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Teorie di gauge e propagatori in spazio-tempo curvo
Gauge Field Theories and Propagators in curved space-time

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Introduction

The study of quantum gauge field theories and gravitation is both an intellectual pursuit and a necessity connected to black hole theory and quantum cosmological models. In this work DeWitt's formalism for field theories is presented; it provides a framework in which the quantization of fields possessing infinite dimensional invariance groups may be carried out in a manifestly covariant (non-Hamiltonian) fashion, even in curved space-time; particular attention is paid to the evaluation of the propagators for scalar field theory and Maxwell's theory.

Another important virtue of DeWitt's approach is that it emphasizes the common features of apparently very different theories such as Yang-Mills theories and General Relativity; moreover, it makes it possible to classify all gauge theories in three categories characterized in a purely geometrical way, i.e., by the algebra which the generators of the gauge group obey; the geometry of such theories is the fundamental reason underlying the emergence of ghost fields in the corresponding quantum theories, too. These "tricky extra particles", as Feynman called them in 1964, contribute to a physical observable such as the stress-energy tensor, which can be expressed in terms of Feynman's Green function itself.

This work is structured in six chapters; in the first one, gauge field theories are introduced in DeWitt's formalism, and the set of all fields is presented as an infinite-dimensional manifold, on which an action functional is defined; then gauge transformations are viewed as flows which leave the action functional unchanged; whenever the gauge group realization is linear (this is the case for Yang-Mills theories and General Relativity), manifest covariance is ensured, i.e., the group transformation laws for the various symbols that appear in the theory may be inferred simply from the position and nature of their indices, and both sides of any equation transform similarly. Last, the theory of small disturbances is discussed, Green's functions are introduced in a general fashion and Peierls brackets are defined, in light of their importance in the quantization procedure.

In the second chapter, the path integral formulation of one-particle Quantum Mechanics is derived from the standard, Schrödinger-equation-based theory; then the path integral for (non-gauge, bosonic) field theory is heuristically derived. In the third chapter, quantization of non-gauge field theories is discussed in the framework of DeWitt's formalism: problems with the heuristic quantization rules are stressed, and Schwinger's variational principle is introduced as a way to get around them; then the operator dynamical equations are presented, with the necessary introduction of the measure functional, which, at the simplest level, may be thought of as correcting the lack of self-adjointness of the "time-ordered" version of the classical dynamical equations; nevertheless, it plays a far deeper role, closely linked to the Wick rotation for the evaluation of divergent integrals. Last, the path integral for non-gauge theories is derived.

In the fourth chapter the treatment is extended to gauge theories: some insight is given about the geometrical structure of the infinite-dimensional manifold of field theories belonging to the same class as Yang-Mills and General Relativity; under the group action, this space is separated into orbits; one could say that it is in the space of orbits that the real physics of the system takes place; then one can choose on the manifold a new set of coordinates made of two parts: the first one labelling the orbit, and the second one labelling a particular field configuration belonging to the specified orbit; when one writes down the path integral and reverts to the original coordinates, one finds out that a new term appears: it is the ghost contribution, which involves two ghost fields; they are fermionic for a bosonic theory and bosonic for a fermionic theory. Therefore, an "extended" space may be introduced, where ghost fields appear too and a new action functional, containing also ghost terms, may be defined: then one is fascinated to find out that the gauge transformations for the original theory correspond to a set of rigid invariance transformations for the theory on the extended space, i.e., the BRST transformations.

The fifth chapter is dedicated to the study of the Green function of free, massive, scalar field, first in flat, then in curved space-time; it is shown that all Green functions can be derived from Feynman's Green function: in flat space-time, one can pass to the "momentum" space and easily verify that the choice of the Green function is the choice of an integration contour which passes around two poles; in curved space-time, where the momentum space is no longer avaiable, one can nevertheless define the Feynman's Green function: it is the one that obeys the same variational law as finite square matrices and is symmetric. In sight of the curved space-time treatment, it is of fundamental importance to derive, even in the flat case, a formula for the Feynman propagator involving space-time coordinates only; therefore, after the introduction of some necessary mathematical tools such as bitensors and some geometric quantities such as the world function, an expansion for the Feynman Green function in curved space-time is obtained, valid for small values of the geodetic distance. It is important to observe that the expansion obtained ceases to exist when the massless limit is taken: the massless case has to be treated with alternative methods.

In the final chapter the Feynman Green function for Maxwell's theory in curved space-time is discussed: both ζ-function regularization method and Fock-Schwinger-DeWitt ansatz are used in order to obtain an asymptotic formula for the Feynman propagator, valid for small values of the geodetic distance; therefore the expansion obtained is verified to exhibit the familiar logarithmic singularity which occurs for massive theories in flat space-time and, generally, even for massless theories but in curved space-time. Then the point-splitting method is presented as a valuable tool for regularizing divergent observables such as the stress-energy tensor: one finds out that it can be completely expressed in terms of second derivatives of the Hadamard Green function, which is also closely linked to the effective action. Last, an original computation is presented: a short, closed formula for the divergent part of the stress-energy tensor, originating from a careful handling of more than two thousand terms; for this purpose, a program has been written in FORM, which is a symbolic manipulation system whose original author is Jos Vermaseren at NIKHEF.

This result may be of interest in several applications, and it may be further generalized, as will be outlined in the conclusions.

Chapter 1

Field Theories in DeWitt's Formulation

1.1 A few words on Space-Time (I)

Basic to the whole of quantum field theory is the assumption that space-time, which we shall denote by M (for manifold), has the topological structure

$$M = \mathcal{R} \times \Sigma \tag{1.1}$$

where $\mathcal{R}$ is the real line and Σ is some connected three-dimensional manifold, compact or non-compact. In particular, space-time will be assumed to be endowed with a hyperbolic metric g which admits a foliation of space-time into spacelike sections, each being a complete Cauchy hypersurface (i.e., a spacelike surface which intersects every non spacelike curve *exactly once*) and a topological copy of Σ. Being characterized by a manifold and a tensor field defined on the manifold, it is more accurate to say that a "space-time" is an equivalence class of pairs (M, g): the equivalence relation is the following:

$$(M, g) \sim (N, h) \longleftrightarrow \exists \psi \in Diff(M) | \; N = \psi(M), \; g = \psi^* h, \tag{1.2}$$

where ψ^* is the pullback map associated to ψ. Throughout this work, the following sign convention will be assumed for the signature of the metric tensor: $(-.+,+,...)$.

1.2 Space of Histories and Functional Differentiation

In this chapter DeWitt's formalism for field theories will be introduced, both bosonic and fermionic ones. It provides a framework within which the quantization of fields possessing infinite dimensional invariance groups may be carried out in a manifestly covariant (non-Hamiltonian) fashion.

Denote by Φ the set, or space, of all possible field histories; it will be useful to view Φ as an infinite-dimensional manifold; in this work the coordinates ϕ^i will be assumed to be real-valued, whether c-type or a-type (see Appendix A).

The concept of differentiation on Φ, on which the idea of tangent space at $\phi \in \Phi$ is based, can be introduced through *functional derivative*:

Let $F : \Phi \ni \phi \mapsto F[\phi] \in \Lambda_\infty$; F is called a supernumber-valued scalar field or *functional* on Φ, and its value at a point ϕ of Φ is denoted by $F[\phi]$. Let $\delta\phi$ be an infinitesimal variation in ϕ; it can be represented by a set of functions $\delta\phi^i$ on the manifold M, where, at each point $x \in M$, the $\delta\phi^i(x)$ are components, in the appropriate chart of Φ, of an infinitesimal vector in its tangent space at the point having coordinates $\phi^i(x)$. Let the $\delta\phi^i(x)$ be C^∞ and have compact support in M, and let $\delta F[\phi]$ denote the change in value that $F[\phi]$ undergoes in shifting from ϕ to $\phi + \delta\phi$. If, for all $\phi \in \Phi$ and for all C^∞ variations $\delta\phi$ of compact support, $\delta F[\phi]$ can be written in the form[1]

$$\delta F[\phi] = \int_M \delta\phi^i(x)\ {}_{i(x),}F[\phi]\ d^n x = \int_M F_{,i(x)}[\phi]\ \delta\phi^i(x)\ d^n x, \tag{1.3}$$

where ${}_{i(x),}F[\phi]$, $F_{,i(x)}[\phi]$ in the integrands are independent of $\delta\phi^i$ and depend at most on ϕ, then F is called a *differentiable functional* on Φ, and ${}_{i(x),}F[\phi]$, $F_{,i(x)}[\phi]$ are called *left* and *right* functional derivatives of F, respectively:

$$\begin{aligned} {}_{i(x),}F[\phi] &\equiv \frac{\overrightarrow{\delta}}{\delta\phi(x)^i} F[\phi], & (1.4)\\ F_{,i(x)}[\phi] &\equiv F[\phi] \frac{\overleftarrow{\delta}}{\delta\phi(x)^i}. & (1.5) \end{aligned}$$

Differentiation will be indicated by a comma followed by one or more indices: Greek indices will denote differentiation with respect to the chart coordinates x^μ in M, while Latin indices will denote differentiation with respect to the field coordinates.

In a repeated functional derivative it does not matter whether the left differentiations or the right differentiations are performed first, but the order of the induced indices on either side is important. That is, although left differentiations commute with right differentiations, left differentiations do not generally commute with each other, nor do right differentiations. The laws for interchanging are

$${}_{ij',}F = (-1)^{ij'}\ {}_{j'i,}F \qquad F_{,ij'} = (-1)^{ij'} F_{,j'i}, \tag{1.6}$$

where the convention is here adopted that an index or symbol appearing in an exponent of (-1) is to be understood as assuming the value 0 or 1 according as the associated quantity is c-type or a-type.

If the functional F is *pure*, i.e., either c-number-valued or a-number-valued, then (1.3) implies that its left and right functional derivatives are related by

$${}_{i,}F = (-1)^{i(F+1)} F_{,i}. \tag{1.7}$$

In the previous equation too, the symbol F in the exponent of (-1) assumes the value 0 if F is a c-type quantity, the value 1 if F is a a-type quantity.

When indices appear in exponents of -1 a special rule must be introduced regarding the summation convention: although an index appearing in an exponent of -1 may participate in the summation induced by its appearance twice elsewhere in a term of a given expression, it may not itself induce a summation.

[1]The summation convention over repeated indices is assumed throughout this work.

1.3 Condensed and Supercondensed Notations

In developing the general formalism of field theory we shall find it often convenient to lump the symbol x with the generic index i and to make the latter do double duty as a discrete label for the field components and as a continuous label for the points of space-time. With this notation, the symbol ${\delta^i}_{j'}$ should be understood as a combined δ-distribution Kronecker delta, while Kronecker deltas shall be ${\underline{\delta}^i}_j$ for the sake of clarity:

$$\phi^i \frac{\overleftarrow{\delta}}{\delta\phi^{j'}} \equiv \phi^i(x) \frac{\overleftarrow{\delta}}{\delta\phi^j(x')} = {\delta^i}_{j'} \equiv {\underline{\delta}^i}_j \; \delta(x, x') \tag{1.8}$$

Therefore, it seems natural to establish a new convention: the summation over repeated field indices includes (by virtue of their role as continuous labels) integration over M. Hence (1.3) takes the form:

$$\delta F[\phi] = \delta\phi^i \;{}_{i,}F[\phi] = F_{,i}[\phi] \; \delta\phi^i. \tag{1.9}$$

Sometimes, even this *condensed* notation is cumbersome, and a *supercondensed* notation is used: the indices themselves are suppressed and the following replacement is made:

$$_{i_r \ldots i_1,}F_{j_1 \ldots j_s} \to \;{}_rF_s. \tag{1.10}$$

Remark 1.1. The condensed and supercondensed notations must be used with care because the associative law of multiplication does not always hold. For example, the value of an expression such as $\chi^i \;{}_{i,}F_{,j}\psi^j$ may depend on which summation-integration is performed first. They give the same results only in certain cases[2]. When the law does not hold, ambiguities in condensed expressions will be removed by the use of parentheses or arrows.

1.4 A few words on Space-Time (II)

Use of the condensed notation underscores the following point: *The manifold M of space-time, independently of any physical fields that may be imposed on it, is an index set.* Its points are labels that may be lumped together with the indices for field components.

When M is viewed in this way the notion that *alternative topologies* for space-time may be alternative dynamical possibilities for a given universe makes no sense. Changing the topology of M corresponds to changing the index set, and one cannot change the index set of a theory in midstream. A different index set means a different theory.

Transitions from one topology to another *could* be followed if space-time were embedded in a higher dimensional manifold endowed with physical properties. But then space-time and its contents would not be all there is; the "universe" would be something bigger. Since nobody has yet developed a successful embedding theory of space-time we shall assume that space-time is the universe and leave its topology fixed.

[2]For example, if F is given by the expression $F[\phi] \equiv \frac{1}{2}\int_M K^{\mu}_{ij}(x)\phi^i(x)\phi^j_{,\mu}(x)d^nx$ then the reader may easily verify that if the i summation-integration is performed first one gets a result that differs by an amount $-\int_M (K^{\mu}_{ij}\phi^i\phi^j)_{,\mu}d^nx$ from that obtained when the j summation-integration is performed first. In order to get one result from the other one has to carry out an integration by parts, and this is legitimate only in certain cases, for example if the intersection of the supports of χ^i and ψ^j is compact in M.

1.5 Action Functional and Dynamical Equations

Throughout this work, the following (fundamental) principle will be postulated: the nature and dynamical properties of a classical dynamical system are completely determined by specifying an *action functional* S for it.

The action functional is a differentiable real-c-number-valued scalar field on the space of histories Φ, i.e., a functionally differentiable mapping $S : \Phi \ni \phi \mapsto S[\phi] \in \mathcal{R}_c$.

The choice of action functional for a given system is not unique but depends on the choice of dynamical variables ϕ^i used to describe the system and on the boundary conditions that one imposes on the ϕ^i at the time limits and at spatial infinity. However, all the possible action functionals for a given system must yield equivalent families of *dynamical histories*. A dynamical history is any stationary point of S, i.e., any point ϕ of Φ that satisfies

$$ {}_{i,}S[\phi] = 0 \quad \text{or, equivalently} \quad S_{,i}[\phi] = 0. \tag{1.11}$$

The set of all stationary points is called the *dynamical subspace* of Φ, or the *dynamical shell*, and all histories satisfying (1.11) are said to be *on shell*.

Equations (1.11) are called the *dynamical equations* of the system. They will be assumed to be *local* in time, i.e., involving no time integrals and not more than a finite number of time derivatives. In a relativistic theory this implies that they must also be local in space. This greatly limits the possible choices for S. Throughout this work S will have the general form

$$S[\phi] = \int_M L(\phi^i, \phi^i_{,\mu}, x)\, d^n x + \text{boundary terms}. \tag{1.12}$$

The integrand of this expression, known as the *Lagrange function* or *Lagrangian*, is a scalar density of unit weight. If the gravitational field is not numbered among the ϕ^i, then L usually has an explicit dependence on a fixed background metric. Expression (1.12) immediately yields

$$0 = {}_{i,}S[\phi] \equiv \frac{\overrightarrow{\delta}}{\delta\phi^i} L[\phi] - \left(\frac{\overrightarrow{\delta}}{\delta\phi^i_{,\mu}} L[\phi] \right)_{,\mu}, \tag{1.13}$$

in which the boundary terms do not appear.

It is worth remarking already at this point that the condition of locality, which is imposed on the dynamical equations largely in order to have easy control over causality, is by no means the only condition that is imposed in practice. Even when the action functional has the structure (1.12) there are additional criteria, of a physical nature, that greatly restrict the Lagrange function itself. For example L must satisfy the constraints imposed by relativistic invariance, either special or general; it should lead to an energy that is bounded from below and the stationary points of the action should be non-trivial.

1.6 Invariance Transformations

For many of the most interesting dynamical systems there exists, on the space of histories Φ, a set of *flows* that leave the value of the action invariant. That is, there exists a set of *nowhere vanishing* vector fields Q_α on Φ such that

$$SQ_\alpha \equiv 0. \tag{1.14}$$

Here the vector fields are written as operators acting from the right. They can be expressed in terms of the basis vectors $\frac{\overleftarrow{\delta}}{\delta\phi^i}$ associated to the coordinate patch on Φ defined by the ϕ^i:

$$Q_\alpha = \frac{\overleftarrow{\delta}}{\delta\phi^i}\,{}^iQ_\alpha. \tag{1.15}$$

In terms of the components ${}^iQ_\alpha$, (1.14) becomes

$$S_{,i}\,{}^iQ_\alpha \equiv 0. \tag{1.16}$$

Alternative forms are

$${}_\alpha QS \equiv 0, \qquad \text{or, equivalently} \qquad {}_\alpha Q^{\sim i}{}_{i,}S \equiv 0, \tag{1.17}$$

where ${}_\alpha Q$ acts from the left, and "$\sim$" denotes the supertranspose:

$${}_\alpha Q = (-1)^\alpha Q_\alpha, \qquad {}_\alpha Q^{\sim i} = (-1)^{\alpha(i+1)}\,{}^iQ_\alpha. \tag{1.18}$$

It will be noted that allowance has been made in the previous equation for the possibility that some of the Q_α may be a-type. The index α is said to be c-type or a-type according as the vector field that it designates is c-type or a-type. It should also be noted that eq. (1.14) will generally lead to difficulties at the boundary of the action integral (1.12) unless, for each α, the Q_α have compact support in M. If the Q_α must be independent of both boundary conditions and any special coordinate frame in space-time, then they can only be δ-distributions or derivatives of δ-distributions times local functions of the fields and their derivatives. This means that the index α, like the index i, must include a space-time point and hence range over a continuous infinity of values.

Because of the invariance equation, the value of the action remains invariant under infinitesimal changes in the dynamical variables of the form

$$\delta\phi^i = {}^iQ_\alpha \delta\xi^\alpha. \tag{1.19}$$

The infinitesimal parameters $\delta\xi^\alpha$ of these transformations are C^∞ functions over space-time, c-number-valued or a-number-valued according as the index α is c-type or a-type. The $\delta\xi^\alpha$ will be assumed to be real (i.e., taking their values in either $\mathcal{R}_c$ or $\mathcal{R}_a$), and since the dynamical variables ϕ^i are real-valued this implies that the vector fields Q_α are real or imaginary according as α is c-type or a-type. The $\delta\xi^\alpha$ are additionally required to have compact support in space-time or else to satisfy such conditions at the time boundaries and at spatial infinity as are needed in order that the integrations in

$$0 \equiv \delta S = S_{,i}\delta\phi^i = S_{,i}\,{}^iQ_\alpha\delta\xi^\alpha \tag{1.20}$$

be performable in any order.

In the following sections, two important examples of such theories will be presented: *Yang-Mills* theories and *General Relativity.*

1.6.1 Yang-Mills Theories (I)

The dynamical object in the N-dimensional Yang-Mills theory is a Lie-algebra-valued 1-form, which can be expressed, in a suitable chart, as follows:

$$\begin{aligned} A &\equiv A^\alpha_\mu\, T_\alpha \otimes dx^\mu &(1.21)\\ &\equiv A_\mu dx^\mu, &(1.22)\end{aligned}$$

where the T_α are a basis for the Lie algebra of $SU(N)$, which will be denoted by $su(N)$; for $N \geq 2$, it is the algebra of square anti-hermitian traceless matrices with Lie bracket the commutator; this algebra can be endowed with the Euclidean metric[3]

$$\gamma_{\alpha\beta} \equiv -tr(T_\alpha T_\beta) \tag{1.23}$$

which will be used to lower/raise Lie algebra indices.

Therefore the following Lie-algebra-valued 2-form is defined:

$$\begin{aligned} F &\equiv \frac{1}{2} F^\alpha_{\mu\nu}\, T_\alpha \otimes (dx^\mu \wedge dx^\nu) &(1.24)\\ &\equiv F^\alpha_{\mu\nu}\, T_\alpha \otimes dx^\mu \otimes dx^\nu &(1.25)\\ &\equiv \left(A^\alpha_{\nu;\mu} - A^\alpha_{\mu;\nu} + f^\alpha{}_{\beta\gamma} A^\beta_\mu A^\gamma_\nu\right) T_\alpha \otimes dx^\mu \otimes dx^\nu, &(1.26)\end{aligned}$$

where the semicolon denotes covariant differentiation associated with the Levi-Civita connection ∇, and the $f^\alpha{}_{\beta\gamma}$ are the *structure constants* of $su(N)$ associated to the basis of the T_α, i.e., they satisfy the equation

$$[T_\alpha, T_\beta] = T_\gamma f^\gamma{}_{\alpha\beta}\,. \tag{1.27}$$

The dynamical equations follow from the action functional

$$S_{YM}[A^\alpha_\mu] \equiv -\frac{1}{4}\int_M \sqrt{|\mathrm{g}|} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha\, d^n x, \tag{1.28}$$

where g is defined to be $\det(g_{\mu\nu})$.

It can be readily seen that such an action is invariant under the transformation

$$A_\mu(x) \mapsto {}^U\!A_\mu(x) \equiv U^\dagger(x) A_\mu(x) U(x) + U^\dagger(x) U_{,\mu}(x) \tag{1.29}$$

for every $U : M \ni x \mapsto U(x) \in SU(N)$. Consider an element $T \equiv \omega_\alpha T^\alpha \in su(N)$; as is well known, *exponentiation* yields an element in $SU(N)$; let now $\omega^\alpha(x)$ be a set of real funcions on M; then, for every $x \in M$, $U(x) \equiv e^{\omega^\alpha(x) T_\alpha} \in SU(N)$; if the $\omega^\alpha(x)$ are infinitesimal, then $U(x)$ is close to the identical transformation, and (1.29) reads

$$\begin{aligned} A^\gamma_\mu &\mapsto {}^U\!A^\gamma_\mu \equiv A^\gamma_\mu + A^\alpha_\mu \omega^\beta f^\gamma{}_{\alpha\beta} + \omega^\gamma_{,\mu}, &(1.30)\\ \delta A^\gamma_\mu &\equiv {}^U\!A^\gamma_\mu - A^\gamma_\mu = A^\alpha_\mu \omega^\beta f^\gamma{}_{\alpha\beta} + \omega^\gamma_{,\mu}. &(1.31)\end{aligned}$$

With DeWitt's notation, eq. (1.31) can be written

$$\begin{aligned} \delta A^\gamma_\mu &= {}^\gamma_\mu Q_\alpha \delta\xi^\alpha, &(1.32)\\ {}^\gamma_\mu Q_\alpha(x,x') &\equiv \delta(x,x')\left(A^\beta_\mu(x) f^\gamma{}_{\beta\alpha} + \underline{\delta}^\gamma_\alpha \partial_{\mu'}\right) &(1.33)\\ &= \delta(x,x') A^\beta_\mu(x) f^\gamma{}_{\beta\alpha} - \underline{\delta}^\gamma_\alpha \delta(x,x')_{,\mu'}. &(1.34)\end{aligned}$$

[3]this metric is nonsingular if and only if the group is semisimple; additionally, whenever the group is compact, as in this case, then it is positive definite.

1.6.2 General Relativity (I)

The dynamical object is the Lorentzian metric tensor defined on a manifold M; it can be expressed, in a suitable chart, as follows:

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{1.35}$$

It is important to observe that, if a theory describes nature in terms of a manifold M and tensor fields $T^{(i)}$ defined on the manifold, then if $\phi : M \to N$ is a diffeomorphism, the solutions $(M, T^{(i)})$ and $(N, \phi^* T^{(i)})$ have physically identical properties. Any physically meaningful statement about $(M, T^{(i)})$ will hold with equal validity for $(N, \phi^* T^{(i)})$. On the other hand, if $(N, \phi^* T^{(i)})$ is not related to $(M, T^{(i)})$ by a diffeomorphism, then $(N, \phi^* T^{(i)})$ will be physically distinguishable from $(M, T^{(i)})$.

Thus, the *diffeomorphisms* comprise the gauge freedom of any theory formulated in terms of tensor fields on a manifold. In particular, diffeomorphisms comprise the gauge freedom of general relativity.

Now consider the case where $N = M$; let $\phi : M \ni x \mapsto \phi(x) \in M$ be a diffeomorphism; its pullback acts on tensors of type $(0, 2)$ as follows:

$$\phi^* : T^*_{\phi(x)} M \otimes T^*_{\phi(x)} M \ni g\big|_{\phi(x)} \mapsto \phi^* g\big|_{\phi(x)} \in T^*_x M \otimes T^*_x M, \tag{1.36}$$

$$\phi^* g\big|_{\phi(x)} = g_{\mu\nu}\big|_{\phi(x)} \frac{\partial \phi^\mu}{\partial x^\rho}\bigg|_x \frac{\partial \phi^\nu}{\partial x^\sigma}\bigg|_x dx^\rho\big|_x \otimes dx^\sigma\big|_x. \tag{1.37}$$

If ϕ is close to the identical diffeomorphism, then it can be always seen as an element of the *flow* $\sigma^\xi(t, \cdot) \equiv \sigma_t$ associated to a *vector field* ξ defined on M; hence, for an "infinitesimal" diffeomorphism, eq. (1.37) reads:

$$\begin{aligned}
\phi^* g\big|_{\phi(x)} &= \sigma^{\xi *}_\epsilon g\big|_{\sigma^\xi_\epsilon(x)} = g_{\mu\nu}\big|_{\sigma^\xi_\epsilon(x)} \frac{\partial {\sigma^\xi_\epsilon}^\mu}{\partial x^\rho}\bigg|_x \frac{\partial {\sigma^\xi_\epsilon}^\nu}{\partial x^\sigma}\bigg|_x dx^\rho\big|_x \otimes dx^\sigma\big|_x \\
&= g\big|_x + \epsilon \left(\xi^\alpha \partial_\alpha g_{\rho\sigma} + g_{\rho\mu} \partial_\sigma \xi^\mu + g_{\mu\sigma} \partial_\rho \xi^\mu\right)\big|_x dx^\rho\big|_x \otimes dx^\sigma\big|_x \\
&= g\big|_x + \epsilon \left(\xi_{\rho;\sigma} + \xi_{\sigma;\rho}\right)\big|_x dx^\rho\big|_x \otimes dx^\sigma\big|_x \\
&= g\big|_x + \epsilon \mathcal{L}_\xi g\big|_x.
\end{aligned} \tag{1.38}$$

Eq. (1.38) shows a remarkable result: the Lie algebra of the diffeomorphism group of M, $Diff(M)$, is the space of vector fields on M with Lie bracket Lie derivative.

Therefore, General Relativity is invariant under the trasformation

$$g \mapsto g + \epsilon \mathcal{L}_\xi g, \tag{1.39}$$

$$\delta g = \epsilon \mathcal{L}_\xi g. \tag{1.40}$$

for every vector field ξ on M.

With DeWitt's notation, eq. (1.40) can be written

$$\begin{aligned}
\delta g_{\mu\nu} &= {}_{\mu\nu}Q_\rho \delta\xi^\rho, \\
{}_{\mu\nu}Q_\rho(x, x') &= \delta(x, x') \left(g_{\mu\rho} \nabla_\nu + g_{\nu\rho} \nabla_\mu\right)\big|_{x'} && (1.41) \\
&= \partial_\rho g_{\mu\nu}\big|_x \delta(x, x') - g_{\mu\rho}\big|_x \delta(x, x')_{,\nu} - g_{\nu\rho}\big|_x \delta(x, x')_{,\mu}. && (1.42)
\end{aligned}$$

1.7 Commutator of Invariance Transformations

Let B be a functional on Φ; by applying two invariance transformations Q_α, Q_β to G, one arrives at

$$\begin{aligned} B\left(Q_\alpha Q_\beta\right) &\equiv \left(BQ_\alpha\right) Q_\beta \\ &= \left(B_{,i}{}^iQ_\alpha\right) Q_\beta \\ &= B_{,i}{}^iQ_{\alpha,j}{}^jQ_\beta + (-1)^{j\alpha+ij} B_{,ij}{}^iQ_\alpha{}^jQ_\beta \end{aligned} \tag{1.43}$$

while

$$\begin{aligned} \left(BQ_\beta\right) Q_\alpha &= B_{,i}{}^iQ_{\beta,j}{}^jQ_\alpha + (-1)^{j\beta+ij} B_{,ij}{}^iQ_\beta{}^jQ_\alpha \\ &= B_{,i}{}^iQ_{\beta,j}{}^jQ_\alpha + (-1)^{i\beta+ji} B_{,ji}{}^jQ_\beta{}^iQ_\alpha \\ &= B_{,i}{}^iQ_{\beta,j}{}^jQ_\alpha + (-1)^{i\beta} B_{,ij}{}^jQ_\beta{}^iQ_\alpha \\ &= B_{,i}{}^iQ_{\beta,j}{}^jQ_\alpha + (-1)^{i\beta+i\beta+ij+\alpha\beta+\alpha j} B_{,ij}{}^iQ_\alpha{}^jQ_\beta \\ &= B_{,i}{}^iQ_{\beta,j}{}^jQ_\alpha + (-1)^{ij+\alpha\beta+\alpha j} B_{,ij}{}^iQ_\alpha{}^jQ_\beta . \end{aligned} \tag{1.44}$$

Hence one obtains

$$\begin{aligned} B\left(Q_\alpha Q_\beta - (-1)^{\alpha\beta} Q_\beta Q_\alpha\right) &= B_{,i}{}^iQ_{\alpha,j}{}^jQ_\beta - (-1)^{\alpha\beta} B_{,i}{}^iQ_{\beta,j}{}^jQ_\alpha \\ &= B_{,i}\left({}^iQ_{\alpha,j}{}^jQ_\beta - (-1)^{\alpha\beta}\,{}^iQ_{\beta,j}{}^jQ_\alpha\right) \end{aligned} \tag{1.45}$$

Thus, given two fields Q_α, Q_β, their *supercommutator* or *super Lie bracket* $[Q_\alpha, Q_\beta]$ is itself a vector field:

$$[Q_\alpha, Q_\beta] \equiv Q_\alpha Q_\beta - (-1)^{\alpha\beta} Q_\beta Q_\alpha , \tag{1.46}$$

$${}^i[Q_\alpha, Q_\beta] = {}^iQ_{\alpha,j}{}^jQ_\beta - (-1)^{\alpha\beta}\,{}^iQ_{\beta,j}{}^jQ_\alpha . \tag{1.47}$$

In the particular case where B is the action functional, it is immediately obvious that

$$S[Q_\alpha, Q_\beta] = 0. \tag{1.48}$$

Hence, the commutator of two invariance transformations is an invariance transformation itself.

It must be pointed out at once that, for *every* dynamical system there exist, on the space of histories Φ, vector fields that, like the Q_α, yield zero when acting on the action, i.e., vector fields V of the form

$$V^i = S_{,j}{}^jT^i , \tag{1.49}$$

where T is any antisupersymmetric tensor field:[4]

$${}^jT^i = -(-1)^{ij}\ {}^iT^j . \tag{1.50}$$

Such vector fields, however, vanish on the dynamical shell and are not true flows. They will be called *skew fields*.

[4]The components of T should also have the necessary support or rate-of-fall-off properties in space-time for the implicit summation integration in (1.49) to converge.

It will be assumed that all true flows can be expressed, at each point of Φ, as linear combinations of the Q_α's and skew fields at that point, i.e., that the Q_α's form a pointwise complete set of flows *modulo* skew fields. Pointwise completeness of the Q_α's and equation (1.48) imply that the supercommutator in (1.46) must have the general structure

$$[Q_\alpha, Q_\beta] = Q_\gamma\, c^\gamma{}_{\alpha\beta} + T_{\alpha\beta}\ {}_1S, \tag{1.51}$$

or, in component form:

$${}^i[Q_\alpha, Q_\beta] = {}^iQ_\gamma\, c^\gamma{}_{\alpha\beta} + {}^iT_{\alpha\beta}{}^j\ {}_{j,}S, \tag{1.52}$$

where the $c^\gamma{}_{\alpha\beta}$ are scalar fields on Φ and the $T_{\alpha\beta}$ are tensor fields, having the symmetries:

$$\begin{aligned} c^\gamma{}_{\alpha\beta} &= -(-1)^{\alpha\beta}\, c^\gamma{}_{\beta\alpha} && (1.53)\\ {}^iT_{\alpha\beta}{}^j &= -(-1)^{\alpha\beta}\, {}^iT_{\beta\alpha}{}^j = -(-1)^{ij+(\alpha+\beta)(i+j)}\, {}^jT_{\alpha\beta}{}^i. && (1.54) \end{aligned}$$

In addition to these symmetries the $c^\gamma{}_{\alpha\beta}$ and $T_{\alpha\beta}$ must satisfy functional differential conditions imposed by the Jacobi identity

$$[Q_\alpha, [Q_\beta, Q_\gamma]]\epsilon^{\gamma\beta\alpha} = 0, \tag{1.55}$$

the $\epsilon^{\alpha\beta\gamma}$ being any coefficients completely antisupersymmetric in their indices:

$$\epsilon^{\alpha\beta\gamma} = -(-1)^{\alpha\beta}\epsilon^{\beta\alpha\gamma} = -(-1)^{\beta\gamma}\epsilon^{\alpha\gamma\beta}. \tag{1.56}$$

1.8 Gauge Algebra, Gauge Groups and Orbits

One may easily verify that the super Lie bracket of any two skew fields is a skew field. By functionally differentiating (1.16), one obtains:

$$\begin{aligned} 0 &= (S_{,j}\, {}^jQ_\alpha)_{,i}\\ &= S_{,j}\, {}^jQ_{\alpha,i} + (-1)^{ij+i\alpha}\, S_{,ji}\, {}^jQ_\alpha\\ &= S_{,j}\, {}^jQ_{\alpha,i} + (-1)^{ij+i\alpha+ij}\, S_{,ij}\, {}^jQ_\alpha\\ &= S_{,j}\, {}^jQ_{\alpha,i} + (-1)^{i\alpha}\, S_{,ij}\, {}^jQ_\alpha,\\ S_{,j}\, {}^jQ_{\alpha,i} &= -(-1)^{i\alpha}\, S_{,ij}\, {}^jQ_\alpha. && (1.57) \end{aligned}$$

By using this identity, one can easily verify that the super Lie Bracket of a Q_α with a skew field is again a skew field:

$$\begin{aligned}
S_{,i}\,{}^{i}[Q_\alpha, S_{,j}{}^{j}T^\bullet] &= S_{,i}\left({}^{i}Q_{\alpha,k}\,S_{,j}\,{}^{j}T^k - (-1)^{\alpha T}\left(S_{,j}\,{}^{j}T^i\right)_{,k}\,{}^{k}Q_\alpha\right) \\
&= S_{,i}\,{}^{i}Q_{\alpha,k}\,S_{,j}\,{}^{j}T^k - (-1)^{\alpha T}\,S_{,i}S_{,j}\,{}^{j}T^i{}_{,k}\,{}^{k}Q_\alpha \\
&\quad -(-1)^{\alpha T+ki+kT+kj}\,S_{,i}S_{,jk}\,{}^{j}T^i\,{}^{k}Q_\alpha \\
&= S_{,i}\,{}^{i}Q_{\alpha,k}\,S_{,j}\,{}^{j}T^k - (-1)^{\alpha T+ki+kT+kj}\,S_{,i}S_{,jk}\,{}^{j}T^i\,{}^{k}Q_\alpha \\
&= S_{,i}\,{}^{i}Q_{\alpha,k}\,S_{,j}\,{}^{j}T^k - (-1)^{\alpha T+\alpha i+\alpha T+\alpha j}\,S_{,i}S_{,jk}\,{}^{k}Q_\alpha\,{}^{j}T^i \\
&= S_{,i}\,{}^{i}Q_{\alpha,k}\,S_{,j}\,{}^{j}T^k + (-1)^{\alpha i}\,S_{,i}S_{,k}\,{}^{k}Q_{\alpha,j}\,{}^{j}T^i \\
&= (-1)^{jk+j\alpha+ji}\,S_{,i}S_{,j}\,{}^{i}Q_{\alpha,k}\,{}^{j}T^k + (-1)^{\alpha i}\,S_{,i}S_{,k}\,{}^{k}Q_{\alpha,j}\,{}^{j}T^i \\
&= (-1)^{ik+i\alpha+ji}\,S_{,j}S_{,i}\,{}^{j}Q_{\alpha,k}\,{}^{i}T^k + (-1)^{\alpha i}\,S_{,i}S_{,j}\,{}^{j}Q_{\alpha,k}\,{}^{k}T^i \\
&= (-1)^{ik+i\alpha}\,S_{,i}S_{,j}\,{}^{j}Q_{\alpha,k}\,{}^{i}T^k + (-1)^{\alpha i}\,S_{,i}S_{,j}\,{}^{j}Q_{\alpha,k}\,{}^{k}T^i \\
&= (-1)^{i\alpha}\left[S_{,i}S_{,j}\,{}^{j}Q_{\alpha,k}\left((-1)^{ik}\,{}^{i}T^k + {}^{k}T^i\right)\right] \\
&= 0. \qquad (1.58)
\end{aligned}$$

From these facts, together with eq. (1.51), it follows that the set of all vector fields on Φ of the form

$${}^{i}Q_\alpha\,\xi^\alpha + {}^{i}T^j{}_{j,}S, \qquad (1.59)$$

the ξ^α being arbitrary (ϕ-dependent) coefficients, T being an arbitrary antisupersymmetric tensor field, form a closed algebra under the super Lie bracket operation. When true flows exist this algebra is called *a gauge algebra.*

The vector fields Q_α characterizing the flows on Φ are evidently not unique. They are defined only up to transformations of the form

$${}^{i}\bar{Q}_\alpha = {}^{i}Q_\beta\,X^\beta{}_\alpha + {}^{i}T_\alpha{}^{j}{}_{j,}S, \qquad (1.60)$$

where the $X^\beta{}_\alpha$ are functionally differentiable scalar fields on Φ which, at each point of Φ, form the elements of an invertible matrix, whose inverse is formed by functionally differentiable scalar fields, too, while ${}^{i}T_\alpha{}^{j}$ obey

$${}^{i}T_\alpha{}^{j} = -(-1)^{ij+(i+j)\alpha}\,{}^{j}T_\alpha{}^{i}. \qquad (1.61)$$

It is easy to see that such transformations leave eq. (1.51) unchanged. It is also easy to see that even when the Q_α are fixed, the $T_{\alpha\beta}$ in eq. (1.51) are not unique but are determined only up to transformations of the form

$${}^{i}\bar{T}_{\alpha\beta}{}^{j} = {}^{i}T_{\alpha\beta}{}^{j} + {}^{i}Q_\gamma\,{}^{\gamma}U_{\alpha\beta}{}^{\delta}{}_{\delta}Q^{\sim j}, \qquad (1.62)$$

where the coefficients ${}^{\gamma}U_{\alpha\beta}{}^{\delta}$ satisfy

$${}^{\gamma}U_{\alpha\beta}{}^{\delta} = -(-1)^{\alpha\beta}\,{}^{\gamma}U_{\beta\alpha}{}^{\delta} = -(-1)^{\gamma\delta+(\gamma+\delta)(\alpha+\beta)}\,{}^{\delta}U_{\alpha\beta}{}^{\gamma}. \qquad (1.63)$$

By carrying out these transformations, one may often simplify the relations satisfied by the Q_α. Three cases may be distinguished:

1.8.1 Type-I

The Q_α and the $T_{\alpha\beta}$ may be chosen in such a way that the latter vanish and the $c^\gamma{}_{\alpha\beta}$ are ϕ-independent; then eq. (1.51) becomes

$$[Q_\alpha, Q_\beta] = Q_\gamma c^\gamma{}_{\alpha\beta}, \tag{1.64}$$

$$c^\gamma{}_{\alpha\beta,i} = 0, \tag{1.65}$$

and the Jacobi identity implies:

$$c^\eta{}_{\alpha\delta} c^\delta{}_{\beta\gamma} \epsilon^{\gamma\beta\alpha} = 0. \tag{1.66}$$

In this case the $c^\gamma{}_{\alpha\beta}$ are the *structure constants* of an infinite dimensional Lie group known as the *gauge group* of the system. The gauge group, or more correctly, the *proper gauge group* is defined as the set of transformations of Φ into itself obtained by exponentiating the transformation (1.19) with ϕ-independent ξ^α and taking products of the resulting exponential maps. The proper gauge group is viewed as acting on Φ, and its actions leave S invariant. The *full gauge group* is obtained by appending to the proper gauge group all other ϕ-independent transformations that leave S invariant and do not arise from global symmetries. Elements of the proper group are sometimes called *little gauge transformations*, while elements of the full group outside the proper group are called *big gauge transformations*. When big gauge transformations exist the gauge group has disconnected components. It should be remarked that for the systems encountered in practice a choice of flow vectors Q_α satisfying (1.64) (1.65) is usually given a priori, and it is not necessary to carry out transformations of the forms (1.60) (1.62) to find them. The closure property expressed by (1.64), which is stronger than eq. (1.51), implies that the gauge group decomposes Φ into subspaces to which the Q_α are tangent. These subspaces are known as orbits, and the point-wise linear independence of the Q_α implies that each orbit is a copy of the gauge group supermanifold. If the gauge group has disconnected components, then so does Φ itself. Φ may be viewed as a principal fibre bundle of which the orbits are the fibres. The base space of this bundle is a supermanifold of which the orbits may be regarded as the points. It is called the *space of orbits*. The action functional is a scalar field on the space of orbits, and one might be tempted to say that it is in this space that the real physics of the system takes place. However, there may exist physical observables that remain invariant under little gauge transformations but not big ones, so a separate "physical" base space should in principle be assigned to each component of Φ. But in practice the amounts by which physical observables change under a big gauge transformation are always dynamically inert. Therefore we shall from now on focus solely on the proper gauge group.

1.8.2 Type-II

The $T_{\alpha\beta}$ can be made to vanish but the $c^\gamma{}_{\alpha\beta}$ cannot be made ϕ-independent globally on Φ. Equation (1.64) continues to hold, and the space of histories is again decomposed into orbits to which the Q_α are tangent, but the orbits are not group supermanifolds. If the Q_α are pointwise linearly independent, then the components of the orbits are all topologically identical, and each is a *parallelizable* supermanifold. The space of histories may again be viewed as a fibre bundle, and the real physics of the system takes place in the space of orbit components. The Jacobi identity, in this case, implies

$$({c^\eta}_{\alpha\delta}{c^\delta}_{\beta\gamma} - {c^\eta}_{\alpha\beta,i}\,{}^iQ_\gamma)\epsilon^{\gamma\beta\alpha} = 0. \tag{1.67}$$

1.8.3 Type-III

The $T_{\alpha\beta}$ cannot be made to vanish globally on Φ. Flow vectors of the form $Q_\alpha\xi^\alpha$, where the ξ^α are ϕ-dependent, do not by themselves form a closed system under the super Lie bracket operation, except on the dynamical shell. Only the dynamical shell, not the full space of histories Φ, is decomposed into orbits. The space of histories cannot be viewed as a fibre bundle; only the dynamical shell can. This means that although the real physics takes place in the space of orbit components as usual, the dynamics cannot be derived from an action functional on this space. The full space Φ is needed.

