L^2(\mathbb{Z}_N))$. In brief, since all the maps involved preserve positivity and are linear, we have that

$$\Lambda = \mathscr{Q}_{D_A} \circ \mathscr{L} \circ \mathscr{D}_{D_B} : \mathcal{B}_2(L^2(\mathbb{Z}_M)) \to \mathcal{B}_2(L^2(\mathbb{Z}_N)) \tag{4.97}$$

is a positive linear map. The converse also holds true, because the Wigner and the Weyl transforms are unitary operators. In particular, by 4.97, we have that

$$\mathscr{L} = \mathscr{D}_{D_A} \circ \Lambda \circ \mathscr{Q}_{D_B} : L^2(\mathbb{Z}_M \times \mathbb{Z}_M) \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N). \tag{4.98}$$

Hence, if $\Lambda$ is positive, since $\mathscr{D}$ and $\mathscr{Q}$ preserves positivity, $\mathscr{L}$ is positive. □

Suppose now $\mathscr{L}$ is a map of $(\mathbb{Z}_M \times \mathbb{Z}_M)$-positive type. Following the proof of proposition 4.4.16, we have that $\Lambda$ is a completely positive linear map. Indeed, if $\mathcal{B}_2(L^2(\mathbb{Z}_M)\otimes L^2(\mathbb{Z}_M))) \ni \rho \geq 0$ and $\chi_\rho$ is the corresponding function of quantum positive type, we have that $(\iota \otimes \mathscr{L})\chi_\rho \in L^2((\mathbb{Z}_M \times \mathbb{Z}_M)\times(\mathbb{Z}_N\times\mathbb{Z}_N))$ is a function of quantum positive type, which corresponds to a positive operator on $L^2(\mathbb{Z}_M) \otimes L^2(\mathbb{Z}_N)$. In other terms, the map

$$\mathrm{Id}\otimes\Lambda = \mathrm{Id}\otimes(\mathscr{Q}_{D_A}\circ\mathscr{L}\circ\mathscr{D}_{D_B}) = (\mathscr{Q}_{D_B}\otimes\mathscr{Q}_{D_A})\circ(\iota\otimes\mathscr{L})\circ(\mathscr{D}_{D_B}\otimes\mathscr{D}_{D_B}) \tag{4.99}$$

is a positive map from $L^2(\mathbb{Z}_M) \otimes L^2(\mathbb{Z}_M)$ to $L^2(\mathbb{Z}_M) \otimes L^2(\mathbb{Z}_N)$ [31], hence $\Lambda$ is $M$-positive and, by Choi's theorem, it is a completely positive map. Conversely, if $\Lambda$ is a $M$-positive linear map, we can suitably define $\mathscr{L}$ in such a way that it is a map of $(\mathbb{Z}_M \times \mathbb{Z}_M)$-positive type, hence it is a map of completely positive type. In short, we have

**Theorem 4.4.17.** *There is a one-to-one correspondence between maps of completely positive type and completely positive maps.*

With these facts in mind we can now see that theorem 4.4.14 corresponds to Horodecki's theorem 4.4.9. Indeed, $\rho \in \mathcal{S}(L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M))$ is separable iff $\chi_\rho \in P_Q((\mathbb{Z}_N\times\mathbb{Z}_N)\times(\mathbb{Z}_M\times\mathbb{Z}_M))$ is separable. Hence, by theorem 4.4.14, given an arbitrary map of positive type $\mathscr{L} : L^2(\mathbb{Z}_M \times \mathbb{Z}_M) \to L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$, $(\iota\otimes\mathscr{L})\chi_\rho$ is a function of quantum positive type on $(\mathbb{Z}_N\times\mathbb{Z}_N)\times(\mathbb{Z}_N\times\mathbb{Z}_N)$. Thus, $\Lambda := \mathscr{Q}_{D_A} \circ \mathscr{L} \circ \mathscr{D}_{D_B}$ is a positive map and $(\mathrm{Id} \otimes \Lambda)\rho$ is a positive operator on $L^2(\mathbb{Z}_N) \otimes L^2(\mathbb{Z}_N)$. Therefore, theorem 4.4.14 implies theorem 4.4.9. Clearly, the converse is also true, since these are all "iff" conditions.