1.8.4 Yang-Mills Theories (II)

In this section it will be shown that Yang-Mills theories are Type-I theories, and their structure functions will be calculated explicitly. By recalling eq. (1.34), one obtains

$$\begin{aligned} {}^\gamma_{\ \mu}Q_{\alpha',}{}^{\nu''}{}_{\delta''} &\equiv {}^\gamma_\mu Q_{\alpha'}\frac{\overleftarrow{\delta}}{\delta A(x'')^\delta_\nu} \\ &= \delta(x,x')\delta^{\nu''}_\mu\,\underline{\delta}^\rho_\delta {f^\gamma}_{\rho\alpha'} \\ &= \delta(x,x')\delta^{\nu''}_\mu {f^\gamma}_{\delta\alpha'}\,. \end{aligned} \tag{1.68}$$

Hence

$$\begin{aligned} &{}^\gamma_{\ \mu}Q_{\alpha',}{}^{\nu''}{}_{\delta''}\ {}^{\delta''}{}_{\nu''}Q_{\beta'''} = \\ = &\int_M dx''\left(\delta(x,x')\delta^{\nu''}_\mu {f^\gamma}_{\delta\alpha'}\right)\left(\delta(x'',x''')A^{\rho''}_{\nu''}{f^\delta}_{\rho''\beta'''} - \delta^{\delta''}_{\beta''',\nu''}\right) \\ = &\ \delta(x,x')\delta(x,x''')A^\rho_\mu {f^\gamma}_{\delta\alpha'}{f^\delta}_{\rho\beta'''} - \delta(x,x'){f^\gamma}_{\delta\alpha'}\delta^\delta_{\beta''',\mu}; \end{aligned} \tag{1.69}$$

by swapping (α',x') and (β''',x'''), one obtains:

$$\begin{aligned} &{}^\gamma_{\ \mu}Q_{\beta''',}{}^{\nu''}{}_{\delta''}\ {}^{\delta''}{}_{\nu''}Q_{\alpha'} = \\ = &\ \delta(x,x')\delta(x,x''')A^\rho_\mu\, {f^\gamma}_{\delta\beta'''}\ {f^\delta}_{\rho\alpha'} - \delta(x,x'''){f^\gamma}_{\delta\beta'''}\ \delta^\delta_{\alpha',\mu}, \end{aligned} \tag{1.70}$$

therefore

$$\begin{aligned} &{}^\gamma_\mu[Q_\alpha,Q_\beta] = \\ = &\ {}^\gamma_{\ \mu}Q_{\alpha',}{}^{\nu''}{}_{\delta''}\ {}^{\delta''}{}_{\nu''}Q_{\beta'''} - {}^\gamma_{\ \mu}Q_{\beta''',}{}^{\nu''}{}_{\delta''}\ {}^{\delta''}{}_{\nu''}Q_{\alpha'} \\ = &\ \delta(x,x')\delta(x,x''')A^\rho_\mu\left({f^\gamma}_{\delta\alpha'}{f^\delta}_{\rho\beta'''} - {f^\gamma}_{\delta\beta'''}{f^\delta}_{\rho\alpha'}\right) \\ &- \left(\delta(x,x'){f^\gamma}_{\delta\alpha'}\delta^\delta_{\beta''',\mu} - \delta(x,x'''){f^\gamma}_{\delta\beta'''}\delta^\delta_{\alpha',\mu}\right). \end{aligned} \tag{1.71}$$

The first term contains ${f^\gamma}_{\delta\alpha'}{f^\delta}_{\rho\beta'''} - {f^\gamma}_{\delta\beta'''}{f^\delta}_{\rho\alpha'}$; by using the Jacobi identity for the structure constants and their antisimmetry in the lower indices, one obtains:

$${f^\gamma}_{\delta\alpha'}{f^\delta}_{\rho\beta'''} - {f^\gamma}_{\delta\beta'''}{f^\delta}_{\rho\alpha'} = {f^\gamma}_{\rho\delta}{f^\delta}_{\alpha'\beta'''}\,, \tag{1.72}$$

therefore the first term is

$$\begin{aligned}
&\delta(x,x')\delta(x,x''')A^{\rho}_{\mu}f^{\gamma}{}_{\rho\delta}\,f^{\delta}{}_{\alpha'\beta'''} = \\
= &\int_M dx''f^{\delta''}{}_{\alpha'\beta'''}\,\delta(x'',x')\delta(x'',x''')\left(\delta(x,x'')A^{\rho}_{\mu}f^{\gamma}{}_{\rho\delta''}\right) \qquad (1.73)
\end{aligned}$$

the second term is, instead:

$$\begin{aligned}
&-\left(\delta(x,x')f^{\gamma}{}_{\delta\alpha'}\,\delta^{\delta}_{\beta''',\mu} - \delta(x,x''')f^{\gamma}{}_{\delta\beta'''}\,\delta^{\delta}_{\alpha',\mu}\right) = \\
= &-\left(\delta(x,x')f^{\gamma}{}_{\beta'''\alpha'}\,\delta(x,x''')_{,\mu} - \delta(x,x''')f^{\gamma}{}_{\alpha'\beta'''}\,\delta(x,x')_{,\mu}\right) \\
= &-f^{\gamma}{}_{\beta'''\alpha'}\,\left(\delta(x,x')\delta(x,x''')_{,\mu} + \delta(x,x')_{,\mu}\delta(x,x''')\right) \\
= &\ f^{\gamma}{}_{\alpha'\beta'''}\,\left(\delta(x,x')\delta(x,x''') + \delta(x,x')\delta(x,x''')\right)_{,\mu} \\
= &\int_M dx''f^{\gamma}{}_{\alpha'\beta'''}\,\delta(x,x'')\left(\delta(x'',x')\delta(x'',x''')\right)_{,\mu} \\
= &-\int_M dx''f^{\gamma}{}_{\alpha'\beta'''}\,\delta(x,x'')_{,\mu}\left(\delta(x'',x')\delta(x'',x''')\right) \\
= &-\int_M dx''f^{\delta''}{}_{\alpha'\beta'''}\,\delta^{\gamma}_{\delta'',\mu}\delta(x'',x')\delta(x'',x''') \\
= &\int_M dx''f^{\delta''}{}_{\alpha'\beta'''}\,\delta(x'',x')\delta(x'',x''')\left(-\delta^{\gamma}_{\delta'',\mu}\right) \qquad (1.74)
\end{aligned}$$

Putting it all together, one obtains, eventually:

$$\begin{aligned}
&{}^{\gamma}_{\mu}[Q_{\alpha},Q_{\beta}] = \\
= &\int_M dx''f^{\delta''}{}_{\alpha'\beta'''}\,\delta(x'',x')\delta(x'',x''')\left(\delta(x,x'')A^{\rho}_{\mu}f^{\gamma}{}_{\rho\delta''} - \delta^{\gamma}_{\delta'',\mu}\right) \\
= &\int_M dx''\,{}^{\gamma}_{\mu}Q_{\delta''}f^{\delta''}{}_{\alpha'\beta'''}\,\delta(x'',x')\delta(x'',x''') \qquad (1.75)
\end{aligned}$$

Eq. (1.75) shows that Yang-Mills theories are Type-I theories, with structure constants

$$c^{\delta''}{}_{\alpha'\beta'''} \equiv f^{\delta}{}_{\alpha\beta}\,\delta(x'',x')\delta(x'',x'''). \qquad (1.76)$$

1.8.5 General Relativity (II)

By recalling the commutation law for the Lie derivative, one obtains

$$[\mathcal{L}_X,\mathcal{L}_Y] = \mathcal{L}_{[X,Y]}; \qquad (1.77)$$

But $[X,Y]\big|_x = (X^{\nu}\partial_{\nu}Y^{\mu} - Y^{\nu}\partial_{\nu}X^{\mu})\,\big|_x\partial_{\mu}\big|_x$ can be expressed as

$$\begin{aligned}
[X,Y]\big|_x &= \int_M dx'\int_M dx''c^{\sigma}{}_{\mu'\nu''}X^{\mu'}Y^{\nu''}, \qquad (1.78)\\
c^{\sigma}{}_{\mu'\nu''} &= \delta^{\sigma}_{\mu';\tau}\delta^{\tau}_{\nu''} - \delta^{\sigma}_{\nu'';\tau}\delta^{\tau}_{\mu'}, \qquad (1.79)
\end{aligned}$$

as is straightforward to verify. These equations show that General Relativity is a Type-I theory too, and its structure functions are given by (1.79).

1.9 Physical Observables

A change in the dynamical variables of the form (1.19) leaves the action functional invariant. Such changes therefore play no role in determining the dynamical shell. Moreover, they map the dynamical shell into itself, as may be seen by varying the dynamical equations and making use of eq. (1.57):

$$\begin{aligned} \delta_{j,}S &= {}_{j,}S_{,i}\delta\phi^i \\ &= {}_{j,}S_{,i}{}^iQ_\alpha\delta\xi^\alpha \\ &= -(-1)^{j\alpha+j}S_{,i}{}^iQ_{\alpha,j}\delta\xi^\alpha, \end{aligned} \tag{1.80}$$

therefore

$$ {}_{j,}S = 0 \implies \delta_{j,}S = 0. \tag{1.81}$$

Hence transformations generated by the Q_α are unphysical. No functional of the dynamical variables that is affected by them can be a physical quantity. Conversely, any functional that *is* invariant under (1.19) will be called a *physical observable.* In the classical theory this nomenclature constitutes an abuse of language because both c-type and a-type quantities can be invariant under (1.19), and of course nobody can observe an a-number. However, the quantum counterpart of a real-valued classical observable, whether c-type or a-type, will, for any valid physical theory, be a self-adjoint linear operator in the super Hilbert space of the full quantum theory, having ordinary real numbers as eigenvalues.

It is useful to distinguish two types of invariants under (1.19): *absolute invariants* and *conditional invariants.* An absolute invariant A is a functional of the ϕ^i that is invariant under (1.19) at all points of Φ. It satisies

$$AQ_\alpha = A_{,i}\,{}^iQ_\alpha = 0 \quad \forall\phi \in \Phi. \tag{1.82}$$

The action functional is always an absolute invariant. A conditional invariant B is a funcional of the ϕ^i that is invariant under (1.19) on shell but not everywhere on Φ. It typically satisfies

$$BQ_\alpha = B_{,i}\,{}^iQ_\alpha = S_{,i}\,{}^ib_\alpha \quad \forall\phi \in \Phi, \tag{1.83}$$

where the ${}^ib_\alpha$ are certain ϕ^i-dependent coefficients. A simple example of a conditional invariant is

$$\bar{A} = A + S_{,i}\,{}^ia, \tag{1.84}$$

where A is an absolute invariant and ${}^ia$ are arbitrary ϕ^i-dependent coefficients.

A physical observable may be either an absolute invariant or a conditional invariant. In a physical situation (i.e., when ϕ is on shell) there is in fact no distinction between the two. As a functional of the ϕ^i a physical observable is really defined only modulo the dynamical equations, i.e., up to transformations of the form (1.84).

1.10 Gauge Groups and Manifest Covariance

Manifest covariance refers to the following facts: the group transformation laws for the various symbols that appear in the theory may be inferred simply from the position and nature of their indices, and both sides of any equation transform similarly. This applies for Type-I theories when the group realization is linear, i.e.

$$ {}^{i}Q_{\alpha,jk} = 0. \tag{1.85} $$

In fact, by deriving (1.64), one obtains:

$$ {}^{i}Q_{\alpha,j}\ {}^{j}Q_{\beta,k} - (-1)^{\alpha\beta}\ {}^{i}Q_{\beta,j}\ {}^{j}Q_{\alpha,k} = (-1)^{k(\alpha+\beta+\gamma)}\ {}^{i}Q_{\gamma,k}\ c^{\gamma}{}_{\alpha\beta}, \tag{1.86} $$

which implies that the matrices $({}^{i}Q_{\alpha,j})$ (which, in view of eq. (1.85), are ϕ-independent) generate a representation of the Lie algebra of the gauge group and, by exponentiation, of the gauge group itself. Call this representation the *defining representation* and call the contragradient representation (generated by the negative (super)transposes of the above matrices) the *co-defining representation*. Similarly, eq. (1.66) implies that $(c^{\gamma}{}_{\alpha\beta})$ generate a representation of the Lie algebra of the gauge group too: call it *adjoint representation*; call *co-adjoint representation* the representation generated by the negative (super)transpose of the structure constants.

Given an absolute invariant A, one can take subsequent derivatives of eq. (1.82) and use (1.85); the first two derivatives yield:

$$ \begin{aligned} (A_{,i}{}^{i}Q_{\alpha})_{,j} &= 0, \\ (-1)^{ij+j\alpha}A_{,ij}{}^{i}Q_{\alpha} &= -A_{,i}{}^{i}Q_{\alpha,j}; \end{aligned} \tag{1.87} $$

$$ \begin{aligned} (A_{,i}{}^{i}Q_{\alpha})_{,jk} &= 0, \\ (A_{,i}{}^{i}Q_{\alpha,j} + (-1)^{ij+j\alpha}A_{,ij}{}^{i}Q_{\alpha})_{,k} &= 0, \\ (-1)^{ij+j\alpha+\alpha k+ik}A_{,ijk}{}^{i}Q_{\alpha} &= -(-1)^{jk+j\alpha+ij}A_{,ik}{}^{i}Q_{\alpha,j} \\ &\quad -(-1)^{ij+j\alpha}A_{,ij}{}^{i}Q_{\alpha,k}. \end{aligned} \tag{1.88} $$

These identities relate functional derivatives of any absolute invariant of adjacent order; when the functional under observation is the action functional, these derivatives are called *vertex functions*, and these identities are called *bare Ward* *identities*. They imply the transformation laws

$$\begin{aligned}
\delta A_{,j} &\equiv A_{,ji}\delta\phi^i \\
&= A_{,ji}{}^iQ_\alpha\delta\xi^\alpha \\
&= (-1)^{ij}A_{,ij}{}^iQ_\alpha\delta\xi^\alpha \\
&= (-1)^{j\alpha}(-1)^{ij+j\alpha}A_{ji}{}^jQ_\alpha\delta\xi^\alpha \\
&= (-1)^{j\alpha}\left[(A_{,i}{}^iQ_\alpha)_{,j} - A_{,i}{}^iQ_{\alpha,j}\right]\delta\xi^\alpha \\
&= -(-1)^{j\alpha}A_{,i}{}^iQ_{\alpha,j}\delta\xi^\alpha; \qquad (1.89)
\end{aligned}$$

$$\begin{aligned}
\delta A_{,jk} &\equiv A_{,jki}\delta\phi^i \\
&= A_{,jki}{}^iQ_\alpha\delta\xi^\alpha \\
&= (-1)^{ij+ik}A_{,ijk}{}^iQ_\alpha\delta\xi^\alpha \\
&= (-1)^{j\alpha+k\alpha}(-1)^{ij+ik+j\alpha+k\alpha}A_{,ijk}{}^iQ_\alpha\delta\xi^\alpha \\
&= (-1)^{j\alpha+k\alpha}[(A_{,i}{}^iQ_\alpha)_{,jk} - (-1)^{jk+j\alpha+ij}A_{,ik}{}^iQ_{\alpha,j} \\
&\quad -(-1)^{ij+j\alpha}A_{,ij}{}^iQ_{\alpha,k}]\delta\xi^\alpha \\
&= -(-1)^{j\alpha+k\alpha}[(-1)^{jk+j\alpha+ij}A_{,ik}{}^iQ_{\alpha,j} \\
&\quad +(-1)^{ij+j\alpha}A_{,ij}{}^iQ_{\alpha,k}]\delta\xi^\alpha \\
&= -(-1)^{k\alpha+kj+ij}A_{,ik}{}^iQ_{\alpha,j}\delta\xi^\alpha - (-1)^{k\alpha}A_{,ji}{}^iQ_{\alpha,k}\delta\xi^\alpha; \qquad (1.90)
\end{aligned}$$

analogous equations hold for higher order derivatives.

The above equations show that the functional derivatives of absolute invariants transform according to direct products of the codefining representation. Equation (1.64) may itself be regarded as a transformation law:

$$\begin{aligned}
\delta\,{}^iQ_\alpha &\equiv {}^iQ_{\alpha,j}\;{}^jQ_\beta\;\delta\xi^\beta \\
&= ({}^i[Q_\alpha, Q_\beta] + (-1)^{\alpha\beta}\;{}^iQ_{\alpha,j}\;{}^jQ_\beta)\delta\xi^\beta \\
&= ({}^iQ_\gamma\;c^\gamma{}_{\alpha\beta} + (-1)^{\alpha\beta}\;{}^iQ_{\beta,j}\;{}^jQ_\alpha)\delta\xi^\beta \\
&= (-{}^iQ_\gamma\;c^\gamma{}_{\beta\alpha} + (-1)^{\alpha\beta}\;{}^iQ_{\beta,j}\;{}^jQ_\alpha)\delta\xi^\beta, \qquad (1.91)
\end{aligned}$$

which says that ${}^iQ_\alpha$ transforms according to the direct product of the defining representation and the coadjoint representation.

Hence, when the group realization is linear, quite generally, field indices (Latin) and group indices (Greek) signal respectively the defining representation and the adjoint representation when they are in the upper position and the contragradient representations when they are in the lower position.

One may then wonder whether the realization can be always made linear for Type-I theories: for Yang-Mills theories and General Relativity, this is possible, as has been shown in the previous sections, but the answer is not known in general. However, certain results in the theory of finite-dimensional compact Lie groups are suggestive in this connection. Palais [24] and Mostow [23] showed that if a manifold is acted on by a compact Lie group with finitely many orbit types, then it can be embedded into some finite-dimensional linear, homogeneous, orthogonal representation. Moreover, results of this kind can usually be extended to the case of finite-dimensional semisimple Lie groups whether compact or not.

If similar results could be extended to *field realizations of gauge groups* (which are infinite-dimensional), then one could simply add enough extra fields to Type-I systems to make the realization linear. The extra fields could be made dynamically innocuous by inclusion of appropriate Lagrange-multiplier fields in the action. There is one difference between the finite-dimensional and field theoretical cases that apparently cannot be eliminated: in the case of fields the variables ϕ^i cannot always be chosen in such a way as to yield a realization that is simultaneously linear and homogeneous. In the case of the Yang-Mills field the infinitesimal gauge transformation law (1.31) includes an inhomogeneous term that cannot be removed by any choice of variables. In the case of the Maxwell field the inhomogeneous term is all there is.

1.11 Equation of small disturbances

Let ϕ^i and $\phi^i + \delta\phi^i$ be two neighboring solutions of the dynamical equations (1.11):

$$\begin{aligned} 0 &= {}_{i,}S[\phi], && (1.92) \\ 0 &= {}_{i,}S[\phi + \delta\phi] = {}_{i,}S[\phi] + {}_{i,}S_{,j}[\phi]\delta\phi^j + ..., && (1.93) \end{aligned}$$

where the dots stand for terms which are at least quadratic in $\delta\phi^i$.

Evidently, to first order in $\delta\phi^i$, we have:

$$ {}_{i,}S_{,j}[\phi]\delta\phi^j = 0. \tag{1.94}$$

This is called *homogeneous equation of small disturbances.* Its solutions are known as *Jacobi fields* relative to the on-shell field ϕ. In the following equations the argument ϕ will often be suppressed. In practice a small disturbance is produced by a weak external agent, which may be described by a small change in the functional form of the action. Let A be a pure real-valued scalar field on Φ and ϵ be an infinitesimal real c-number or imaginary a-number according as A is c-type or a-type; therefore $A\epsilon$ is a real valued c-type scalar field, and the following change in the functional form of the action is admissible:

$$S[\phi] \mapsto S[\phi] + A[\phi]\epsilon \tag{1.95}$$

Let $\delta\phi^i$ be a solution of

$$ {}_{i,}S_{,j}\delta\phi^j = -{}_{i,}A\epsilon. \tag{1.96}$$

It is easy to see that, neglecting higher order terms, $\phi^i + \delta\phi^i$ satisfies the dynamical equations of the system $S + A\epsilon$ if and only if ϕ^i satisfies those of the system S:

$$\begin{aligned} {}_{i,}(S + A\epsilon)[\phi + \delta\phi] &= {}_{i,}S[\phi + \delta\phi] + {}_{i,}A[\phi + \delta\phi]\epsilon \\ &= {}_{i,}S[\phi] + {}_{i,}S_{,j}[\phi]\, \delta\phi^j + {}_{i,}A[\phi]\, \epsilon + {}_{i,}A_{,j}[\phi]\, \delta\phi^j\, \epsilon + ... \\ &= {}_{i,}S + ({}_{i,}S_{,j}\delta\phi^j + {}_{i,}A\epsilon) + ... && (1.97) \end{aligned}$$

Eq. (1.96) is called *inhomogeneous equation of small disturbances.* Its general solution is obtained by adding to a particular solution an arbitrary Jacobi field.

When the dynamical equation (1.11) is satisfied, eq. (1.57) reads:

$$_{\alpha}Q^{\sim i}{}_{i,}S_{,j} = 0 \quad (S_{,j} = 0) \tag{1.98}$$

When applied to (1.96), by virtue of the arbitrariness of ϵ, this equation implies

$$_{\alpha}Q^{\sim i}{}_{i,}A = 0 \quad (S_{,j} = 0). \tag{1.99}$$

Therefore eq. (1.96) is seen to be inconsistent unless A is a conditional invariant, i.e., a physical observable. If this is not the case, then the solution of $_{i,}(S+A\epsilon) = 0$ cannot differ from ϕ by infinitesimal amounts. Evidently small changes in the action functional will produce small changes in the on-shell dynamical variables only if they leave intact the flow invariances of the theory.

1.11.1 Supplementary conditions

Eq. (1.98) implies that, on the dynamical shell

$$_{i,}S_{,j}{}^{j}Q_{\alpha}\delta\xi^{\alpha} = 0 \quad (S_{,j} = 0) \tag{1.100}$$

for every $\delta\xi^{\alpha}$ of compact support in space-time: this implies that $_{1}S_{1}$ is not an invertible operator; hence, it has no Green's functions.

When ${}^{i}Q_{\alpha}\delta\xi^{\alpha}$ is added to a solution of eq. (1.96), the result is another solution: however, they are physically identical, since they differ merely by an invariance transformation (1.19). It is convenient to remove this redundancy by imposing a differential *supplementary condition* on the small disturbances $\delta\phi^{i}$, of the form[5]

$$_{\alpha}P_{i}\,\delta\phi^{i} = 0. \tag{1.101}$$

The supplementary condition is effective as long as the operator

$$_{\alpha}\mathcal{F}_{\beta} \equiv {}_{\alpha}P_{j}{}^{j}Q_{\beta} \tag{1.102}$$

is nonsingular and has Green's functions; in fact, given a Jacobi field $\delta\phi^{i}$, all the physically identical solutions can be written as:

$$\delta\phi^{i} + {}^{i}Q_{\alpha}\delta\xi^{\alpha}; \tag{1.103}$$

by imposing the supplementary condition (1.101), one obtains

$$\begin{aligned}
{}_{\alpha}P_{i}\,(\delta\phi^{i} + {}^{i}Q_{\beta}\delta\xi^{\beta}) &= 0 \\
{}_{\alpha}P_{i}\,\delta\phi^{i} + {}_{\alpha}P_{i}\,{}^{i}Q_{\beta}\delta\xi^{\beta} &= 0 \\
{}_{\alpha}P_{i}\,\delta\phi^{i} + {}_{\alpha}\mathcal{F}_{\beta}\,\delta\xi^{\beta} &= 0.
\end{aligned} \tag{1.104}$$

Being $_{\alpha}\mathcal{F}_{\beta}$ invertible, this equation determines $\delta\xi^{\alpha}$ and, therefore, the solution $\delta\phi^{i} + {}^{i}Q_{\alpha}\delta\xi^{\alpha}$.

Let η be a local, continuous, nonsingular, supersymmetric matrix whose elements are:

$$\eta^{\alpha\beta} = (-1)^{\alpha+\beta}\eta^{\beta\alpha}; \tag{1.105}$$

[5] Position and nature of the indices of the auxiliary distributions introduced throughout this work is not accidental: when dealing with Type-I theories with linear gauge group realization, they show how these distributions must transform under gauge transformations.

introduce the following differental operator:

$$ {}_iF_j \equiv {}_{i,}S_{,j} + {}_iP^{\sim}{}_\alpha \eta^{\alpha\beta}{}_\beta P_j \,, \tag{1.106}$$

where ${}_iP^{\sim}{}_\alpha$ is the supertranspose of ${}_\alpha P_i$:

$$ {}_iP^{\sim}{}_\alpha = (-1)^{i+\alpha+i\alpha}\,{}_\alpha P_i \,. \tag{1.107}$$

It is easy to see that ${}_iF_j$ has the same supersymmetry properties as ${}_{i,}S_{,j}$, i.e. it is supersymmetric:

$$ {}_iF_j = (-1)^{i+j+ij}{}_jF_i \,. \tag{1.108}$$

From now on, we will assume that the kernel of the linear differential operator ${}_{i,}S_{,j}$ consists of the fields of the form ${}^iQ_\alpha \delta\xi^\alpha$, with $\delta\xi^\alpha$ of compact support in space-time: this is true in all the practical cases. Hence, the folloing holds:

Theorem 1.2. *If ${}_\alpha\mathcal{F}_\beta$ is nonsingular, then ${}_iF_j$ is nonsingular too.*

Proof. Suppose that there is a set of functions X^j of compact support such that

$$ {}_iF_j X^j = 0. \tag{1.109}$$

By suppressing all indices and recalling the definition (1.106), one obtains:

$$ ({}_1S_1 + P^{\sim}\eta P)X = 0. \tag{1.110}$$

By taking the supertranspose and recalling the supersymmetry properties stated above, one can write:

$$ X^{\sim}({}_1S_1 + P^{\sim}\eta P) = 0. \tag{1.111}$$

By applying Q_α from the right and using (1.98), which holds on shell, one arrives at:

$$ X^{\sim}({}_1S_1 + P^{\sim}\eta P)Q_\alpha = 0, \tag{1.112}$$

$$ X^{\sim}P^{\sim}\eta P Q_\alpha = 0. \tag{1.113}$$

By restoring indices and recalling the definition (1.102), one can write:

$$ X^{\sim j}{}_jP^{\sim}{}_\alpha \eta^{\alpha\beta}{}_\beta\mathcal{F}_\gamma = 0. \tag{1.114}$$

Being $\eta^{\alpha\beta}$, ${}_\beta\mathcal{F}_\gamma$ nonsingular, the previous equation implies

$$ X^{\sim j}{}_jP^{\sim}{}_\alpha = 0, \tag{1.115}$$

or equivalently

$$ {}_\alpha P_i X^i = 0, \tag{1.116}$$

$$ PX = 0. \tag{1.117}$$

Hence, from the first equation (1.109), it follows

$$ \begin{aligned} FX &= ({}_1S_1 + P^{\sim}\eta P)X \\ &= {}_1S_1 X \\ &= 0. \end{aligned} \tag{1.118}$$

But the kernel of ${}_1S_1$ consists of the fields of the form ${}^iQ_\alpha\delta\xi^\alpha$, with $\delta\xi^\alpha$ of compact support in space-time, then X must be of the form $Q\xi$; therefore one can write again eq. (1.117) and recall the definition (1.102):

$$\begin{aligned} PX &= 0, \\ PQ\xi &= 0, \\ \mathcal{F}\xi &= 0. \end{aligned} \tag{1.119}$$

Since $\mathcal{F}$ is non singular, this equation implies $\xi = 0$ and, as a consequence, $X = 0$. Therefore the kernel of the operator ${}_iF_j$ consists of the null field $X^i = 0$ only: then ${}_iF_j$ is nonsingular.

This ends the proof. □

1.11.2 Retarded and Advanced Green's function of F

When the supplementary condition (1.101) is satisfied, eq. (1.96) may be replaced by

$${}_iF_j\ \delta\phi^j = -{}_{i,}A\epsilon. \tag{1.120}$$

In fact, if both (1.101) and (1.96) are satisfied, it is obvious that (1.120) is satisfied too; on the other hand, if (1.120) is satisfied, by applying ${}_\alpha Q^{\sim i}$ from the left one obtains:

$$\begin{aligned} {}_\alpha Q^{\sim i}\ {}_iF_j\ \delta\phi^j &= {}_\alpha Q^{\sim i}(-{}_{i,}A\epsilon) \\ {}_\alpha Q^{\sim i}({}_{i,}S_{,j} + {}_iP^{\sim}{}_\alpha\ \eta^{\alpha\beta}\ {}_\beta P_j)\delta\phi^j &= {}_\alpha Q^{\sim i}(-{}_{i,}A\epsilon) \\ {}_\alpha Q^{\sim i}\ {}_iP^{\sim}{}_\alpha\ \eta^{\alpha\beta}\ {}_\beta P_j\ \delta\phi^j &= 0 \\ {}_\alpha\mathcal{F}^{\sim}{}_\beta\ \eta^{\alpha\beta}\ {}_\beta P_j\ \delta\phi^j &= 0, \end{aligned} \tag{1.121}$$

where (1.98) have been used, plus the fact that A is a conditional invariant; Being $\eta^{\alpha\beta}$, ${}_\beta\mathcal{F}_\gamma$ nonsingular, the previous equation implies

$${}_\beta P_j\ \delta\phi^j = 0, \tag{1.122}$$

i.e., the supplementary condition is satisfied. Hence, the following holds:

$$\begin{aligned} {}_iF_j\ \delta\phi^j &= -{}_{i,}A\epsilon \\ {}_{i,}S_{,j}\ \delta\phi^j + {}_iP^{\sim}{}_\alpha\ \eta^{\alpha\beta}\ {}_\beta P_j\ \delta\phi^j &= -{}_{i,}A\epsilon \\ {}_{i,}S_{,j}\ \delta\phi^j &= -{}_{i,}A\epsilon \end{aligned} \tag{1.123}$$

i.e., the inhomogeneous equation of small disturbances is satisfied too.

Since F is a nonsingular operator this equation has unique solutions for given boundary conditions. These solutions can be expressed in terms of Green's functions.

We shall consider in this chapter only retarded and advanced boundary conditions. Denote by $\delta^-\phi^i$ and $\delta^+\phi^i$ respectively the corresponding solutions. Then

$$\delta^\pm\phi^i = G^{\pm ij}{}_{j,}A\epsilon \tag{1.124}$$

where G^{-ij} and G^{+ij} are the retarded and advanced Green's functions of ${}_iF_j$, respectively :

$$
{}_iF_k\, G^{\pm kj} \;=\; -{}_i\delta^j, \tag{1.125}
$$

$$
G^{-ij} = 0 \quad \text{if} \quad i < j, \qquad\qquad G^{+ij} = 0 \quad \text{if} \quad i > j, \tag{1.126}
$$

where "$i < j$" means "the time associated with the index i lies in the past of the time associated with the index j" and "$i > j$" means "the time associated with the index i lies in the future of the time associated with the index j". Consequently the kinematical conditions (1.126) imply that $G^{-ij}(G^{+ij}))$ is non-vanishing only when the space-time point associated with i lies on or inside the future (past) light cone emanating from the space-time point associated with j.

A minus sign appears on the right of eq. (1.125) for historical reasons, and the symbol ${}_i\delta^j$ represents a combined Kronecker delta δ-distribution. In the supercondensed notation (1.125) is written

$$
FG^{\pm} = -1. \tag{1.127}
$$

It should be remarked that the summation-integration involved on the right side of eq. (1.124) will generally not converge unless the functional form of A is such that the functions ${}_{j,}A$ do not increase in magnitude too rapidly toward the past or future. A sufficient condition for convergence, of course, is that supp ${}_{j,}A$ be limited in time, where "${}_{j,}A$" denotes the union of the supports of all the ${}_{j,}A$. If (1.124) does not converge, then the solutions of ${}_{i,}S_{,j}\,\delta\phi^j + {}_{i,}A\epsilon = 0$ do not lie close (in Φ) to those of ${}_{i,}S = 0$ no matter how small ϵ may be chosen.

1.11.3 Equality of Left and Right Green's Functions

In eqs. (1.125), (1.127) the $G^{\pm}$ appear as *right* Green's functions. They are also *left* Green's functions. To prove this let us temporarily distinguish left Green's functions from right Green's functions by employing subscripts L and R. Let X^i be arbitrary functions of compact support in space-time and let

$$
Y^i \equiv (G_R^{-ij} - G_L^{-ij})_j\, F_k\, X^k. \tag{1.128}
$$

Since the X^k have compact support it does not matter whether the j summation-integration or the k summation-integration is performed first in this expression. That is, ${}_jF_k$ may be regarded as acting either to the right or to the left. By performing the j summation-integration first and using $G_L^{-ij}\,{}_jF_k \;=\; -{}^i\delta_k$, one obtains

$$
\begin{aligned}
Y^i &= G_R^{-ij}\;{}_jF_k\;X^k + {}^i\delta_k\;X^k \\
&= G_R^{-ij}\;{}_jF_k\;X^k + X^i.
\end{aligned} \tag{1.129}
$$

By applying ${}_mF_i$ from the the left, performing the i summation-integration first and using ${}_mF_i\,G_R^{-ij} = -{}_m\delta^j$, one obtains:

$$
\begin{aligned}
{}_mF_i\,Y^i &= {}_mF_i\,G_R^{-ij}\,{}_jF_k\,X^k + {}_mF_i\,X^i \\
{}_mF_i\,Y^i &= -{}_m\delta^j\,{}_jF_k\,X^k + {}_mF_i\,X^i \\
{}_mF_i\,Y^i &= -{}_mF_k\,X^k + {}_mF_i\,X^i \\
{}_mF_i\,Y^i &= 0.
\end{aligned} \tag{1.130}
$$

But Y^i vanishes if $i <$ supp $X \equiv \cup_j$supp X^j. This means that the boundary data for the above equation vanish to the past of supp X , and hence Y^i must vanish *everywhere*. Since the X^i are arbitrary it follows that

$$\begin{aligned} 0 &= (G_R^{-ij} - G_L^{-ij})_j F_k \\ &= G_R^{-ij}{}_j F_k + {}^i\delta_k, \\ G_R^{-ij}{}_j F_k &= -{}^i\delta_k. \end{aligned} \tag{1.131}$$

But this is just the condition that G_R^{-ij} be a left Green's function. Therefore $G_R^{-ij} = G_L^{-ij}$. In a similar manner one may show that $G_R^{+ij} = G_L^{+ij}$. Thus eqs. (1.125), (1.127) imply

$$G^{\pm ik}{}_k F_j = -{}^i\delta_j \tag{1.132}$$

or, in supercondensed notation

$$G^{\pm}\overleftarrow{F} = -1. \tag{1.133}$$

It is important to stress that this proof holds regardless of the symmetry of F.

1.11.4 Reciprocity Relations

The actual supersymmetry of F gives rise to simple relations between the retarded and advanced Green's functions. Consider the expression

$$(-1)^{ki} G^{-ki}{}_k F_l\, G^{+lj}. \tag{1.134}$$

Because of the kinematical conditions (1.126), the intersection of the supports of G^{-ki} and G^{+lj}, with i and j held fixed, is compact, since it is the intersection of a forward light cone with a backward light cone. Therefore it makes no difference whether the k summation-integration or the l summation-integration is performed first. Using this fact, together with the supersymmetry law (1.108), one obtains:

$$\begin{aligned} 0 &= (-1)^{ki} G^{-ki} ({}_k F_l - (-1)^{k+l+kl}{}_l F_k) G^{+lj} \\ &= -(-1)^{ki} G^{-ki}{}_k \delta^j - (-1)^{ki+k+l+kl+il+ik+kl+k}{}_l F_k\, G^{-ki} G^{+lj} \\ &= -(-1)^{ji} G^{-ji} + (-1)^{l+il}{}_l\delta^i G^{+lj} \\ &= -(-1)^{ji} G^{-ji} + G^{+ij}. \end{aligned} \tag{1.135}$$

Therefore

$$(-1)^{ji} G^{-ji} = G^{+ij}, \tag{1.136}$$

or, equivalently

$$G^{\pm ij} = (-1)^{ij} G^{\mp ji}. \tag{1.137}$$

Equations (1.137) are called *reciprocity relations for the Green's functions.* In the supercondensed notations, they take the form:

$$G^{\pm} = G^{\mp\sim}. \tag{1.138}$$

1.11.5 Relation between Green's Functions of $\mathcal{F}$ and F

In this section an important relation between the Green's Functions of $\mathcal{F}$ and F will be derived; attention will be confined, for now, to the retarded and advanced Green's functions, those of $\mathcal{F}$ being denoted by $\mathcal{G}^-$ and $\mathcal{G}^+$, respectively:

$$ {}_\alpha\mathcal{F}_\gamma \mathcal{G}^{\pm\gamma\beta} = -{}_\alpha\delta^\beta. \tag{1.139}$$

Also in this case, $\mathcal{G}^\pm$ are both right and left Green's functions, and obey similar reciprocity relations to the ones shown for G.

By writing down the definition of F (1.106) and using (1.98), that is valid on shell, one obtains:

$$\begin{aligned} {}_jF_k{}^kQ_\beta &= {}_jS_k{}^kQ_\beta + {}_jP^\sim{}_\alpha\eta^{\alpha\gamma}{}_\gamma P_k{}^kQ_\beta \\ &= 0 + {}_jP^\sim{}_\alpha\eta^{\alpha\gamma}{}_\gamma\mathcal{F}_\beta . \end{aligned} \tag{1.140}$$

Multiplying this equation on the left by $G^\pm$ and on the right by $\mathcal{G}^\pm$, and noting that the intersection of the supports of these extra factors (with the outer suppressed indices held fixed) is compact so that F and $\mathcal{F}$ may act in either direction, one gets

$$\begin{aligned} G^{\pm ij}{}_jF_k{}^kQ_\beta\mathcal{G}^{\pm\beta\theta} &= G^{\pm ij}{}_jP^\sim{}_\alpha\eta^{\alpha\gamma}{}_\gamma\mathcal{F}_\beta\mathcal{G}^{\pm\beta\theta}, \\ {}^i\delta_k{}^kQ_\beta\mathcal{G}^{\pm\beta\theta} &= G^{\pm ij}{}_jP^\sim{}_\alpha\eta^{\alpha\gamma}{}_\gamma\delta^\theta, \\ {}^iQ_\beta\mathcal{G}^{\pm\beta\theta} &= G^{\pm ij}{}_jP^\sim{}_\alpha\eta^{\alpha\theta}, \end{aligned} \tag{1.141}$$

or, equivalently:

$$ Q\mathcal{G}^\pm = G^\pm P^\sim\eta. \tag{1.142}$$

Now, if η is chosen to be *ultralocal*, i.e., in η no undifferentiated δ distributions appear, then its negative inverse λ is unique and supersymmetric; on the other hand, if η is not ultralocal, then its negative inverse is not unique: they are Green's functions, and they will be assumed to obey the same kinematical relations as $G^\pm$, $\mathcal{G}^\pm$; its elements will be indicated as

$$ {}_\alpha\lambda_\beta = (-1)^{\alpha+\beta+\alpha\beta}{}_\beta\lambda_\alpha. \tag{1.143}$$

Hence eq. (1.142) may be written

$$ -Q\mathcal{G}^\pm\lambda = G^\pm P^\sim, \tag{1.144}$$

or, taking the supertranspose and using (1.137) and the symmetry properties:

$$ -\lambda\mathcal{G}^{\mp\sim}Q^\sim = PG^\pm. \tag{1.145}$$

Given this equation, one can prove in another way that (1.124) is the solution for the inhomogeneous equation of small disturbances which obeys the supplementary conditions:

$$\begin{aligned} {}_\alpha P_i\,\delta^\pm\phi^i &= {}_\alpha P_i\,G^{\pm ij}{}_{j,}A\epsilon \\ &= -{}_\alpha\lambda_\beta\mathcal{G}^{\pm\beta\gamma}{}_\gamma Q^{\sim j}{}_{j,}A\epsilon \\ &= 0, \end{aligned} \tag{1.146}$$

where the last line follows from the fact that A is a physical observable.

1.11.6 Landau Green's Functions

Given $\phi \in \Phi_0$, the entire tangent space at ϕ, $T_\phi\Phi$, is spanned by vectors of the form $G^{\pm ij}{}_{j,}D$, with D of compact support. Eq. (1.146) shows that, in order to get the subspace of $T_\phi\Phi$ obeying the supplementary conditions, only D which are physical observables have to be considered.

Another possible choice to obtain the same result without imposing conditions on D is to "modify" the Green's function: the task is complete if one finds an object $B^{\pm ij}$ such that

$$\begin{cases} P(G^\pm + B^\pm) = 0. \\ {}_1S_1 B^\pm = 0 \end{cases} \tag{1.147}$$

But, by using (1.145), the first equation reads

$$PB^\pm = \lambda \mathcal{G}^{\mp\sim} Q^\sim. \tag{1.148}$$

Using $PQ\mathcal{G} = \mathcal{F}\mathcal{G} = -1$ and (1.98), it is easy to see that

$$B^\pm = -Q\mathcal{G}^\pm \lambda \mathcal{G}^{\mp\sim} Q^\sim \tag{1.149}$$

is a solution which satisfies the second equation of the system, too. Therefore one is led to define *Landau Green's functions* $G^\pm_\infty$

$$\begin{aligned} G^\pm_\infty &\equiv G^\pm + B^\pm \\ &= G^\pm - Q\mathcal{G}^\pm \lambda \mathcal{G}^{\mp\sim} Q^\sim, \end{aligned} \tag{1.150}$$

or, with restored indices:

$$\begin{aligned} G^{\pm ij}_\infty &\equiv G^{\pm ij} + B^{\pm ij} \\ &= G^{\pm ij} - {}^iQ_\alpha \mathcal{G}^{\pm\alpha\beta}{}_\beta\lambda_\gamma \mathcal{G}^{\mp\sim\gamma\delta}{}_\delta Q^{\sim j}. \end{aligned} \tag{1.151}$$

The Landau Green's functions are defined only on shell and, when applied to a physical observable, $G^\pm$ and $G^\pm_\infty$ yield trivially the same results.

It is important to observe that $G^\pm_\infty$ is no longer a negative inverse for F; in fact

$$\begin{aligned} G^\pm_\infty F &= (G^\pm - Q\mathcal{G}^\pm \lambda \mathcal{G}^{\mp\sim} Q^\sim)F \\ &= -1 - Q\mathcal{G}^\pm \lambda \mathcal{G}^{\mp\sim} Q^\sim F \\ &= -1 - Q\mathcal{G}^\pm \lambda \mathcal{G}^{\mp\sim} Q^\sim ({}_1S_1 + P^\sim \eta P) \\ &= -1 - Q\mathcal{G}^\pm \lambda \mathcal{G}^{\mp\sim} Q^\sim P^\sim \eta P \\ &= -1 - Q\mathcal{G}^\pm \lambda \mathcal{G}^{\mp\sim} \mathcal{F}^\sim \eta P \\ &= -1 + Q\mathcal{G}^\pm \lambda \eta P \\ &= -1 - Q\mathcal{G}^\pm P. \end{aligned} \tag{1.152}$$

Call this operator $\Pi^\pm \equiv -G^\pm_\infty F$; it can be easily seen that it is a projection operator whose kernel is the subspace of $T_\phi\Phi$ that is tangent to the invariance flows Q_α:

$$\begin{aligned}
\Pi^{\pm 2} &= \Pi^{\pm}\Pi^{\pm} \\
&= (1 + Q\mathcal{G}^{\pm}P)(1 + Q\mathcal{G}^{\pm}P) \\
&= 1 + 2Q\mathcal{G}^{\pm}P + Q\mathcal{G}^{\pm}PQ\mathcal{G}^{\pm}P \\
&= 1 + 2Q\mathcal{G}^{\pm}P + Q\mathcal{G}^{\pm}\mathcal{F}\mathcal{G}^{\pm}P \\
&= 1 + 2Q\mathcal{G}^{\pm}P - Q\mathcal{G}^{\pm}P \\
&= 1 + Q\mathcal{G}^{\pm}P \\
&= \Pi^{\pm},
\end{aligned} \tag{1.153}$$

$$\begin{aligned}
\Pi^{\pm}Q &= (1 + Q\mathcal{G}^{\pm}P)Q \\
&= Q + Q\mathcal{G}^{\pm}PQ \\
&= Q + Q\mathcal{G}^{\pm}\mathcal{F} \\
&= Q - Q \\
&= 0,
\end{aligned} \tag{1.154}$$

and, obviously

$$\begin{aligned}
P\Pi^{\pm} &= P(1 + Q\mathcal{G}^{\pm}P) \\
&= P + PQ\mathcal{G}^{\pm}P \\
&= P + \mathcal{F}\mathcal{G}^{\pm}P \\
&= P - P \\
&= 0.
\end{aligned} \tag{1.155}$$

Finally, noticing that $\Pi^{\pm} \equiv -G^{\pm}_{\infty}F = -G^{\pm}_{\infty}{}_1S_1 = -G^{\pm}{}_1S_1$, and recalling that the restriction of a projection operator on its image is the identity operator, it can be said that the Landau Green's functions are the negative inverses of the restriction on $Ran(\Pi^{\pm})$ of the singular operator ${}_1S_1$.

1.12 Disturbances in Physical Observables

Let B be a physical observable; call $\delta^{\pm}B$ the changes in value of B under the disturbance (1.124) caused by the change in the action functional (1.95). Then

$$\begin{aligned}
\delta^{\pm}B &= B_{,i}\delta^{\pm}\phi^i \\
&= B_{,i}(G^{\pm ij}{}_{j,}A)\epsilon \\
&= (-1)^{AB}(A_{,j}G^{\mp ji})_{i,}B\epsilon,
\end{aligned} \tag{1.156}$$

in which eq. (1.7) and the reciprocity relations (1.137) have been used in obtaining the final expression. Parentheses have been inserted because it is not guaranteed that if the summation-integration over i is performed before the summation-integration over j the same result will be obtained. We shall assume that the functions $B_{,i}$ do not increase in magnitude too rapidly for convergence either in the past or in the future. In fact we shall assume that these functions are well enough behaved that the parentheses may be removed. The following are some sufficient (although not necessary) conditions for the parentheses to be absent:

1. With the retarded solution $\delta^-\phi$, if $(\cup_j \text{supp }_{j,}A) \cap (\cup_i \text{supp } B_{,i})$ is compact and there exist spacelike hypersurfaces Σ_+, Σ_- such that $(\cup_i \text{supp } B_{,i}) < \Sigma_+$ and $(\cup_j \text{supp }_{j,}A) > \Sigma_-$. In this case $\delta^- B$ vanishes unless $\Sigma_+ > \Sigma_-$.

2. With the advanced solution $\delta^+\phi$, if $(\cup_j \text{supp }_{j,}A) \cap (\cup_i \text{supp } B_{,i})$ is compact and there exist spacelike hypersurfaces Σ_+, Σ_- such that $(\cup_j \text{supp }_{j,}A) < \Sigma_+$ and $(\cup_i \text{supp } B_{,i}) > \Sigma_-$. In this case $\delta^+ B$ vanishes unless $\Sigma_+ > \Sigma_-$.

3. With either solution, if $(\cup_j \text{supp }_{j,}A)$ and $(\cup_i \text{supp } B_{,i})$ are both compact.

Remark 1.3. When, as now, we are working with ϕ on shell, a question arises regarding the meaning of the expressions supp $_{j,}A$ and supp$B_{,i}$. When ϕ is on shell the functional form of a physical observable is defined only *modulo* the dynamical equations, and hence the expressions supp $_{j,}A$ and supp $B_{,i}$ would seem to be ambiguous. The following clarification is necessary: every physical observable has an expression in terms of the fields ϕ^i the functional form of which is independent of that of S. *This* is the form that is to be understood in the expressions supp $_{j,}A$ and supp $B_{,i}$. This form remains invariant under the change (1.95) in the action. Only its value changes, because the values of the dynamical variables themselves change.

1.12.1 The Reciprocity Relation for Physical Observables

It will be useful to introduce the notation

$$D_A^{\pm} B \equiv A_{,i} G^{\mp ij}{}_{j,} B \tag{1.157}$$

Equation (1.156) may then be written:

$$\begin{aligned} \delta^{\pm} B &= D_{A\epsilon}^{\pm} B \\ &= (A\epsilon)_{,i} G^{\mp ij}{}_{j,} B, \end{aligned} \tag{1.158}$$

as may be seen by noting that A and ϵ have the same type.

Colloquially, $D_A^- B$ may be called the "retarded effect of A on B" and $D_A^+ B$ the "advanced effect of A on B". It is a consequence of the reciprocity relations (1.137) that

$$\begin{aligned} D_A^{\pm} B &\equiv A_{,i} G^{\mp ij}{}_{j,} B \\ &= (-1)^{AB} B_{,i} G^{\pm ij}{}_{j,} A \\ &= (-1)^{AB} D_B^{\mp} A. \end{aligned} \tag{1.159}$$

In words: *The retarded effect of A on B equals $(-1)^{AB}$ times the advanced effect of B on A (and vice versa).* This is known as the reciprocity relation for physical observables.

1.12.2 Off Shell Relations

So far we have used the distributions ${}_{\alpha}P_i$, $\eta^{\alpha\beta}$ only on shell. However, they, like the ${}^iQ_{\alpha}$, ${}_iS_j$, etc. have specific functional forms (as functionals of ϕ) and are defined also off shell. Thus the operators ${}_iF_j$ and ${}_{\alpha}\mathcal{F}_{\beta}$ and Green's functions $G^{\pm ij}$ and $\mathcal{G}^{\pm\alpha\beta}$ are defined off shell as well. Off shell, eq. (1.98) no longer necessarily holds and the relations (1.142) (1.144) fail to be satisfied generically. However, the operators ${}_iF_j$ and ${}_\alpha\mathcal{F}_\beta$ continue to be nonsingular and the Green's functions $G^{\pm ij}$ and $\mathcal{G}^{\pm\alpha\beta}$ continue to exist, at least in an open neighborhood of the dynamical shell.

We need $G^{\pm ij}$ and $\mathcal{G}^{\pm\alpha\beta}$ off shell because we need to be able to calculate their functional derivatives, in order to discuss the invariance properties of $D_A^\pm B$. We begin by considering an arbitrary infinitesimal variation δF in the operator F. This variation may arise either by shifting the point ϕ in Φ, or by varying the functional forms of P, η, or even the action S (the vector fields Q_α will be left untouched). It leads to corresponding variations $\delta G^\pm$ in the advanced and retarded Green's functions.

The $\delta G^\pm$ satisfy a differential equation that is obtained by varying eq. (1.127):

$$\begin{aligned} \delta(FG^\pm) &= \delta(-1), \\ \delta F G^\pm + F\delta G^\pm &= 0, \\ F\delta G^\pm &= -\delta F G^\pm. \end{aligned} \tag{1.160}$$

This equation has the following unique solution:

$$\delta G^\pm = G^\pm \delta F G^\pm \tag{1.161}$$

which is determined by the kinematical conditions (1.126) that the Green's functions satisfy. It will be noted that the intersection of the supports of the two factors $G^\pm$ (with outer suppressed indices fixed) in the previous equation is compact, so that the operator δF in this equation may act in either direction.

Equation (1.161) is just the equation one would get if F were a finite square matrix and $G^\pm$ were its negative inverse. There are important differences, however, between the present case and the case of finite matrices. First, F has many "inverses", or Green's functions, not just one. Second, most of its Green's functions do *not* satisfy variational equations having the structure (1.161). For example, the average

$$\bar{G} \equiv \tfrac{1}{2}(G^+ + G^-) \tag{1.162}$$

is a Green's function of F:

$$\begin{aligned} F\bar{G} &= \tfrac{1}{2}F(G^+ + G^-) \\ &= \tfrac{1}{2}FG^+ + \tfrac{1}{2}FG^- \\ &= -\tfrac{1}{2} - \tfrac{1}{2} \\ &= -1, \end{aligned} \tag{1.163}$$

but it satisfies

$$\delta\bar{G} = \tfrac{1}{2}(G^+\delta F G^+ + G^-\delta F G^-), \tag{1.164}$$

which is not equal to $\bar{G}\delta F\bar{G}$.

If a Green's function does satisfy

$$\delta G = G\delta F G \tag{1.165}$$

it will be called a *coherent* Green's function. In many of the equations of this work the operators F, $\mathcal{F}$, ${}_1S_1$, δF, etc. will appear sandwiched between factors, the intersections of the supports of which are not compact. These operators will nevertheless be able to act in either direction because the Green's functions in these factors are all in the same *coherence class* and establish an over-all set of boundary conditions that are preserved regardless of the direction of action.

Suppose ϕ is on shell and suppose the variation δF arises from variations δP and $\delta\eta$ in P and η: then

$$\delta F = \delta P^{\sim}\eta P + P^{\sim}\delta\eta P + P^{\sim}\eta\delta P. \tag{1.166}$$

Inserting this expression in (1.161) and making use of eqs. (1.144) and (1.145), as well as

$$\delta\eta = \eta\ \delta\lambda\ \eta, \tag{1.167}$$

one obtains

$$\begin{aligned} \delta G^{\pm} &= G^{\pm}(\delta P^{\sim}\eta P + P^{\sim}\delta\eta P + P^{\sim}\eta\delta P)G^{\pm} \\ &= G^{\pm}\delta P^{\sim}\eta P G^{\pm} + G^{\pm}P^{\sim}(\eta\ \delta\lambda\ \eta)PG^{\pm} + G^{\pm}P^{\sim}\eta\delta P G^{\pm} \\ &= -G^{\pm}\delta P^{\sim}\eta\lambda\mathcal{G}^{\mp\sim}Q^{\sim} - Q\mathcal{G}^{\pm}\lambda(\eta\ \delta\lambda\ \eta)(-\lambda\mathcal{G}^{\mp\sim}Q^{\sim}) - Q\mathcal{G}^{\pm}\lambda\eta\delta P G^{\pm} \\ &= G^{\pm}\delta P^{\sim}\mathcal{G}^{\mp\sim}Q^{\sim} + Q\mathcal{G}^{\pm}\delta\lambda\mathcal{G}^{\mp\sim}Q^{\sim} + Q\mathcal{G}^{\pm}\delta P G^{\pm}. \end{aligned} \tag{1.168}$$

If the variation δF arises instead from a variation in ϕ, then one is led to the formula

$$\begin{aligned} \delta G^{\pm ij} &= G^{\pm ij}_{\ \ ,k}\ \delta\phi^{k} \\ &= G^{\pm il}\ {}_{l}\delta F_{m}\ G^{\pm mj} \\ &= G^{\pm il}\ {}_{l}F_{m,k}\ \delta\phi^{k}\ G^{\pm mj} \\ &= (-1)^{km+kj}\ G^{\pm il}\ {}_{l}F_{m,k}\ G^{\pm mj}\ \delta\phi^{k}, \end{aligned} \tag{1.169}$$

and then

$$G^{\pm ij}_{\ \ ,k} = (-1)^{km+kj}\ G^{\pm il}\ {}_{l}F_{m,k}\ G^{\pm mj} \tag{1.170}$$

Of course this formula is valid off shell. Going on shell one can proceed just as in the derivation of eq.(1.168) and obtain:

$$\begin{aligned} G^{\pm ij}_{\ \ ,k} &= (-1)^{km+kj}\ G^{\pm il}({}_{l,}S_{,m} + {}_{l}P^{\sim}{}_{\alpha}\ \eta^{\alpha\beta}\ {}_{\beta}P_{m})_{,k}\ G^{\pm mj} \\ &= (-1)^{km+kj}\ G^{\pm il}({}_{l,}S_{,mk} \\ &\quad + {}_{l}P^{\sim}{}_{\alpha}\ \eta^{\alpha\beta}\ {}_{\beta}P_{m,k} \\ &\quad + (-1)^{km+k\beta}\ {}_{l}P^{\sim}{}_{\alpha}\ \eta^{\alpha\beta}{}_{,k}\ {}_{\beta}P_{m} \\ &\quad + (-1)^{km+k\alpha}\ {}_{l}P^{\sim}{}_{\alpha,k}\ \eta^{\alpha\beta}\ {}_{\beta}P_{m})\ G^{\pm mj} \\ &= (-1)^{km+kj}\ [G^{\pm il}\ {}_{l,}S_{,mk}\ G^{\pm mj} \\ &\quad + {}^{i}Q_{\alpha}\ \mathcal{G}^{\pm\alpha\beta}\ {}_{\beta}P_{m,k}\ G^{\pm mj} \\ &\quad - (-1)^{km+k\beta}\ {}^{i}Q_{\alpha}\ \mathcal{G}^{\pm\alpha\beta}\ {}_{\beta}\eta_{\gamma}\ \eta^{\gamma\theta}{}_{,k}\ {}_{\theta}\lambda_{\iota}\ \mathcal{G}^{\mp\sim\iota\kappa}\ {}_{\kappa}Q^{\sim j} \\ &\quad + (-1)^{km+k\alpha}\ G^{\pm il}\ {}_{l}P^{\sim}{}_{\alpha,k}\ \mathcal{G}^{\mp\sim\alpha\beta}\ {}_{\beta}Q^{\sim j}] \\ &= (-1)^{km+kj}[G^{\pm il}\ {}_{l,}S_{,mk}\ G^{\pm mj} \\ &\quad + {}^{i}Q_{\alpha}\ \mathcal{G}^{\pm\alpha\beta}\ {}_{\beta}P_{m,k}\ G^{\pm mj} \\ &\quad + (-1)^{km+k\beta}\ {}^{i}Q_{\alpha}\ \mathcal{G}^{\pm\alpha\beta}\ {}_{\beta}\lambda_{\gamma,k}\ \mathcal{G}^{\mp\sim\gamma\theta}\ {}_{\theta}Q^{\sim j} \\ &\quad + (-1)^{km+k\alpha}\ G^{\pm il}\ {}_{l}P^{\sim}{}_{\alpha,k}\ \mathcal{G}^{\mp\sim\alpha\beta}\ {}_{\beta}Q^{\sim j}]. \end{aligned} \tag{1.171}$$

It will be noted that summation-integrations can be performed in any order in the previous equations because all the Green's functions are in the same coherence class.

1.12.3 Invariance Properties of $D_A^{\pm}B$

Expression (1.157) for $D_A^{\pm}B$ involves the Green's functions $G^{\pm}$, which depend on specific choice for the operators P and η. Since $D_A^{\pm}B$ represent the physical effects of physical changes in the action they must be P- and η- independent. To verify this write the variation of (1.157), under changes in P and η, in the supercondensed notation:

$$\delta D_A^{\pm}B = A_{,i}\delta G^{\mp ij}{}_{j,}B. \tag{1.172}$$

If ϕ is on shell, one can insert (1.168) into the right-hand side. The result is immediately seen to vanish,

$$\delta D_A^{\pm}B = 0, \tag{1.173}$$

because of the on shell invariance conditions

$$A_1 Q = 0 \qquad Q^{\sim}{}_1 B = 0 \tag{1.174}$$

satisfied by A and B as physical observables.

For each on shell ϕ the *values* of the $D_A^{\pm}B$ are physical, i.e., P- and η-independent. However, as *functionals of the* ϕ (off shell as well as on) the $D_A^{\pm}B$ turn out *not* to be physical observables unless both A and B are absolute invariants. To see this introduce parameters ξ^{α} of compact support in M, and make use of the supercondensed notation. Then if one evaluates the following quantity on shell, one obtains:

$$\begin{aligned}(D_A^{\pm}B)_1 Q\xi &= (A_1\, G^{\mp}\,{}_1B)_1 Q\xi \\ &= A_1\, G^{\mp}\,{}_1B_1 Q\xi + A_1\, G^{\mp}{}_1\, Q\xi\,{}_1B + \xi^{\sim}Q^{\sim}\,{}_1A_1\, G^{\mp}\,{}_1B \\ &= A_1\, G^{\mp}\,{}_1B_1 Q\xi + A_1\, G^{\mp}\,{}_1S_2\, Q\xi\, G^{\mp}\,{}_1B + \xi^{\sim}Q^{\sim}\,{}_1A_1\, G^{\mp}\,{}_1B,\end{aligned} \tag{1.175}$$

where eqs. (1.171) and (1.174) have been used.

In order to evaluate ${}_1B_1 Q\xi$ and $\xi^{\sim}Q^{\sim}\,{}_1A_1$, multiply

$$A_1\, Q = S_1\, a \qquad B_1\, Q = S_1\, b \tag{1.176}$$

by ξ and functionally differentiate; the result is (on shell):

$${}_1A_1\, Q\xi = -{}_1(\xi^{\sim}Q^{\sim})\,{}_1A + {}_1S_1\, a\xi \qquad {}_1B_1\, Q\xi = -{}_1(\xi^{\sim}Q^{\sim})\,{}_1B + {}_1S_1\, b\xi. \tag{1.177}$$

Functionally differentiating twice $S_1\, Q\xi = 0$ and then going on shell, one obtains the third Ward identity:

First functional derivative:

$${}_1S_1\, Q\xi = -{}_1(\xi^{\sim}Q^{\sim})\,{}_1S \tag{1.178}$$

Second functional derivative:

$${}_1S_2\, Q\xi + {}_1S_1\,(Q\xi)_1 = -{}_1(\xi^{\sim}Q^{\sim})\,{}_1S_1 - {}_1(\xi^{\sim}Q^{\sim})_1\,{}_1S \tag{1.179}$$

Going on shell:

$$ {}_1S_2\, Q\xi = -{}_1S_1\, (Q\xi)_1 - {}_1(\xi^{\sim} Q^{\sim})\, {}_1S_1. \tag{1.180} $$

Hence, inserting eqs. (1.177) (1.180) in (1.175), one obtains:

$$ \begin{aligned} (D_A^{\pm} B)_1 Q\xi &= A_1\, G^{\mp}\, (-{}_1(\xi^{\sim} Q^{\sim})\, {}_1B + {}_1S_1\, b\xi) \\ &\quad + A_1\, G^{\mp}\, (-{}_1S_1\, (Q\xi)_1 - {}_1(\xi^{\sim} Q^{\sim})\, {}_1S_1) G^{\mp}\, {}_1B \\ &\quad + (-\, A_1\, (Q\xi)_1 + (\xi^{\sim} a^{\sim})\, {}_1S_1)\, G^{\mp}\, {}_1B \\ &= -A_1\, G^{\mp}\, {}_1(\xi^{\sim} Q^{\sim})\, {}_1B + A_1\, (-\Pi^{\mp})\, b\xi \\ &\quad - A_1\, (-\Pi^{\mp})\, (Q\xi)_1\ \ G^{\mp}\, {}_1B - A_1\, G^{\mp}\, {}_1(\xi^{\sim} Q^{\sim})\, (-\Pi^{\pm\sim})\, {}_1B \\ &\quad -\, A_1\, (Q\xi)_1\, G^{\mp}\, {}_1B + \xi^{\sim} a^{\sim}\, (-\Pi^{\pm\sim})\, {}_1B \\ &= -A_1\, b\xi - \xi^{\sim} a^{\sim}\, {}_1B, \end{aligned} \tag{1.181} $$

in which the properties of the projection operator $\Pi^{\pm}$ and eq. (1.174) have been exploited in obtaining the final expression.

This remarkably simple expression has two important properties. First, it vanishes if A and B are absolute invariants, i.e., if a and b vanish. Second, it is independent of the $\pm$ signs, being the same for both retarded and advanced disturbances.

Another result that is independent of the $\pm$ signs is the following, which shows explicitly that $D_A^{\pm} B$ is not invariant under changes in A and B of the form (1.84); on shell:

$$ \begin{aligned} D_{\bar{A}}^{\pm} \bar{B} - D_A^{\pm} B &= (A + a^l{}_{l,}S)_{,i}\, G^{\mp ij}{}_{j,}(B + S_{,k}{}^k b) - A_{,i} G^{\mp ij}{}_{j,}B \\ &= (A + a^{\sim}\, {}_1S)_1\, G^{\mp}\, {}_1(B + S_1\, b) - A_1\, G^{\mp}\, {}_1B \\ &= A_1\, G^{\mp}\, {}_1(S_1\, b) \\ &\quad + (a^{\sim}\, {}_1S)_1\, G^{\mp}\, {}_1(B) \\ &\quad + (a^{\sim}\, {}_1S)_1\, G^{\mp}\, {}_1(S_1\, b) \\ &= A_1\, G^{\mp}\, {}_1S_1\, b + A_1\, G^{\mp}\, S_1\, {}_1b \\ &\quad + a\, {}_1S_1\, G^{\mp}\, {}_1B + a^{\sim}{}_1\, {}_1S\, G^{\mp}\, {}_1B \\ &\quad + a^{\sim}\, {}_1S_1\, G^{\mp}\, {}_1S_1\, b \\ &\quad + a^{\sim}\, {}_1S_1\, G^{\mp}\, S_1\, {}_1b \\ &\quad + a^{\sim}{}_{11}S\, G^{\mp}\, {}_1S_1\, b \\ &\quad + a^{\sim}{}_{11}S\, G^{\mp}\, S_1\, {}_1b \\ &= A_1\, (-\Pi^{\mp})\, b \\ &\quad + a^{\sim}\, (-\Pi^{\pm\sim})\, {}_1B \\ &\quad + a^{\sim}\, {}_1S_1\, (-\Pi^{\mp})\, b \\ &= -A_1\, b - a^{\sim}\, {}_1B - a^{\sim}\, {}_1S_1\, b, \end{aligned} \tag{1.182} $$

where

$$ \bar{A} = A + a^i{}_{i,}S \qquad \bar{B} = B + S_{,k}{}^k b. \tag{1.183} $$

Here a^i and ${}^k b$ are assumed to have the properties (e.g., rapid fall-off in the past and future) that are necessary for the associative law of multiplication to hold.

1.12.4 Peierls Bracket, Supercommutator Function

Let A and B be two physical observables. Their *Peierls Bracket* is defined to be

$$(A, B) \equiv D_A^- B - (-1)^{AB} D_B^- A. \tag{1.184}$$

Using the definition (1.157) and the reciprocity relation for physical observables (1.159) one may re-express this bracket in the form

$$\begin{aligned}(A, B) &= D_A^- B - D_A^+ B \\ &= A_{,i}\, G^{+ij}\ {}_{j,}B - A_{,i}\, G^{-ij}\ {}_{j,}B \\ &= A_{,i}\, \tilde{G}^{ij}\ {}_{j,}B,\end{aligned} \tag{1.185}$$

where

$$\tilde{G}^{ij} \equiv G^{+ij} - G^{-ij}. \tag{1.186}$$

In anticipation of its role in quantum theory $\tilde{G}$ wll be called the *supercommutator function*; it has the symmetry properties

$$\begin{aligned}\tilde{G}^{ji} &= G^{+ji} - G^{-ji} \\ &= (-1)^{ij}(G^{-ij} - G^{+ji}) \\ &= -(-1)^{ij}\tilde{G}^{ij},\end{aligned} \tag{1.187}$$

or, in supercondensed notation

$$\tilde{G}^{\sim} = -\tilde{G}. \tag{1.188}$$

Unlike $D_A^{\pm} B$ *the Peierls bracket*, as a functional of ϕ, *is always a physical observable* no matter whether A and B are absolute invariants or conditional invariants. This follows from (1.175), which yields

$$\begin{aligned}(A, B)_1 Q\xi &= (D_A^- B)_1 Q\xi - (D_A^+ B)_1 Q\xi \\ &= 0.\end{aligned} \tag{1.189}$$

Moreover, we also have, using (1.182)

$$\begin{aligned}(\bar{A}, \bar{B}) &= D_{\bar{A}}^- \bar{B} - D_{\bar{A}}^+ \bar{B} \\ &= (D_A^- B - A_1\, b - a^{\sim}\ {}_1 B - a^{\sim}\ {}_1 S_1\ b) \\ &\quad -(D_A^+ B - A_1\, b - a^{\sim}\ {}_1 B - a^{\sim}\ {}_1 S_1\ b) \\ &= D_A^- B - D_A^+ B \\ &= (A, B),\end{aligned} \tag{1.190}$$

where $\bar{A}$, $\bar{B}$ are given by (1.183). Eq. (1.190) shows that *it is immaterial whether the dynamical equations are used before or after computing the Peierls bracket. That is, use of Peierls bracket commutes with use of any on shell conditions or restrictions.*

If the dynamical system possesses no invariant flows, then the ϕ are themselves physical observables, and eq. (1.185) implies

$$(\phi^i, \phi^j) = \tilde{G}^{ij} \tag{1.191}$$

When invariant flows are present the Peierls bracket of the ϕ is not defined. However, in computing the brackets of observables one may proceed *as if* it were given by eq. (1.191).

1.12.5 The Bracket Identities

Let A^α and B^α be any two families of physical observables, and let $U(A)$ and $V(B)$ be two any functions of these families. Equation (1.185) has the immediate corollary

$$(U(A), V(B)) = U(A)\frac{\overleftarrow{\partial}}{\partial A^\alpha}(A^\alpha, B^\beta)\frac{\overrightarrow{\partial}}{\partial B^\beta}V(B). \tag{1.192}$$

The following properties are called *simple* identities satisfied by the Peierls Bracket; their proof is straightforward:

$$(A, B + C) = (A, B) + (A, C) \tag{1.193}$$

$$(A, BC) = (A, B)C + (-1)^{AB}B(A, C) \tag{1.194}$$

$$(A, B) = -(-1)^{AB}(B, A) \tag{1.195}$$

The Peierls Bracket also satisfies the *Jacobi identity*, which can be expressed in the form:

$$\epsilon_{\alpha\beta\gamma}(A^\gamma, (A^\beta, A^\alpha)) = 0. \tag{1.196}$$

Chapter 2

Path Integral in QM and QFT with no Invariance Flows: a Naive Approach

2.1 Quantum Mechanics

We begin our discussion about path integral quantization deriving it from the Schrödinger formulation of quantum mechanics.

As everybody knows, the Schrödinger equation is:

$$\hat{H}|\psi(t)\rangle = i\frac{d}{dt}|\psi(t)\rangle \tag{2.1}$$

where $\hat{H}$ is the Hamiltonian operator, $|\psi(t)\rangle$ is the quantum state of the system at time t, and $\hbar = 1$ for notational convenience.

The Hamiltonian operator is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x}) \tag{2.2}$$

where $\hat{p}$, $\hat{x}$ are momentum and position operators, respectively; they obey $[\hat{x}, \hat{p}] = i$. Position and momentum eigenstates obey the following relations:

$$\begin{aligned}
\hat{x}|x\rangle &= x|x\rangle, &(2.3)\\
\langle x|x'\rangle &= \delta(x, x'), &(2.4)\\
\hat{1} &= \int_{\mathcal{R}} dx\ \ |x\rangle\langle x|, &(2.5)\\
\hat{p}|p\rangle &= p|p\rangle, &(2.6)\\
\langle p|p'\rangle &= \delta(p, p'), &(2.7)\\
\hat{1} &= \int_{\mathcal{R}} dp\ \ |p\rangle\langle p|, &(2.8)\\
\langle x|p\rangle &= \frac{1}{\sqrt{2\pi}}e^{ipx}, &(2.9)
\end{aligned}$$

where $\delta(\cdot, \cdot)$ is the Dirac delta distribution.

We shall take the potential $V(x)$ of the form

$$V(x) = \sum_j \frac{g_j}{|x - x_j|} + V_1(x), \tag{2.10}$$

where $g_j \in \mathcal{R}$ and $V_1(x)$ has the following properties:

- it is continuous except at most at some surfaces, on which it is finitely discontinuous;
- it is bounded from below;
- at infinity, it grows at most like a polynomial.

Under these assumptions, it can be shown that $\hat{H}$ is *symmetric* and *essentially self-adjoint* on the domain $\mathcal{D}_0(\hat{H})$[1]

$$\mathcal{D}_0(\hat{H}) \equiv \left\{ f(x) \in \mathcal{L}^2(\mathcal{R}^3) | f(x) \in C^2(\mathcal{R}^3), (-\frac{1}{2m}\triangle + V(x))f(x) \in \mathcal{L}^2(\mathcal{R}^3) \right\}. \tag{2.12}$$

Hence, it admits only one self-adjoint extension, $\hat{H}^\dagger$. We shall always extend $\hat{H}$ from its original domain $\mathcal{D}_0(\hat{H})$ to $\mathcal{D}(\hat{H}^\dagger)$ and we shall set

$$\mathcal{D}(\hat{H}) \equiv \mathcal{D}(\hat{H}^\dagger), \quad \hat{H} = \hat{H}^\dagger. \tag{2.13}$$

Therefore, by this extension $\hat{H}$ is *self-adjoint*; this property is crucial for the following statements.

It is well known that every such Hamiltonian operator $\hat{H}$ can be associated with a time-evolution operator $\hat{U}(t, t_0)$, with the following properties:

(A) it is unitary: $\hat{U}^\dagger(t, t_0)\hat{U}(t, t_0) = \hat{U}(t, t_0)\hat{U}^\dagger(t, t_0) = \hat{1}$,

(B) it obeys the composition rule: $\hat{U}(t, t_1)\hat{U}(t_1, t_0) = \hat{U}(t, t_0)$,

(C) it reduces to the identity operator for $t = t_0$: $\hat{U}(t_0, t_0) = \hat{1}$,

(D) it provides the time evolution for the solutions of Schrödinger equation: $|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle$

for every $t \geq t_1 \geq t_0$ and initial state $|\psi(t_0)\rangle$.
Whenever the Hamiltonian operator is time-independent, the associated time-evolution operator has the form:

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)} \tag{2.14}$$

Now, let us recall Heisenberg's picture: physical states do not evolve, while (time-indipendent) observables and their eigenstates evolve according to:

$$\begin{aligned} A(t) &= \hat{U}^\dagger(t, t_0)A(t_0)\hat{U}(t, t_0) && (2.15) \\ |a, t\rangle &= \hat{U}^\dagger(t, t_0)|a, t_0\rangle && (2.16) \end{aligned}$$

[1] For the sake of clarity, the operator appearing in (2.12) is not $\hat{H}$, but its representative in the position auto-kets basis $\{|x\rangle\}$:

$$(-\frac{1}{2m}\triangle + V(x))f(x) = \langle x|\hat{H}|f\rangle. \tag{2.11}$$

That being said, we can project the equation (D) on a position eigenstate and use (2.5):

$$\begin{aligned}\langle x|\psi(t)\rangle \equiv \psi(t,x) &= \langle x|\hat{U}(t,t_0)|\psi(t_0)\rangle \\ &= \langle x,t|\int_{\mathcal{R}} dx' \quad |x',t_0\rangle\langle x',t_0|\psi(t_0)\rangle \\ &= \int_{\mathcal{R}} dx' \langle x,t|x',t_0\rangle \psi(t_0,x') \qquad (2.17)\end{aligned}$$

Our next step is the evaluation of $\langle x,t|x',t_0\rangle$, that will be called kernel of the Schrödinger equation.

It is important to recall Trotter's product formula:

Theorem 2.1. *If $\hat{T}$, $\hat{V}$ are self-adjoint operators, and their sum $\hat{T}+\hat{V}$ is also self-adjoint, the following holds:*

$$e^{-i(t-t_0)(\hat{T}+\hat{V})} = \lim_{N\to+\infty}(e^{\frac{-i(t-t_0)\hat{T}}{N}} \ e^{\frac{-i(t-t_0)\hat{V}}{N}})^N \qquad (2.18)$$

Hence, first we evaluate $\langle x|e^{\frac{-i(t-t_0)\hat{T}}{N}} \ e^{\frac{-i(t-t_0)\hat{V}}{N}}|x'\rangle$, with $\hat{T} \equiv \frac{1}{2m}\hat{p}^2$; by using (2.8) and (2.9), we obtain

$$\begin{aligned}&\langle x|e^{\frac{-i(t-t_0)\hat{T}}{N}} \ e^{\frac{-i(t-t_0)\hat{V}}{N}}|x'\rangle \\ =& \int_{\mathcal{R}} dp' \quad \langle x|e^{-i\hat{T}\delta t}|p'\rangle\langle p'|e^{-i\hat{V}\delta t}|x'\rangle \\ =& \int_{\mathcal{R}} dp' \quad e^{\frac{-i}{2m}p'^2\delta t}\langle x|p'\rangle e^{-iV(x')\delta t}\langle p'|x'\rangle \\ =& \frac{e^{-iV(x')\delta t}}{2\pi}\int_{\mathcal{R}} dp' \quad e^{\frac{-i}{2m}p'^2\delta t}e^{ip'(x-x')} \\ =& \frac{e^{-iV(x')\delta t}}{2\pi}\sqrt{\frac{2\pi m}{i\delta t}}e^{-\frac{m(x-x')^2}{2i\delta t}} \\ =& e^{i\delta t[\frac{m}{2}(\frac{(x-x')^2}{\delta t^2}-V(x')]}\sqrt{\frac{m}{2i\pi\delta t}} \qquad (2.19)\end{aligned}$$

where $\delta t \equiv \frac{t-t_0}{N}$ has been defined and, on the fourth line, the following Gaussian integral has been used:

$$\int_{\mathcal{R}} dp' \quad e^{-ap'^2+bp'} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \qquad (2.20)$$

with $a = \frac{i\delta t}{2m}$, $b = i(x-x')$.

Thus, using $N-1$ times (2.5) and Trotter's formula, we obtain:

$$
\begin{aligned}
& \langle x, t | x', t_0 \rangle \\
= \; & \langle x | e^{-i\hat{H}(t-t_0)} | x' \rangle \\
= \; & \lim_{N\to+\infty} \langle x | (e^{\frac{-i(t-t_0)\hat{T}}{N}} \; e^{\frac{-i(t-t_0)\hat{V}}{N}})^N | x' \rangle \\
= \; & \lim_{N\to+\infty} \int_{\mathcal{R}} dx_{N-1} \ldots \int_{\mathcal{R}} dx_1 \prod_{j=0}^{N-1} \left[e^{i\delta t [\frac{m}{2} \frac{(x_{j+1}-x_j)^2}{\delta t^2} - V(x_j)]} \sqrt{\frac{m}{2i\pi\delta t}} \right] \\
= \; & \lim_{N\to+\infty} \int_{\mathcal{R}} dx_{N-1} \ldots \int_{\mathcal{R}} dx_1 \quad \bullet \\
& \bullet \left(\frac{m}{2i\pi\delta t} \right)^{N/2} \exp \left[i\delta t \left(\sum_{j=0}^{N-1} \frac{m}{2} \frac{(x_{j+1}-x_j)^2}{\delta t^2} - V(x_j) \right) \right], \qquad (2.21)
\end{aligned}
$$

where $x_N \equiv x$, $x_0 \equiv x'$, $\delta t \equiv \frac{t-t_0}{N}$. Now, we define $t_k \equiv t_0 + k\delta t$, for $k = 1, \ldots, N-1$, and $t_0 \equiv t'$, $t_N \equiv t$.

Then, let us consider the following polygonal chain, with vertices $\{x_k\}_{k=0}^N$:

$$
\Gamma_N(t) = \begin{cases} \ldots \\ x_k + t \left(\dfrac{x_{k+1}-x_k}{t_{k+1}-t_k} \right) & \text{if } t_k \le t \le t_{k+1} \\ \ldots \end{cases} \qquad (2.22)
$$

Therefore we have

$$
\frac{d}{dt}\Gamma_N(t) \equiv \dot{\Gamma}_N(t) = \begin{cases} \ldots \\ \left(\dfrac{x_{k+1}-x_k}{t_{k+1}-t_k} \right) = \left(\dfrac{x_{k+1}-x_k}{\delta t} \right) & \text{if } t_k < t < t_{k+1} \\ \ldots \end{cases} \qquad (2.23)
$$

Hence the argument in the exponential (2.21) can be written as a "discrete action":

$$
S_N[\Gamma_N] \equiv \int_{t_0}^{t_N} dt \quad \left[\frac{m}{2} \, \dot{\Gamma}_N(t)^2 - V(\Gamma_N(t)) \sum_{k=0}^{N-1} \delta(t, t_k) \right] \qquad (2.24)
$$

When N approaches infinity, $\Gamma_N(t)$ tends to a continuous curve $\gamma(t)$ with endpoints x', x, and the discrete action S_N tends to the classic action:

$$
S[\gamma] \equiv \int_{t_0}^{t_N} dt \;\; L(\gamma(t), \dot{\gamma}(t)), \qquad (2.25)
$$

where $L \equiv T - V$ is the Langrange function of the system.

On reverting to (2.21), when the limit $N \to \infty$ is taken, integrating on the midpoint variables $x_1, \ldots x_{N-1}$ means taking the sum on all continuous paths[2]

[2]Altough our "derivation" of the path integral applies to paths obtained as limits of polygonal chains, i.e., continuous paths, Feynman's formulation involves all paths, both the continuous and the discontinuous ones.

with endpoints x', x. Therefore, with the formal definition of the path measure:

$$\int_{t_0,x'}^{t,x} \mathcal{D}\gamma \equiv \lim_{N\to+\infty} \int_{\mathcal{R}} dx_{N-1} \,...\, \int_{\mathcal{R}} dx_1 \left(\frac{m}{2i\pi\delta t}\right)^{N/2} \tag{2.26}$$

(2.21) reads:

$$\langle x, t|x', t_0\rangle = \int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]} \tag{2.27}$$

where the limits of integration denote that the sum is taken on the paths $\gamma(t)$ such that $\gamma(t_0) = x'$, $\gamma(t) = x$. This result is extremely significant, and can be generalized; let us consider the matrix element of the position operator $\hat{x}$ at time t_1 such that $t_0 \le t_1 \le t$:

$$\begin{aligned}
\langle x, t|\hat{x}(t_1)|x', t_0\rangle &= \int_{\mathcal{R}}\int_{\mathcal{R}} dy_2\, dy_1 \;\; \langle x, t|y_2, t_1\rangle\langle y_2, t_1|\hat{x}(t_1)|y_1, t_1\rangle\langle y_1, t_1|x', t_1\rangle \\
&= \int_{\mathcal{R}}\int_{\mathcal{R}} dy_2\, dy_1 \;\; \langle x, t|y_2, t_1\rangle y_1 \delta(y_1, y_2)\langle y_1, t_1|x', t_1\rangle \\
&= \int_{\mathcal{R}} dy \;\; \langle x, t|y, t_1\rangle y \langle y, t_1|x', t_1\rangle \\
&= \int_{\mathcal{R}} dy \int_{t_1,y}^{t,x} \mathcal{D}\gamma_2 \int_{t_0,x'}^{t_1,y} \mathcal{D}\gamma_1 \;\; e^{iS[\gamma_2]} e^{iS[\gamma_1]} y \\
&= \int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]}\gamma(t_1),
\end{aligned} \tag{2.28}$$

where (2.3), (2.5), (2.16), (2.27) have been used and the further (reasonable) assumption has been made:

since γ_1, γ_2 are paths such that $\gamma_1(t_1) = \gamma_2(t_1) = y$, $\gamma_1(t_0) = x'$, $\gamma_2(t) = x$, their union is a path γ such that $\gamma(t_0) = x'$, $\gamma(t) = x$, $\gamma(t_1) = y$ and the following holds:

$$\int_{\mathcal{R}} dy \int_{t_1,y}^{t,x} \mathcal{D}\gamma_2 \int_{t_0,x'}^{t_1,y} \mathcal{D}\gamma_1 = \int_{t_0,x'}^{t,x} \mathcal{D}\gamma, \tag{2.29}$$

while it is obvious that

$$S[\gamma] = S[\gamma_1] + S[\gamma_2]. \tag{2.30}$$

Now, let us consider the following expression:

$$\int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]}\gamma(t_1)\gamma(t_2), \tag{2.31}$$

with $t_0 \le t_1, t_2 \le t$;

if $t_1 \le t_2$, (2.29) implies

$$\int_{t_0,x'}^{t,x} \mathcal{D}\gamma = \int_{\mathcal{R}} dz \int_{\mathcal{R}} dy \int_{t_2,z}^{t,x} \mathcal{D}\gamma_3 \int_{t_1,y}^{t_2,z} \mathcal{D}\gamma_2 \int_{t_0,x'}^{t_1,y} \mathcal{D}\gamma_1. \tag{2.32}$$

Hence, we obtain

$$\begin{aligned}
&\int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]}\gamma(t_1)\gamma(t_2) = \\
= \;& \int_{\mathcal{R}} dz \int_{\mathcal{R}} dy \int_{t_2,z}^{t,x} \mathcal{D}\gamma_3 \int_{t_1,y}^{t_2,z} \mathcal{D}\gamma_2 \int_{t_0,x'}^{t_1,y} \mathcal{D}\gamma_1 \; e^{iS[\gamma_1]} e^{iS[\gamma_2]} e^{iS[\gamma_3]} yz \\
= \;& \int_{\mathcal{R}} dz \int_{t_2,z}^{t,x} \mathcal{D}\gamma_3 \; e^{iS[\gamma_3]} z \int_{\mathcal{R}} dy \int_{t_1,y}^{t_2,z} \mathcal{D}\gamma_2 \; e^{iS[\gamma_2]} \int_{t_0,x'}^{t_1,y} \mathcal{D}\gamma_1 \; e^{iS[\gamma_1]} y \\
= \;& \int_{\mathcal{R}} dz \langle x,t|\hat{x}(t_2)|z,t_2\rangle\langle z,t_2|\hat{x}(t_1)|x',t_0\rangle \\
= \;& \langle x,t|\hat{x}(t_2)\hat{x}(t_1)|x',t_0\rangle.
\end{aligned} \tag{2.33}$$

On the other hand, if $t_2 \leq t_1$, the same reasoning leads to:

$$\int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]}\gamma(t_1)\gamma(t_2) = \langle x,t|\hat{x}(t_1)\hat{x}(t_2)|x',t_0\rangle. \tag{2.34}$$

Thus the following holds, for every $t_0 \leq t_1, t_2 \leq t$:

$$\int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]}\gamma(t_1)\gamma(t_2) = \langle x,t|\mathcal{T}(\hat{x}(t_1)\hat{x}(t_2))|x',t_0\rangle, \tag{2.35}$$

where $\mathcal{T}$ is the time-ordering operator.