Lastly, we quickly discuss the PPT criterion, which in its standard formulation assumes the following form:

**Theorem 4.4.18** (PPT criterion)**.** *If $\rho \in \mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_B)$ is a separable state, then the operator $\rho^{T_B} := (\mathrm{Id} \otimes T)\rho$ is positive, where $T$ is the transposed operator on $\mathcal{H}_B$ such that $T\rho_B = \rho_B^T,\ \rho_B \in \mathcal{S}(\mathcal{H}_B)$.*

Its phase space analogous is quite simple. Indeed, observe that

$$\chi_\rho(-j,-k) = \frac{1}{N}\operatorname{tr}(D(-j,-k)^*\rho) = \frac{1}{N}\operatorname{tr}(D(j,k)\rho) = \overline{\chi_\rho(j,k)} \quad (4.100)$$

(recall that $\mu((j,k),(-j,-k)) = 1$, so that $D(-j,-k) = D(j,k)^*$).
Hence, recalling separability condition 4.4.6, we have [29, 31]

**Proposition 4.4.19.** *If $\chi_\rho \in P_Q(\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_M)$ is a separable function of quantum positive type, then $\tilde{\chi}_\rho(j_1,k_1,j_2,k_2) := \chi_\rho(j_1,k_1,-j_2,-k_2)$ is a function of quantum positive type.*

The link with the usual formulation is given by the following [29]

**Proposition 4.4.20.** *Given a finite quantum state $\rho$ in $L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M)$, $\rho^{T_B}$ is a positive operator if and only if $\tilde{\chi}_\rho$ is a function of quantum positive type.*

The proof of this fact is straightforward if we recall that any representation determines a *contragredient representation*, defined as $\overline{D(j,k)} := D(-j,-k)^T$, whose matrix coefficient are the complex conjugate of those of $D(j,k)$ (such representation depends on the choice of the base because of the transpose operator) [18]. Indeed, we have that $\overline{D}$ is still a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ whose multiplier is the complex conjugate of the multiplier of $D$:

$$\begin{aligned}
\overline{D(j+j',k+k')} &= D(-j'-j,-k'-k)^T \\
&= \mu((-j',-k'),(-j,-k))\left(D(-j',-k')D(-j,-k)\right)^T \\
&= \overline{\mu\left((j,k),(j',k')\right)}\;\overline{D(j,k)}\;\overline{D(j',k')}.
\end{aligned}$$

We also notice that $\overline{D(j,k)}$ is irreducible, because so it is $D$ [29]. In this way, we have that

$$\tilde{\chi}_\rho(j,k) = \frac{1}{N}\operatorname{tr}(D(-j,-k)^*\rho) = \frac{1}{N}\operatorname{tr}\Big(\overline{D(j,k)^*}\rho^T\Big). \quad (4.101)$$

Hence, returning to $L^2(\mathbb{Z}_N)\otimes L^2(\mathbb{Z}_M)$, by Weyl transform, we have that

$$\begin{aligned}
\rho^{T_B} &= (\mathrm{Id}\otimes T)\rho \\
&= \frac{1}{NM}\sum_{j_1,k_1\in\mathbb{Z}_N}\sum_{j_2,k_2\in\mathbb{Z}_M}\tilde{\chi}_\rho(j_1,k_1,j_2,k_2)D_A(j_1,k_1)\otimes\overline{D_B(j_2,k_2)},
\end{aligned} \quad (4.102)$$

therefore, $\rho^{T_B}$ is positive if and only if $\tilde{\chi}_\rho$ is a function of quantum positive type.