The previous equation can be further generalized; the result is:

$$\int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]} \prod_{k=1}^{l} \gamma(t_k) = \langle x,t|\mathcal{T}\left(\prod_{k=1}^{l} \hat{x}(t_k)\right)|x',t_0\rangle. \tag{2.36}$$

The next step is to derive another expression for the lhs of (2.36); to accomplish this, we shall follow Srednicki's approach [30] a "source term" is added to the kernel:

$$\langle x,t|x',t_0\rangle[f] \equiv \int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; \exp\left[iS[\gamma] + i\int_{t_0}^{t} f(t)\gamma(t)\right]. \tag{2.37}$$

In fact, it is straightforward to verify that

$$-i\frac{\delta}{\delta f(t_1)}\langle x,t|x',t_0\rangle[f]\Big|_{f=0} = \int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]}\gamma(t_1), \tag{2.38}$$

and, more generally

$$(-i)^l\frac{\delta}{\delta f(t_l)} \cdots \frac{\delta}{\delta f(t_1)}\langle x,t|x',t_0\rangle[f]\Big|_{f=0} = \int_{t_0,x'}^{t,x} \mathcal{D}\gamma \;\; e^{iS[\gamma]} \prod_{k=1}^{l} \gamma(t_k). \tag{2.39}$$

Therefore, (2.36) now reads

$$(-i)^l\frac{\delta}{\delta f(t_l)} \cdots \frac{\delta}{\delta f(t_1)}\langle x,t|x',t_0\rangle[f]\Big|_{f=0} = \langle x,t|\mathcal{T}\left(\prod_{k=1}^{l} \hat{x}(t_k)\right)|x',t_0\rangle. \tag{2.40}$$

In sight of the quantum field theory, the further step is to use energy auto-kets insted of position auto-kets; to accomplish this, we observe that, given $\epsilon > 0$, if we replace $\hat{H}$ with $\hat{H}(1 - i\epsilon)$:

$$\begin{aligned} \lim_{t_0 \to -\infty} |x', t_0\rangle &= \lim_{t_0 \to -\infty} e^{-i\hat{H}(t-t_0)}|x, t\rangle \\ &= |0, -\infty\rangle, \end{aligned} \tag{2.41}$$

$$\begin{aligned} \lim_{t \to +\infty} \langle x, t| &= \lim_{t \to +\infty} \langle x, t_0|e^{-i\hat{H}(t-t_0)} \\ &= \langle 0, +\infty|, \end{aligned} \tag{2.42}$$

under the assumption that the spectrum of $\hat{H}$ is bounded from below, the lower bound being 0.

Hence, taking the limits $t \to +\infty$, $t_0 \to -\infty$ with $\epsilon > 0$ in the above formulas, then letting $\epsilon \to 0$, we obtain

$$(-i)^l \frac{\delta}{\delta f(t_l)} \cdots \frac{\delta}{\delta f(t_1)} \langle 0, +\infty|0, -\infty\rangle[f]\bigg|_{f=0} = \langle 0, +\infty|\mathcal{T}\left(\prod_{k=1}^{l} \hat{x}(t_k)\right)|0, -\infty\rangle, \tag{2.43}$$

where

$$\langle 0, +\infty|0, -\infty\rangle[f] \equiv \int_{-\infty, H=0}^{+\infty, H=0} \mathcal{D}\gamma \ \exp\left[iS[\gamma] + i\int_{-\infty}^{+\infty} dt\ f(t)\gamma(t)\right]. \tag{2.44}$$

It should be stressed that our choice for the "regularization scheme" is equivalent to imposing the ground state as both initial and final state.[3]

2.2 Quantum Field Theory

The same reasoning can be used in any bosonic quantum field theory described by an action principle whenever no invariance flow is present; the dictionary we need is:

$$x(t) \mapsto \phi(t, x), \tag{2.45}$$

$$\hat{x}(t) \mapsto \hat{\phi}(t, x), \tag{2.46}$$

$$f(t) \mapsto J(t, x). \tag{2.47}$$

Hence the following holds:

$$(-i)^l \tfrac{\delta}{\delta J(x_l)} \cdots \tfrac{\delta}{\delta J(x_1)} \langle 0, +\infty|0, -\infty\rangle[J]\bigg|_{J=0} = \langle 0, +\infty|\mathcal{T}\left(\prod_{k=1}^{l} \hat{\phi}(x_k)\right)|0, -\infty\rangle, \tag{2.48}$$

where

$$\langle 0, +\infty|0, -\infty\rangle[J] \equiv \int_{-\infty, H=0}^{+\infty, H=0} \mathcal{D}\phi \ \exp\left[iS[\phi] + i\int_{\mathcal{R}^4} d^4x J(x)\phi(x)\right]. \tag{2.49}$$

[3]Rigorous results concerning functional integration and path integral can be found in the following textbooks: B. Simon [29], J. Glimm, A. Jaffe [18], P. Cartier, C. DeWitt-Morette [5].

Chapter 3

Path Integral in QM and QFT with no Invariance Flows: a Less Naive Approach

3.1 Problems with heuristic quantization rules

Given a classical field theory, how does one pass from the classical theory to the quantum theory? Traditionally, one attempts to answer the first question by starting from a *classical* (or *super*classical) dynamical system and obtaining from it a corresponding quantum system. Each real dynamical variable ϕ is replaced by a self-adjoint linear operator $\hat{\phi}$ of the same type (c-type or a-type), and these operators are assumed to satisfy differential equations similar, if not identical, in form to the dynamical equations of the classical theory. The only differences are: (i) a particular choice of factor ordering may have to be made, (ii) renormalization constants may have to be inserted, and (iii) in order to maintain consistency one may have to add extra terms that do not appear in the (super)classical theory.

In the same way, a functional $R[\phi]$ on Φ is replaced by an operator $\hat{R} \equiv R[\hat{\phi}]$. The super Hilbert or Fock space on which the operators act is not given a priori but is constructed in such a way as to yield a representation of the operator superalgebra satisfied by the $\hat{\phi}$. This superalgebra is always determined in some way by the *heuristic quantization rule*

$$(\phi^j, \phi^k) \mapsto (\hat{\phi}^j, \hat{\phi}^k) \equiv -i[\hat{\phi}^j, \hat{\phi}^k] \quad (\hbar = 1), \tag{3.1}$$

which tries to identify, up to a factor i, each Peierls bracket with a supercommutator. When the (super)classical action S possesses invariant flows, the variables $\hat{\phi}^i$, and hence the supercommutator (3.1), are defined only *modulo* invariance transformations. If there are no invariant flows one may in principle write

$$[\hat{\phi}^j, \hat{\phi}^k] = i\hat{\tilde{G}}^{jk}, \tag{3.2}$$

but there is a difficulty. When the dynamical equations are nonlinear the quantum supercommutator function $\hat{\tilde{G}}^{ij}$ is not just the identity operator times the classical $\tilde{G}^{ij}$ but depends on the $\hat{\phi}^i$. It is usually difficult if not impossible to give a simple factor-ordering prescription for passing from $\tilde{G}^{ij}$ to $\hat{\tilde{G}}^{ij}$. The difficulty is even greater with the more general quantization rule

$$[\hat{A}, \hat{B}] = i(\hat{A}, \hat{B}) \stackrel{?}{=} i\hat{A}_{,j}\, \hat{\tilde{G}}^{jk}\,{}_{k,}\hat{B}, \tag{3.3}$$

which is applicable in principle to all physical observables (i.e., flow invariants). Even though simple factor-ordering prescriptions may exist for defining $\hat{A}$ and $\hat{B}$ there will often be no simple prescription for passing from the classical $A_{,i}\, \tilde{G}^{ij}\,{}_{j,}B$ to its quantum analog.

A possible way out of these difficulties will be discussed in the following.

3.2 Transition Amplitudes

In classical physics the dynamical equations are of central importance because their solutions correspond directly to reality. In quantum physics the situation is different. Solutions of the dynamical equations represent the system only in a generic sense. Correspondence to reality can be set up only when the state vector has been specified.

Instead of making direct use of the operator dynamical equations one can express the dynamical content of the quantum theory in another form, which brings the state vector into the picture and which is often more useful in applications. Let $\hat{A}$, $\hat{B}$ be any two physical observables of a given system which satisfy

$$\text{supp}\ \hat{A}_{,i} > \text{supp}\ \hat{B}_{,i}. \tag{3.4}$$

That is, $\hat{A}$ is constructed out of $\hat{\phi}$ taken from a region of space-time that lies to the future of the region from which the $\hat{\phi}$ making up $\hat{B}$ are taken. Let $|a\rangle$ and $|b\rangle$ be normalized physical eigenvectors of $\hat{A}$ and $\hat{B}$ respectively, corresponding to physical eigenvalues a and b. The inner product $\langle a|b\rangle$ is often called a *transition amplitude*. If the state vector of the system is $|b\rangle$, then $\langle a|b\rangle$ is the *probability amplitude* for the system to be found in the state represented by $|a\rangle$, i.e., for the value a to be obtained when $\hat{A}$ is measured. The probability itself is $|\langle a|b\rangle|^2$.

3.3 The Schwinger Variational Principle

Suppose the action functional of the classical theory suffers an infinitesimal change δS; the quantum theory will change accordingly, and we shall postulate that the associated change $\delta\hat{S}$ in the quantum action $\hat{S}$ is self-adjoint. This produces a change in the quantum dynamical equations and hence a change in their solutions $\hat{\phi}^i$. Suppose the forms of $\hat{A}$ and $\hat{B}$ as functional of the $\hat{\phi}^i$ remain unchanged. As operators, $\hat{A}$ and $\hat{B}$ will nevertheless be changed because the $\hat{\phi}^i$ have changed. Denote these changes by $\delta\hat{A}$ and $\delta\hat{B}$ respectively. The eigenvectors $|a\rangle$ and $|b\rangle$ too will suffer changes $\delta|a\rangle$ and $\delta|b\rangle$. The precise nature of these changes will depend on boundary conditions.

Suppose $\delta\hat{S}$ satisfies the condition

$$\text{supp}\ \hat{A}_{,i} > \delta\hat{S}_{,i} > \text{supp}\ \hat{B}_{,i}. \tag{3.5}$$

That is, suppose $\delta \hat{S}$ is constructed out of $\hat{\phi}$'s taken from a region of space-time that lies to the past of the region associated with $\hat{A}$ and to the future of the region associated with $\hat{B}$. Suppose furthermore that retarded boundary conditions are adopted. Then at times to the past of the region associated with $\delta \hat{S}$ the dynamical variables $\hat{\phi}^i$ will remain unchanged. This means that

$$\delta \hat{B} = 0. \tag{3.6}$$

The observable $\hat{A}$, on the other hand, suffers a change which, with DeWitt's notation, can be written as

$$\delta \hat{A} = D^-_{\delta \hat{S}} \hat{A}. \tag{3.7}$$

In view of the kinematical relation (3.5) one has also:

$$D^-_{\hat{A}} \delta \hat{S} = 0, \tag{3.8}$$

and hence

$$\delta \hat{A} = D^-_{\delta \hat{S}} \hat{A} - D^-_{\hat{A}} \delta \hat{S} \equiv (\delta \hat{S}, \hat{A}). \tag{3.9}$$

Therefore, imposing the heuristic quantization rule, one finds

$$\delta \hat{A} = -i[\delta \hat{S}, \hat{A}]. \tag{3.10}$$

Since the relation between Peierls brackets and supercommutators is only heuristic, the derivation of eq. (3.10) is hardly rigorous. Indeed, if an arbitrary operator ordering is chosen for the dynamical equations, eq. (3.10) need *not* hold. However, there is an inevitability and elegance about this equation which suggests that one turn the problem around and demand that the dynamics be such that it *does* hold, whatever operator ordering may be chosen for $\delta \hat{S}$ as a functional of the $\hat{\phi}$'s. Note that if it holds for $\hat{A} = \hat{\phi}^j$, with $j > \delta \hat{S}_{,i}$, (3.10) reads

$$\begin{aligned} \hat{\phi}^j + \delta \hat{\phi}^j &= \hat{\phi}^j - i[\delta \hat{S}, \hat{\phi}^j] \\ &\equiv -i \delta \hat{S} \hat{\phi}^j + i \hat{\phi}^j \delta \hat{S} \\ &= \hat{u}^{-1} \hat{\phi}^j \hat{u}, \end{aligned} \tag{3.11}$$

with

$$\hat{u} \equiv 1 + i \delta \hat{S}, \tag{3.12}$$

hence (3.11) is a unitary transformation, and (3.10) holds for all $\hat{A}$ satisfying the kinematical inequality (3.5):

$$\begin{aligned} \hat{A} + \delta \hat{A} &= A[\hat{\phi} + \delta \hat{\phi}] \\ &= A[\hat{u}^{-1} \hat{\phi} \hat{u}] \\ &= \hat{u}^{-1} A[\hat{\phi}] \hat{u} \\ &= \hat{A} - i[\delta \hat{S}, \hat{A}]. \end{aligned} \tag{3.13}$$

In this work the previous equations will be postulated as rigorous statements of quantum dynamics; moreover, we shall try to costrain the structure $\delta \hat{S}$ so that corresponding to each classical theory (i.e., to each action functional S) there is a virtually unique quantum theory or at most a unique *family* of quantum theories.

Modulo an ignorable phase change $i\delta\theta|a\rangle$, with $\delta\theta$ real c-number, the change (3.10) induces a change in the eigenvector $|a\rangle$ given by

$$|a\rangle + \delta|a\rangle = \hat{u}^{-1}|a\rangle, \tag{3.14}$$

as is straightforward to verify:

$$\begin{aligned}(\hat{A} + \delta\hat{A})(\hat{u}^{-1}|a\rangle) &= (\hat{u}^{-1}A[\hat{\phi}]\hat{u})(\hat{u}^{-1}|a\rangle) \\ &= \hat{u}^{-1}A[\hat{\phi}]|a\rangle \\ &= a(\hat{u}^{-1}|a\rangle). \end{aligned}\tag{3.15}$$

Eq. (3.14) is equivalent to

$$\delta|a\rangle = -i\delta\hat{S}|a\rangle. \tag{3.16}$$

Equation (3.6), on the other hand, implies:

$$\delta|b\rangle = 0 \tag{3.17}$$

modulo a similar ignorable phase change. Hence

$$\begin{aligned}\delta\langle a|b\rangle &= (\delta\langle a|)|b\rangle + \langle a|(\delta|b\rangle) \\ &= i\langle a|\delta\hat{S}|b\rangle \end{aligned}\tag{3.18}$$

Equation (3.18) is known as the *Schwinger Variational Principle.* Although the Schwinger variational principle was "derived" through imposition of retarded boundary conditions, it is in fact independent of boundary conditions. For example, if advanced boundary conditions are imposed and use is made of the reciprocity relation (1.159), then eqs. (3.10) and (3.6) get replaced by

$$\delta\hat{A} = 0, \tag{3.19}$$

$$\delta\hat{B} = D^{+}_{\delta\hat{S}}\hat{B} = D^{-}_{\hat{B}}\delta\hat{S} - D^{-}_{\delta\hat{S}}\hat{B} = (\hat{B}, \delta\hat{S}) = -i[\hat{B}, \delta\hat{S}], \tag{3.20}$$

which imply

$$\delta|a\rangle = 0, \qquad \delta|b\rangle = i\delta\hat{S}|b\rangle \tag{3.21}$$

again leading to (3.18).

Whether one imposes retarded or advanced boundary conditions, or something in between, the following statements are always true:

1. The unperturbed dynamical equations continue to hold in the regions to the past and to the future of supp $\delta\hat{S}_{,i}$.

2. The $\hat{\phi} + \delta\hat{\phi}$ in these regions are related to the unperturbed $\hat{\phi}$ by unitary transformations.

Remark 3.1. Being the variation (3.18) a unitary transformation the Schwinger principle is guaranteed to preserve both the probability interpretation of the quantum theory and the unit normalization of total probability.

Remark 3.2. The particular choice of physical observables $\hat{A}$ and $\hat{B}$ in the statement of the Schwinger principle is irrelevant. Only the condition (3.5) is important. Since the eigenvalues of more than one observable usually have to be specified in order to determine a quantum state uniquely, it will be convenient from now on to replace (3.18) by the more general statement

$$\delta\langle \text{out}|\text{in}\rangle = i\langle \text{out}|\delta\hat{S}|\text{in}\rangle, \tag{3.22}$$

where $|\text{in}\rangle$ and $|\text{out}\rangle$ are state supervectors determined by some unspecified conditions on the dynamics in regions respectively to the past and to the future of the region in which one may wish to vary the action.

3.4 External Sources and Chronological Products

When the action possesses no invariant flows, a particularly convenient way to vary $\hat{S}$ is to append to it a term of the form $J_i\hat{\phi}^i$, where the J_i are pure supernumber-valued functions over space-time, c-type and real when $\hat{\phi}^i$ is c-type, a-type and imaginary when $\hat{\phi}^i$ is a-type. The J_i are called *external sources.*

Let the external sources suffer variations δJ_i, whose supports are confined to the space-time region lying, in time, between the regions associated with the state supervectors $|\text{in}\rangle$ and $|\text{out}\rangle$. Then the transition amplitude $\langle\text{out}|\text{in}\rangle$ suffers the change

$$\delta\langle \text{out}|\text{in}\rangle = i\langle \text{out}|\delta J_j\hat{\phi}^j|\text{in}\rangle, \tag{3.23}$$

which implies

$$\frac{\overrightarrow{\delta}}{i\delta J_j}\langle \text{out}|\text{in}\rangle = (-1)^{jF}\langle \text{out}|\hat{\phi}^j|\text{in}\rangle, \tag{3.24}$$

where F is the fermionic number of $|\text{out}\rangle$, i.e., F is 0 or 1 according as $|\text{out}\rangle$ is c-type or a-type. Let $|\phi\rangle$ be a complete set of normalized physical eigenvectors of $\hat{\phi}^j$, corresponding to the eigenvalues ϕ^j. Such eigenvectors exist since, when no invariant flows are present, the $\hat{\phi}^j$ are physical observables. The previous equation may then be rewritten in the form

$$\frac{\overrightarrow{\delta}}{i\delta J_j}\langle \text{out}|\text{in}\rangle = (-1)^{jF}\sum\langle \text{out}|\phi\rangle\phi^j\langle\phi|\text{in}\rangle, \tag{3.25}$$

where the summation is over all the $|\phi\rangle$. Now let δJ_i be a second variation in the sources, and suppose supp $\delta J_i > j$. Then the factor $\langle\phi|\text{in}\rangle$ in (3.25) remains unchanged, and

$$\begin{aligned}
\delta\frac{\overrightarrow{\delta}}{i\delta J_j}\langle \text{out}|\text{in}\rangle &= (-1)^{jF}\sum(\delta\langle \text{out}|\phi\rangle)\phi^j\langle\phi|\text{in}\rangle \\
&= (-1)^{jF}\sum(\langle \text{out}|\delta J_k\hat{\phi}^k|\phi\rangle)\phi^j\langle\phi|\text{in}\rangle \\
&= (-1)^{jF}\langle \text{out}|\delta J_k\hat{\phi}^k\hat{\phi}^j \text{in}\rangle.
\end{aligned} \tag{3.26}$$

Therefore

$$\frac{\overrightarrow{\delta}}{i\delta J_i}\frac{\overrightarrow{\delta}}{i\delta J_j}\langle \text{out}|\text{in}\rangle = (-1)^{(i+j)F}\langle \text{out}|\hat{\phi}^i\hat{\phi}^j|\text{in}\rangle. \tag{3.27}$$

If, on the other hand, supp $\delta J_i < j$, then

$$\begin{aligned}
\delta \frac{\overrightarrow{\delta}}{i\delta J_j}\langle \text{out}|\text{in}\rangle &= (-1)^{jF}\sum \langle \text{out}|\phi\rangle\phi^j(\delta\langle\phi|\text{in}\rangle) \\
&= i(-1)^{jF}\sum \langle \text{out}|\phi\rangle\phi^j(\langle\phi|\delta J_k\hat{\phi}^k|\text{in}\rangle) \\
&= i(-1)^{jF}\langle \text{out}|\hat{\phi}^j\delta J_k\hat{\phi}^k|\text{in}\rangle) \\
&= i(-1)^{jF}\langle \text{out}|\hat{\phi}^j\delta J_k\hat{\phi}^k|\text{in}\rangle), \qquad (3.28)
\end{aligned}$$

and

$$\frac{\overrightarrow{\delta}}{i\delta J_i}\frac{\overrightarrow{\delta}}{i\delta J_j}\langle \text{out}|\text{in}\rangle = (-1)^{(i+j)F+ij}\langle \text{out}|\hat{\phi}^j\hat{\phi}^i|\text{in}\rangle. \qquad (3.29)$$

Continuing in this manner one obtains, quite generally,

$$\frac{\overrightarrow{\delta}}{i\delta J_{i_1}}\cdots\frac{\overrightarrow{\delta}}{i\delta J_{i_n}}\langle \text{out}|\text{in}\rangle = (-1)^{(i_1+\ldots i_n)F}\langle \text{out}|\mathcal{T}(\hat{\phi}^{i_1}\ldots\hat{\phi}^{i_n})|\text{in}\rangle \qquad (3.30)$$

where $\mathcal{T}$ is the *chronological ordering operator*, which rearranges the factors $\hat{\phi}_{i_1}\ldots\hat{\phi}_{i_n}$ so that the times associated with the indices appear in chronological sequence, increasing from right to left, and which inserts an additional factor -1 for each interchange of a pair of a-type indices that occurs in the carrying out of this rearrangement.

In the above derivation of (3.30) the times associated with the indices were assumed to be in a well defined chronological order. However, we shall ultimately need to give meaning to the *chronological product* $\mathcal{T}(\hat{\phi}_{i_1}\ldots\hat{\phi}_{i_n})$ for arbitrary relative orientations of the space-time points associated with the indices. When the points associated with an index pair i, j are separated by a spacelike interval there is no ambiguity in the chronological product because the supercommutator function $\hat{\tilde{G}}^{ij}$ then vanishes. Problems arise in the limit when two points coincide: it will be seen later that these problems are all resolved by requiring the $\mathcal{T}$-operation to commute with both differentiation and integration with respect to space-time coordinates: this requirement is equivalent to first imposing the linearity condition

$$\mathcal{T}((\alpha\hat{\phi}^i + \beta\hat{\phi}^j)\hat{\phi}^k\hat{\phi}^l\ldots) = \alpha\mathcal{T}(\alpha\hat{\phi}^i\hat{\phi}^k\hat{\phi}^l\ldots) + \beta\mathcal{T}(\hat{\phi}^j\hat{\phi}^k\hat{\phi}^l\ldots) \qquad (3.31)$$

for all $\alpha, \beta \in \Lambda_\infty$ and then requiring the $\mathcal{T}$-operation to commute with certain operations involving passage to a limit.

Remark 3.3. The above requirements have the consequence that an expression like $\langle \text{out}|\mathcal{T}(\hat{S}_{,i}\hat{\phi}^i\hat{\phi}^j)|\text{in}\rangle$ does not generally vanish despite the fact that $\hat{S}_{,i}$ is zero when the operators that compose it are ordered appropriately for the operator dynamical equations.

Now let $A[\phi]$ be any functional of the classical ϕ for which supp $A_{,i}$ lies between the "in" and "out" regions, and which possesses a functional Taylor expansion about $\phi = 0$ with a nonzero radius of convergence:

$$A[\phi] = A[0] + A_1[0]\phi + \frac{1}{2}A_2[0]\phi\phi + \ldots . \qquad (3.32)$$

It then follows from eq. (3.30) and the requirements illustrated by eq. (3.31) that

$$\langle \mathrm{out}|\mathcal{T}(A[\hat{\phi}])|\mathrm{in}\rangle = (-1)^{AF} A\left[\frac{\overrightarrow{\delta}}{i\delta J}\right]\langle \mathrm{out}|\mathrm{in}\rangle \quad , \tag{3.33}$$

$$A\left[\frac{\overrightarrow{\delta}}{i\delta J}\right] \equiv A[0] + A_1[0]\frac{\overrightarrow{\delta}}{i\delta J} + \frac{1}{2}A_2[0]\frac{\overrightarrow{\delta}}{i\delta J}\frac{\overrightarrow{\delta}}{i\delta J} + \ldots \quad . \tag{3.34}$$

The previous equations provide a way of associating an operator, i.e., $\mathcal{T}(A[\hat{\phi}])$, with each classical functional $A[\phi]$ having appropriate properties. If $A[\phi]$ has only a finite radius of convergence, then, strictly speaking, $\mathcal{T}(A[\hat{\phi}])$ is not defined by eqs. (3.33) and (3.34), but it can often be given a meaning, for given "in" and "out" states, by analytic continuation. One can also frequently give $\mathcal{T}(A[\hat{\phi}])$ a meaning, even when $A[\phi]$ is singular at $\phi = 0$, by expanding about a different point and continuing analytically. Equation (3.33) finds a wide variety of applications in quantum field theory.

3.5 The Operator Dynamical Equations and the Measure Functional

The association established by eq. (3.33), between a classical functional and a quantum operator, suggests that when $A[\phi]$ is a classical observable the operator $\mathcal{T}(A[\hat{\phi}])$ might be taken as its quantum counterpart. This suggestion is often valid when $\mathcal{T}(A[\hat{\phi}])$ is self-adjoint. However, *the chronological product is not self-adjoint in general, even when $A[\phi]$ is real because the operation of taking the adjoint reverses the order of all factors, placing them in antichronological order.*

In order for $\mathcal{T}(A[\hat{\phi}])$ to be self-adjoint $A[\phi]$ must usually be *local*, i.e., built out of ϕ's and their derivatives all taken at the same space-time point. But this is not sufficient to guarantee the self-adjointness of $\mathcal{T}(A[\hat{\phi}])$. For example, $\mathcal{T}(S_{,i}[\hat{\phi}])$ is not generally self-adjoint (or anti-self-adjoint if the index i is a-type rather than c-type). Hence the operator dynamical equations of the quantum theory should not be taken in the form $\mathcal{T}(S_{,i}[\hat{\phi}]) = 0$ or, when external sources are present, in the form $\mathcal{T}(S_{,i}[\hat{\phi}]) = -J_i$. What we shall assume instead is the validity of the following

Postulate 1. *There exists a functional $\mu[\phi]$, determined by the classical action $S[\phi]$, such that the operator dynamical equations take the form*

$$\mathcal{T}\left(\{S[\hat{\phi}] - i\log\mu[\hat{\phi}]\}\frac{\overleftarrow{\delta}}{\delta\phi^k}\right) = -J_k. \tag{3.35}$$

The functional $\mu[\phi]$ is known as the *measure functional.* At the simplest level it may be thought of as correcting for the lack of self-adjointness or anti-self-adjointness of $\mathcal{T}(S_{,i}[\hat{\phi}])$. But it plays a far deeper role than this: once it is chosen the quantum theory is completely determined up to a finite-parameter family. Establishment of a correspondence between each classical theory and a unique quantum theory (or family of quantum theories) is therefore achieved by making $\mu[\phi]$ depend in a definite way on $S[\phi]$. How this dependence is itself to be chosen is a major question, which has to be approached in steps, but there is an easy argument that leads quickly to at least an approximate answer.

In the quantum theory, as in the classical theory, it is often convenient to separate the dynamical variables $\hat{\phi}^i$ into a background ϕ_0^i and a remainder $\hat{\phi}_1^i$. If the background ϕ_0^i is classical, i.e., it is a pure supernumber-valued function times the identity operator, then the $\hat{\phi}_1^i$ satisfy the same supercommutation relations as the $\hat{\phi}^i$:

$$[\hat{\phi}_1^j, \hat{\phi}_1^k] = i\hat{\tilde{G}}^{jk}. \tag{3.36}$$

In terms of the $\hat{\phi}_1^i$ the operator dynamical equations take the form

$$S_{,i}[\phi_0] + S_{,ik}[\phi_0]\hat{\phi}_1^k + \frac{1}{2}S_{,ikl}[\phi_0]\hat{\phi}_1^l\hat{\phi}_1^k + O(\hbar^2) + O(\hat{\phi}_1^3) = -J_i \tag{3.37}$$

The terms of order 0 and 1 in $\hat{\phi}_1$ on the left-hand side are unambiguous. Terms of higher order are not unambiguous since we do not yet know the complete factor-oredering rules. The coefficients of the higher order terms will generally *not* be just the classical coefficients $S_{,ijk\ldots}$. However, they will differ from the classical coefficients by terms of order $\hbar^2$, which arise when the $\hat{\phi}_1$'s are ordered as in (3.37) and which will be dropped in the present approximate analysis.

Taking the supercommutator of (3.37) with $\hat{\phi}_1^j$, remembering that the sources J_i are supernumber-valued, and moving the index i to the left, one finds

$$\begin{aligned}
&S_{,ik}[\phi_0][\hat{\phi}_1^k, \hat{\phi}_1^j] + \frac{1}{2}S_{,ikl}[\phi_0][\hat{\phi}_1^l\hat{\phi}_1^k, \hat{\phi}_1^j] + \ldots = 0,\\
&iS_{,ik}[\phi_0]\hat{\tilde{G}}^{kj} - (-1)^{j(k+l)}\frac{1}{2}S_{,ikl}[\phi_0][\hat{\phi}_1^j, \hat{\phi}_1^l\hat{\phi}_1^k] + \ldots. = 0,\\
&iS_{,ik}[\phi_0]\hat{\tilde{G}}^{kj} - (-1)^{j(k+l)}\frac{1}{2}S_{,ikl}[\phi_0]([\hat{\phi}_1^j, \hat{\phi}_1^l]\hat{\phi}_1^k\\
&\quad +(-1)^{jl}\hat{\phi}_1^l[\hat{\phi}_1^j, \hat{\phi}_1^k]) + \ldots = 0,\\
&iS_{,ik}[\phi_0]\hat{\tilde{G}}^{kj} - (-1)^{j(k+l)}\frac{1}{2}S_{,ikl}[\phi_0]\left(i\hat{\tilde{G}}^{jl}\hat{\phi}_1^k + i(-1)^{jl}\hat{\phi}_1^l\hat{\tilde{G}}^{jk}\right) + \ldots = 0,\\
&S_{,ik}[\phi_0]\hat{\tilde{G}}^{kj} - (-1)^{j(k+l)}\frac{1}{2}S_{,ikl}[\phi_0]\left(\hat{\tilde{G}}^{jl}\hat{\phi}_1^k + (-1)^{jl}\hat{\phi}_1^l\hat{\tilde{G}}^{jk}\right) + \ldots = 0,\\
&{}_{i,}S_{,k}[\phi_0]\hat{\tilde{G}}^{kj} - (-1)^{j(k+l)}\frac{1}{2}{}_{i,}S_{,kl}[\phi_0]\hat{\tilde{G}}^{jl}\hat{\phi}_1^k\\
&\quad -(-1)^{j(k+l)+jl}\frac{1}{2}{}_{i,}S_{,kl}[\phi_0]\hat{\phi}_1^l\hat{\tilde{G}}^{jk} + \ldots = 0,\\
&{}_{i,}S_{,k}[\phi_0]\hat{\tilde{G}}^{kj} + (-1)^{jk}\frac{1}{2}{}_{i,}S_{,kl}[\phi_0]\hat{\tilde{G}}^{lj}\hat{\phi}_1^k + \frac{1}{2}{}_{i,}S_{,kl}[\phi_0]\hat{\phi}_1^l\hat{\tilde{G}}^{kj} + \ldots = 0.
\end{aligned} \tag{3.38}$$

The quantum supercommutator function, like the classical supercommutator function, can be expressed as the difference between an advanced Green's function and a retarded Green's function, both now operator-valued:

$$\hat{\tilde{G}}^{ij} = \hat{G}^{+ij} - \hat{G}^{-ij}, \tag{3.39}$$

where

$$
\begin{aligned}
&{}_{i,}S_{,k}[\hat{\phi}]\hat{G}^{\pm kj} = -{}_{i}\delta^{j}, \\
&{}_{i,}S_{,k}[\phi_0]\hat{G}^{\pm kj} + (-1)^{jk}\frac{1}{2}{}_{i,}S_{,kl}[\phi_0]\hat{G}^{\pm lj}\hat{\phi}_1^k + \\
&\quad + \frac{1}{2}{}_{i,}S_{,kl}[\phi_0]\hat{\phi}_1^l\hat{G}^{\pm kj} + \ldots = -{}_{i}\delta^{j}.
\end{aligned}
\tag{3.40}
$$

The previous equation, as is straightforward to verify, can be solved by iteration, yielding:

$$
\begin{aligned}
\hat{G}^{\pm ij} &= G^{\pm ij}[\phi_0] + \frac{1}{2}(-1)^{jk}G^{\pm im}[\phi_0]\ {}_{m,}S_{,kl}\ G^{\pm lj}[\phi_0]\hat{\phi}_1^k + \\
&\quad + \frac{1}{2}G^{\pm im}[\phi_0]\ {}_{m,}S_{,kl}\ \hat{\phi}_1^l G^{\pm kj}[\phi_0] + \ldots \\
&= G^{\pm ij}[\phi_0] + G^{\pm ij}_{,k}[\phi_0]\hat{\phi}_1^k + \ldots\ .
\end{aligned}
\tag{3.41}
$$

Again it is not possible to specify the forms of the higher terms in the expansion. The coefficients of these terms will generally *not* be just functional derivatives $G^{\pm ij}_{,kl\ldots}[\phi_0]$ of the classical Green's functions.

Introduce now the space-time generalization of the step function:

$$
\theta(i,j') \equiv \begin{cases} 1 & \text{if } x^0 > x'^0 \\ \frac{1}{2} & \text{if } x^0 = x'^0 \\ 0 & \text{if } x^0 < x'^0 \end{cases}
\tag{3.42}
$$

where x^0 is a global timelike coordinate and the submanifolds x^0 = constant are complete spacelike Cauchy hypersurfaces. If the points x and x' are not in the immediate vicinity of one another, then

$$
\begin{aligned}
\hat{\phi}^i\hat{\phi}^{j'} - \mathcal{T}(\hat{\phi}^i\hat{\phi}^{j'}) &= [\theta(i,j') + \theta(j',i)]\,\hat{\phi}^i\hat{\phi}^{j'} - \theta(i,j')\hat{\phi}^i\hat{\phi}^{j'} + \\
&\quad - (-1)^{ij'}\theta(j',i)\hat{\phi}^{j'}\hat{\phi}^i \\
&= \theta(j',i)\left[\hat{\phi}^i, \hat{\phi}^{j'}\right] \\
&= i\theta(j',i)\hat{\tilde{G}}^{ij'} \\
&= i\hat{G}^{+ij'}.
\end{aligned}
\tag{3.43}
$$

We shall assume that this equation in fact holds for *all* x, x', at least up to the order needed in our analysis of the expanded dynamical equations. Equation (3.37) may then be written in the form

$$
\begin{aligned}
-J_l &= S_{,l}[\phi_0] + S_{,lk}[\phi_0]\hat{\phi}_1^k + \frac{1}{2}S_{,ljk}[\phi_0]\left[\mathcal{T}(\hat{\phi}^k\hat{\phi}^j) + i\hat{G}^{+kj}\right] + \ldots \\
&= \mathcal{T}(S_{,l}[\hat{\phi}] + \frac{1}{2}iS_{,ljk}[\hat{\phi}]G^{+kj}[\hat{\phi}] + \ldots),
\end{aligned}
\tag{3.44}
$$

where the dots in the second line stands not only for the unwritten terms in the first line but also for the error in replacing $S_{ijk}[\phi_0]\hat{G}^{+kj}$ by $\mathcal{T}(S_{,ijk}[\hat{\phi}]G^{+kj}[\hat{\phi}])$. The previous expression can be simplified by recalling that ${}_1S_1$ is a negative inverse of G^+; therefore:

$$\begin{aligned} S_{,ijk}\ G^{+kj} &= -S_{,jk}\ G^{+kj}{}_{,i} + (S_{,jk}\ G^{+kj})_{,i} \\ &= -S_{,jk}\ G^{+kj}{}_{,i} + (-\delta_k{}^k)_{,i} \\ &= -S_{,jk}\ G^{+kj}{}_{,i} \\ &= -(-1)^j{}_{j,}S_{,k}\ G^{+kj}{}_{,i} \\ &= \log|\mathrm{sdet}G^+|)_{,i}. \end{aligned} \tag{3.45}$$

Thus

$$-J_k = \mathcal{T}(S_{,k}[\hat{\phi}] + \frac{1}{2}i(\log|\mathrm{sdet}G^+[\hat{\phi}]|)_{,k} + ...). \tag{3.46}$$

Comparison of eqs. (3.35) and (3.46) yields

$$\mu[\phi] \approx \mathrm{const} \cdot |\mathrm{sdet}G^+[\phi]|^{-\frac{1}{2}}. \tag{3.47}$$

3.6 Functional Fourier Analysis. The Feynman Functional Integral

The transition amplitude $\langle \mathrm{out}|\mathrm{in}\rangle$ is a functional of the external sources. Let us try to express it as a functional Fourier integral:

$$\langle \mathrm{out}|\mathrm{in}\rangle = \int X[\phi]e^{iJ\phi}[d\phi] = \int e^{iJ\phi}X[\phi][d\phi], \tag{3.48}$$

$$[d\phi] \equiv \prod_i d\phi^i. \tag{3.49}$$

Here the ϕ^i are supernumber-valued variables of integration, and the product in eq. (3.49) is a continuous infinite one. Both it and the integral itself are thus formal expressions. Integration over the c-number variables is to be understood as patterned on ordinary integration. Integration over the a-number variables is to be understood as an infinite limit of a multiple Berezin integral (see Appendix A). The integral is to be understood as taken over a certain subspace of the space of field histories Φ, whose properties will be indicated below. Assuming the validity of integrating by parts, and making use of eqs. (3.33) and (3.35), one may write

$$
\begin{aligned}
&\int X[\phi]\frac{\overleftarrow{\delta}}{i\delta\phi^k}e^{iJ\phi}[d\phi] = \\
= \quad & -\int X(e^{iJ\phi}\frac{\overleftarrow{\delta}}{i\delta\phi^k})[d\phi] \\
= \quad & -\int X(e^{iJ\phi}\frac{\overleftarrow{\delta}}{i\delta\phi^k})[d\phi] \\
= \quad & -\int Xe^{iJ\phi}J_k[d\phi] \\
= \quad & -\int XJ_k e^{iJ\phi}[d\phi] \\
= \quad & -(-1)^{kX}J_k\int Xe^{iJ\phi}[d\phi] \\
= \quad & -(-1)^{kX}J_k\int Xe^{iJ\phi}[d\phi] \\
= \quad & -(-1)^{kX}J_k\langle\mathrm{out}|\mathrm{in}\rangle \\
= \quad & (-1)^{k(X+F)}\langle\mathrm{out}|-J_k|\mathrm{in}\rangle \\
= \quad & (-1)^{k(X+F)}\langle\mathrm{out}|\mathcal{T}\left(\{S[\hat{\phi}]-i\log\mu[\hat{\phi}]\}\frac{\overleftarrow{\delta}}{\delta\phi^k}\right)|\mathrm{in}\rangle \\
= \quad & (-1)^{kX}\{S_{,k}\left[\overrightarrow{\delta}/i\delta J\right]-i\mu^{-1}\left[\overrightarrow{\delta}/i\delta J\right]\mu_{,k}\left[\overrightarrow{\delta}/i\delta J\right]\}\langle\mathrm{out}|\mathrm{in}\rangle \\
= \quad & (-1)^{kX}\{S_{,k}\left[\overrightarrow{\delta}/i\delta J\right]-i\mu^{-1}\left[\overrightarrow{\delta}/i\delta J\right]\mu_{,k}\left[\overrightarrow{\delta}/i\delta J\right]\}\int e^{iJ\phi}X[\phi][d\phi] \\
= \quad & (-1)^{kX}\int\{S[\phi]-i(\log\mu[\phi])\}\frac{\overleftarrow{\delta}}{\delta\phi^k}e^{iJ\phi}X[\phi][d\phi].
\end{aligned}
\tag{3.50}
$$

Because of the uniqueness of Fourier integral representations the integrands in the first and last lines must be equal:

$$
X[\phi]\frac{\overleftarrow{\delta}}{i\delta\phi^k} = (-1)^{kX}\{S[\phi]-i(\log\mu[\phi])\}\frac{\overleftarrow{\delta}}{\delta\phi^k}X[\phi]. \tag{3.51}
$$

One possible solution of this equation is

$$
X[\phi] = Ne^{iS[\phi]}\mu[\phi], \tag{3.52}
$$

where N is a constant of integration. This leads to

$$
\langle\mathrm{out}|\mathrm{in}\rangle = N\int e^{i(S[\phi]+J\phi)}\mu[\phi][d\phi]. \tag{3.53}
$$

3.7 The Schwinger Variational Principle Revisited

Expression (3.53), when combined with eq. (3.33), yields immediately a functional integral expression for the "in-out" matrix elements of chronological products:

$$\langle \text{out}|\mathcal{T}(A[\hat{\phi}])|\text{in}\rangle = (-1)^{A(F+N)} N \int A[\phi] e^{i(S[\phi]+J\phi)} \mu[\phi][d\phi]. \qquad (3.54)$$

The previous equation can be used to obtain a partial check on the consistency of the Feynman integral with the Schwinger variational principle which was used to derive it in the first place. Under a variation δS in the functional form of the action, such that supp $\delta S_{,i}$ lies in the "in between" region, eqs. (3.53) and (3.54) yield

$$\begin{aligned}
\delta\langle \text{out}|\text{in}\rangle &= N \int \left(i\delta S[\phi] e^{i(S[\phi]+J\phi)} \mu[\phi] + e^{i(S[\phi]+J\phi)} \delta\mu[\phi] \right) [d\phi] \\
&= N \int \left(i\delta S[\phi] e^{i(S[\phi]+J\phi)} \mu[\phi] + e^{i(S[\phi]+J\phi)} \mu[\phi] \delta \log \mu[\phi] \right) [d\phi] \\
&= iN \int \left(\delta S[\phi] - i\delta \log \mu[\phi] \right) e^{i(S[\phi]+J\phi)} \mu[\phi] \\
&= i\langle \text{out}|\mathcal{T}(\delta S[\hat{\phi}] - i\delta \log \mu[\hat{\phi}])|\text{in}\rangle. \qquad (3.55)
\end{aligned}$$

But from (3.47) one has

$$\begin{aligned}
\delta \log \mu[\phi] &= \mu^{-1}[\phi] \delta\mu[\phi] \\
&\approx |\text{sdet} G^{+}[\phi]|^{\frac{1}{2}} \delta |\text{sdet} G^{+}[\phi]|^{-\frac{1}{2}} \\
&= |\text{sdet} G^{+}[\phi]|^{\frac{1}{2}} \left(-\frac{1}{2}\right) |\text{sdet} G^{+}[\phi]|^{-\frac{3}{2}} \delta |\text{sdet} G^{+}[\phi]| \\
&= |\text{sdet} G^{+}[\phi]|^{\frac{1}{2}} \left(-\frac{1}{2}\right) |\text{sdet} G^{+}[\phi]|^{-\frac{3}{2}} |\text{sdet} G^{+}[\phi]| \text{str}(G^{+}[\phi] \delta_1 S_1[\phi]) \\
&= -\frac{1}{2} \text{str}(G^{+}[\phi] \delta_1 S_1[\phi]). \qquad (3.56)
\end{aligned}$$

Hence

$$\delta\langle \text{out}|\text{in}\rangle = i\langle \text{out}|\mathcal{T}(\delta S[\hat{\phi}] + \frac{i}{2} \text{str}(G^{+}[\hat{\phi}] \delta_1 S_1[\hat{\phi}]) + \ldots)|\text{in}\rangle. \qquad (3.57)$$

By the same kind of rearrangement as was used in obtaining eq. (3.46) one easily sees that the chronological product is, at least approximately, just $\delta S[\hat{\phi}]$ in its self-adjoint operator form.

3.8 Expressions involving $S_{,i}$

Let $A[\phi]$ be an arbitrary functional of ϕ such that the support of $A_{,j}$ lies between the "in" and "out" regions. Since the functional integral respects the procedure of integration by parts, the following identity holds:

$$\begin{aligned}
0 &= (-1)^{(A+i)(F+N)} N \int [d\phi] \left(A[\phi] \mu[\phi] e^{i(S[\phi]+J\phi)} \right) \frac{\overleftarrow{\delta}}{\delta \phi^k} \\
&= \langle \text{out}|\mathcal{T}(A_{,k}[\hat{\phi}] + A[\hat{\phi}](\log \mu)_{,k}[\hat{\phi}] + iA[\hat{\phi}] S_{,k}[\hat{\phi}])|\text{in}\rangle. \qquad (3.58)
\end{aligned}$$

Since the "in" and "out" state supervectors may be chosen arbitrarily this is in fact a statement about chronological products of operators involving $\hat{S}_{,i}$ when the sources vanish:

$$\mathcal{T}\left((S[\hat{\phi}] - i\log\mu[\hat{\phi}])\frac{\overleftarrow{\delta}}{\delta\phi^k}A[\hat{\phi}]\right) = i(-1)^{iA}\mathcal{T}\left(A_{,k}[\hat{\phi}]\right). \tag{3.59}$$

This expression does not generally vanish despite eq. (3.35).

Chapter 4

Path Integral in QFT with Invariance Flows

4.1 Structure of the Space of Field Histories

The task of extending the previous treatment to gauge theories, i.e., field theories whose action functional features invariance flows, requires a basic understanding of the properties of the space Φ of field histories, its structure and its geometry. We shall only focus on Type-I theories in which the coefficients $c^{\alpha}{}_{\beta\gamma}$ are structure constants of an infinite dimensional Lie group, the gauge group, which we shall denote by $\mathcal{G}$. As already stated, Φ may be viewed as a principal bundle having $\mathcal{G}$ as its typical fibre. Real physics takes place in the base space of this bundle, i.e., the space of orbits (or fibres), denoted by $\Phi/\mathcal{G}$.

Since $\mathcal{G}$ is a group manifold it admits an invariant Riemannian or pseudo-Riemannian metric. This metric can be extended in an infinity of ways to a group (or flow) invariant metric on Φ. But it turns out that if one requires the extended metric to be *ultralocal* (a requirement that greatly simplifies the analysis of any formalism in which it is used) then, up to a scale factor, it is unique in the case of the Yang-Mills field and belongs to a one-parameter family in the case of gravity, these being the two primary gauge systems of interest.

Let us denote this metric tensor by γ and its components in the chart specified by the dynamical variables ϕ^i by ${}_i\gamma_j$. Let x and x' be the space-time points specified by the indices i and j. Ultralocality of γ is the condition that ${}_i\gamma_j$ be equal to the undifferentiated delta distributon $\delta(x, x')$ times a coefficient that involves no space-time derivatives of fields. Group invariance of γ is the statement

$$\mathcal{L}_{Q_\alpha}\gamma = 0. \tag{4.1}$$

The Q_α are Killing vectors for the metric γ and vertical vector fields for the principal bundle Φ. In the following, we shall rely on the following convention: in gauge theories the indices from the first part of the Greek alphabet are always c-type. Therefore they need never appear in exponents of (-1). Indices of type a from the middle of the Latin alphabet refer to fermion fields that may be coupled to the basic gauge fields.

Choice of an invariant metric on Φ immediately singles out a natural family of connection 1-forms ω^α on Φ:

$$ {}_i\omega^\alpha \equiv {}_iQ_\beta \, \mathcal{N}^{\beta\alpha}, \tag{4.2} $$

where ${}_iQ_\alpha = {}_i\gamma_j \; {}^jQ_\alpha$ and $\mathcal{N}^{\beta\alpha}$ is any coherent Green's function of the real self-adjoint (and hence symmetric) operator

$$ \mathcal{M}_{\alpha\beta} \equiv -{}^iQ_\alpha \; {}_i\gamma_j \; {}^jQ_\beta, \tag{4.3} $$

which, in all cases of interest, turns out to be globally nonsingular, i.e., over the whole of Φ. In view of the relation $\mathcal{M}_{\alpha\beta}\mathcal{N}^{\beta\gamma} = -\delta_\alpha{}^\gamma$ one easily sees that

$$ \begin{aligned} {}^iQ_\alpha \; {}_i\omega^\beta &= {}^iQ_\alpha \; {}_iQ_\gamma \, \mathcal{N}^{\gamma\beta} \\ &= {}^iQ_\alpha \; {}_i\gamma_j \; {}^jQ_\gamma \, \mathcal{N}^{\gamma\beta} \\ &= -\mathcal{M}_{\alpha\gamma} \, \mathcal{N}^{\gamma\beta} \\ &= \delta_\alpha{}^\beta, \end{aligned} \tag{4.4} $$

and that horizontal vectors on Φ are those that are perpendicular (under the metric γ) to the fibres. A horizontal vector may be obtained from any vector by application of the *horizontal projection operator*:

$$ \Pi^i{}_j \equiv \delta^i{}_j - {}^iQ_\alpha \, \omega^\alpha{}_j, \quad \omega^\alpha{}_i \equiv (-1)^i \, {}_i\omega^\alpha. \tag{4.5} $$

The name is justified by the following properties:

$$ \begin{aligned} \Pi^i{}_j \, \Pi^j{}_k &= (\delta^i{}_j - {}^iQ_\alpha \, \omega^\alpha{}_j)(\delta^j{}_k - {}^jQ_\alpha \, \omega^\alpha{}_k) \\ &= \delta^i{}_k - {}^iQ_\alpha \, \omega^\alpha{}_k - {}^iQ_\alpha \, \omega^\alpha{}_k + {}^iQ_\alpha \, \omega^\alpha{}_j \; {}^jQ_\beta \, \omega^\beta{}_k \\ &= \delta^i{}_k - 2 \, {}^iQ_\alpha \, \omega^\alpha{}_k + (-1)^j \; {}^iQ_\alpha \; {}_jQ_\gamma \, \mathcal{N}^{\gamma\alpha} \; {}^jQ_\beta \, \omega^\beta{}_k \\ &= \delta^i{}_k - 2 \, {}^iQ_\alpha \, \omega^\alpha{}_k + (-1)^j \; {}^iQ_\alpha \; {}_j\gamma_l \; {}^lQ_\gamma \, \mathcal{N}^{\gamma\alpha} \; {}^jQ_\beta \, \omega^\beta{}_k \\ &= \delta^i{}_k - 2 \, {}^iQ_\alpha \, \omega^\alpha{}_k + (-1)^{j+j} \; {}^iQ_\alpha \; {}^jQ_\beta \; {}_j\gamma_l \; {}^lQ_\gamma \, \mathcal{N}^{\gamma\alpha} \, \omega^\beta{}_k \\ &= \delta^i{}_k - 2 \, {}^iQ_\alpha \, \omega^\alpha{}_k - {}^iQ_\alpha \, \mathcal{M}_{\beta\gamma} \, \mathcal{N}^{\gamma\alpha} \, \omega^\beta{}_k \\ &= \delta^i{}_k - 2 \, {}^iQ_\alpha \, \omega^\alpha{}_k + {}^iQ_\alpha \, \delta_\beta{}^\alpha \, \omega^\beta{}_k \\ &= \delta^i{}_k - 2 \, {}^iQ_\alpha \, \omega^\alpha{}_k + {}^iQ_\alpha \, \omega^\alpha{}_k \\ &= \delta^i{}_k - {}^iQ_\alpha \, \omega^\alpha{}_k \\ &= \Pi^i{}_k, \end{aligned} \tag{4.6} $$

$$ \begin{aligned} \omega^\alpha{}_i \, \Pi^i{}_j &= \omega^\alpha{}_i \, (\delta^i{}_j - {}^iQ_\beta \, \omega^\beta{}_j) \\ &= \omega^\alpha{}_j - \omega^\alpha{}_i \; {}^iQ_\beta \, \omega^\beta{}_j \\ &= \omega^\alpha{}_j - (-1)^{i+j} \; {}_i\omega^\alpha \; {}^iQ_\beta \; {}_j\omega^\beta \\ &= \omega^\alpha{}_j - (-1)^j \; {}^iQ_\beta \; {}_i\omega^\alpha \; {}_j\omega^\beta \\ &= \omega^\alpha{}_j - (-1)^j \, \delta_\beta{}^\alpha \; {}_j\omega^\beta \\ &= \omega^\alpha{}_j - (-1)^j \; {}_j\omega^\alpha \\ &= \omega^\alpha{}_j - \omega^\alpha{}_j \\ &= 0, \end{aligned} \tag{4.7} $$

$$\begin{aligned}\Pi^i{}_j\,{}^jQ_\alpha &= (\delta^i{}_j - {}^iQ_\beta\,\omega^\beta{}_j)\,{}^jQ_\alpha \\ &= {}^iQ_\alpha - {}^iQ_\beta\,\omega^\beta{}_j\,{}^jQ_\alpha \\ &= {}^iQ_\alpha - (-1)^j\,{}^iQ_\beta\,{}_j\omega^\beta\,{}^jQ_\alpha \\ &= {}^iQ_\alpha - (-1)^{j+j}\,{}^iQ_\beta\,{}^jQ_\beta\,{}_j\omega^\alpha \\ &= {}^iQ_\alpha - {}^iQ_\beta\,\delta_\alpha{}^\beta \\ &= {}^iQ_\alpha - {}^iQ_\alpha \\ &= 0. \end{aligned} \tag{4.8}$$

4.2 Fibre-Adapted Coordinate Patches

When dealing with Φ it is convenient to consider a transformation

$$\phi^i \to I^A, K^\alpha \tag{4.9}$$

to a set of *fibre-adapted coordinates* I^A, K^α. Here the I 's label the fibres (i.e., the points in $\Phi/\mathcal{G}$) and are gauge invariant (i.e., flow invariant):

$$I^A Q_\alpha = I^A{}_{,i}\,{}^iQ_\alpha = 0. \tag{4.10}$$

The K's label the points within each fibre. Because there is no canonical way of associating points on one fibre with those on another, transformations between fibre-adapted coordinate patches have the general structure

$$I'^A = I'^A[I], \quad K'^\alpha = K'^\alpha[I,K], \tag{4.11}$$

which is still special enough so that the Jacobian of the transformation splits into factors:

$$\frac{\delta(I',K')}{\delta(I,K)} = \frac{\delta(I')}{\delta(I)}\frac{\delta(K')}{\delta(K)}. \tag{4.12}$$

One often makes specific choices for the K's. One usually singles out a *base point* ϕ_* in Φ and chooses the K's to be local functionals of the ϕ's of such a form that the matrix

$$\underline{\mathcal{M}}^\alpha_\beta \equiv K^\alpha Q_\beta = K^\alpha{}_{,i}\,{}^iQ_\beta \tag{4.13}$$

is a nonsingular differential operator at and in a neighborhood of ϕ_*. Typical convenient choices for ϕ_* are $A^\alpha_{\mu *}(x) = 0$ in pure Yang-Mills theory and $g_{\mu\nu *}(x)$ = some well studied background metric (Minkowski, Friedmann-Lemaître-Robertson-Walker, black hole, etc.) in pure gravity theory.

One can also make specific choices for the I's, but in general such invariants depend nonlocally on the ϕ's and are clumsy to work with; no specific choices will be made here, so in what follows the I's will remain purely conceptual.

In the region of Φ where the operator $\underline{\mathcal{M}}$ is nonsingular it is easy to show that

$$\frac{\overleftarrow{\delta}}{\delta K^\alpha} = -Q_\beta \underline{\mathcal{N}}^\beta_\alpha, \tag{4.14}$$

where $\underline{\mathcal{N}}$ is a Green's function of $\underline{\mathcal{M}}$. In fact, by multiplying (4.13) first on the right by $\underline{\mathcal{N}}$ and then on the left by $\frac{\overleftarrow{\delta}}{\delta K}$, and using (4.10), one obtains

$$\begin{aligned}
\underline{\mathcal{M}}^{\alpha}_{\beta}\underline{\mathcal{N}}^{\beta}_{\gamma} &= K^{\alpha}{}_{,i}\,{}^{i}Q_{\beta}\,\underline{\mathcal{N}}^{\beta}_{\gamma}, \\
-\delta^{\alpha}{}_{\gamma} &= K^{\alpha}{}_{,i}\,{}^{i}Q_{\beta}\,\underline{\mathcal{N}}^{\beta}_{\gamma}, \\
-\frac{\overleftarrow{\delta}}{\delta K^{\alpha}}\,\delta^{\alpha}{}_{\gamma} &= \frac{\overleftarrow{\delta}}{\delta K^{\alpha}}\,K^{\alpha}{}_{,i}\,{}^{i}Q_{\beta}\,\underline{\mathcal{N}}^{\beta}_{\gamma}, \\
-\frac{\overleftarrow{\delta}}{\delta K^{\gamma}} &= \frac{\overleftarrow{\delta}}{\delta K^{\alpha}}\,K^{\alpha}\frac{\overleftarrow{\delta}}{\delta\phi^{i}}\,{}^{i}Q_{\beta}\,\underline{\mathcal{N}}^{\beta}_{\gamma}, \\
-\frac{\overleftarrow{\delta}}{\delta K^{\gamma}} &= \left(\frac{\overleftarrow{\delta}}{\delta K^{\alpha}}\,K^{\alpha}\frac{\overleftarrow{\delta}}{\delta\phi^{i}} + \frac{\overleftarrow{\delta}}{\delta I^{A}}\,I^{A}\frac{\overleftarrow{\delta}}{\delta\phi^{i}}\right){}^{i}Q_{\beta}\,\underline{\mathcal{N}}^{\beta}_{\gamma}, \\
-\frac{\overleftarrow{\delta}}{\delta K^{\gamma}} &= \frac{\overleftarrow{\delta}}{\delta\phi^{i}}\,{}^{i}Q_{\beta}\,\underline{\mathcal{N}}^{\beta}_{\gamma}, \\
-\frac{\overleftarrow{\delta}}{\delta K^{\gamma}} &= Q_{\beta}\underline{\mathcal{N}}^{\beta}_{\gamma}.
\end{aligned} \tag{4.15}$$

It is important to stress that when $\mathcal{G}$ is non-Abelian it is impossible for the K^{α} to be valid coordinates globally: in fact if they were, then the $\overleftarrow{\delta}/\delta K^{\alpha}$, which are vertical vector fields that commute with each other, would generate Abelian orbits (fibres). This means that $\underline{\mathcal{M}}$, unlike $\mathcal{M}$, cannot be nonsingular globally on Φ.

4.3 Functional Integration for "In-Out" Amplitudes

Since the physics of a gauge theory takes place in the base space $\Phi/\mathcal{G}$ it is natural to try to write a functional integral for "in-out" amplitudes in the form

$$\langle \text{out}|\text{in}\rangle = \int \mu_I[I][dI]e^{iS[I]}. \tag{4.16}$$

All functional integrals encountered in previous sections were purely formal. Eq. (4.16) is even *more* formal, for the following reasons:

1. The labels I^A are not chosen explicitly but used only conceptually.
2. Since all known usable explicit choices depend nonlocally on the ϕ's it is hard to know what one can mean by an advanced Green's function of the Jacobi field operator ${}_{A,}S_{,B}$ (or its superdeterminant) and hence how to determine the measure $\mu_I[I]$ even approximately.
3. It is also hard to know how to set boundary conditions.

To bring the local variables ϕ^i into the theory one must first introduce the remaining variables K^{α} of a fibre-adapted coordinate system and then transform to the ϕ's. Let $\Omega[I, K]$ be a real scalar function(al) on Φ such that the integral

$$\Delta[I] \equiv \int e^{i\Omega[I,K]}\mu_K[I, K][dK] \tag{4.17}$$

exists and is nonvanishing for all I, the measure $\mu_K[I,K]$ being assumed to transform under changes (generally I-dependent) of the fibre-adapted coordinates K^α according to

$$\mu_{K'}[I,K'] = \mu_K[I,K]\frac{\delta K}{\delta K'}. \tag{4.18}$$

Then $\Delta[I]$ is invariant under such coordinate changes and one may write:

$$\langle \text{out}|\text{in}\rangle = \int [dI]\int [dK] e^{i(S[I]+\Omega[I,K])}\Delta[I]^{-1}\mu_{I,K}[I,K], \tag{4.19}$$

where

$$\mu_{I,K}[I,K] \equiv \mu_I[I]\mu_K[I,K]. \tag{4.20}$$

In a similar way one may write the analog of eq. (3.33) in the forms

$$\langle \text{out}|\mathcal{T}(A[I])|\text{in}\rangle = \int \mu_I[I][dI]A[I]e^{iS[I]} \tag{4.21}$$

$$= \int [dI]\int [dK] A[I]e^{i(S[I]+\Omega[I,K])}\Delta[I]^{-1}\mu_{I,K}[I,K]. \tag{4.22}$$

Under changes of fibre-adapted coordinates the measure $\mu_I[I]$ must obviously transform according to

$$\mu_{I'}[I'] = \mu_I[I]\frac{\delta I}{\delta I'}, \tag{4.23}$$

and hence the total measure $\mu_{I,K}[I,K]$ transforms as it should:

$$\mu_{I',K'}[I',K'] = \mu_{I,K}[I,K]\frac{\delta I}{\delta I'}\frac{\delta K}{\delta K'} = \mu_{I,K}[I,K]\frac{\delta(I,K)}{\delta(I',K')}. \tag{4.24}$$

To make the transformation from the I^A, K^α to the local coordinates ϕ^i one must include also the formal Jacobian

$$J[\phi] \equiv \frac{\delta(I,K)}{\delta\phi} = \text{sdet}\begin{pmatrix} I^A_{,i} \\ K^\alpha_{,i}\end{pmatrix}. \tag{4.25}$$

Then the functional integrals (4.19) and (4.22) take the forms:

$$\langle \text{out}|\text{in}\rangle = \int [d\phi] e^{i(S[\phi]+\Omega[\phi])}\Delta[\phi]^{-1}J[\phi]\mu_{I,K}[\phi], \tag{4.26}$$

$$\langle \text{out}|\mathcal{T}(A[\phi])|\text{in}\rangle = \int [d\phi]A[\phi]e^{i(S[\phi]+\Omega[\phi])} \bullet \bullet\, \Delta[\phi]^{-1}J[\phi]\mu_{I,K}[\phi], \tag{4.27}$$

in which we have abused notation somewhat by simply writing $\Delta[I] = \Delta[\phi]$, $S[I] = S[\phi]$, $\mu_{I,K}[I] = \mu_{I,K}[\phi]$, $\Omega[I,K] = \Omega[\phi]$ and $A[I] = A[\phi]$. The last abuse in fact allows a certain generalization of the formalism. In eqs. (3.33), (3.34) the functional A was an invariant, i.e., a physical observable. The integral (4.27) may be regarded as a generalized average which can give meaning to $\langle \text{out}|\mathcal{T}(A[\phi])|\text{in}\rangle$ even when A is not gauge invariant. True physical amplitudes, of course, only involve A's that are gauge invariant. Note that when (and only when) A is gauge invariant the average (4.27) is completely independent of the choice of the functional $\Omega[\phi]$.

4.4 Properties of the Jacobian $J[\phi]$

Suppose we carry out an infinitesimal transformation of the fibre-adapted coordinates K^α:

$$K^{'\alpha} = K^\alpha + \delta K^\alpha[I, K]. \tag{4.28}$$

Formally this will produce the following change in the Jacobian $J[\phi]$:

$$\begin{aligned} \delta J[\phi] &= \delta \text{ sdet} \begin{pmatrix} I^A_{,i} \\ K^\alpha_{,i} \end{pmatrix} \\ &= J \text{ str} \left[\begin{pmatrix} I^A_{,j} \\ K^\alpha_{,j} \end{pmatrix}^{-1} \begin{pmatrix} \delta I^A_{,i} \\ \delta K^\alpha_{,i} \end{pmatrix} \right] \\ &= J \text{ str} \left[\begin{pmatrix} \phi^j_{,A} \\ \phi^j_{,\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ \delta K^\alpha_{,i} \end{pmatrix} \right] \\ &= (-1)^i J \phi^i_{,\alpha}\, \delta K^\alpha_{,i}, \end{aligned} \tag{4.29}$$

Here we encounter an immediate problem: the meaning to be given to the factor $\phi^i_{,\alpha}$. If we apply the operator (4.14) to the fields ϕ^i we get

$$\phi^i_{,\alpha} = -{}^iQ_\beta\, \underline{\mathcal{N}}^\beta_\alpha, \tag{4.30}$$

and we have to decide which Green's function of $\underline{\mathcal{M}}$ to use. Different choices correspond to different possible interpretations of the Jacobian itself. Tentatively we choose $\underline{\mathcal{N}}$ and J to be coherent with the boundary conditions appropriate to the functional integral.

Now note that

$$\begin{aligned} \delta \log J &= J^{-1} \delta J \\ &= (-1)^i \phi^i_{,\alpha}\, \delta K^\alpha_{,i} \\ &= -(-1)^i\, {}^iQ_\beta\, \underline{\mathcal{N}}^\beta_\alpha\, \delta K^\alpha_{,i} \\ &= -\underline{\mathcal{N}}^\beta_\alpha\, \delta K^\alpha_{,i}\, {}^iQ_\beta \\ &= -\underline{\mathcal{N}}^\beta_\alpha\, \delta \underline{\mathcal{M}}^\alpha_\beta \\ &= -\delta \log \det \underline{\mathcal{N}}; \end{aligned} \tag{4.31}$$

therefore

$$\delta\, (J \det \underline{\mathcal{N}}) = 0, \tag{4.32}$$

i.e.,the product $J \det \underline{\mathcal{N}}$ is independent of how the coordinates K^α are chosen. The same is also true of the products $J^\pm \det \underline{\mathcal{N}}^\pm$, where the $J^\pm$ are the Jacobians interpreted according to advanced or retarded boundary conditions. This does *not* automatically mean that these products depend only on the I's and are hence gauge invariant: in fact, these products are not scalar functionals, but scalar densities of unit weight; therefore:

$$\begin{aligned}
(J\ \det\underline{\mathcal{N}})\overleftarrow{\mathcal{L}}_{Q_\alpha} &= J\ \det\underline{\mathcal{N}}\ \mathrm{str}^i Q_{\alpha,k} + (J\ \det\underline{\mathcal{N}})Q_\alpha \\
&= (-1)^i\ J\ \det\underline{\mathcal{N}}\ {}^iQ_{\alpha,i} + (J\ \det\underline{\mathcal{N}})_{,i}\,{}^iQ_\alpha \\
&= (-1)^i\ J\ \det\underline{\mathcal{N}}\ {}^iQ_{\alpha,i} \\
&\quad +(J\ \det\underline{\mathcal{N}})_{,A}\ I^A_{,i}\ {}^iQ_\alpha \\
&\quad +(J\ \det\underline{\mathcal{N}})_{,\beta}\ K^\beta_{,i}\ {}^iQ_\alpha \\
&= (-1)^i\ J\ \det\underline{\mathcal{N}}\ {}^iQ_{\alpha,i} \\
&\quad +(J\ \det\underline{\mathcal{N}})_{,A}\ (I^A\ Q_\alpha) \\
&\quad +(J\ \det\underline{\mathcal{N}})_{,\beta}\ K^\beta_{,i}\ {}^iQ_\alpha \\
&= (-1)^i\ J\ \det\underline{\mathcal{N}}\ {}^iQ_{\alpha,i},
\end{aligned} \tag{4.33}$$

where the following facts have been used:

1. the I 's are flow invariant, i.e., $I^A\ Q_\alpha = 0$,
2. $J\ \det\underline{\mathcal{N}}$ is independent of the K 's, i.e., $(J\ \det\underline{\mathcal{N}})_{,\beta} = 0$,
3. $\overleftarrow{\delta}/\delta\phi^i = \overleftarrow{\delta}/\delta I^A\ I^A_{,i} + \overleftarrow{\delta}/\delta K^\beta\ K^\beta_{,i}$.

Taking the Lie derivative of $J\ \det\underline{\mathcal{N}}$ with respect to Q_α is seen to be the same as multiplying it by $(-1)^i\ {}^iQ_{\alpha,i}$, which is a constant. This constant, which essentially describes a rescaling of $J\ \det\underline{\mathcal{N}}$ as one moves up and down the fibres, depends in no way on the choice made for the K's. For *practical* purposes it may be taken as *zero*, for two reasons:

1. In Yang-Mills theory the zero value may follow formally from the compactness of the finite dimensional Lie group with which it is associated, which implies $f^\gamma{}_{\gamma\alpha} = 0$:

$$\begin{aligned}
{}^\gamma{}_\mu Q_{\alpha'},{}^\mu{}_\gamma &= \int dx\ \delta(x,x')\delta(x,x)\underline{\delta}^\mu_\mu\ f^\gamma{}_{\gamma\alpha} \\
&= N\delta(x',x')f^\gamma{}_{\gamma\alpha} \\
&= 0.
\end{aligned} \tag{4.34}$$

2. In both Yang-Mills and gravity theories it is also formally either a δ-distribution, or the derivative of a δ-distribution, with coincident arguments. In dimensional regularization such formal expressions vanish.

From now on therefore we set

$$(-1)^i\ {}^iQ_{\alpha,i} = 0, \tag{4.35}$$

and similarly

$$c^\beta{}_{\alpha\beta} = 0. \tag{4.36}$$

4.5 A Special Choice for $\Omega[\phi]$ and a New Measure Functional

Although we know that the K^α cannot be global coordinates when the gauge group is non-Abelian, in the loop expansion we can pretend that they are. That is, we pretend that they are coordinates in a tangent space. A favorite choice for the functional $\Omega[I, K]$ is then

$$\Omega = \frac{1}{2}\kappa_{\alpha\beta} K^\alpha K^\beta \tag{4.37}$$

where $(\kappa_{\alpha\beta})$ is a symmetric ultralocal invertible continuous real matrix which can be chosen either to be constant or to depend on the base point ϕ_* in the neighborhood of which the operator $\underline{\mathcal{M}}^\alpha_\beta$ of (4.13) is nonsingular. The K's themselves may be chosen to vanish at this base point.

Since we are staying in a single chart it is simplest to choose

$$\mu_K[I, K] = 1 \tag{4.38}$$

so that (4.17) reduces to

$$\Delta = \text{const} \cdot (\det\kappa)^{-1/2}. \tag{4.39}$$

Equations (4.26) and (4.27) then take the forms

$$\langle \text{out}|\text{in}\rangle = \int \mu[\phi][d\phi] e^{i(S[\phi]+\frac{1}{2}\kappa_{\alpha\beta}K^\alpha K^\beta}(\det\underline{\mathcal{N}})^{-1}, \tag{4.40}$$

$$\langle \text{out}|\mathcal{T}(A[\phi])|\text{in}\rangle = \int \mu[\phi][d\phi] A[\phi] e^{i(S[\phi]+\frac{1}{2}\kappa_{\alpha\beta}K^\alpha K^\beta}(\det\underline{\mathcal{N}})^{-1}, \tag{4.41}$$

where

$$\mu[\phi] = \text{const} \cdot \mu_I[\phi](\det\kappa)^{1/2} J[\phi]\det\underline{\mathcal{N}} \tag{4.42}$$

The previous expression may be regarded as a new measure functional, which is to be used when the integration is carried out over the whole space of histories Φ rather than just the base space $\Phi/\mathcal{G}$. By virtue of eq. (4.33), the constancy of κ, and the fact that $\mu_I[\phi]$ depends only on the I's, it follows that this measure satisfies

$$\mu \overleftarrow{\mathcal{L}}_{Q_\alpha} = 0. \tag{4.43}$$

4.6 Ghosts and BRST Symmetry

The functional integrals (4.40) and (4.41) do not differ greatly in form from expressions (3.53) and (3.54). The chief difference is the presence of the K's and κ's, and the curious factor $(\det\underline{\mathcal{N}})^{-1}$ which comes ultimately from the Jacobian J. By introducing the a-type *ghost fields* χ_α, ψ^β, one obtains

$$(\det\underline{\mathcal{N}})^{-1} = \int [d\chi] \int [d\psi]\; e^{i\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta}. \tag{4.44}$$

Therefore eq. (4.40) may be written

$$\langle \text{out}|\text{in}\rangle = \int \mu[\phi][d\phi] \int [d\chi] \int [d\psi]\; e^{i(S[\phi]+\frac{1}{2}\kappa_{\alpha\beta}K^\alpha K^\beta + \chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta)}. \tag{4.45}$$

It is important to emphasize that the ghost fields arise entirely from the fiber-bundle structure of Φ, from the Jacobian of the transformation from the fiber-adapted coordinates to the conventional local fields ϕ^i. [1]

Hence, the theory may be seen as a field theory on an "extended" space of field histories $\bar{\Phi}$, where the ghost field χ_α, ψ^β appears in addition to ϕ's; however, the new fields have to be considered non-physical, since their "fermionic number" is the opposite of the one suggested by their indices, i.e., they are fermionic fields altough α, β are bosonic indices. It is clear that the full argument of the exponential in (4.45) is no longer invariant under gauge transformations, because of the "gauge-averaging" term $\frac{1}{2}\kappa_{\alpha\beta}K^\alpha K^\beta$ and the ghost term $\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta$.

Nevertheless, both the full action functional

$$\bar{S}[\phi, \chi, \psi] \equiv S[\phi] + \tfrac{1}{2}\kappa_{\alpha\beta}K^\alpha K^\beta + \chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta \tag{4.46}$$

and the measure

$$\bar{\mu}[\phi, \chi, \psi][d\phi][d\chi][d\psi] \equiv \mu[\phi][d\phi][d\chi][d\psi] \tag{4.47}$$

are invariant under a group of *global transformations* whose infinitesimal form is

$$\begin{aligned} \delta\phi^i &= {}^iQ_\alpha\ \psi^\alpha \delta\lambda, & (4.48)\\ \delta\chi_\alpha &= \kappa_{\alpha\beta}K^\beta \delta\lambda, & (4.49)\\ \delta\psi^\alpha &= -\tfrac{1}{2}c^\alpha{}_{\beta\gamma}\ \psi^\beta\psi^\gamma\delta\lambda, & (4.50) \end{aligned}$$

where $\delta\lambda$ is an infinitesimal a-number. They are called Becchi-Rouet-Stora-Tyutin (BRST) transformations.

As is clear, for the fields ϕ^i, they are gauge transformations with an a-type parameter; therefore, from the invariance of $\mu[\phi]$ under gauge transformations, $\bar{\mu}[\phi, \chi, \psi]$ is invariant under BRST transformations; one can show that $\bar{S}[\phi, \chi, \psi]$ is invariant too:

$$\delta S[\phi] = 0, \tag{4.51}$$

$$\begin{aligned} \delta(\tfrac{1}{2}\kappa_{\alpha\beta}K^\alpha K^\beta) &= \tfrac{1}{2}\cdot 2\kappa_{\alpha\beta}\delta K^\alpha K^\beta \\ &= \kappa_{\alpha\beta}(K^\alpha_{,i}\ \delta\phi^i)K^\beta \\ &= \kappa_{\alpha\beta}(K^\alpha_{,i}\ {}^iQ_\gamma\ \psi^\gamma\delta\lambda)K^\beta \\ &= \kappa_{\alpha\beta}(K^\alpha_{,i}\ {}^iQ_\gamma\ \psi^\gamma)K^\beta\delta\lambda \\ &= \kappa_{\alpha\beta}\underline{\mathcal{M}}^\alpha_\gamma\ \psi^\gamma K^\beta\delta\lambda, \end{aligned} \tag{4.52}$$

[1] These "tricky extra particles" were first introduced by R.P. Feynman [13] as a way of compensating for the propagation of nonphysical modes in one-loop order.

$$\begin{aligned}
\delta(\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta) &= (\delta\chi_\alpha)\underline{\mathcal{M}}^\alpha_\beta \psi^\beta + \chi_\alpha \underline{\mathcal{M}}^\alpha_\beta (\delta\psi^\beta) + \chi_\alpha(\delta\underline{\mathcal{M}}^\alpha_\beta)\psi^\beta \\
&= \kappa_{\alpha\gamma} K^\gamma \delta\lambda \underline{\mathcal{M}}^\alpha_\beta \psi^\beta + \chi_\alpha \underline{\mathcal{M}}^\alpha_\beta (-\tfrac{1}{2} c^\beta{}_{\alpha\gamma}\ \psi^\alpha \psi^\gamma \delta\lambda) \\
&\quad + \chi_\alpha(\delta K^\alpha_{,i}\ {}^iQ_\beta)\psi^\beta + \chi_\alpha(K^\alpha_{,i}\ \delta {}^iQ_\beta)\psi^\beta \\
&= -\kappa_{\alpha\beta}\underline{\mathcal{M}}^\alpha_\gamma \psi^\gamma K^\beta \delta\lambda - \tfrac{1}{2}\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda \\
&\quad + \chi_\alpha(K^\alpha_{,ij}\delta\phi^j\ {}^iQ_\beta)\psi^\beta + \chi_\alpha(K^\alpha_{,i}\ {}^iQ_{\beta,j})\ \delta\phi^j\psi^\beta \\
&= -\kappa_{\alpha\beta}\underline{\mathcal{M}}^\alpha_\gamma \psi^\gamma K^\beta \delta\lambda - \tfrac{1}{2}\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda \\
&\quad + \chi_\alpha(K^\alpha_{,ij}\ {}^jQ_\eta\ \psi^\eta\delta\lambda\ {}^iQ_\beta)\psi^\beta + \chi_\alpha(K^\alpha_{,i}\ {}^iQ_{\beta,j})\ {}^jQ_\zeta\ \psi^\zeta\delta\lambda\psi^\beta \\
&= -\kappa_{\alpha\beta}\underline{\mathcal{M}}^\alpha_\gamma \psi^\gamma K^\beta \delta\lambda - \tfrac{1}{2}\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda \\
&\quad + \chi_\alpha K^\alpha_{,ij}\ {}^jQ_\eta\ {}^iQ_\beta\ \psi^\beta\psi^\eta\delta\lambda \\
&\quad + \chi_\alpha K^\alpha_{,i}\ {}^iQ_{\beta,j}\ {}^jQ_\zeta\ \psi^\beta\psi^\zeta\delta\lambda. \qquad (4.53)
\end{aligned}$$

In the last equation, consider the third term:

$$\begin{aligned}
\chi_\alpha K^\alpha_{,ij}\ {}^jQ_\eta\ {}^iQ_\beta\ \psi^\beta\psi^\eta\delta\lambda &= \chi_\alpha K^\alpha_{,ji}\ {}^iQ_\eta\ {}^jQ_\beta\ \psi^\beta\psi^\eta\delta\lambda \\
&= (-1)^{ij}\ \chi_\alpha K^\alpha_{,ij}\ {}^iQ_\eta\ {}^jQ_\beta\ \psi^\beta\psi^\eta\delta\lambda \\
&= \chi_\alpha K^\alpha_{,ij}\ {}^jQ_\beta\ {}^iQ_\eta\ \psi^\beta\psi^\eta\delta\lambda \\
&= -\chi_\alpha K^\alpha_{,ij}\ {}^jQ_\beta\ {}^iQ_\eta\ \psi^\eta\psi^\beta\delta\lambda \\
&= -\chi_\alpha K^\alpha_{,ij}\ {}^jQ_\eta\ {}^iQ_\beta\ \psi^\beta\psi^\eta\delta\lambda, \qquad (4.54)
\end{aligned}$$

therefore

$$\chi_\alpha K^\alpha_{,ij}\ {}^jQ_\eta\ {}^iQ_\beta\ \psi^\beta\psi^\eta\delta\lambda = 0. \qquad (4.55)$$

Consider again (4.53); adding the second and the fourth terms, one obtains

$$\begin{aligned}
& -\tfrac{1}{2}\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda + \chi_\alpha K^\alpha_{,i}\ {}^iQ_{\beta,j}\ {}^jQ_\zeta\ \psi^\beta\psi^\zeta\delta\lambda \\
& = -\tfrac{1}{2}\chi_\alpha\ K^\alpha_{,i}\ {}^iQ_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda + \chi_\alpha K^\alpha_{,i}\ {}^iQ_{\zeta,j}\ {}^jQ_\gamma\ \psi^\zeta\psi^\gamma\delta\lambda \\
& = -\tfrac{1}{2}\chi_\alpha\ K^\alpha_{,i}\ {}^iQ_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda \\
& + \tfrac{1}{2}\chi_\alpha K^\alpha_{,i}\ {}^iQ_{\zeta,j}\ {}^jQ_\gamma\ \psi^\zeta\psi^\gamma\delta\lambda - \tfrac{1}{2}\chi_\alpha K^\alpha_{,i}\ {}^iQ_{\gamma,j}\ {}^jQ_\zeta\ \psi^\zeta\psi^\gamma\delta\lambda \\
& = -\tfrac{1}{2}\chi_\alpha\ K^\alpha_{,i}\ {}^iQ_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda \\
& + \tfrac{1}{2}\chi_\alpha K^\alpha_{,i}({}^iQ_{\zeta,j}\ {}^jQ_\gamma - {}^iQ_{\gamma,j}\ {}^jQ_\zeta)\psi^\zeta\psi^\gamma\delta\lambda \\
& = -\tfrac{1}{2}\chi_\alpha\ K^\alpha_{,i}\ {}^iQ_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda \\
& + \tfrac{1}{2}\chi_\alpha K^\alpha_{,i}\ {}^iQ_\beta\ c^\beta{}_{\zeta\gamma}\ \psi^\zeta\psi^\gamma\delta\lambda \\
& = 0. \qquad (4.56)
\end{aligned}$$

Noting that the first term in (4.53) is the opposite of the rhs of (4.52), it is proven that

$$\delta\bar{S} = \delta S + \delta(\tfrac{1}{2}\kappa_{\alpha\beta}K^\alpha K^\beta) + \delta(\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta) = 0. \qquad (4.57)$$

Moreover, BRST invariance is often a good substitute for the original gauge invariance. For example, if A is a functional of ϕ, but not of the ghost field, it is easy to see that A is gauge invariant if and only if it is BRST invariant:

$$\begin{aligned}
\delta_{BRST} A &= A_{,i}\ \delta_{BRST}\phi^i \\
&= A_{,i}\ {}^iQ_\alpha\ \psi^\alpha\delta\lambda; \qquad (4.58)
\end{aligned}$$

In this case, since the ψ^α are arbitrary functions on space-time (although not necessarily of compact support) BRST invariance of A implies gauge invariance in the original sense, and vice versa.

Therefore, any Type-I gauge theory with action functional $S[\phi]$ may be viewed as a non-gauge, BRST-symmetric theory on an extendend space of field histories where ghost fields appear, with action functional $\bar{S}[\phi, \chi, \psi]$ given by (4.46).

Remark 4.1. In the general case when no assumption on the bosonic nature of the indices from the first part of the Greek alphabet, $\mathrm{sdet}(\underline{N}^\alpha_\beta)^{-1}$ appears in place of $\det(\underline{N}^\alpha_\beta)^{-1}$; therefore (see Appendix A) if the indices from the first part of the Greek alphabet are bosonic, one obtains

$$\mathrm{sdet}(\underline{N}^\alpha_\beta)^{-1} = \det(\underline{N}^\alpha_\beta)^{-1} = \int [d\chi] \int [d\psi]\, e^{i\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta}, \tag{4.59}$$

where χ_α, ψ^β are fermionic fields, as shown in the previous section. On the other hand, if the indices from the first part of the Greek alphabet are fermionic, then

$$\mathrm{sdet}(\underline{N}^\alpha_\beta)^{-1} = \det(\underline{N}^\alpha_\beta) = \int [d\chi] \int [d\psi]\, e^{i\chi_\alpha \underline{\mathcal{M}}^\alpha_\beta \psi^\beta}, \tag{4.60}$$

where, in this case, χ_α, ψ^β are bosonic fields.

Hence, we infer that the fermionic nature of the ghost fields is *always* opposite to the one suggested by their indices, and this is another clue that ghost fields do not represent physical particles.

4.7 A few words on the Measure Functional

The measure functional was introduced as a device for correcting the possible failure of chronological ordering to yield Hermitian (or skew-Hermitian) operator field equations. It arose from the noncommutativity (or nonanticommutativity) of field operators and hence is a purely quantum construct. But the measure functional plays a far deeper role, and we shall briefly outline the reason in this section.

As is well known, the main tool to evaluate transition amplitudes in an interacting field theory is renormalized perturbation theory; in order to obtain renormalized observables, one has to choose a renormalization scheme and has to deal with divergent Feynman diagrams, up to a chosen order. Consider now one-loop perturbation theory for some field in Minkowski space-time; in Minkowski space-time one can use the Fourier transform and pass to momentum space; therefore the task is to evaluate a graph consisting of a single closed loop with r external prongs. Let the momenta assigned to the internal lines all have the same orientation around the loop. Then, making use of the so-called "Feynman's trick" to combine the factors contributed by the internal lines, i.e., by the propagators, and appropriately shifting the integration zero point, one finds for the Feynman-propagator contribution to the value of the graph an expression having the general form

$$I(C) = \text{constant} \cdot \int d^{r-1}y \int d^n k\, \frac{P_m(y,k,p)}{[k^2 - i\epsilon + Q_m(y,k,p)]^r} \tag{4.61}$$

$$= \text{constant} \cdot \int d^{r-1}y \int_C d^n k\, \frac{P_m(y,k,p)}{[k^2 + Q_m(y,k,p)]^r} \tag{4.62}$$

in which external space-time and/or spinor indices have been suppressed. Here "y" denotes the parameters $y_1, y_2, ..., y_{r-1}$ needed to implement "Feynman's trick" and $\int d^{r-1}y$ is a schematic symbol for the integrations in which these parameters are involved. The incoming momenta at the external prongs are $(-p_2 - p_3 - ... - p_r), p_2, ..., p_r$. Q_m is a quadratic function of these momenta, which also depends on the y's and on the masses m associated with the internal lines. P_m is a polynomial in the k's and p's, which depends on the y's and m's. C in (4.62) denotes the contour in the complex plane of the time component k^0 of the k-variable which is appropriate to the Feynman propagator: this contour runs from $-\infty$ to 0 below the negative real axis (in the complex k^0-plane) and from 0 to $+\infty$ above the positive real axis: it is the same as integrating on the real line with the $i\epsilon$ prescription used in (4.61). *If the integral were convergent*, the contour could be rotated so that it would run along the imaginary axis. One would set $k^0 = ik^n$, and (4.62) would become an integral over Euclidean momentum-n-space. Generically, however, this rotation, which is known as *Wick rotation*, is not legitimate. Contributions from arcs at infinity, which themselves diverge or are nonvanishing, have to be included. *These contributions cannot be handled by dimensional regularization.*

When the measure is included it contributes to the generic one-loop graph an amount equal to the negative of the integral

$$\begin{aligned} I(C^+) &= \text{constant} \cdot \int d^{r-1}y \int d^n k \, \frac{P_m(y,k,p)}{[-(k^0 - i\epsilon)^2 + (\vec{k})^2 + Q_m(y,k,p)]^r} \\ &= \text{constant} \cdot \int d^{r-1}y \int_{C^+} d^n k \, \frac{P_m(y,k,p)}{[k^2 + Q_m(y,k,p)]^r} \end{aligned} \tag{4.63}$$

where C^+ is the contour (in the complex k^0-plane) appropriate to the advanced Green function: it runs from $-\infty$ to $+\infty$ below the real axis; it is the same as integrating on the real line with the $i\epsilon$ prescription used on the first line of the previous equation. These two contributions, taken together, yield $I(C) - I(C^+)$ as the correct value of the graph. This corresponds to taking a contour that runs from $+\infty$ to 0 below the positive real axis and then back to $+\infty$ again above the positive real axis, and yields an integral that *can* be handled by dimensional regularization.