We conclude with two remarks. First, we observe that only positive maps are interesting for Horodecki's theorem 4.4.14, because $(\iota \otimes \mathscr{L})\chi_\rho$ is always a function of quantum positive type if $\mathscr{L}$ is a map of completely positive type. Secondly, Horodecki's theorem, in a certain sense, generalizes PPT criterion 4.4.19: we obtain a characterization of separable functions of quantum positive type when we take into account all the maps $\mathscr{L}$ such that $(\iota \otimes \mathscr{L})\chi_\rho$ is a function of quantum positive type. If we merely require that $\tilde{\chi}$ is a function of quantum positive type, as in the PPT criterion, we have a necessary condition for separability only.

# Conclusions

In the present work we have discussed of the phase space quantization, an alternative formulation of quantum mechanics which is built on group theory. In particular, we have introduced a formulation of quantum mechanics on finite Hilbert spaces solely relying on the irreducible projective representations of a discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$, suitably chosen in such a way that the group structure $G \times \hat{G}$, where $G$ is Abelian, is preserved.
We have accomplished this task following the steps usually taken to construct quantum mechanics on phase space. Hence, a preliminar discussion on representation theory of l.c.s.c. groups has been necessary: we have introduced unitary and projective representations, the seconds which have been fundamental because, by Schur's lemma, the only irreducible unitary representations of the continuous and the discrete phase space are one-dimensional, hence physically trivial.
At this point, the necessary tools to develop quantum mechanics on discrete phase space has been discussed in analogy with the standard phase space, which has always been discussed first as a guide. In particular, in the case of $\mathbb{R}^n \times \mathbb{R}^n$ (respectively, $\mathbb{Z}_N \times \mathbb{Z}_N$), by a suitable central extension, we have linked its projective representations with the unitary ones of the Heisenberg-Weyl group $\mathbb{H}_n(\mathbb{R})$ (respectively, $\mathbb{H}(\mathbb{Z}_N)$). In this context, the symplectic structure of the phase space has arisen looking at the classification of its multipliers only. Moreover, by means of such representations, we have been able to introduce Weyl systems, which allow a mathematically consistent formulation of the CCRs.
However, due to finiteness, the irreducible representations of $\mathbb{H}(\mathbb{Z}_N)$ behaves in a slightly different way with respect to the ones of $\mathbb{H}_n(\mathbb{R})$: some irreducible representations (labeled by a parameter $\lambda \in \mathbb{Z}_N$ which resembles $\hbar$) could bring a rescaling of the finite system considered. In particular, the definition of finite Weyl system resembles a continuous irreducible Weyl system if $\gcd(\lambda, N) = 1$, otherwise it behaves as a continuous reducible one when acting on a $N$-dimensional Hilbert space. Regarding Stone-von Neumann's theorem, analogous considerations hold true.

Next, we have introduced the main tools of harmonic analysis on phase space, by means of which on one hand the standard Wigner function has been retrieved, while on the other hand we have been able to deal with the discrete Wigner function. The symplectic Fourier transform, which entails in its character the symplectic structure of $G \times \hat{G}$, and the twisted convolution, namely the $\star$-product of functions on phase space, has been discussed. Then, we have defined the Gabor transform, the fundamental tool in time-frequency analysis, that turned out to be a Wigner function disguised. At last, we have discussed of the notion of square integrable representation, on which the phase space approach relies entirely. Indeed, we have seen that we can define a Wigner transform for every l.c.s.c. group $G$ that admits a square integrable (projective) representation. Then the Weyl transform is introduced as the pseudo-inverse map of the Wigner transform. As a result, the space $L^2(G)$ has been provided of a non-commutative associative product, the $\star$-product (which corresponds to twisted convolution for unimodular groups), with respect to it is a Banach $*$-algebra. Then, the standard Wigner function on $\mathbb{R}^n \times \mathbb{R}^n$ has been introduced by virtue of square integrability of Weyl systems, as well as the discrete one on $\mathbb{Z}_N \times \mathbb{Z}_N$. We have also compared this Wigner function with the ones defined by means of phase-point operators. In particular, we have observed that, in the discrete case, the two definitions coincide for a discrete phase space whose order is odd only, because in the even case the phase-point operators does not form a basis in $\mathcal{M}_N(\mathbb{C})$. About that, a possible development of this work may be a group theoretical resolution of the inconsistency of the Wigner function defined in terms of phase point operators when $N$ is even, introducing some redundancy as in [34]. As a last topic, we have discussed the separability problem of quantum states of a finite bipartite system, which can be regarded as the product of two discrete phase space. In particular, we have seen how functions of quantum positive type on $L^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ allow us to give a phase space formulation of some famous criteria, such as PPT criterion and Horodecki's theorem.