The remarkable fact is that $I(C) - I(C^+)$ is equal precisely to the value that is obtained by Wick rotation. This means that *the measure justifies the Wick-rotation procedure.* Although it has never been proved, one may speculate that the exact measure functional, whatever it is, will justify the Wick rotation to all orders and will establish a rigorous connection between quantum field theory in Minkowski space-time and its corresponding euclideanized version.

Chapter 5

Green's Functions: Neutral Scalar Meson

5.1 Integral Representations in Minkowski Space-Time

Green's functions have been introduced in chapter I : in gauge theories, they are the negative inverses of the differential operator ${}_iF_j$, while in field theories with no gauge transformations, they are the negative inverses of the non-singular operator ${}_1S_1$. The prototypes of the Green's functions of interest in quantum field theory are those of the neutral scalar meson in Minkowski space-time; by spin-statistics theorem, it has to be a bosonic particle, and its Lagrangian is

$$L = -\tfrac{1}{2}\left(\phi_{,}{}^{\mu}\phi_{,\mu} + m^2\phi^2\right). \tag{5.1}$$

Therefore

$$S_1 = {}_1S = \frac{\delta}{\delta\phi(x)}S = \left(\phi_{,}{}^{\mu}{}_{\mu} - m^2\phi\right), \tag{5.2}$$

and

$$\begin{aligned} {}_1S_1 &= {}_2S \\ &= \frac{\delta}{\delta\phi(x')}\frac{\delta}{\delta\phi(x)}S \\ &= \left(\delta(x,x')_{,}{}^{\mu}{}_{\mu} - m^2\delta(x,x')\right) \\ &= \left(\partial_\mu\partial^\mu - m^2\right)\delta(x,x'). \end{aligned} \tag{5.3}$$

Hence (1.125) takes the form

$$\begin{aligned} &\int dy\ \left[\left(\partial_\mu\partial^\mu - m^2\right)\delta(x,y)\right]G(y,x') \\ &= \left(\partial_\mu\partial^\mu - m^2\right)G(x,x') = -\delta(x,x'). \end{aligned} \tag{5.4}$$

This equation is most easily solved in "momentum" space, i.e., using the Fourier transform; as is well known, it is a linear, invertible operator which turns derivative operators into multiplication operators, i.e., the Fourier transform of a linear, differential equation for a function is a linear, algebraic equation which is easily solved; therefore the desired function can be obtained by inverse Fourier transform; in our case, the result is

$$G(x,x') = \frac{1}{(2\pi)^4}\int dp\,\frac{e^{ip(x-x')}}{m^2+p^2}. \tag{5.5}$$

In the previous equation the contours in the p^1, p^2, p^3 planes are confined to the real axis and the choice of Green's function is determined by selecting a contour in the p^0 plane which passes in an appropriate fashion around the poles at $\pm E$ where

$$\begin{aligned} E &\equiv \sqrt{m^2+\vec{p}^2} \equiv \omega, & (5.6)\\ \vec{p} &\equiv (p^1,p^2,p^3), & (5.7)\\ (\vec{p})^2 &\equiv (p^1)^2+(p^2)^2+(p^3)^2. & (5.8) \end{aligned}$$

The most important contours are shown in Figure 5.1. From these contours the following relations between the various Green's functions are easily established:

$$\begin{aligned} \bar{G} &= \tfrac{1}{2}\left(G^+ + G^-\right) = \tfrac{1}{2}\tilde{G} + G^- = -\tfrac{1}{2}\tilde{G} + G^+, & (5.9)\\ \tilde{G} &= G^+ - G^- = G^{(+)} + G^{(-)}, & (5.10)\\ G^{(1)} &= i\left(G^{(+)} - G^{(-)}\right), & (5.11)\\ G &= \bar{G} + \tfrac{1}{2}iG^{(1)} = G^- + G^{(-)} = G^+ - G^{(+)}, & (5.12)\\ G^* &= \bar{G} - \tfrac{1}{2}iG^{(1)} = G^- + G^{(+)} = G^+ - G^{(-)}. & (5.13) \end{aligned}$$

By closing the contours for G^+ and G^- at infinity it is easy to see that these functions satisfy the kinematical conditions (1.126) and hence are the advanced and retarded Green's functions. Their uniqueness is also evident. We may therefore write the further relations

$$\begin{aligned} G^+(x,x') &= 2\theta(x',x)\bar{G}(x,x') = \theta(x',x)\tilde{G}, & (5.14)\\ G^-(x,x') &= 2\theta(x,x')\bar{G}(x,x') = -\theta(x,x')\tilde{G}, & (5.15)\\ \tilde{G}(x,x') &= -2\epsilon(x,x')\bar{G}(x,x'), & (5.16)\\ \bar{G}(x,x') &= -\tfrac{1}{2}\epsilon(x,x')\tilde{G}(x,x'), & (5.17) \end{aligned}$$

where $\theta(x,x')$ and $\epsilon(x,x')$ are the step functions, defined by

$$\begin{aligned} \theta(x,x') &\equiv \begin{cases} 1 & \text{for } x > x', \\ 0 & \text{for } x < x', \end{cases} & (5.18)\\ &= 1-\theta(x',x) = \tfrac{1}{2}\left[1+\epsilon(x,x')\right], & (5.19)\\ \epsilon(x,x') &= \theta(x,x') - \theta(x',x) = \begin{cases} 1 & \text{for } x > x', \\ -1 & \text{for } x < x', \end{cases} \quad = -\epsilon(x',x). & (5.20) \end{aligned}$$

We also have

$$\begin{aligned} G(x,x') &= -\theta(x,x')G^{(+)}(x,x') + \theta(x',x)G^{(-)}(x,x'), & (5.21)\\ G^*(x,x') &= \theta(x',x)G^{(+)}(x,x') - \theta(x,x')G^{(-)}(x,x'), & (5.22)\\ G^{(+)}(x,x') &= -\theta(x,x')G(x,x') + \theta(x',x)G^*(x,x'), & (5.23)\\ G^{(-)}(x,x') &= \theta(x',x)G(x,x') - \theta(x,x')G^*(x,x'), & (5.24) \end{aligned}$$

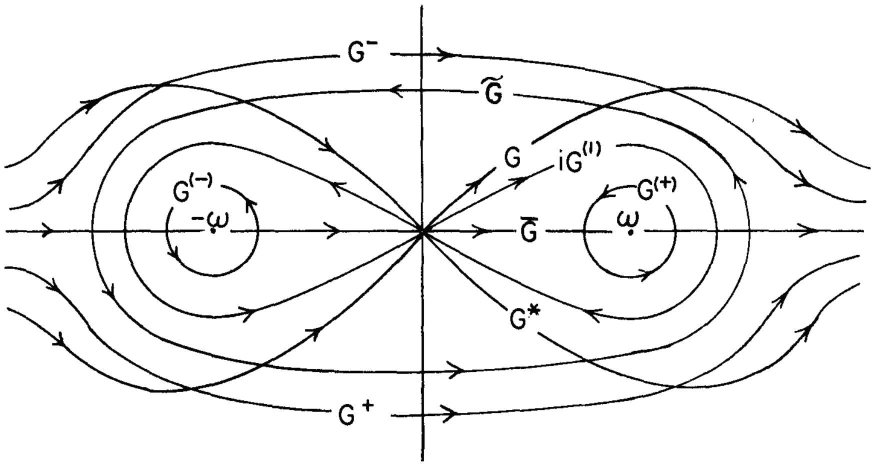

Figure 5.1: Contours in the complex p^0-plane for the integral representation of the Green's function of the neutron scalar meson

which follow from (5.10), (5.16), (5.17), (5.19), and the identities

$$\theta(x,x')\theta(x',x) = 0 \tag{5.25}$$

$$[\theta(x,x')]^2 = \theta(x,x') \tag{5.26}$$

$$[\epsilon(x,x')]^2 = 1 \tag{5.27}$$

Care should be exercised in the use of the step functions. Strictly speaking, all the equations where $\theta(x,x')$, $\epsilon(x,x')$ appear and their corollaries can be inferred to hold only when one of the two points x, x' is clearly to the future or the past of the other. When the two points are separated by a space-like interval further investigation is needed. We shall see presently that the functions $G^{\pm}(x,x')$ vanish for finite space-like separations, and hence the investigation reduces to a study of the behavior of the Green's functions when x' is in the immediate neighborhood of x. The study is complicated by the fact that the Green's functions are actually *distributions* rather than ordinary functions. It turns out, in the present case, that the above relations are in fact valid everywhere. Analogous relations, for the Green's functions of systems more complicated than the neutral scalar meson, however, do not always similarly hold when $x = x'$. In this work we shall avoid this difficulty by using the step functions only when $x \neq x'$. We may also remark that there will never be any ambiguity about the Green's functions themselves. In the present case they are well defined by the integral representation (5.5), once the contour is chosen.

5.2 Symmetries of Green's functions

For the neutral scalar meson, the reciprocity relations (1.137) and (1.187) read

$$G^{\pm}(x,x') = G^{\mp}(x',x), \tag{5.28}$$
$$\tilde{G}(x,x') = -\tilde{G}(x',x), \tag{5.29}$$
$$\bar{G}(x,x') = \bar{G}(x',x). \tag{5.30}$$

The contour for the function $\bar{G}$ corresponds to performing a *principal value* integration along the real axis. In light of the reality of all the integration variables in this case, and because of the symmetry (in p) of the denominator of the integrand of (5.5), we may infer the reality of $\bar{G}$:

$$\bar{G}^* = \bar{G}. \tag{5.31}$$

Similarly, by performing the transformation $p \mapsto -p$, paying attention to the contour and using the previous relations, we may infer

$$G(x,x') = G(x',x), \tag{5.32}$$
$$G^{(1)}(x,x') = G^{(1)}(x',x), \tag{5.33}$$
$$G^{(\pm)}(x,x') = -G^{(\mp)}(x',x), \tag{5.34}$$

and

$$G^{\pm *} = G^{\pm}, \tag{5.35}$$
$$\tilde{G}^* = \tilde{G}, \tag{5.36}$$
$$G^{(1)*} = G^{(1)}, \tag{5.37}$$
$$G^{(\pm)*} = -G^{(\pm)}, \tag{5.38}$$

i.e., $G^{\pm}, \tilde{G}, G^{(1)}$ are all real, while $G^{(\pm)}$ is imaginary; it follows that G^*, as defined above, is the complex conjugate of G.

We note, finally, the differential equations satisfied by the various functions:

$$\begin{aligned}\left(\partial_\mu\partial^\mu - m^2\right) G(x,x') &= \left(\partial_\mu\partial^\mu - m^2\right) \bar{G}(x,x') \\ &= \left(\partial_\mu\partial^\mu - m^2\right) G^{\pm}(x,x') \\ &= -\delta(x,x'), \end{aligned} \tag{5.39}$$
$$\begin{aligned}\left(\partial_\mu\partial^\mu - m^2\right) \tilde{G}(x,x') &= \left(\partial_\mu\partial^\mu - m^2\right) G^{(1)}(x,x') \\ &= \left(\partial_\mu\partial^\mu - m^2\right) G^{(\pm)}(x,x') \\ &= 0. \end{aligned} \tag{5.40}$$

5.3 The Feynman Propagator

In harmony with chapter I, $\tilde{G}$ is known as the *commutator function*, and $G^{(+)}$, $G^{(-)}$ are called its *positive and negative frequency parts*, respectively. $G^{(1)}$ is known as *Hadamard's elementary function*, and G is called the Feynman propagator. From the relations given above it may be seen that all of the functions which we have introduced may be obtained from the Feynman propagator by splitting it into its real, imaginary, advanced, and retarded parts, and recombining these parts in various ways. It suffices therefore to evaluate the Feynman propagator in order to obtain all the rest. From Fig. 5.1 it is not hard to see that the contour for the Feynman propagator may be displaced to the real axis provided we give to the mass m in equation (5.5) an infinitesimal negative imaginary part,i.e., we move up the negative pole and move down the positive one in the complex p^0-plane. We therefore write

$$G(x, x') = \frac{1}{(2\pi)^4} \int dp \, \frac{e^{ip(x-x')}}{m^2 - i\epsilon + p^2}, \quad \epsilon > 0. \tag{5.41}$$

with the understanding that the limit $\epsilon \to 0$ has to be taken at the end of all integrations. It is important to stress that the Feynman propagator can be obtained by analytic continuation from the *unique* Green's function which the operator $(\partial_\mu \partial^\mu - m^2)$ possesses when the x-manifold has a positive definite metric. This fact is responsible for many of the remarkable properties which characterize the Feynman propagator, and the analytic continuation method is often employed to obtain it.

Making use of the integral identities

$$\int_0^{+\infty} ds \, e^{-is(\xi - i\epsilon)} = \frac{1}{i(\xi - i\epsilon)}, \quad \epsilon > 0 \tag{5.42}$$

$$\int_{-\infty}^{+\infty} dx \, e^{iax^2} = \sqrt{(\pi/|a|)} e^{i\mathrm{sgn}(a)(\pi/4)}, \quad a \text{ real}, \tag{5.43}$$

(where sgn is the signum function: $\mathrm{sgn}(a) \equiv a/|a|$), one obtains

$$
\begin{aligned}
G(x,x') &= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds \int dp\; e^{ip(x-x')} e^{-is(m^2+p^2-i\epsilon)} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds \int dp\; e^{-i\left[(m^2+p^2-i\epsilon)s-p(x-x'))\right]} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)} \int dp\; e^{-isp^2+ip(x-x')} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)} \int dp\; e^{-isp^2+ip(x-x')-i(\frac{x-x'}{2\sqrt{s}})^2} e^{i(\frac{x-x'}{2\sqrt{s}})^2} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)} e^{i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \int dp\; \bullet \\
&\quad \bullet\; e^{-i\left(sp^2-p(x-x')+\left(\frac{x-x'}{2\sqrt{s}}\right)^2\right)} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)+i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \int dp\; e^{-i\left(\sqrt{s}p-\frac{x-x'}{2\sqrt{s}}\right)^2} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)+i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \int dp\; e^{-is\left(p-\frac{x-x'}{2s}\right)^2} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)+i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \;\bullet \\
&\quad \bullet \left(\sqrt{(\pi/s)}\right)^4 \left(e^{i\mathrm{sgn}(-s)(\pi/4)}\right)^2 \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)+i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \frac{\pi^2}{s^2} e^{-i(\pi/4)2} \\
&= \frac{i}{(2\pi)^4}\int_0^{+\infty} ds\; e^{-is(m^2-i\epsilon)+i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \frac{\pi^2}{s^2} e^{-i\pi/2} \\
&= \frac{-i^2}{(4\pi)^2}\int_0^{+\infty} ds\; \frac{1}{s^2} e^{-is(m^2-i\epsilon)+i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \\
&= \frac{1}{(4\pi)^2}\int_0^{+\infty} ds\; \frac{1}{s^2} e^{-is(m^2-i\epsilon)+i\left(\frac{x-x'}{2\sqrt{s}}\right)^2} \\
&= \frac{1}{(4\pi)^2}\int_0^{+\infty} ds\; \frac{1}{s^2} e^{-i\left(m^2 s-\frac{(x-x')^2}{4s}\right)}. \qquad (5.44)
\end{aligned}
$$

In the final form the negative imaginary part $-i\epsilon$ attached to m^2 has been dropped, with the understanding that $G(x,x')$ has to be regarded as the *boundary value* (on the real axis) of a function of m^2 and $(x-x')^2$ which is analytic in the lower half m^2 plane and in the upper half $(x-x')^2$ plane. The fact that $G(x,x')$ depends on x and x' only through the combination $(x-x')^2$ is a consequence of Lorentz invariance and the homogeneity of flat space-time. We shall see later that in a curved space-time the dependence of $G(x,x')$ on x and x' will not be so simple.

When $(x-x')^2 < 0$ it is convenient to introduce the new variables

$$z^2 = -m^2(x-x')^2 > 0, \quad z > 0 \tag{5.45}$$

$$u = -2im^2\tfrac{s}{z}, \tag{5.46}$$

which convert (5.44) to

$$\begin{aligned} G(x,x') &= \frac{1}{16\pi^2}\int_0^{-i\infty} du\, \left(-\frac{z}{2im^2}\right)\left(-\frac{4m^4}{u^2z^2}\right) e^{-i\left(-m^2\frac{uz}{2im^2}+\frac{im^2(x-x')^2}{2uz}\right)} \\ &= \frac{1}{16\pi^2}\int_{-i\infty}^{0} du\, \left(\frac{iz}{2m^2}\right)\left(\frac{4m^4}{u^2z^2}\right) e^{\left(\frac{uz}{2}+\frac{m^2(x-x')^2}{2uz}\right)} \\ &= \frac{1}{16\pi^2}\frac{2im^2}{z}\int_{-i\infty}^{0} du\, \frac{1}{u^2}e^{\frac{z}{2}\left(u-\frac{1}{u}\right)} \\ &= \frac{im^2}{8\pi^2}\frac{1}{z}\int_{-i\infty}^{0} du\, \frac{1}{u^2}e^{\frac{z}{2}\left(u-\frac{1}{u}\right)} \end{aligned} \tag{5.47}$$

The contour of integration may be deformed in the manner shown in Figure ... (da inserire), and in virtue of the well known integral representation

$$H_1^{(2)}(z) = \frac{1}{i\pi}\int_C du\, \frac{1}{u^2}e^{\frac{z}{2}\left(u-\frac{1}{u}\right)} \tag{5.48}$$

of the Hankel function of the second kind, of order 1, we finally have

$$G(x,x') = -\frac{m^2}{8\pi}\frac{H_1^{(2)}(z)}{z}. \tag{5.49}$$

5.4 Series Expansions and Singularities

For small values of z (i.e., near the *light cone*) it is convenient to use the power series expansions

$$H_1^{(2)}(z) = J_1(z) - iY_1(z), \tag{5.50}$$

$$J_1(z) = \frac{z}{2} - \frac{z^3}{2^2 4} + \frac{z^5}{2^2 4^2 6} + \ldots, \tag{5.51}$$

$$\begin{aligned} Y_1(z) &= \frac{2}{\pi}\Big[-\frac{1}{z}J_0(z) + (\gamma + \log\tfrac{z}{2})J_1(z) \\ &\quad -\frac{z}{2} + \frac{z^3}{2^2 4}(1+\tfrac{1}{2}) - \frac{z^5}{2^2 4^2 6}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \ldots\Big], \end{aligned} \tag{5.52}$$

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \ldots, \tag{5.53}$$

$$\gamma = 0,5772\ldots . \tag{5.54}$$

Remembering that analytic continuation should be performed in the lower half z^2 plane, and making use of the identities, which hold $\forall z^2 \in \mathcal{R}$:

$$\lim_{\epsilon\to 0^+}\frac{1}{z^2 - i\epsilon} = \frac{1}{z^2} + i\pi\delta(z^2), \tag{5.55}$$

$$\lim_{\epsilon\to 0^+}\log(z^2 - i\epsilon) = \log|z^2| - i\pi\theta(-z^2), \tag{5.56}$$

we find, on splitting the Feynman propagator into its real and imaginary parts

$$\begin{aligned}\bar{G}(x,x') = \mathrm{Re}(G)(x,x') = \frac{1}{4\pi}\delta((x-x')^2) \\ -\frac{m^2}{8\pi}\theta(-(x-x')^2)\left[\tfrac{1}{2}+\frac{m^2(x-x')^2}{2^2 4}+\frac{m^4(x-x')^4}{2^2 4^2 6}+...\right],\end{aligned} \tag{5.57}$$

$$\begin{aligned}G^{(1)}(x,x') = 2\mathrm{Im}(G)(x,x') = \frac{m^2}{2\pi^2}\left\{\frac{1}{m^2(x-x')^2}\right. \\ +\left[\gamma-\log 2+\log m+\tfrac{1}{2}\log|(x-x')^2|\right]\left[\tfrac{1}{2}+\frac{m^2(x-x')^2}{2^2 4}+...\right] \\ \left.-\tfrac{1}{4}-\frac{m^2(x-x')^2}{2^2 4}(1+\tfrac{1}{4})+\frac{m^4(x-x')^4}{2^2 4^2 6}(1+\tfrac{1}{2}+\tfrac{1}{6})-...\right\}.\end{aligned} \tag{5.58}$$

The Green's function G (and hence also G^+ and G^-) is seen to have a δ-distribution type singularity on the light cone $[(x-x')^2=0]$ and to vanish outside the light cone $[(x-x')^2>0]$. It also vanishes inside the light cone when $m=0$[1]. In this case we have

$$\bar{G}(x,x') = \frac{1}{4\pi}\delta((x-x')^2), \tag{5.59}$$

$$G^{(1)}(x,x') = \frac{1}{2\pi^2(x-x')^2}, \tag{5.60}$$

whence, in virtue of equations (5.14) and (5.15) and the identity

$$\delta(\xi^2-a^2) = \frac{1}{2a}\left[\delta(\xi-a)+\delta(\xi+a)\right], \quad a>0 \tag{5.61}$$

we obtain

$$G^{\pm}(x,x') = \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x'}|}\delta(x^0-x'^0\pm|\vec{x}-\vec{x'}|). \tag{5.62}$$

5.5 Curved Space-Time Formalism

Consider now the scalar field theory obtained by applying the *minimal coupling* to the gravitational field to the field theory examined in the previous sections:

1. Replace the Minkowski metric $\eta_{\mu\nu}$ by $g_{\mu\nu}$.
2. Replace ordinary space-time derivatives by the covariant derivatives associated to the (unique) Levi-Civita connection determined by the metric.
3. Multiply the Lagrange function by $|\mathrm{g}|^{1/2}$, where $\mathrm{g}=\det(g_{\mu\nu})$.

Then the action functional for the theory is

$$S[\phi] = -\tfrac{1}{2}\int dx\,|\mathrm{g}|^{1/2}\left(\phi_{;}^{\mu}\phi_{;\mu}+m^2\phi^2\right). \tag{5.63}$$

Hence

$$S_1[\phi] = |\mathrm{g}|^{1/2}\left(\phi_{;\ \mu}^{\ \mu}-m^2\phi\right), \tag{5.64}$$

[1]This property no longer holds when space-time is curved.

and

$$
\begin{aligned}
{}_1S_1 &= |\mathrm{g}(x)|^{1/2}\left(\delta(x,x')_{;}{}^{\mu}{}_{\mu} - m^2\delta(x,x')\right) \\
&= |\mathrm{g}(x)|^{1/2}\left(\nabla^\mu\nabla_\mu - m^2\right)\delta(x,x'). \qquad (5.65)
\end{aligned}
$$

Therefore the equation for the Green's functions is

$$
|\mathrm{g}(x)|^{1/2}\left(\nabla^\mu\nabla_\mu - m^2\right)G(x,x') = -\delta(x,x'). \qquad (5.66)
$$

A very elegant method for solving this equation exists, which is due to Schwinger. One regards the Green's function as the matrix element of an operator G in an abstract (nonphysical) Hilbert space:

$$
G(x,x') = \langle x|G|x'\rangle\,, \qquad (5.67)
$$

the basis vectors $|x'\rangle$ being eigenvectors of a commuting set of Hermitian operators x^μ

$$
x^\mu|x'\rangle = x'^\mu|x'\rangle, \quad \langle x''|x'\rangle = \delta(x'',x'). \qquad (5.68)
$$

The differential equation (5.66) may then be recast in the operator form

$$
(p_\mu|\mathrm{g}|^{1/2}g^{\mu\nu}p_\nu + m^2|\mathrm{g}|^{1/2})G = 1, \qquad (5.69)
$$

where the p_μ are Hermitian operators which satisfy the commutation relations

$$
[x^\mu, p_\nu] = i\underline{\delta}^\mu_\nu, \quad [p_\mu, p_\nu] = 0. \qquad (5.70)
$$

5.6 General Definition of the Feynman Propagator

In order to solve the operator equation (5.66) we must first decide which Green's function we want. As in the previous sections, we shall choose the Feynman propagator as the basic Green's function of interest. However, this immediately begs the question of what we *mean* by the Feynman propagator when space-time is curved and non-empty. In a flat empty space-time the Feynman propagator can be defined as that Green's function which propagates positive frequencies into the future and negative frequencies into the past (see eq. (5.21)). The same definition can be used when space-time is curved provided it becomes asymptotically flat at large space-like and time-like distances and the words "future" and "past" are replaced by "remote future" and "remote past" respectively. Under these circumstances the same variational law holds for the Feynman propagator as well as the retarded and advanced Green's functions:

$$
\delta G^{ij} = G^{ik}\,{}_k\delta F_l\; G^{lj}, \qquad (5.71)
$$

for this law immediately permits the expansion about the flat-empty-space-time values, ${}^0F$ and ${}^0G$, of the operators F and G,

$$
\begin{aligned}
G - {}^0G &= \delta G = ({}^0G + \delta G)(F - {}^0F)({}^0G + \delta G) \\
&= {}^0G\;U\;{}^0G + \delta G\;F\;{}^0G + {}^0G\;F\;\delta G + \delta G\;F\;\delta G \\
&= {}^0G\;U\;{}^0G + {}^0G\;U\;{}^0GU\;{}^0G + \ldots \qquad (5.72)
\end{aligned}
$$

where $U \equiv F - {}^0F$; hence one obtains

$$\begin{aligned} G &= {}^0G + {}^0G\, U\, {}^0G + {}^0G\, U\, {}^0G\, U\, {}^0G + ... \\ &= {}^0G\,(1 - U\,{}^0G)^{-1} \qquad (5.73) \\ &= (1 - {}^0GU)^{-1}\, {}^0G. \qquad (5.74) \end{aligned}$$

We see that the first ${}^0G$ standing on the left and the last ${}^0G$ standing on the right, in each term of the expansion, do indeed ensure that ultimately only pure positive frequencies are found in the remote future and pure negative frequencies in the remote past, owing to the effectively limited domain over which U is non-vanishing.

A word is perhaps in order at this point regarding the very special properties the Feynman propagator possesses. When F is symmetric (and we have always assumed it is) the Feynman propagator is symmetric. Since it also satisfies the variational law (5.71) *it is the only Green's function which, when regarded as a continuous matrix, obeys all the rules of finite matrix theory.* In a certain sense it may therefore be regarded as *the* inverse of the matrix $(-{}_iF_j)$. In flat space-time its special properties stem from the fact (already noted) that it may be obtained by analytic continuation from the *unique* inverse which $(-{}_iF_j)$ possesses in a Euclidean space. When space-time is curved these properties may themselves be used to define the Feynman propagator *even when space-time is not asymptotically flat.*

5.7 Integral Representation (I)

From the results of the previous sections, we shall obtain the Feynman propagator, in curved space-times as well as flat, simply by giving the mass parameter m an infinitesimal negative imaginary part. This has the effect of rendering the operator in (5.69) nonsingular so that inverses may be taken in a simple and direct fashion. It also emphasizes once again that Green's functions are boundary values of analytic functions. Multiplying equation (5.69) on the left by $|g|^{-1/4}$ and on the right by $|g|^{1/4}$, we obtain

$$\begin{aligned} & |g|^{-1/4}(p_\mu |g|^{1/2} g^{\mu\nu} p_\nu + m^2 |g|^{1/2}) G |g|^{1/4} = |g|^{-1/4}|g|^{1/4}, \\ & |g|^{-1/4} p_\mu |g|^{1/2} g^{\mu\nu} p_\nu G |g|^{1/4} + m^2 |g|^{1/4} G |g|^{1/4} = 1, \\ & |g|^{-1/4} p_\mu |g|^{1/2} g^{\mu\nu} p_\nu |g|^{-1/4} |g|^{1/4} G |g|^{1/4} + m^2 |g|^{1/4} G |g|^{1/4} = 1, \\ & (|g|^{-1/4} p_\mu |g|^{1/2} g^{\mu\nu} p_\nu |g|^{-1/4} + m^2) |g|^{1/4} G |g|^{1/4} = 1, \\ & (H + m^2)|g|^{1/4} G |g|^{1/4} = 1, \qquad (5.75) \end{aligned}$$

where

$$H \equiv |g|^{-1/4} p_\mu |g|^{1/2} g^{\mu\nu} p_\nu |g|^{1/4}. \qquad (5.76)$$

Therefore, with the correct prescription for the Feynman propagator:

$$|g|^{1/4} G |g|^{1/4} = \frac{1}{H + m^2 - i\epsilon} = i \int_0^{+\infty} ds\; e^{-is(H+m^2)}. \qquad (5.77)$$

Taking matrix elements of the previous equation we obtain

$$\begin{aligned} |\mathrm{g}(\mathrm{x}')|^{1/4} G |\mathrm{g}(x'')|^{1/4} &= i\int_0^{+\infty} ds\ \left\langle x'|e^{-isH}|x''\right\rangle e^{-ism^2}, \\ |\mathrm{g}'|^{1/4} G |\mathrm{g}''|^{1/4} &= i\int_0^{+\infty} ds\ \langle x', s|x'', 0\rangle\, e^{-ism^2}, \end{aligned} \tag{5.78}$$

with

$$\langle x', s|x'', 0\rangle \equiv \left\langle x'|e^{-isH}|x''\right\rangle . \tag{5.79}$$

Thus we are led to an associated dynamical problem governed by the "Hamiltonian" H.

The "transition amplitude" $\langle x', s|x'', 0\rangle$ satisfies the Schrödinger equation

$$i\frac{\partial}{\partial s}\langle x', s|x'', 0\rangle = \langle x', s|H|x'', 0\rangle = -\langle x', s|x'', 0\rangle_{;\mu'}{}^{\mu'} \tag{5.80}$$

and the boundary condition

$$\langle x', 0|x'', 0\rangle = \delta(x', x''). \tag{5.81}$$

In flat empty space-time, this equation is solved by

$$\langle x', s|x'', 0\rangle_{\text{Minkowski}} = \frac{-i}{16\pi^2}\frac{1}{s^2}e^{i\frac{(x'-x'')^2}{4s}}, \tag{5.82}$$

which agrees with (5.44).

In order to discuss the generalization to curved space-time, some insight on auxiliary geometric quantities is necessary.

5.8 Auxiliary Geometric Quantities

5.8.1 k-point tensors

As is well known, a (r, s)-tensor field T on a manifold M is a map which assigns to every point $p \in M$ an element from the direct product of the tangent space T_pM, taken r times, and the cotangent space T_p^*M, taken s times:

$$\begin{aligned} &T : M \to (TM)^r \otimes (T^*M)^s , \\ &p \mapsto T(p) = T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s}(p)\frac{\partial}{\partial x^{\mu_1}}\bigg|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{\mu_r}}\bigg|_p \otimes dx^{\nu_1}\bigg|_p \otimes \ldots \otimes dx^{\nu_s}\bigg|_p . \end{aligned} \tag{5.83}$$

The tensor field concept can be generalized in this way: we shall call k-point tensor on a manifold M a map which assigns to every k-tuple of points in M an element from the direct product of the tensor spaces built upon those points:

$$
\begin{aligned}
&T : M^k \to [(TM)^{r_1} \otimes (T^*M)^{s_1}] \otimes ... \otimes [(TM)^{r_k} \otimes (T^*M)^{s_k}] \,, \\
&(p^{(1)}, p^{(2)}, ..., p^{(k)}) \mapsto \\
&\quad T^{\mu_1^{(1)} ... \mu_r^{(1)} \quad ... \quad \mu_1^{(k)} ... \mu_{rk}^{(k)}}{}_{\nu_1^{(1)} ... \nu_s^{(1)} \quad ... \quad \nu_1^{(k)} ... \nu_{s^{(k)}}^{(k)}} (p^{(1)}, p^{(2)}, ..., p^{(k)}) \cdot \\
&\quad \cdot \frac{\partial}{\partial x^{\mu_1^{(1)}}}\bigg|_{p^{(1)}} \otimes ... \otimes \frac{\partial}{\partial x^{\mu^{(1)}_{r^{(1)}}}}\bigg|_{p^{(1)}} \otimes dx^{\nu_1^{(1)}}\bigg|_{p^{(1)}} \otimes ... \otimes dx^{\nu^{(1)}_{s^{(1)}}}\bigg|_{p^{(1)}} \otimes ... \\
&\quad \otimes \frac{\partial}{\partial x^{\mu_1^{(k)}}}\bigg|_{p^{(k)}} \otimes ... \otimes \frac{\partial}{\partial x^{\mu^{(k)}_{r^{(k)}}}}\bigg|_{p^{(k)}} \otimes dx^{\nu_1^{(k)}}\bigg|_{p^{(k)}} \otimes ... \otimes dx^{\nu^{(k)}_{s^{(k)}}}\bigg|_{p^{(k)}} \,.
\end{aligned}
\tag{5.84}
$$

Roughly speaking, whenever $k-1$ points (however chosen) are held fixed, a k-point tensor becomes a tensor field.

5.8.2 Geodesics

For a detailed discussion on geodesics on a Riemannian manifold, see Milnor, Spivak, Wells [22]; for a Lorentzian manifold, see Hawking, Ellis [20]. Here we will only introduce tools necessary for later treatise. As is well known, given a connection ∇ on a manifold M, there is exactly one parallel transport on M, i.e., exactly one way to parallel transport a given vector along any curve; one shall define *geodesic* (associated to that connection) any curve whose tangent vector is parallel transported along the curve. If M is a (pseudo-) Riemannian manifold with metric tensor g, and ∇ is the unique Levi-Civita connection associated to that metric, then, given two close enough points, the curve for which the length functional (associated to the metric g) is stationary (on the curves which connect those points) is a geodesic.

In fact the equations for a geodesic $x(\tau)$ with affine parameter,

$$
\ddot{x}^\mu(\tau) + \Gamma^\mu_{\rho\sigma}(x(\tau))\dot{x}^\rho(\tau)\dot{x}^\sigma(\tau) = 0 \tag{5.85}
$$

are precisely the Euler-Lagrange equations associated to the functional

$$
\underline{S}[x(\tau)] = \text{Length}[x(\tau)] = \int d\tau \ \underline{L}(x(\tau), \dot{x}(\tau) = \int d\tau \ \left[\pm \dot{x}^\rho g_{\rho\sigma} \dot{x}^\sigma\right]^{1/2}
$$

:

$$
\begin{aligned}
& \frac{\delta \underline{S}}{\delta x^{\mu}} = \frac{\partial \underline{L}}{\partial x^{\mu}} - \frac{d}{d\tau}\frac{\partial \underline{L}}{\partial \dot{x}^{\mu}} = \\
= \quad & \tfrac{1}{2}\left[\pm \dot{x}^{\rho} g_{\rho\sigma} \dot{x}^{\sigma}\right]^{-1/2} \left[\pm g_{\rho\sigma,\mu} \dot{x}^{\rho} \dot{x}^{\sigma} \mp \frac{d}{d\tau}\left(2 g_{\mu\sigma} \dot{x}^{\sigma}\right)\right] \\
= \quad & \mp \frac{1}{\underline{L}}\left[\frac{d}{d\tau}\left(g_{\mu\sigma}\dot{x}^{\sigma}\right) - \tfrac{1}{2} g_{\rho\sigma,\mu}\dot{x}^{\rho}\dot{x}^{\sigma}\right] \\
= \quad & \mp \frac{1}{\underline{L}}\left(g_{\mu\sigma}\ddot{x}^{\sigma} + \frac{d}{d\tau}\left(g_{\mu\sigma}\right)\dot{x}^{\sigma} - \tfrac{1}{2} g_{\rho\sigma,\mu}\dot{x}^{\rho}\dot{x}^{\sigma}\right) \\
= \quad & \mp \frac{1}{\underline{L}}\left(g_{\mu\sigma}\ddot{x}^{\sigma} + g_{\mu\sigma,\rho}\dot{x}^{\rho}\dot{x}^{\sigma} - \tfrac{1}{2} g_{\rho\sigma,\mu}\dot{x}^{\rho}\dot{x}^{\sigma}\right) \\
= \quad & \mp \frac{1}{\underline{L}}\left(g_{\mu\sigma}\ddot{x}^{\sigma} + \tfrac{1}{2} g_{\mu\sigma,\rho}\dot{x}^{\rho}\dot{x}^{\sigma} + \tfrac{1}{2} g_{\mu\rho,\sigma}\dot{x}^{\rho}\dot{x}^{\sigma} - \tfrac{1}{2} g_{\rho\sigma,\mu}\dot{x}^{\rho}\dot{x}^{\sigma}\right) = 0, \\
& g_{\mu\sigma}\ddot{x}^{\sigma} + \tfrac{1}{2} g_{\mu\sigma,\rho}\dot{x}^{\rho}\dot{x}^{\sigma} + \tfrac{1}{2} g_{\mu\rho,\sigma}\dot{x}^{\rho}\dot{x}^{\sigma} - \tfrac{1}{2} g_{\rho\sigma,\mu}\dot{x}^{\rho}\dot{x}^{\sigma} = 0. \qquad (5.86)
\end{aligned}
$$

Raising the index μ, we obtain

$$\ddot{x}^{\mu} + \tfrac{1}{2} g^{\mu\nu}\left(g_{\nu\sigma,\rho} + g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu}\right)\dot{x}^{\rho}\dot{x}^{\sigma} = 0, \qquad (5.87)$$

which are exactly the geodesic equations, being

$$\Gamma^{\mu}_{\sigma\rho} = \tfrac{1}{2} g^{\mu\nu}\left(g_{\nu\sigma,\rho} + g_{\nu\rho,\sigma} - g_{\rho\sigma,\nu}\right). \qquad (5.88)$$

As is straightforward to verify, the same equations are obtained considering the functional $S[x(\tau)] = \int d\tau\ L(x(\tau), \dot{x}(\tau)) = \int d\tau\ \tfrac{1}{2}\dot{x}^{\rho} g_{\rho\sigma}\dot{x}^{\sigma}$.

Since a variational principle has been introduced, the theory of geodesics may be viewed *formally* as a dynamical theory, and all the results of Hamilton-Jacobi theory can be immediately applied to it. The "conjugate momenta" and "Hamiltonian" are given by

$$
\begin{aligned}
p_{\mu} \quad &\equiv \quad \frac{\partial L}{\partial \dot{x}^{\mu}} = g_{\mu\nu}\dot{x}^{\nu} \equiv \dot{x}_{\mu}, && (5.89)\\
H \quad &\equiv \quad p_{\mu}\dot{x}^{\mu} - L = \dot{x}_{\mu}\dot{x}^{\mu} - \tfrac{1}{2}\dot{x}^{\mu} g_{\mu\nu}\dot{x}^{\nu} = L. && (5.90)
\end{aligned}
$$

Hence we are led to the "energy integral" for the geodesics:

$$H = L = \tfrac{1}{2}\dot{x}^{\rho} g_{\rho\sigma}\dot{x}^{\sigma} = \tfrac{1}{2}\left(\frac{ds}{d\tau}\right)^{2} = \text{const}, \qquad (5.91)$$

where s is the arc length defined on the curve; the action functional on a solution, i.e., a geodesic, whose endpoints are $x(\tau) \equiv x$, $x(\tau') \equiv x'$, reduces to

$$\begin{aligned} S(x,\tau|x',\tau') &= \int_{\tau'}^{\tau} d\tau''\ \tfrac{1}{2}\dot{x}^{\rho} g_{\rho\sigma}\dot{x}^{\sigma} \\ &= \int_{\tau'}^{\tau} d\tau''\ \tfrac{1}{2}\left(\frac{ds}{d\tau''}\right)^2 \\ &= \tfrac{1}{2}\int_{x'}^{x} ds \frac{d\tau''}{ds}\left(\frac{ds}{d\tau''}\right)^2 \\ &= \tfrac{1}{2}\int_{x'}^{x} ds \left(\frac{ds}{d\tau''}\right) \\ &= \tfrac{1}{2}\left(\frac{ds}{d\tau}\right)\int_{x'}^{x} ds \\ &= \tfrac{1}{2}\left(\frac{ds}{d\tau}\right)(s(x)-s(x')) \\ &= \tfrac{1}{2}\left(\frac{s(x)-s(x')}{\tau-\tau'}\right)(s(x)-s(x')) \\ &= \frac{\sigma(x,x')}{\tau-\tau'}, \end{aligned} \tag{5.92}$$

where the *bi-scalar* $\sigma(x,x')$, which we shall call *geodetic interval* or *world function*, is equal to one half the square of the distance along the geodesic between x and x'.

The bi-scalar of geodetic interval satisfies an important differential equation which follows immediately from the Hamilton-Jacobi equation for the action S; we have

$$p_\mu = \frac{\partial S}{\partial x^\mu} = \frac{\sigma_{;\mu}}{\tau-\tau'}, \tag{5.93}$$

$$0 = \frac{\partial S}{\partial \tau} + H = -\frac{\sigma(x,x')}{(\tau-\tau')^2} + \tfrac{1}{2}p_\mu p^\mu, \tag{5.94}$$

where p_μ is now the "momentum" at x corresponding to the geodesic defined by the endpoints x, x'; therefore the world function is the solution of the Cauchy problem

$$\begin{cases} \tfrac{1}{2}\sigma_{;\mu}\sigma^{;\mu} = \sigma, \\ \sigma(x',x') = 0. \end{cases} \tag{5.95}$$

Obviously, the Hamilton-Jacobi equation holds on the other endpoint too; then

$$\tfrac{1}{2}\sigma_{;\mu}\sigma^{;\mu} = \tfrac{1}{2}\sigma_{;\mu'}\sigma^{;\mu'} = \sigma. \tag{5.96}$$

In other words, $\sigma_{;\mu}$ is a vector of length equal to the distance along the geodesic between x and x', tangent to the geodesic at x, and oriented in the direction $x' \to x$, while $\sigma_{;\mu'}$ is a vector of equal length, tangent to the geodesic at x', and oriented in the opposite direction. The geodetic interval itself is obviously a symmetric function of x and x':

$$\sigma(x,x') = \sigma(x',x). \tag{5.97}$$

5.8.3 Caustic Surfaces

In a general Riemannian manifold the geodetic interval is not single-valued, except when x and x' are sufficiently close to one another. The geodesics emanating from a given point will often, beyond a certain distance, begin to cross over one another. The locus of points at which the onset of overlap occurs forms an envelope of the family of geodesics, known as a *caustic surface*. The equation for the caustic surface relative to a given point can be expressed in terms of the quantity $\det(\sigma_{;\mu\nu'})$.

In fact, a geodesic can be specified by means of its endpoints or by means of one of its endpoints together with a tangent vector at that point having a length equal to the length of the geodesic. Therefore we can vary $\sigma_{;\mu'}$ helding x' fixed, and evaluate the resulting variation in x; it is straightforward to obtain

$$\delta\sigma_{;\mu'} = \sigma_{;\mu'\nu}\delta x^{\nu}; \tag{5.98}$$

therefore

$$\delta x^{\mu} = -D^{-1\mu\nu'}\delta\sigma_{;\nu'}, \tag{5.99}$$

where $D^{-1\mu\nu'}$ is the inverse transpose of the finite matrix having the elements $D_{\rho\nu'} = -\sigma_{;\nu'\rho}$, i.e.,

$$D^{-1\mu\nu'}D_{\rho\nu'} = -D^{-1\mu\nu'}\sigma_{;\nu'\rho} = \underline{\delta}^{\mu}_{\rho}. \tag{5.100}$$

When $D^{-1\mu\nu'}$ is a singular matrix, it is possible to choose a variation in $\sigma_{;\mu'}$ which produces no variation in the x. The point x then lies on the caustic surface relative to x', and the condition for this is evidently $D^{-1} = 0$, where

$$D = -\det(D_{\mu\nu'}), \tag{5.101}$$

the minus sign expressing a convention appropriate to the metric of space-time. In 4-dimensional space-time the caustic surface will usually be a 3-dimensional hypersurface, but degenerate forms having fewer dimensions, including zero (focal points) can occur. It will be noted that variations of $\sigma_{;\mu'}$ which leave x unchanged must be orthogonal to $\sigma_{;\mu'}$; that is, the length of the geodesic itself must remain unchanged. This may be inferred by taking the derivative of the Hamilton-Jacobi equation (5.95):

$$\sigma^{;\mu}\sigma_{;\mu\nu'} = \sigma_{;\nu'}, \tag{5.102}$$

and recasting it in the form

$$-D^{-1\mu\nu'}\sigma_{;\nu'} = \sigma^{;\mu} \neq 0, \tag{5.103}$$

which shows, together with (5.99) that changing the length of $\sigma^{;\mu'}$ without changing its direction necessarily shifts x a proportional distance: in fact, by taking $\delta\sigma_{;\mu'} = \epsilon\sigma_{;\mu'}$, one obtains

$$\begin{aligned} \delta x^{\mu} &= -D^{-1\mu\nu'}\delta\sigma_{;\nu'} \\ &= -\epsilon D^{-1\mu\nu'}\sigma_{;\nu'} \\ &= -\epsilon\sigma^{;\mu}. \end{aligned} \tag{5.104}$$

5.8.4 Divergence of Geodesics

The determinant D is a bi-density, of unit weight at both x and x'. Not surprisingly it plays a fundamental role in the description of the *rate* at which geodesics emanating from fixed points diverge from or converge toward one another. If we differentiate equation (5.102) with respect to x^ρ and we note that the indices μ and ρ commute, we get

$$\begin{aligned} \sigma_{;\ \rho}^{\ \mu}\sigma_{;\mu\nu'} + \sigma^{;\mu}\sigma_{;\mu\nu'\rho} &= \sigma_{;\nu'\rho}, \\ \sigma_{;\ \rho}^{\ \mu}\sigma_{;\mu\nu'} + \sigma^{;\mu}\sigma_{;\rho\nu'\mu} &= \sigma_{;\nu'\rho}, \\ -\sigma_{;\ \rho}^{\ \mu}D_{\mu\nu'} - \sigma^{;\mu}D_{\nu'\rho;\mu} &= -D_{\rho\nu'}, \\ D_{\rho\nu'} &= \sigma_{;\ \rho}^{\ \mu}D_{\mu\nu'} + \sigma^{;\mu}D_{\nu'\rho;\mu}, \end{aligned} \tag{5.105}$$

which, on multiplication by $D^{-1\rho\nu'}$, gives

$$\begin{aligned} D_{\rho\nu'}D^{-1\rho\nu'} &= \sigma_{;\ \rho}^{\ \mu}D_{\mu\nu'}D^{-1\rho\nu'} + \sigma^{;\mu}D_{\nu'\rho;\mu}D^{-1\rho\nu'} \\ -\underline{\delta}_\rho^\rho &= -\sigma_{;\ \rho}^{\ \mu}\underline{\delta}_\mu^\rho - \sigma^{;\mu}DD_{;\mu} \\ 4 &= \sigma_{;\ \mu}^{\ \mu} + \sigma^{;\mu}D^{-1}D_{;\mu} \end{aligned} \tag{5.106}$$

$$D^{-1}\left(D\sigma^{;\mu}\right)_{;\mu} = 4. \tag{5.107}$$

The significance of this equation may be made transparent by first replacing D with the bi-scalar

$$\Delta \equiv |\mathrm{g}|^{-1/2}D|\mathrm{g}'|^{-1/2} \tag{5.108}$$

and observing that the operator $\sigma^{;\mu}\partial_\mu$ gives the derivative of any function along the geodesic from x'. Thus

$$\sigma^{;\mu}\partial_\mu f = (\tau - \tau')\dot{f} \tag{5.109}$$

where f is any scalar. Arbitrarily setting $\tau' = 0$, we may recast equation (5.107) in the form

$$\sigma_{;\ \mu}^{\ \mu} = 4 - \frac{d(\log\Delta)}{d(\log\tau)}. \tag{5.110}$$

In fact

$$\begin{aligned} \sigma^{;\mu}D^{-1}D_{;\mu} &= \sigma^{;\mu}\left(|\mathrm{g}|^{1/2}\Delta|\mathrm{g}'|^{1/2}\right)^{-1}\left(|\mathrm{g}|^{1/2}\Delta|\mathrm{g}'|^{1/2}\right)_{;\mu} \\ &= \sigma^{;\mu}|\mathrm{g}|^{-1/2}\Delta^{-1}|\mathrm{g}'|^{-1/2}|\mathrm{g}|^{1/2}\Delta_{;\mu}|\mathrm{g}'|^{1/2} \\ &= \sigma^{;\mu}\Delta^{-1}\Delta_{;\mu} \\ &= \sigma^{;\mu}(\log\Delta)_{;\mu} \\ &= \tau\frac{d}{d\tau}(\log\Delta) \\ &= \frac{d(\log\Delta)}{d(\log\tau)}. \end{aligned} \tag{5.111}$$

From (5.110) it follows immediately that Δ increases or decreases along each geodesic from x' according as the rate of divergence of the neighboring geodesics from x', which is measured by $\sigma_{;\ \mu}^{\ \mu}$, is less than or greater than 4, the rate in flat space-time. If the divergence rate becomes negatively infinite a caustic surface develops and Δ blows up.

5.8.5 Geodetic Parallel Displacement

Another geometrical quantity of fundamental importance is the *geodetic parallel displacement bi-vector*, $g_{\mu\nu'}$, which is defined by the differential equations

$$\sigma_{;}{}^{\eta} g_{\mu\nu';\eta} = 0 \tag{5.112}$$

together with the boundary condition

$$\lim_{x'\to x} g_{\mu\nu'} = g_{\mu\nu}. \tag{5.113}$$

The bi-vector $g_{\mu\nu'}$ gets its name from the fact that the result of applying it, for example, to a local contravariant vector $A^{\mu'}$ at x', is to obtain the covariant form or the vector which results from displacing $A^{\mu'}$ in a parallel fashion along the geodesic from x' to x. This follows from the defining equation (5.112), which requires the covariant derivative of $g_{\mu\nu'}$ to vanish in directions tangent to the geodesic: in fact

$$\begin{aligned} \frac{d}{d\tau}\left(g_{\mu\nu'}A^{\nu'}\right) &= \frac{\sigma_{;}{}^{\rho}}{\tau}\left(g_{\mu\nu'}A^{\nu'}\right)_{;\rho} \\ &= \tfrac{1}{\tau}\sigma_{;}{}^{\rho} g_{\mu\nu';\rho}A^{\nu'} \\ &= 0, \end{aligned} \tag{5.114}$$

and

$$\begin{aligned} \lim_{x\to x'} g^{\mu\rho} g_{\rho\nu'}A^{\nu'} &= g^{\mu'\rho'} g_{\rho'\nu'}A^{\nu'} \\ &= \underline{\delta}^{\mu'}_{\nu'}A^{\nu'} \\ &= A^{\mu'}. \end{aligned} \tag{5.115}$$

From its geometrical significance and the fact that tangents to a geodesic are self-parallel the following properties of $g_{\mu\nu'}$ are obvious:

$$g_{\mu}{}^{\nu'}\sigma_{;\nu'} = -\sigma_{;\mu}, \qquad g^{\nu}{}_{\mu'}\sigma_{;\nu} = -\sigma_{;\mu'}, \tag{5.116}$$

$$\sigma_{;}{}^{\eta'} g_{\mu\nu';\eta'} = 0, \tag{5.117}$$

$$g_{\mu\nu'} = g_{\nu'\mu}, \tag{5.118}$$

$$g_{\mu\rho'}g_{\nu'}{}^{\rho'} = g_{\mu\nu}, \qquad g_{\rho\mu'}g^{\rho}{}_{\nu'} = g_{\mu'\nu'}, \tag{5.119}$$

$$\det(-g_{\mu\nu'}) = |\mathrm{g}|^{1/2}|\mathrm{g}'|^{1/2}. \tag{5.120}$$

In a similar manner one may define a *geodetic parallel displacement bi-spinor* $I(x, x')$ which satisfies

$$\sigma_{;}{}^{\mu} I_{;\mu} = 0, \tag{5.121}$$

$$\lim_{x'\to x} I = \text{unity matrix}, \tag{5.122}$$

and which transforms like $\psi \equiv \psi(x)$ at x and like $\psi' \equiv \psi(x')$ at x'.

5.9 Integral Representation (II)

Now we are ready to propose an ansatz to solve (5.80), (5.81): in (5.82), replace

$$\frac{1}{s^2} \mapsto \frac{D(x, x')^{1/2}}{s^2} \tag{5.123}$$

$$(x - x')^2 \mapsto 2\sigma(x, x') \tag{5.124}$$

and multiplicate by a power series in (is):

$$\sum_{n=0}^{+\infty} a_n(x,x')(is)^n, \tag{5.125}$$
$$\lim_{x'\to x} a_0(x,x') = 1, \tag{5.126}$$

Then the ansatz is

$$\langle x', s|x'', 0\rangle = \frac{-i}{16\pi^2}\frac{D(x,x')^{1/2}}{s^2} e^{i\frac{\sigma(x,x')}{2s}} \sum_{n=0}^{+\infty} a_n(x,x')(is)^n, \tag{5.127}$$
$$\lim_{x'\to x} a_0(x,x') = 1. \tag{5.128}$$

A requirement for the ansatz to be meaningful is the existence of a recurrence relation for the unknown a_n's. Inserting (5.127) in (5.80), the lhs is:

$$\begin{aligned}
i\frac{\partial}{\partial s}\langle x', s|x'', 0\rangle &= i\frac{\partial}{\partial s}\left(\frac{-i}{16\pi^2}\frac{D^{1/2}}{s^2} e^{i\frac{\sigma}{2s}} \sum_{n=0}^{+\infty} a_n(is)^n\right) \\
&= \frac{D^{1/2}}{16\pi^2}\frac{\partial}{\partial s}\left(\frac{1}{s^2} e^{i\frac{\sigma}{2s}} \sum_{n=0}^{+\infty} a_n(is)^n\right) \\
&= \frac{D^{1/2}}{16\pi^2} e^{i\frac{\sigma}{2s}}\Bigg(-\frac{2}{s^3}\sum_{n=0}^{+\infty} a_n(is)^n + \frac{1}{s^2}\frac{-i\sigma}{2s^2}\sum_{n=0}^{+\infty} a_n(is)^n \\
&\quad + \frac{1}{s^2}\sum_{n=1}^{+\infty} ina_n(is)^{n-1}\Bigg) \\
&= \frac{D^{1/2}}{16\pi^2} e^{i\frac{\sigma}{2s}}\Bigg(\sum_{n=0}^{+\infty}(-2i^3 a_n)(is)^{n-3} + \sum_{n=0}^{+\infty}(-ii^4\sigma a_n/2)(is)^{n-4} \\
&\quad + \sum_{n=1}^{+\infty}(ii^2 n a_n)(is)^{n-3}\Bigg) \\
&= \frac{iD^{1/2}}{16\pi^2} e^{i\frac{\sigma}{2s}}\Bigg(\sum_{n=0}^{+\infty}(2a_n)(is)^{n-3} + \sum_{n=0}^{+\infty}(-\sigma a_n/2)(is)^{n-4} \\
&\quad + \sum_{n=1}^{+\infty}(-n a_n)(is)^{n-3}\Bigg),
\end{aligned} \tag{5.129}$$

while the rhs is

$$
\begin{aligned}
&-\langle x',s|x'',0\rangle_{;\mu}{}^{\mu} = -\left(\frac{-i}{16\pi^2}\frac{D^{1/2}}{s^2}e^{i\frac{\sigma}{2s}}\sum_{n=0}^{+\infty}a_n(is)^n\right)_{;\mu}{}^{\mu}\\
&= \frac{i}{16\pi^2}\frac{1}{s^2}\left(D^{1/2}e^{i\frac{\sigma}{2s}}\sum_{n=0}^{+\infty}a_n(is)^n\right)_{;\mu}{}^{\mu}\\
&= \frac{i}{16\pi^2}\frac{1}{s^2}e^{i\frac{\sigma}{2s}}\left(D^{1/2}{}_{;\mu}{}^{\mu}\sum_{n=0}^{+\infty}a_n(is)^n\right.\\
&\quad +D^{1/2}\frac{i}{2s}\sigma_{;\mu}{}^{\mu}\sum_{n=0}^{+\infty}a_n(is)^n + D^{1/2}\left(\frac{i}{2s}\right)^2\sigma_{;\mu}\sigma^{;\mu}\sum_{n=0}^{+\infty}a_n(is)^n\\
&\quad +D^{1/2}\sum_{n=0}^{+\infty}a_{n;\mu}{}^{\mu}(is)^n + 2D^{1/2}{}_{;\mu}\left(\frac{i}{2s}\right)\sigma^{;\mu}\sum_{n=0}^{+\infty}a_n(is)^n\\
&\quad \left.+2D^{1/2}{}_{;\mu}\sum_{n=0}^{+\infty}a_n{}^{;\mu}(is)^n + 2D^{1/2}\left(\frac{i}{2s}\right)\sigma_{;\mu}\sum_{n=0}^{+\infty}a_n{}^{;\mu}(is)^n\right)\\
&= -\frac{iD^{1/2}}{16\pi^2}e^{i\frac{\sigma}{2s}}\left(\sum_{n=0}^{+\infty}\frac{D^{1/2}{}_{;\mu}{}^{\mu}}{D^{1/2}}a_n(is)^{n-2}\right.\\
&\quad +\sum_{n=0}^{+\infty}\frac{-\sigma_{;\mu}{}^{\mu}}{2}a_n(is)^{n-3} + \sum_{n=0}^{+\infty}\frac{\sigma_{;\mu}\sigma^{;\mu}}{4}a_n(is)^{n-4}\\
&\quad +\sum_{n=0}^{+\infty}a_{n;\mu}{}^{\mu}(is)^n + \sum_{n=0}^{+\infty}\frac{-D^{1/2}{}_{;\mu}}{D^{1/2}}\sigma^{;\mu}a_n(is)^{n-3}\\
&\quad \left.+\sum_{n=0}^{+\infty}\frac{2D^{1/2}{}_{;\mu}}{D^{1/2}}a_n{}^{;\mu}(is)^{n-2} + \sum_{n=0}^{+\infty}(-\sigma_{;\mu}a_n{}^{;\mu})(is)^{n-3}\right);
\end{aligned}
\tag{5.130}
$$

exploiting (5.95) and (5.106), one obtains

$$
\begin{aligned}
&-\langle x',s|x'',0\rangle_{;\mu}{}^{\mu} = -\frac{iD^{1/2}}{16\pi^2}e^{i\frac{\sigma}{2s}}\left(\sum_{n=0}^{+\infty}\frac{D^{1/2}{}_{;\mu}{}^{\mu}}{D^{1/2}}a_n(is)^{n-2}\right.\\
&\quad +\sum_{n=0}^{+\infty}(-2a_n(is)^{n-3} + \sum_{n=0}^{+\infty}\frac{\sigma}{2}a_n(is)^{n-4}\\
&\quad +\sum_{n=0}^{+\infty}a_{n;\mu}{}^{\mu}(is)^{n-2} + \sum_{n=0}^{+\infty}\frac{2D^{1/2}{}_{;\mu}}{D^{1/2}}a_n{}^{;\mu}(is)^{n-2}\\
&\quad \left.+\sum_{n=0}^{+\infty}(-\sigma_{;\mu}a_n{}^{;\mu})(is)^{n-3}\right).
\end{aligned}
\tag{5.131}
$$

Finally, equating lhs and rhs one obtains

$$
\begin{aligned}
0 \quad = \quad & \frac{iD^{1/2}}{16\pi^2} e^{i\frac{\sigma}{2s}} \Bigg(\sum_{n=0}^{+\infty} (-na_n)(is)^{n-3} + \\
& \sum_{n=0}^{+\infty} \Delta^{-1/2} (\Delta^{1/2} a_n)_{;\mu}{}^{\mu} (is)^{n-2} + \\
& \sum_{n=0}^{+\infty} (-\sigma_{;\mu} a_n{}^{;\mu})(is)^{n-3} \Bigg). \qquad (5.132)
\end{aligned}
$$

The necessary and sufficient condition for this equation to be satisfied for every s is the multiplicative coefficient of every monomial $(is)^k$ be zero; therefore

$$
\begin{aligned}
& (is)^{-3} : \sigma_{;\mu} a_0{}^{;\mu} = 0, \qquad (5.133) \\
& (is)^k, \; k > -3 : \\
& \quad \sigma_{;\mu} a_{n+1}{}^{;\mu} + (n+1) a_{n+1} = \Delta^{-1/2} (\Delta^{1/2} a_n)_{;\mu}{}^{\mu}. \qquad (5.134)
\end{aligned}
$$

In view of (5.121) and (5.122) the equations for a_0 are solved by

$$a_0(x, x') = I(x, x'), \qquad (5.135)$$

while the recurrence relation (5.134) may be solved by integrating along each geodesic emanating from x'; in fact, multiplying the lhs by τ''^{n}, where τ'' is the parameter labeling a point x'' on the geodesic between $x'(\tau'' = 0)$ and $x(\tau'' = \tau)$, one obtains

$$
\begin{aligned}
& \tau''^{n} \sigma_{;\mu} a_{n+1}{}^{;\mu} (x(\tau''), x') + (n+1) \tau''^{n} a_{n+1}(x(\tau''), x') = \\
= \; & \tau''^{n+1} \frac{d}{d\tau''} a(x(\tau''), x') + \left(\frac{d}{d\tau''} \tau''^{n+1} \right) a_{n+1}(x(\tau''), x') \\
= \; & \frac{d}{d\tau''} \left(\tau''^{n+1} a_{n+1}(x(\tau''), x') \right), \qquad (5.136)
\end{aligned}
$$

therefore

$$a_{n+1}(x, x') = \tau^{-n-1} \int_0^{\tau} d\tau'' \; \tau''^{n} \Delta''^{-1/2} (\Delta''^{1/2} a''_n)_{;\mu''}{}^{\mu''} \qquad (5.137)$$

5.10 Series Expansions

Inserting (5.127) in (5.78), we now get

$$
\begin{aligned}
G(x, x') \quad = \quad & \frac{\Delta^{1/2}}{(4\pi)^2} \int_0^{\infty} ds \; \frac{1}{s^2} e^{-i\left(m^2 s - \frac{\sigma}{2s}\right)} \sum_{n=0}^{+\infty} a_n (is)^n \\
= \quad & \frac{\Delta^{1/2}}{(4\pi)^2} \sum_{n=0}^{+\infty} a_n \left(-\frac{\partial}{\partial m^2} \right)^n \int_0^{\infty} ds \; e^{-i\left(m^2 s - \frac{\sigma}{2s}\right)}. \qquad (5.138)
\end{aligned}
$$

The latter integral has already been evaluated (equations (5.44) and (5.49)). Breaking the Feynman propagator into its real and imaginary parts and making use of the expansions (5.57) and (5.58), we obtain, upon carrying out the differentiations with respect to m^2,

$$
\begin{aligned}
G(x,x') &= \bar{G}(x,x') + \frac{1}{2} i G^{(1)}(x,x'),\\
\bar{G}(x,x') &= \frac{\Delta^{1/2} I}{8\pi}\delta(\sigma) - \frac{\Delta^{1/2}}{8\pi}\theta(-\sigma)\Big[\tfrac{1}{2}(m^2 I - a_1)\\
&\quad + \frac{2\sigma}{2^2 4}(m^4 I - 2m^2 a_1 + 2a_2)\\
&\quad + \frac{(2\sigma)^2}{2^2 4^2 6}(m^6 I - 3m^4 a_1 + 6m^2 a_2 - 6a_3) + ...\Big], \qquad (5.139)\\
G^{(1)}(x,x') &= \frac{\Delta^{1/2} I}{4\pi^2 \sigma}\\
&\quad + \frac{\Delta^{1/2}}{2\pi^2}(\gamma - \tfrac{1}{2}\mathrm{log}2 + \tfrac{1}{2}\mathrm{log}|2m^2\sigma|) \bullet\\
&\quad \bullet \Big[\tfrac{1}{2}(m^2 I - a_1) + \frac{2\sigma}{2^2 4}(m^4 I - 2m^2 a_1 + 2a_2) + ...\Big]\\
&\quad - \frac{\Delta^{1/2}}{2\pi^2}\Big[\tfrac{1}{4} m^2 I + \frac{2\sigma}{2^2 4}(\tfrac{5}{4} m^4 I - 2m^2 a_1 + a_2)\\
&\quad + \frac{(2\sigma)^2}{2^2 4^2 6}\left(\frac{5}{3} m^6 I - \frac{9}{2} m^4 a_1 + \frac{15}{2} m^2 a_2 - \frac{9}{2} a_3\right) + ...\Big]\\
&\quad + \frac{\Delta^{1/2}}{2\pi^2}\Big[\left(\frac{a_2}{m^2} + \frac{a_3}{4m^4} + \frac{a_4}{2m^6} + ...\right)\\
&\quad - \frac{2\sigma}{2^2 4}\left(\frac{a_3}{m^2} + \frac{a_4}{m^4} + ...\right) + ...\Big]. \qquad (5.140)
\end{aligned}
$$

Several comments must be made about these expansions. First, there is the obvious remark that they are useful only for small values of σ. However, this is precisely the domain in which we are often interested, particularly in renormalization theory. We note that the Feynman propagator has, at $\sigma = 0$, the same types of singularity in the presence of a gravitational field as it has in a flat empty space-time. The Green's function $\bar{G}$, which can be split into the advanced and retarded Green's functions, has a δ-distribution singularity on the light cone and vanishes outside. We note, however, that when $m = 0$, it no longer vanishes inside the light cone as it does when space-time is flat and empty. Instead, we have

$$
\bar{G} = \frac{\Delta^{1/2} I}{8\pi}\delta(\sigma) + \frac{\Delta^{1/2}}{16\pi}\theta(-\sigma)\left(a_1 - \frac{\sigma}{2} a_2 + \frac{\sigma^2}{2\cdot 4} a_3 - ...\right). \qquad (5.141)
$$

lt is important to observe in this connection that although the expansion in terms of the a's can be used for $\bar{G}$ when $m = 0$, it cannot be used for $G^{(1)}$. This may be seen from the last line of (5.140) which shows that an expansion in inverse powers of m^2 is involved. When m is vanishing, alternative methods, based either on special properties of the fields or on perturbation theory, must be found for evaluating the Feynman propagator.

Chapter 6

Green's Functions: Photon

6.1 Quantum Maxwell Theory in Curved Space-Time

In the following sections, the task will be the evaluation of the photon Green's functions in curved space-time. Following DeWitt's approach and using the minimal coupling, the full action functional for the theory is

$$\bar{S} = \int d^4x \, |\mathrm{g}|^{1/2} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\left(A^{\mu}{}_{;\mu}\right)^2}{2\xi} - \frac{\chi \Box \psi}{\sqrt{\xi}} \right). \tag{6.1}$$

The first term is the classical action functional for the field A_μ, while the second and the third ones are the gauge-averaging and the ghost ones, respectively; moreover, ξ is a real parameter which accounts for the behaviour under rescaling of the gauge-fixing coordinate K^α:

$$\begin{aligned} K^\alpha &\mapsto \frac{1}{\sqrt{\xi}} K^\alpha \\ &\Longrightarrow \begin{cases} \Omega \mapsto \frac{1}{\xi} \, \Omega, \\ \underline{M}^{\alpha}_{\beta} \mapsto \frac{1}{\sqrt{\xi}} \underline{M}^{\alpha}_{\beta}. \end{cases} \end{aligned} \tag{6.2}$$

The full action functional (6.1) can be put in the form

$$\bar{S} = \int d^4x \, |\mathrm{g}|^{1/2} \left(-\tfrac{1}{2} A_\mu P^{\mu\nu}(\xi) A_\nu + \tfrac{1}{\sqrt{\xi}} \chi P_0 \psi \right), \tag{6.3}$$

where

$$P^{\mu\nu}(\xi) = -g^{\mu\nu}\Box + R^{\mu\nu} + (1 - \tfrac{1}{\xi}) \nabla^\mu \nabla \nu, \tag{6.4}$$

$$P_0 = -\Box = -g^{\nu\mu} \nabla_\mu \nabla_\nu. \tag{6.5}$$

Hence the equation for the photon Green's functions is

$$|\mathrm{g}|^{1/2} P^{\lambda}_{\mu}(\xi) G^{(\xi)}_{\lambda\nu}(x, x') = g_{\mu\nu}(x) \delta(x, x'). \tag{6.6}$$

By analogy with the methods of the previous sections, we introduce two non-physical Hilbert spaces spanned by the basis kets $|x, \mu\rangle$, $|x\rangle$, respectively; for them, the following orthogonality relations hold:

$$\langle x, \mu | x', \nu \rangle = g_{\mu\nu}(x)\delta(x, x'), \tag{6.7}$$

$$\langle x | x' \rangle = \delta(x, x'). \tag{6.8}$$

Of course, $|x\rangle$ is the familiar Dirac notation for the eigenfunctionals of the position operator which has continuous spectrum, while the index of both $|x, \mu\rangle$ and the associated "bra" $\langle x, \mu|$ is viewed as that of a covariant vector density of weight 1/2. The "Hamiltonian " operators $H(\xi)$, H_0 associated with $P^{\mu\nu}(\xi)$ and P_0, respectively, are defined by

$$\langle x, \mu | H(\xi) | x', \nu \rangle = P_\mu^\lambda(\xi) \langle x, \lambda | x', \nu \rangle, \tag{6.9}$$

$$\langle x | H_0 | x' \rangle = P_0 \langle x | x' \rangle. \tag{6.10}$$

Then the "operator" solution with the Feynman prescription is

$$|\mathrm{g}|^{1/4} G^{(\xi)} |\mathrm{g}|^{1/4} = \frac{1}{H(\xi) - i\epsilon} = i \int d\tau \, e^{-i\tau H(\xi)}. \tag{6.11}$$

Taking matrix elements of the previous equation we obtain

$$|\mathrm{g(x)}|^{1/4} G^{(\xi)}_{\mu\nu}(x, x') |\mathrm{g}(x')|^{1/4} = i \int_0^{+\infty} d\tau \, \left\langle x, \mu | e^{-i\tau H(\xi)} | x', \nu \right\rangle^{(\xi)},$$

$$|\mathrm{g(x)}|^{1/4} G^{(\xi)}_{\mu\nu}(x, x') |\mathrm{g}(x')|^{1/4} = i \int_0^{+\infty} d\tau \, \langle x, \mu; \tau | x', \nu; 0 \rangle^{(\xi)}, \tag{6.12}$$

where

$$\langle x, \mu; \tau | x', \nu; 0 \rangle \equiv \left\langle x, \mu | e^{-i\tau H(\xi)} | x', \nu \right\rangle. \tag{6.13}$$

Thus the "transition amplitude" $\langle x, \mu; \tau | x', \nu; 0 \rangle^{(\xi)}$ satisfies the Schrödinger equation associated to $H(\xi)$:

$$i \frac{\partial}{\partial \tau} \langle x, \mu; \tau | x', \nu; 0 \rangle^{(\xi)} = H(\xi) \langle x, \mu; \tau | x', \nu; 0 \rangle^{(\xi)}, \tag{6.14}$$

with the initial condition

$$\langle x, \mu; 0 | x', \nu; 0 \rangle^{(\xi)} = g_{\mu\nu}(x)\delta(x, x'). \tag{6.15}$$

For arbitrary values of ξ, the operator $P^{\mu\nu}(\xi)$, as well as $P_\mu^\lambda \equiv g_{\rho\mu} P^{\rho\lambda}(\xi)$, is non-minimal, i.e. the wavelike-operator part $-g_{\mu\nu}\Box + R^{\mu\nu}$ is spoiled by $(1 - 1/\xi)\nabla^\mu \nabla \nu$. Nevertheless, if one knows $\langle x, \mu; \tau | x', \nu; 0 \rangle^{(\xi)}$ at $\xi = 1$, one can use this solution, here denoted by $\langle x, \mu; \tau | x', \nu; 0 \rangle^{(1)}$, to evaluate $\langle x, \mu; \tau | x', \nu; 0 \rangle^{(\xi)}$ according to the Endo [12] formula

$$\langle x, \mu; \tau | x', \nu; 0 \rangle^{(\xi)} = \langle x, \mu; \tau | x', \nu; 0 \rangle^{(1)} + i \int_\tau^{\tau/\xi} dy \, \nabla_\mu \nabla^\lambda \langle x, \lambda; y | x', \nu; 0 \rangle^{(1)}. \tag{6.16}$$

The previous equation plays a key role in evaluating the regularized photon Green function, as we shall see in the following.

Once the transition amplitude is known, equation (6.12) describes the *massless limit of the Feynman propagator* (for which one would have to add an infinitesimal negative imaginary mass). At this stage, formula (6.12) needs a suitable regularization. Following Endo, we use ζ-function regularization and introduce a regularization parameter μ^A defining (any suffix to denote regularization of the photon Green function is omitted for simplicity of notation):

$$|\mathrm{g(x)}|^{1/4} G^{(\xi)}_{\mu\nu}(x,x')|\mathrm{g}(x')|^{1/4} \equiv \lim_{s\to 0} \frac{\mu_A^{2s}\, i^{s+1}}{\Gamma(s+1)} \int_0^{+\infty} d\tau\, \tau^s \left\langle x,\mu;\tau|x',\nu;0\right\rangle^{(\xi)} . \tag{6.17}$$

It should be stressed that the limit $s \to 0$ should be taken at the very end of all calculations, and cannot be brought within the integral.

The transition amplitude $\langle x,\mu;\tau|x',\nu;0\rangle^{(1)}$ is known as $\tau \to 0$ and as

$$\sigma(x,x') \to 0$$

through its Fock-Schwinger-DeWitt asymptotic expansion (DeWitt [8], [11], Fock [14], Schwinger [28], Christensen [7])

$$\langle x,\mu;\tau|x',\nu;0\rangle^{(1)} \sim \frac{i}{16\pi^2}|\mathrm{g}|^{\frac{1}{4}}\sqrt{\Delta}|\mathrm{g}'|^{\frac{1}{4}} e^{\frac{i\sigma}{2\tau}} \sum_{n=0}^{\infty}(i\tau)^{n-2} b_{n\ \mu\nu'}, \tag{6.18}$$

$$\lim_{x'\to x} b_{0\ \mu\nu'} = g_{\mu\nu}(x), \tag{6.19}$$

the coefficient bivectors $b_{n\ \mu\nu'}$ are evaluated by solving a recursion formula obtained upon insertion of (6.18) into eq. (6.14) (the same procedure has been applied to the scalar theory in the previous sections); such a recursion formula reads

$$\sigma^{;\lambda} b_{0\ \mu\nu';\lambda} = 0, \tag{6.20}$$

$$\sigma^{;\lambda} b_{n+1\ \mu\nu';\lambda} + (n+1) b_{n+1\ \mu\nu'} = \frac{1}{\sqrt{\Delta}}\left(\sqrt{\Delta} b_{n\ \mu\nu'}\right)_{;\lambda}{}^{\lambda} - R^{\lambda}_{\mu} b_{n\ \lambda\nu'}. \tag{6.21}$$

In view of eqs. (5.112), (5.113), the equations (6.19) and (6.20) are solved by $b_{0\ \mu\nu'} = g_{\mu\nu'}$.

Now pay attention to the integral in (6.17); using Endo's formula, we can split it in the following way:

$$\begin{aligned} I_{\mu\nu'}(s,\xi) &\equiv \int_0^{+\infty} d\tau\, \tau^s\ \langle x,\mu;\tau|x',\nu;0\rangle^{(\xi)} \\ &= \int_0^{+\infty} d\tau\, \tau^s\ \langle x,\mu;\tau|x',\nu;0\rangle^{(1)} \\ &\quad + i\int_0^{+\infty} d\tau\, \tau^s \int_\tau^{\tau/\xi} dy\ \nabla_\mu \nabla^\lambda \langle x,\lambda;y|x',\nu;0\rangle^{(1)} \\ &\equiv I^A_{\mu\nu'}(s) + I^B_{\mu\nu'}(s,\xi), \end{aligned} \tag{6.22}$$

where

$$I^A_{\mu\nu'}(s) \sim \frac{1}{16\pi^2}|\mathrm{g}|^{1/4}\sqrt{\Delta}|\mathrm{g}'|^{\frac{1}{4}}\sum_{n=0}^{\infty} i^{n-1} b_{n\ \mu\nu'} F_{s,n}(x,x'), \tag{6.23}$$

$$I^B_{\mu\nu'}(s,\xi) \sim \frac{1}{16\pi^2}\sum_{n=0}^{\infty} i^n \nabla_\mu \nabla^\lambda \left(|\mathrm{g}|^{1/4}\sqrt{\Delta}|\mathrm{g}'|^{\frac{1}{4}} b_{n\ \lambda\nu'} \tilde{F}_{s,n}(x,x';\xi)\right) \tag{6.24}$$

and the following auxiliary functions have been introduced:

$$F_{s,n}(x,x') = \left(\frac{\sigma(x,x')}{2}\right)^{s+n-1}\int_0^\infty dy\ y^{-s-n}e^{iy}, \tag{6.25}$$

$$\tilde{F}_{s,n}(x,x';\xi) = \left(\frac{\sigma(x,x')}{2}\right)^{n-1}\int_0^\infty d\tau\ \tau^s \int_{\xi\sigma/2\tau}^{\sigma/2\tau} dy\ y^{-n}e^{iy}. \tag{6.26}$$

At this stage, the Feynman photon Green function has the asymptotic expansion

$$G^{(\xi)}_{\mu\nu'} \sim |\mathrm{g}|^{-1/4}\lim_{s\to 0}\frac{\mu_A^{2s}\ i^{s+1}}{\Gamma(s+1)}\left(I^A_{\mu\nu'}(s) + I^B_{\mu\nu'}(s,\xi)\right)|\mathrm{g}'|^{-1/4}. \tag{6.27}$$

6.2 Regularized Integrals

We are now going to evaluate the regularized integrals occurring in the space-time covariant form of the Feynman Green's function. For this purpose, we point out that the integral in the expression of $F_{s,n}(x,x')$ (eq. (6.25)) is a particular case of the integral

$$I(\beta) \equiv \int_0^\infty dy\ y^{-\beta}e^{i\beta} = i\Gamma(1-\beta)e^{-i\frac{\pi}{2}\beta}. \tag{6.28}$$

Recall now that the Γ-function $\Gamma(z) \equiv \int_0^\infty dy\ y^{z-1}e^{-y}$, originally defined on the half-plane $\mathrm{Re}(z) > 0$, can be analytically extended to a meromorphic function, with first-order poles at $0, -1, -2, ..., -\infty$. With this understanding, we write that

$$F_{s,n}(x,x') = i\left(\frac{\sigma(x,x')}{2}\right)^{s+n-1}\Gamma(1-s-n)e^{-i\frac{\pi}{2}(s+n)}, \tag{6.29}$$

where $\Gamma(1-s-n)$ has first order poles at $1-s-n=-k$, with $k = 0, 1, 2, ..., \infty$.

In order to evaluate the double integral occurring in (6.26), we first exploit the identity

$$\int_{\xi\sigma/2\tau}^{\sigma/2\tau} dy\ y^{-n}e^{iy} = i^{3n+1}\left[\Gamma\left(1-n, -i\frac{\xi\sigma}{2\tau}\right) - \Gamma\left(1-n, -i\frac{\sigma}{2\tau}\right)\right], \tag{6.30}$$

where in square brackets we have the incomplete Γ-function

$$\Gamma(a,x) \equiv \int_x^\infty du\ u^{a-1}e^{-u}. \tag{6.31}$$

Hence we re-express $\tilde{F}_{s,n}(x,x';\xi)$ in the form

$$\tilde{F}_{s,n}(x,x';\xi) = \left(\frac{\sigma(x,x')}{2}\right)^{n-1} i^{3n+1}\left[I^{x,x'}_{s,n}(\xi) - I^{x,x'}_{s,n}(1)\right], \tag{6.32}$$

where

$$I_{s,n}^{x,x'}(\xi) \equiv \int_0^\infty d\tau \; \tau^s \Gamma\left(1-n, -i\frac{\xi\sigma}{2\tau}\right). \tag{6.33}$$

At this stage, we are led to consider the integral

$$J(\beta,\nu,c) \equiv \int_0^\infty dx \; x^{\beta-1}\Gamma(\nu,cx). \tag{6.34}$$

On setting $y \equiv cx$, if $\mathrm{Re}(c) > 0$, and exploiting the Leibniz rule amd the fundamental theorem of calculus one finds (Prudnikov, Brychkov, Marichev, Romer [25])

$$J = \frac{1}{\beta c^\beta}\int_0^\infty dy \; \left(\frac{d}{dy}y^\beta\right)\left(\int_y^\infty du \; u^{\nu-1}e^{-u}\right) = \frac{\Gamma(\beta+\nu)}{\beta c^\beta}, \tag{6.35}$$

because, for $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\beta+\nu) > 0$, the total derivative of

$$y^\beta \int_y^\infty du \; u^{\nu-1}e^{-u}$$

yields a vanishing contribution.

We can however consider the analytic extension of $\Gamma(\beta+\nu)$, after changing the variable in the integral (6.33) according to $1/\tau \equiv T$, which yields:

$$I_{s,n}^{x,x'}(\xi) \equiv \lim_{\epsilon\to 0}\int_0^\infty dT \; T^{-(s+2)}\Gamma\left(1-n, \left(\epsilon - i\frac{\xi\sigma}{2}\right)T\right), \tag{6.36}$$

where a small positive $\epsilon > 0$ has been considered so as to apply the result (6.35). In our case, $\beta = -(s+1)$, $\nu = 1-n$, $c = \epsilon - i\xi\sigma/2$, and after making the analytic extension of $\Gamma(\beta+\nu)$ we find

$$\tilde{F}_{s,n}(x,x';\xi) = -\left(\tfrac{\sigma(x,x')}{2}\right)^{s+n} i^{3(s+n)}\frac{\Gamma(-s-n)}{(s+1)}(\xi^{s+1}-1), \tag{6.37}$$

where $\Gamma(-s-n)$ has first-order poles at $s+n=k$ for all $k = 0,1,2,...,\infty$.

6.3 The Feynman Photon Green Function

Our formulae (6.29) and (6.37) should be inserted into (6.23) and (6.24) to work out the asymptotic expansion of the Feynman photon Green function: For this purpose it's crucial to take the limit as $s \to 0$ at the last stage. Hence we find, as x' approaches x (which implies $\sigma(x,x') \to 0$),

$$G_{\mu\nu'}^{(\xi)} \sim \frac{i}{16\pi^2}\lim_{s\to 0}\frac{\mu_A^{2s}}{\Gamma(s+1)}\mathcal{G}_{\mu\nu'}^{(\xi)}(s), \tag{6.38}$$

where, after having defined

$$U_{n\;\mu}{}^{\lambda}(s;\xi) \equiv \frac{2}{\sigma(x,x')}\delta_\mu^\lambda + \frac{(\xi^{s+1}-1)}{(s+n)(s+1)}\nabla_\mu\nabla^\lambda, \tag{6.39}$$

$$B_{n\;\lambda\nu'}(s) \equiv b_{n\;\lambda\nu'}\sqrt{\Delta}(x,x')(\sigma(x,x')/2)^{s+n}, \tag{6.40}$$

we write

$$\mathcal{G}^{(\xi)}_{\mu\nu'}(s) \equiv \sum_{n=0}^{\infty} \Gamma(1-s-n) U_{n\ \mu}{}^{\lambda}(s;\xi) B_{n\ \lambda\nu'}(s). \tag{6.41}$$

What is crucial for us is the $s \to 0$ limit of the sum in the previous equation. Indeed, on studying first, for simplicity, the case when the gauge-field operator reduces to a minimal (wavelike) operator, i.e., at $\xi = 1$, one finds

$$\mathcal{G}^{(1)}_{\mu\nu'}(s) = \frac{2\sqrt{\Delta}(x,x')}{\sigma(x,x')} \sum_{n=0}^{\infty} f_{n\ \mu\nu'}(s), \tag{6.42}$$

having defined

$$f_{n\ \mu\nu'}(s) \equiv \Gamma(1-s-n) b_{n\ \mu\nu'} (\sigma(x,x')/2)^{s+n}. \tag{6.43}$$

Since $b_{0\ \mu\nu'} = g_{\mu\nu'}$, we therefore find

$$\mathcal{G}^{(1)}_{\mu\nu'}(0) = \frac{2\sqrt{\Delta}(x,x')}{\sigma(x,x')} g_{\mu\nu'} + \frac{2\sqrt{\Delta}(x,x')}{\sigma(x,x')} \lim_{s\to 0} \sum_{n=1}^{\infty} f_{n\ \mu\nu'}(s), \tag{6.44}$$

which is very encouraging, since the first term on the rhs is precisely the first term in the Hadamard asymptotic expansion at small $\sigma(x,x')$ (Christensen [7]). On the other hand, the Hadamard Green function is, apart from a factor 2, the imaginary part of the Feynman Green function, in agreement with formula (6.38). Eventually, we find therefore, at small $\sigma(x,x')$,

$$\begin{aligned} G^{(\xi)}_{\mu\nu'} \quad \sim \quad & \frac{1}{8\pi^2} \frac{\sqrt{\Delta}(x,x')}{\sigma(x,x') + i\epsilon} g_{\mu\nu'} + \frac{i}{16\pi^2} \lim_{s\to 0} \left[\frac{(\xi-1)}{s(s+1)} \nabla_{\mu} \nabla^{\lambda} B_{0\ \lambda\nu'}(s) \right. \\ & \left. + \sum_{n=1}^{\infty} \Gamma(1-s-n) U_{n\ \mu}{}^{\lambda}(s;\xi) B_{n\ \lambda\nu'}(s) \right], \end{aligned} \tag{6.45}$$

i.e., the "flat" Feynman propagator, with the $i\epsilon$ term restored, plus correction resulting from the gauge parameter ($\xi \neq 1$ leading to a non-minimal operator) and from the non-vanishing curvature.