In summary, in this thesis work we have emphatized the central role of harmonic analysis in the phase space approach to quantum mechanics, whose interest in has been renewed in the last years due to various applications in quantum information and quantum computing. In particular, the square-integrability of the representations considered has played a crucial role in the generalization of the Weyl-Wigner correspondence to $\mathbb{Z}_N \times \mathbb{Z}_N$, which also served at introducing a $\star$-product of functions.

Of course we have only been able to study the most fundamental facts of the Weyl-Wigner correspondence applied on discrete phase space and the possible future developments are various. A first route to explore may

be the study of the behaviour of the discrete Wigner function under the automorphisms of $\mathbb{H}(\mathbb{Z}_N)$, among them we find the (discrete) symplectic transformations, which are particularly insidious to deal with because of the underlying finite geometry of $\mathbb{Z}_N \times \mathbb{Z}_N$; this topic is also closely connected with the study of mutually unbiased bases [46].
On the other hand, there are many things left behind that can be analyzed more deeply. First of all, we may employ the $\star$-product to describe the dynamical evolution of a quantum state in terms of Wigner function. Infact, here we have only observed that the $\star$-product induced by the generalized Wigner transform is the twisted convolution on $\mathbb{Z}_N \times \mathbb{Z}_N$; clearly the Grönewold-Moyal product can also be introduced in the picture via the discrete Wigner function. Moreover, only the discrete Wigner function induced by a $N$-dimensional Weyl system has been studied. However, if we consider a Weyl system whose dimension is strictly less than $N$, we may lose unitarity of the Weyl and Wigner transforms, and things become trickier. For instance, some proofs concerning functions of quantum positive type here presented rely heavily on unitarity of the Wigner function; further investigations in this direction may be interesting.
The discussion on entanglement can be naturally deepened. Firstly, we have discussed of bipartite systems only and further developments can be made in this direction, for example trying to find new entanglement criteria relying on the powerful tools of harmonic analysis. On the other hand, the language of functions of quantum positive type may be applied to multipartite systems. At any rate, the deep bound with time-frequency analysis may allow us to investigate these interesting topics in an alternative way than the usual approaches to finite quantum mechanics.

Anyhow, the discrete phase space is not the only generalized phase space of physical interest on which quantum mechanics can be built. For instance, the group $\mathbb{Z} \times \mathbb{T}$ is a suitable phase space candidate and represents a considerable playground for some quantum systems [1] which, at least as far as we know, is approached via the phase-point operators only [25].

From a more abstract point of view, progresses can be made too.
The problem of unitarity of the Wigner transform can be approached in a more general way, for example investigating wether the general phase space $G \times \hat{G}$ always admits a square integrable representation with respect to the induced dequantization map is unitary.
Lastly, we observe that, from a physical point of view, the Weyl-Wigner correspondence may be rather a limited approach in some areas of research, because it relies entirely on the notion of square integrable representation. For instance, the Wigner dequantization scheme cannot be applied in the case of the Poincaré group, because the latter does not admit square in-

tegrable representations [8]. In such a case, we must relax our hypothesis releasing the square integrability of the representations. Then, a possible way to deal with the Poincaré group can be the introduction of a weak wavelet transform [3], which can be used to achieve a representation of the quantum relativistic vector state on the space of square integrable functions on the latter group.

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