A further crucial check is whether our infinite sum (6.41) is also able to recover the familiar $\log \sigma(x,x')$ singularity, which occurs for massive theories in flat space-time and, more generally, even for *massless* theories (as in our case) but in curved space-time. For this purpose, it is enough to set $\xi = 1$ and focus on the sum in (6.42). Such a formula can be studied with the help of the Euler-Maclaurin formula (see Appendix B and Wong [34]), which provides, among the others, a term given by the integral (herafter, since the discrete summation index n is replaced by the continuous variable z, we consider the coefficients $b_{z\ \mu\nu'}$, functions of z that reduce to the coefficient bivectors $b_{n\ \mu\nu'}$ for $z = n$)

$$\begin{aligned} & \frac{J_{\mu\nu'}(s)}{\sqrt{\Delta}(x,x')} \\ & \equiv \int_0^{\infty} dz\ \Gamma(1-s-z) b_{z\ \mu\nu'} (\sigma(x,x')/2)^{s+z-1} \\ & = (\sigma(x,x')/2)^s \left\{ \int_0^1 dz\ \Gamma(1-s-z) b_{z\ \mu\nu'} e^{(z-1)\log(\sigma(x,x')/2)} \right. \\ & \left. + \int_1^{\infty} dz\ \Gamma(1-s-z) b_{z\ \mu\nu'} e^{(z-1)\log(\sigma(x,x')/2)} \right\}. \end{aligned} \tag{6.46}$$

At this stage we set $z-1 \equiv y$ in the second integral in curly brackets on the rhs of the previous equation; then it becomes

$$\tilde{J}_{\mu\nu'}(s) = \left(\int_0^{y^*} dy \; + \int_{y^*}^{\infty} dy \right) \Gamma(-s-y) b_{y+1\;\mu\nu'} e^{y \log(\sigma(x,x')/2)}. \tag{6.47}$$

We begin to understand what happens: at small $\sigma(x,x')$, the integrand in (6.47) becomes exponentially damped, so that the resulting asymptotic expansion is obtained from integration in the interval $[0, y^*]$ for some y^* in a small neighbourhood of the origin. Here we first expand $e^{y \log(\sigma(x,x')/2)}$ at small y for fixed $\sigma(x,x')$, and eventually take the $\sigma(x,x') \to 0$ limit. Such a procedure yields the asymptotic expansion

$$\tilde{J}_{\mu\nu'}(s) \sim \log(\sigma(x,x')/2) \int_0^{y^*} dy\, \Gamma(-s-y) b_{y+1\;\mu\nu'}. \tag{6.48}$$

On taking the $s \to 0$ limit we therefore recover the familiar $\log(\sigma(x,x'))$ singularity of the photon Green function, which results from the non-vanishing Riemann curvature (in Minkowski space-time, the corresponding $b_{y+1\;\mu\nu'}$ would instead vanish).

6.4 Second Derivatives, Stress-Energy Tensor, Effective Action

In this section it will be shown that, given a field theory (in the present case, quantum electrodynamics), there is a close relation between second derivatives of the Hadamard Green's functions, stress-energy tensor and effective action, following the footsteps of Christensen [7] and DeWitt [9]. In a classical field theory, the stress-energy tensor can be obtained as

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{6.49}$$

For a free theory, this object is quadratic in the fields; when dealing with the associated quantum theory, the expectation value of this observable is, in general, divergent: this happens because the two field operators are taken at the same space-time point. An useful way to regularize the expectation value of the stress-energy tensor is to insert into the formal expression for $T^{\mu\nu}$ not the field operators themselves but operators that have been smeared out by means of a smooth function $s(x)$ of compact support:

$$\phi_s(x) \equiv \int d^4y\; s(x-y)\phi(y). \tag{6.50}$$

The resulting operator is well defined and the behaviour of its (finite) expectation value may be studied as the size of the support of $s(x)$ tends to zero. A regularization method equivalent to the one of the smearing method but easier to apply in practice is simply to separate the points at which the two fiels in $T^{\mu\nu}$ are taken and then to examine the tensor as the points are brought together again.

For the action functional (6.1), the stress-energy tensor is

$$T^{\mu\nu} = \frac{2}{\sqrt{|\mathrm{g}|}} \frac{\delta \bar{S}}{\delta g_{\mu\nu}} = T^{\mu\nu}_{\mathrm{Maxwell}} + \tfrac{1}{\xi} T^{\mu\nu}_{\mathrm{gauge}} + \tfrac{1}{\sqrt{\xi}} T^{\mu\nu}_{\mathrm{ghost}}, \tag{6.51}$$

where

$$T^{\mu\nu}_{\mathrm{Maxwell}} \equiv \frac{\delta S_{\mathrm{Maxwell}}}{\delta g_{\mu\nu}} \equiv \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \left(-\frac{1}{4} |\mathrm{g}(x)|^{1/2} F_{\mu\nu} F^{\mu\nu} \right), \tag{6.52}$$

$$T^{\mu\nu}_{\mathrm{gauge}} \equiv \frac{\delta S_{\mathrm{gauge}}}{\delta g_{\mu\nu}} \equiv \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \left(-\frac{1}{2} |\mathrm{g}(x)|^{1/2} A^{\mu}{}_{;\mu} \right), \tag{6.53}$$

$$T^{\mu\nu}_{\mathrm{ghost}} \equiv \frac{\delta S_{\mathrm{ghost}}}{\delta g_{\mu\nu}} \equiv \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \left(-|\mathrm{g}(x)|^{1/2} \chi \Box \psi \right) \tag{6.54}$$

$T^{\mu\nu}_{\mathrm{Maxwell}}$ calculations:

$$\begin{aligned}
& \delta(-\tfrac{1}{4}|\mathrm{g}|^{1/2} F_{\mu\nu} F^{\mu\nu}) \\
= \;& -\tfrac{1}{4} F_{\mu\nu} F_{\rho\sigma} (g^{\mu\rho})(g^{\nu\sigma})(\delta |\mathrm{g}|^{1/2}) \\
& -\tfrac{1}{4} F_{\mu\nu} F_{\rho\sigma} (\delta g^{\mu\rho})(g^{\nu\sigma})(|\mathrm{g}|^{1/2}) \\
& -\tfrac{1}{4} F_{\mu\nu} F_{\rho\sigma} (g^{\mu\rho})(\delta g^{\nu\sigma})(|\mathrm{g}|^{1/2}) \\
= \;& -\tfrac{1}{4} F_{\mu\nu} F_{\rho\sigma} (g^{\mu\rho})(g^{\nu\sigma})(\tfrac{1}{2}|\mathrm{g}|^{1/2} g^{\alpha\beta} \delta g_{\alpha\beta}) \\
& -\tfrac{1}{4} F_{\mu\nu} F_{\rho\sigma} (-g^{\mu\alpha} g^{\rho\beta} \delta g_{\alpha\beta})(g^{\nu\sigma})(|\mathrm{g}|^{1/2}) \\
& -\tfrac{1}{4} F_{\mu\nu} F_{\rho\sigma} (g^{\mu\rho})(-g^{\nu\alpha} g^{\sigma\beta} \delta g_{\alpha\beta})(|\mathrm{g}|^{1/2}) \\
= \;& -\tfrac{1}{8} F_{\mu\nu} F^{\mu\nu} |\mathrm{g}|^{1/2} g^{\alpha\beta} \delta g_{\alpha\beta} \\
& -\tfrac{1}{2} F^{\alpha}{}_{\rho} F^{\rho\beta} |\mathrm{g}|^{1/2} \delta g_{\alpha\beta}.
\end{aligned} \tag{6.55}$$

Therefore

$$T^{\mu\nu}_{\mathrm{Maxwell}} = -F^{\mu}{}_{\rho} F^{\rho\nu} - \tfrac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu}. \tag{6.56}$$

$T^{\mu\nu}_{\rm gauge}$ calculations:

$$
\begin{aligned}
& \delta\left(-\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\left(A^{\mu}{}_{;\mu}\right)^2\right) \\
= \quad & -\tfrac{1}{2}(\delta|\mathrm{g}(x)|^{1/2})\left(A^{\mu}{}_{;\mu}\right)^2 \\
& -(|\mathrm{g}(x)|^{1/2})A^{\eta}{}_{;\eta}\left[\delta(g^{\mu\nu}\nabla_\nu A_\mu)\right] \\
= \quad & -\tfrac{1}{2}(\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}g^{\alpha\beta}\delta g_{\alpha\beta})\left(A^{\mu}{}_{;\mu}\right)^2 \\
& -(|\mathrm{g}(x)|^{1/2})A^{\eta}{}_{;\eta}\left[\delta g^{\mu\nu}\nabla_\nu A_\mu + g^{\mu\nu}\delta(\partial_\nu A_\mu - \Gamma^{\rho}_{\mu\nu}A_\rho)\right] \\
= \quad & -\tfrac{1}{4}|\mathrm{g}(x)|^{1/2}\left(A^{\mu}{}_{;\mu}\right)^2 g^{\alpha\beta}\delta g_{\alpha\beta} \\
& +|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\nabla_\nu A_\mu \\
& +|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}g^{\mu\nu}\delta\Gamma^{\rho}_{\mu\nu}A_\rho \\
= \quad & -\tfrac{1}{4}|\mathrm{g}(x)|^{1/2}\left(A^{\mu}{}_{;\mu}\right)^2 g^{\alpha\beta}\delta g_{\alpha\beta} \\
& +|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}\delta g_{\alpha\beta}A^{\alpha;\beta} \\
& +|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}g^{\mu\nu}\left[\tfrac{1}{2}g^{\rho\alpha}(\delta g_{\alpha\nu;\mu}+\delta g_{\alpha\mu;\nu}-\delta g_{\mu\nu;\alpha})\right]A_\rho \\
= \quad & -\tfrac{1}{4}|\mathrm{g}(x)|^{1/2}\left(A^{\mu}{}_{;\mu}\right)^2 g^{\alpha\beta}\delta g_{\alpha\beta} \\
& +|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}\delta g_{\alpha\beta}A^{\alpha;\beta} \\
& +|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}g^{\mu\nu}A^{\alpha}\delta g_{\alpha\nu;\mu} - \tfrac{1}{2}|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}A^{\alpha}g^{\mu\nu}\delta g_{\mu\nu;\alpha}.
\end{aligned}
$$

Integrating by parts on the last line, one obtains

$$
\begin{aligned}
& \delta\left(-\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\left(A^{\mu}{}_{;\mu}\right)^2\right) \\
= \quad & -\tfrac{1}{4}|\mathrm{g}(x)|^{1/2}\left(A^{\mu}{}_{;\mu}\right)^2 g^{\alpha\beta}\delta g_{\alpha\beta} \\
& +|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}A^{\alpha;\beta}\delta g_{\alpha\beta} \\
& -|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta\mu}g^{\mu\nu}A^{\alpha}\delta g_{\alpha\nu} - |\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}g^{\mu\nu}A^{\alpha}{}_{;\mu}\delta g_{\alpha\nu} \\
& +\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta\alpha}A^{\alpha}g^{\mu\nu}\delta g_{\mu\nu} + \tfrac{1}{2}|\mathrm{g}(x)|^{1/2}A^{\eta}{}_{;\eta}A^{\alpha}{}_{;\alpha}g^{\mu\nu}\delta g_{\mu\nu} \\
= \quad & \left(\tfrac{1}{4}\left(A^{\mu}{}_{;\mu}\right)^2 + \tfrac{1}{2}A^{\mu}{}_{;\nu\rho}A^{\rho}\right)|\mathrm{g}(x)|^{1/2}g^{\alpha\beta}\delta g_{\alpha\beta} \\
& -\tfrac{1}{2}\left(A_{\eta}{}^{;\eta\alpha}A^{\beta} + A_{\eta}{}^{;\eta\beta}A^{\alpha}\right)|\mathrm{g}(x)|^{1/2}\delta g_{\alpha\beta}.
\end{aligned}
\tag{6.57}
$$

Therefore

$$
T^{\mu\nu}_{\rm gauge} = \left(\tfrac{1}{2}\left(A^{\alpha}{}_{;\alpha}\right)^2 + A^{\alpha}{}_{;\alpha\beta}A^{\beta}\right)g^{\mu\nu} - \left(A_{\alpha}{}^{;\alpha\mu}A^{\nu} + A_{\alpha}{}^{;\alpha\nu}A^{\mu}\right). \tag{6.58}
$$

$T^{\mu\nu}_{\text{ghost}}$ calculations:

$$
\begin{aligned}
& \delta(-|\mathrm{g}(x)|^{1/2}\chi\Box\psi) \\
= \; & -\delta(|\mathrm{g}(x)|^{1/2})\chi\Box\psi - |\mathrm{g}(x)|^{1/2}\chi\delta(\Box\psi) \\
= \; & -\tfrac{1}{2}|\mathrm{g}(x)|^{1/2})\chi\Box\psi g^{\mu\nu}\delta g_{\mu\nu} \\
& -|\mathrm{g}(x)|^{1/2}\chi\delta\left(g^{\nu\mu}\partial_\mu\partial_\nu\psi - g^{\nu\mu}\Gamma^\rho_{\mu\nu}\partial_\rho\psi\right) \\
= \; & -\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi\Box\psi g^{\mu\nu}\delta g_{\mu\nu} \\
& +|\mathrm{g}(x)|^{1/2}\chi g^{\nu\alpha}g^{\mu\beta}\delta g_{\alpha\beta}\partial_\mu\partial_\nu\psi \\
& -|\mathrm{g}(x)|^{1/2}\chi g^{\nu\alpha}g^{\mu\beta}\delta g_{\alpha\beta}\Gamma^\rho_{\mu\nu}\partial_\rho\psi \\
& +|\mathrm{g}(x)|^{1/2}\chi g^{\nu\mu}\delta\Gamma^\rho_{\mu\nu}\partial_\rho\psi \\
= \; & -\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi\Box\psi g^{\mu\nu}\delta g_{\mu\nu} \\
& +|\mathrm{g}(x)|^{1/2}\chi g^{\nu\alpha}g^{\mu\beta}\delta g_{\alpha\beta}\psi_{;\nu\mu} \\
& +|\mathrm{g}(x)|^{1/2}\chi g^{\nu\mu}(\tfrac{1}{2}g^{\rho\eta}(\nabla_\mu\delta g_{\nu\eta} + \nabla_\nu\delta g_{\mu\eta} - \nabla_\eta\delta g_{\mu\nu}))\partial_\rho\psi \\
= \; & -\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi\Box\psi g^{\mu\nu}\delta g_{\mu\nu} \\
& +|\mathrm{g}(x)|^{1/2}\chi g^{\nu\alpha}g^{\mu\beta}\delta g_{\alpha\beta}\psi_{;\nu\mu} \\
& +|\mathrm{g}(x)|^{1/2}\chi g^{\nu\mu}g^{\rho\eta}\delta g_{\nu\eta;\mu}\partial_\rho\psi - \tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi g^{\nu\mu}g^{\rho\eta}\delta g_{\mu\nu;\eta}\partial_\rho\psi.
\end{aligned}
$$

Integrating by parts on the last line, one obtains

$$
\begin{aligned}
= \; & -\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi\Box\psi g^{\mu\nu}\delta g_{\mu\nu} \\
& +|\mathrm{g}(x)|^{1/2}\chi g^{\nu\alpha}g^{\mu\beta}\delta g_{\alpha\beta}\psi_{;\nu\mu} \\
& -|\mathrm{g}(x)|^{1/2}\chi g^{\nu\mu}g^{\rho\eta}\delta g_{\nu\eta}\psi_{;\rho\mu} - |\mathrm{g}(x)|^{1/2}\chi_{;\mu}g^{\nu\mu}g^{\rho\eta}\delta g_{\nu\eta}\psi_{;\rho} \\
& +\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi_{;\eta}g^{\nu\mu}g^{\rho\eta}\delta g_{\mu\nu}\psi_{;\rho} + \tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi g^{\nu\mu}g^{\rho\eta}\delta g_{\mu\nu}\psi_{;\rho\eta} \\
= \; & -\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi\psi_{;\alpha}{}^{\alpha}g^{\mu\nu}\delta g_{\mu\nu} \\
& +|\mathrm{g}(x)|^{1/2}\chi\psi^{;\alpha\beta}\delta g_{\alpha\beta} \\
& -|\mathrm{g}(x)|^{1/2}\chi\psi^{;\eta\nu}\delta g_{\nu\eta} - |\mathrm{g}(x)|^{1/2}\chi^{;\mu}\psi^{;\eta}\delta g_{\mu\eta} \\
& +\tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi_{;\eta}\psi^{;\eta}g^{\nu\mu}\delta g_{\mu\nu} + \tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi g^{\nu\mu}\psi_{;\eta}{}^{\eta}\delta g_{\mu\nu} \\
= \; & \tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\chi_{;\alpha}\psi^{;\alpha}g^{\mu\nu}\delta g_{\mu\nu} - \tfrac{1}{2}|\mathrm{g}(x)|^{1/2}\left(\chi^{;\mu}\psi^{;\nu} + \chi^{;\nu}\psi^{;\mu}\right)\delta g_{\mu\nu}.
\end{aligned}
$$

Therefore

$$T^{\mu\nu}_{\text{ghost}} = \chi_{;\alpha}\psi^{;\alpha}g^{\mu\nu} - \left(\chi^{;\mu}\psi^{;\nu} + \chi^{;\nu}\psi^{;\mu}\right). \tag{6.59}$$

In order to exploit the point split method, we write

$$
\begin{aligned}
& A_{\mu;\alpha}A_{\rho;\sigma} \\
= \; & \tfrac{1}{2}A_{\mu;\alpha}A_{\rho;\sigma} + \tfrac{1}{2}A_{\rho;\sigma}A_{\mu;\alpha} \\
= \; & \tfrac{1}{2}\left[A_{\mu;\alpha}, A_{\rho;\sigma}\right]_+ \\
= \; & \lim_{x'\to x}\left\{\tfrac{1}{4}\left[A_{\mu;\alpha}, A_{\rho';\sigma'}\right]_+ + \tfrac{1}{4}\left[A_{\mu';\alpha'}, A_{\rho;\sigma}\right]_+\right\},
\end{aligned} \tag{6.60}
$$

where $[\ ,\ ]_+$ is the anti-commutator. On the last line, we can pass from the classical fields to the quantum operators; then, evaluating (6.60) between $\langle$out, vac and $|\mathrm{in}, \mathrm{vac}\rangle$ and recalling that, up to a numerical factor, the matrix element of the anticommutator function is the Hadamard Green's function, one obtains[1]

$$A_{\mu;\alpha} A_{\rho;\sigma} \mapsto \lim_{x' \to x} \left\{ \frac{1}{4} G^{(H)}_{\mu\rho';\alpha\sigma'} + \frac{1}{4} G^{(H)}_{\rho\mu';\sigma\alpha'} \right\}, \tag{6.61}$$

where, in order to avoid any confusion with the Feynman Green function in the minimal case (ξ=1), the Hadamard Green function shall be named $G^{(H)}_{\mu\nu'}$. In a similar manner

$$\begin{aligned} A_{\mu;\nu\rho} A_\sigma &= \frac{1}{4}[A_{\mu;\nu\rho}, A_\sigma]_+ + \frac{1}{4}[A_{\mu';\nu'\rho'}, A_\sigma]_+ \\ &\mapsto \lim_{x' \to x} \left\{ \frac{1}{4} G^{(H)}_{\mu\sigma';\nu\rho} + \frac{1}{4} G^{(H)}_{\sigma\mu';\nu'\rho'} \right\} \end{aligned} \tag{6.62}$$

and

$$\chi_{;\mu} \psi_{;\nu} \mapsto \lim_{x' \to x} \left\{ \frac{1}{4} G^{(H)}_{;\mu\nu'} + \frac{1}{4} G^{(H)}_{;\nu\mu'} \right\}, \tag{6.63}$$

where $G^{(H)}(x, x')$ is the Hadamard Green's function for the ghost fields.

Hence the point-split method yields

$$\begin{aligned} \langle T^{\alpha\beta} \rangle^{\mathrm{matrix}}_{\mathrm{Maxwell}} = \frac{1}{4} \lim_{x' \to x} \Big\{ & g^{\mu\sigma} (g^{\alpha\gamma} g^{\beta\rho} - \frac{1}{4} g^{\alpha\beta} g^{\gamma\rho}) \bullet \\ & \bullet\, (G^{(H)}_{\sigma\mu';\rho\gamma'} + G^{(H)}_{\mu\sigma';\gamma\rho'} + G^{(H)}_{\rho\gamma';\sigma\mu'} + G^{(H)}_{\gamma\rho';\mu\sigma'} \\ & - G^{(H)}_{\rho\mu';\sigma\gamma'} - G^{(H)}_{\mu\rho';\gamma\sigma'} - G^{(H)}_{\sigma\gamma';\rho\mu'} - G^{(H)}_{\gamma\sigma';\mu\rho'}) \Big\}, \end{aligned} \tag{6.64}$$

$$\begin{aligned} \langle T^{\alpha\beta} \rangle^{\mathrm{matrix}}_{\mathrm{gauge}} = \lim_{x' \to x} \Big\{ & -\frac{1}{4} g^{\mu\rho} E^{\alpha\beta\nu\sigma} (G^{(H)}_{\rho\sigma';\mu\nu} + G^{(H)}_{\rho\sigma';\mu'\nu'}) \\ & + \frac{1}{8} g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} (G^{(H)}_{\rho\sigma';\mu\nu'} + G^{(H)}_{\sigma\rho';\nu\mu'}) \Big\}, \end{aligned} \tag{6.65}$$

$$\langle T^{\alpha\beta} \rangle^{\mathrm{matrix}}_{\mathrm{ghost}} = -\frac{1}{4} \lim_{x' \to x} \Big\{ E^{\alpha\beta\mu\nu} (G^{(H)}_{;\mu\nu'} + G^{(H)}_{;\nu\mu'}) \Big\}, \tag{6.66}$$

where the DeWitt supermetric has been introduced:

$$E^{\mu\nu\rho\tau} \equiv g^{\mu\rho} g^{\nu\tau} + g^{\mu\tau} g^{\nu\rho} - g^{\mu\nu} g^{\rho\tau}. \tag{6.67}$$

We should now specify in which order the various operations we rely upon are performed. Indeed, in the evaluation of the Feynman Green function, we first sum over n and then take the $s \to 0$ limit. Here, we eventually obtain

[1]It should be pointed out that this is an abuse of notation: given for example two 2-point tensors B, C, whose components are $B_{\mu\nu'}$, $C_{\mu\nu'}$, respectively, you can only sum the homologous components pertaining to the same point: in fact

$$B_{\mu\nu'} + C_{\nu\mu'}$$

does *not* define a 2-point tensor; therefore, throughtout the following,

$$\lim_{x' \to x} \{B_{\mu\nu'} + C_{\nu\mu'}\}$$

stands for

$$\left(\lim_{x' \to x} B_{\mu\nu'} \right) + \left(\lim_{x' \to x} C_{\mu\nu'} \right)$$

which is, of course, a well defined tensor at x.

the energy-momentum tensor of the quantum theory according to the point-splitting procedure, *with the understanding that the coincidence limit* $\lim_{x'\to x}$ *is the last operation to be performed.*

It is clear from (6.64), (6.65), (6.66) that our analysis of the stress-energy tensor is virtually completed if we can provide a closed expression for the coincidence limit of the second derivatives of the Hadamard Green function. It is easy to see that divergences appear; in the minimal case ($\xi = 1$), the divergent part of $G^{(H)}_{\gamma\beta';\rho\tau'}$ is of the form (Bimonte, Calloni, Di Fiore, Esposito, Milano, Rosa [3])

$$\lim_{\epsilon\to 0}\left\{\Gamma(\epsilon)([b_{1\ \gamma\beta';\rho\tau'}] - \tfrac{1}{6}[b_{1\ \gamma\beta'}]R_{\rho\tau} - \tfrac{1}{2}\Gamma(\epsilon-1)[b_{2\ \gamma\beta'}]g_{\rho\tau}\right\}, \tag{6.68}$$

with the convention that the coincidence limit $\lim_{x'\to x}$ has to be taken for the quantities in square brackets.

In many applications, we are interested in finding

$$\langle T^{\alpha\beta}\rangle^{\rm vac} \equiv \langle {\rm in}, {\rm vac}|T^{\alpha\beta}|{\rm in}, {\rm vac}\rangle \tag{6.69}$$

the vacuum expectation value of the stress-energy tensor in the vacuum state defined prior to any dynamics in the background gravitational field. This quantity, properly regularized and renormalized, gives us all the information we want about particle production and vacuum polarization. It is the object we choose to use as source in the semiclassical gravitational field equations,

$$G_{\mu\nu} = \langle T_{\mu\nu}\rangle^{\rm vac}, \tag{6.70}$$

when doing a back-reaction problem. So why are $\langle T^{\alpha\beta}\rangle^{\rm matrix}$, the Green functions and their divergences interesting? The answer is the following: DeWitt [9] showed that

$$\langle T^{\alpha\beta}\rangle^{\rm vac} = \langle T^{\alpha\beta}\rangle^{\rm matrix} + \langle T^{\alpha\beta}\rangle^{\rm finite}, \tag{6.71}$$

where $\langle T^{\alpha\beta}\rangle^{\rm finite}$ is zero when there is no particle production, is always finite, and satisfies the conservation equation $\langle T^{\alpha\beta}\rangle^{\rm finite}_{;\beta} = 0$. The divergences appearing in $\langle T^{\alpha\beta}\rangle^{\rm vac}$ and $\langle T^{\alpha\beta}\rangle^{\rm matrix}$ are identical. Regularize $\langle T^{\alpha\beta}\rangle^{\rm matrix}$ and you have regularized $\langle T^{\alpha\beta}\rangle^{\rm vac}$. Regularizing $\langle T^{\alpha\beta}\rangle^{\rm vac}$ gives:

$$\langle T^{\alpha\beta}\rangle^{\rm matrix} = \langle T^{\alpha\beta}\rangle^{\rm div} + \langle T^{\alpha\beta}\rangle^{\rm matrix,ren}, \tag{6.72}$$

where $\langle T^{\alpha\beta}\rangle^{\rm div}$ contains the infinite pieces which we will renormalize away by adding infinite counterterms onto the classical action for the gravitational field and $\langle T^{\alpha\beta}\rangle^{\rm matrix,ren}$ is the remaining finite physical part of the matrix element. Renormalizing $\langle T^{\alpha\beta}\rangle^{\rm div}$ away also gives us a renormalized $\langle T^{\alpha\beta}\rangle^{\rm vac}$

$$\begin{aligned}\langle T^{\alpha\beta}\rangle^{\rm vac,ren} &= \langle T^{\alpha\beta}\rangle^{\rm vac} - \langle T^{\alpha\beta}\rangle^{\rm div} \\ &= \langle T^{\alpha\beta}\rangle^{\rm matrix,ren} + \langle T^{\alpha\beta}\rangle^{\rm finite},\end{aligned} \tag{6.73}$$

to be used as the source in (6.70).

Another important fact is the close link between Feynman (and Hadamard) Green's functions and the so-called effective action: DeWitt showed that

$$\langle T^{\alpha\beta}\rangle^{\rm matrix} = 2|{\rm g}|^{-1/2}\frac{\delta W_{\rm eff}}{\delta g_{\alpha\beta}}, \tag{6.74}$$

where

$$W_{\rm eff} = -i \log\langle {\rm out}, {\rm vac}|{\rm in}, {\rm vac}\rangle. \tag{6.75}$$

Having defined

$$W_{\rm eff} = \int d^4x \; L_{\rm eff}, \tag{6.76}$$

he also found that

$$L_{\rm eff} = {\rm Im} \lim_{x'\to x} {\rm tr} \frac{\partial}{\partial\sigma} \left(|{\rm g}|^{1/4}(x) G(x,x') |{\rm g}|^{1/4}(x') \right). \tag{6.77}$$

Thus the renormalized effective langrangian is

$$L_{\rm eff,ren} = L_{\rm eff} - L_{\rm div}, \tag{6.78}$$

where $L_{\rm div}$ can be evaluated by means of the divergent part of the asymptotic expansion of Feynman and Hadamard Green's functions.

Interestingly, the divergent part of the one-loop effective action for the quantum version of Einstein gravity has been recently discussed by Giacchini et al. [17] in relation to renormalization group equations for the Newton constant and the cosmological constant; the reader may find there an up-todate discussion of the concepts just introduced in our chapter.

6.5 $\langle T^{\alpha\beta}\rangle^{\rm div}$ Calculations

In this section we will show the results of the calculation pertaining to the divergent part of the coincidence limit of the Hadamard Green function and their application to the evaluation of the divergent part of $\langle T^{\alpha\beta}\rangle$; the divergent part of the Hadamard Green function is (see (6.68)):

$$\left[G^{(H)}_{\gamma\beta';\rho\tau'}\right]^{\rm div} = \lim_{\epsilon\to 0} \left\{ \Gamma(\epsilon)([b_{1\;\gamma\beta';\rho\tau'}] - \tfrac{1}{6}[b_{1\;\gamma\beta'}]R_{\rho\tau} - \tfrac{1}{2}\Gamma(\epsilon-1)[b_{2\;\gamma\beta'}]g_{\rho\tau} \right\}. \tag{6.79}$$

Thus, defining

$$S_{\lambda\mu\nu\rho} \equiv -\tfrac{1}{3}(R_{\lambda\nu\mu\rho} + R_{\lambda\rho\mu\nu}), \tag{6.80}$$

using

$$\Gamma(\epsilon - k) = \frac{1}{\epsilon}\frac{(-1)^k}{k!} + O(1), \quad \text{for } k = 0,1,2,... \Longrightarrow \begin{cases} \Gamma(\epsilon) = \frac{1}{\epsilon} + O(1), \\ \Gamma(\epsilon-1) = -\frac{1}{\epsilon} + O(1) \end{cases} \tag{6.81}$$

and the following coincidence limits (see Appendices C, D for the derivation of some of these and Synge [32], [31], Christensen [6], Birrell and Davies [4]):[2]

$$[b_{1\ \mu\nu';\alpha\beta'}] = -[b_{1\ \mu\nu';\alpha\beta}] + [b_{1\ \mu\nu';\alpha}]_{;\beta}, \tag{6.82}$$

$$[b_{1\ \mu\nu'}] = \frac{1}{6}Rg_{\mu\nu} - R_{\mu\nu}, \tag{6.83}$$

$$\begin{aligned}[b_{2\ \mu\nu'}] = &-\frac{1}{6}RR_{\mu\nu} - \frac{1}{6}\Box R_{\mu\nu} + \frac{1}{2}R_{\mu\rho}R^{\rho}_{\nu} \\ &-\frac{1}{12}R^{\lambda\sigma\rho}{}_{\nu}R_{\lambda\sigma\lambda\psi} + (\frac{1}{72}R^2 + \frac{1}{30}\Box R \\ &-\frac{1}{180}R^{\rho\sigma}R_{\rho\sigma} + \frac{1}{180}R^{\rho\sigma\lambda\psi}R_{\rho\sigma\lambda\psi})g_{\mu\nu},\end{aligned} \tag{6.84}$$

$$[b_{1\ \mu\nu';\rho}] = \frac{1}{24}(R_{\ \rho} + 2R^{\lambda}_{\rho;\lambda})g_{\mu\nu} - \frac{1}{2}R_{\mu\nu;\rho} - \frac{1}{6}R_{\mu\nu}{}^{\psi}{}_{\rho;\psi}, \tag{6.85}$$

$$\begin{aligned}[b_{1\ \mu\nu';\rho\omega}] = &-\frac{1}{3}[b_{1\ \lambda\nu'}]R_{\mu}{}^{\lambda}{}_{\rho\omega} + \frac{1}{3}\{-\frac{1}{36}Rg_{\mu\nu}R_{\rho\omega} - R_{\mu\nu;\rho\omega} \\ &+\frac{1}{2}R^{\lambda}_{\mu}R_{\lambda\nu\rho\omega} - \frac{1}{12}RR_{\mu\nu\rho\omega} - \frac{1}{6}R^{\lambda}_{\rho}R_{\mu\nu\lambda\omega} \\ &-\frac{1}{6}R^{\lambda}_{\omega}R_{\mu\nu\lambda\rho} + g_{\mu\nu}g^{\lambda\psi}[\sqrt{\Delta}_{;\lambda\psi\rho\omega}] + g^{\lambda\psi}[g_{\mu\nu';\lambda\psi\rho\omega}]\},\end{aligned} \tag{6.86}$$

$$\begin{aligned}[\sqrt{\Delta}_{;\alpha\beta\gamma\delta}] = &-\frac{1}{8}\{[\sigma^{;\rho}{}_{\rho\alpha\beta\gamma\delta}] \\ &-\frac{1}{3}(R_{\alpha\rho}R^{\rho}{}_{\beta\gamma\delta} + R_{\beta\rho}R^{\rho}{}_{\alpha\gamma\delta} + R_{\gamma\rho}R^{\rho}{}_{\alpha\beta\delta} + R_{\delta\rho}R^{\rho}{}_{\alpha\beta\gamma}) \\ &+\frac{1}{3}(R_{\alpha\rho}S^{\rho}{}_{\beta\gamma\delta} + R_{\beta\rho}S^{\rho}{}_{\alpha\gamma\delta} + R_{\gamma\rho}S^{\rho}{}_{\alpha\beta\delta} + R_{\delta\rho}S^{\rho}{}_{\alpha\beta\gamma}) \\ &-\frac{2}{9}(R_{\alpha\beta}R_{\gamma\delta} + R_{\alpha\gamma}R_{\beta\delta} + R_{\alpha\delta}R_{\beta\gamma})\},\end{aligned} \tag{6.87}$$

$$\begin{aligned}[g_{\alpha\beta';\mu\nu\sigma\tau}] = &-\frac{1}{4}(R_{\alpha\beta\mu\nu;\sigma\tau} + R_{\alpha\beta\mu\sigma;\nu\tau} + R_{\alpha\beta\mu\tau;\nu\sigma}) \\ &\frac{1}{8}(R_{\alpha\beta\rho\tau}S^{\rho}{}_{\mu\nu\sigma} + R_{\alpha\beta\rho\sigma}S^{\rho}{}_{\mu\nu\tau} + R_{\alpha\beta\rho\nu}S^{\rho}{}_{\mu\sigma\tau} + \\ &+R_{\alpha\beta\rho\mu}S^{\rho}{}_{\nu\sigma\tau}) \\ &-\frac{1}{8}(R^{\rho}{}_{\beta\sigma\tau}R_{\rho\alpha\mu\nu} + R^{\rho}{}_{\beta\mu\nu}R_{\rho\alpha\sigma\tau} + R_{\alpha\beta\rho\tau}R^{\rho}{}_{\mu\nu\sigma} \\ &+R_{\alpha\beta\rho\sigma}R^{\rho}{}_{\mu\nu\tau} + R^{\rho}{}_{\beta\mu\tau}R_{\rho\alpha\nu\sigma} + R^{\rho}{}_{\beta\nu\sigma}R_{\rho\alpha\mu\tau} \\ &+R^{\rho}{}_{\beta\nu\tau}R_{\rho\alpha\mu\sigma} + R^{\rho}{}_{\beta\mu\sigma}R_{\rho\alpha\nu\tau} + R_{\alpha\beta\rho\nu}R^{\rho}{}_{\mu\sigma\tau} \\ &+R_{\alpha\beta\mu\rho}R^{\rho}{}_{\nu\sigma\tau}),\end{aligned} \tag{6.88}$$

$$\begin{aligned}[\sigma^{;\rho}{}_{\rho\alpha\beta\gamma\delta}] = &-\frac{2}{5}(R_{\alpha\beta;\gamma\delta} + R_{\alpha\gamma;\beta\delta} + R_{\alpha\delta;\beta\gamma} \\ &+R_{\beta\gamma;\alpha\delta} + R_{\beta\delta;\alpha\gamma} + R_{\gamma\delta;\alpha\beta}) + \\ &-\frac{1}{5}(R^{\rho}_{\alpha}S_{\rho\beta\gamma\delta} + R^{\rho}_{\beta}S_{\rho\alpha\gamma\delta} + R^{\rho}_{\gamma}S_{\rho\alpha\beta\delta} \\ &+R^{\rho}_{\delta}S_{\rho\alpha\beta\gamma}) - \frac{2}{15}(R_{\delta\rho}R^{\rho}{}_{\alpha\beta\gamma} + R_{\gamma\rho}R^{\rho}{}_{\alpha\beta\delta} \\ &-\frac{2}{5}(S^{\rho\tau}{}_{\alpha\beta}S_{\rho\tau\gamma\delta} + S^{\rho\tau}{}_{\alpha\gamma}S_{\rho\tau\beta\delta} + S^{\rho\tau}{}_{\alpha\delta}S_{\rho\tau\beta\gamma}) \\ &-\frac{1}{15}(S^{\rho}{}_{\alpha\beta\gamma}R_{\rho\delta} + S^{\rho}{}_{\alpha\beta\delta}R_{\rho\gamma} + S^{\rho}{}_{\alpha\gamma\delta}R_{\rho\beta} \\ &S^{\rho}{}_{\beta\gamma\delta}R_{\rho\alpha}),\end{aligned} \tag{6.89}$$

[2]The author's thanks go to Dr S. M. Christensen for his kind help in the derivation of eqs. (6.88), (6.89).

one obtains

$$
\begin{aligned}
\left[G^{(H)}_{\gamma\beta';\rho\tau'}\right]^{\rm div} &= \frac{1}{\epsilon}\bullet \\
\bullet\quad & [g_{\beta\gamma}\, g_{\rho\tau}(-\tfrac{1}{360}R_{\alpha_1\alpha_2}R^{\alpha_1\alpha_2}+\tfrac{1}{144}R^2 \\
&+\tfrac{1}{360}R_{\alpha_1\alpha_2\alpha_3\alpha_4}R^{\alpha_1\alpha_2\alpha_3\alpha_4}+\tfrac{1}{60}\Box R) \\
&+g_{\beta\gamma}(-\tfrac{1}{135}R_{\alpha_1\alpha_2\alpha_3\rho}R^{\alpha_1\alpha_2\alpha_3}{}_{\tau}-\tfrac{1}{135}R_{\alpha_1\alpha_2\alpha_3\rho}R^{\alpha_1\alpha_3\alpha_2}{}_{\tau} \\
&-\tfrac{13}{90}R_{\alpha_1\rho}R^{\alpha_1}_{\tau}+\tfrac{1}{40}R_{;\rho\tau}-\tfrac{1}{60}\Box R_{\rho\tau}+\tfrac{1}{18}R\,R_{\rho\tau} \\
&+\tfrac{7}{45}R^{\alpha_1\alpha_2}R_{\alpha_2\rho\alpha_1\tau}+\tfrac{1}{20}R_{\rho\alpha_1;}{}^{\alpha_1}{}_{\tau}-\tfrac{1}{60}R_{\tau\alpha_1;}{}^{\alpha_1}{}_{\rho}) \\
&+g_{\rho\tau}(\tfrac{1}{4}R_{\alpha_1\beta}R^{\alpha_1}_{\gamma}-\tfrac{1}{12}\Box R_{\beta\gamma}-\tfrac{1}{12}R\,R_{\beta\gamma} \\
&-\tfrac{1}{24}R_{\alpha_1\alpha_2\alpha_3\beta}R^{\alpha_1\alpha_2\alpha_3}{}_{\gamma})+\tfrac{1}{6}R_{\beta\gamma}\,R_{\rho\tau} \\
&+\tfrac{1}{12}R^{\alpha_1}{}_{\rho}{}^{\alpha_2}{}_{\beta}R_{\alpha_1\tau\alpha_2\gamma}+\tfrac{1}{12}R^{\alpha_1}{}_{\rho}{}^{\alpha_2}{}_{\gamma}R_{\alpha_1\tau\alpha_2\beta} \\
&-\tfrac{1}{12}R_{\gamma\beta\alpha_1\rho;}{}^{\alpha_1}{}_{\tau}+\tfrac{1}{12}R_{\gamma\beta\alpha_1\tau;}{}^{\alpha_1}{}_{\rho}+\tfrac{1}{3}R^{\alpha_1}_{\beta}R_{\alpha_1\gamma\rho\tau} \\
&-\tfrac{1}{6}R^{\alpha_1}_{\gamma}R_{\alpha_1\beta\rho\tau}-\tfrac{1}{6}R_{\beta\gamma;\rho\tau}-\tfrac{1}{12}R\,R_{\beta\gamma\rho\tau}]. \qquad (6.90)
\end{aligned}
$$

In order to obtain the divergent part of $\langle T^{\alpha\beta}\rangle$, one also needs:

1. Equation (1.142), i.e., a relation between field Green's functions and ghost Green's function; in our case, (1.142) yields the following equations, also known as *quantum Ward identities*:

$$-G^{(H)}_{;\mu} = G^{(H)}{}_{\mu\nu';}{}^{\nu'}, \qquad (6.91)$$
$$-G^{(H)}_{;\nu'} = G^{(H)}{}_{\mu\nu';}{}^{\mu}. \qquad (6.92)$$

   Therefore, taking another covariant derivative, one obtains:

$$-G^{(H)}_{;\mu\rho'} = G^{(H)}{}_{\mu\nu';}{}^{\nu'}{}_{\rho'}, \qquad (6.93)$$
$$-G^{(H)}_{;\nu'\rho} = G^{(H)}{}_{\mu\nu';}{}^{\mu}{}_{\rho}. \qquad (6.94)$$

2. An expression for the divergent part of the coincidence limit of $G^{(H)}_{\rho\sigma';\mu\nu}$ and $G^{(H)}_{\rho\sigma';\mu'\nu'}$ in terms of $G^{(H)}_{\rho\sigma';\mu\nu'}$; they are easily obtained using the parallel displacement matrix:

$$
\begin{aligned}
\left[G^{(H)}_{\rho\sigma';\mu\nu}\right] &= \left[G^{(H)}_{\rho\sigma';\mu\beta'}g^{\beta'}{}_{\nu}\right] \\
&= \left[G^{(H)}_{\rho\sigma';\mu\beta'}\right]\left[g^{\beta'}{}_{\nu}\right] \\
&= \left[G^{(H)}_{\rho\sigma';\mu\beta'}\right]\underline{\delta}^{\beta}_{\nu} \\
&= \left[G^{(H)}_{\rho\sigma';\mu\nu'}\right], \qquad (6.95)
\end{aligned}
$$

$$
\begin{aligned}
\left[G^{(H)}_{\rho\sigma';\mu'\nu'}\right] &= \left[G^{(H)}_{\rho\sigma';\beta\nu'}g^{\beta}{}_{\mu'}\right] \\
&= \left[G^{(H)}_{\rho\sigma';\beta\nu'}\right]\left[g^{\beta}{}_{\mu'}\right] \\
&= \left[G^{(H)}_{\rho\sigma';\beta\nu'}\right]\underline{\delta}^{\beta}_{\mu} \\
&= \left[G^{(H)}_{\rho\sigma';\mu\nu'}\right]. \qquad (6.96)
\end{aligned}
$$

Then the final result is:

$$
\begin{aligned}
\langle T^{\mu\nu}\rangle^{\rm div} \;=\; & \frac{1}{\epsilon}\bullet \\
\bullet\; & [g^{\mu\nu}(-\tfrac{17}{45}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{7}{144}R^2 + \tfrac{1}{24}R_{\alpha\beta\gamma\delta}R^{\alpha\gamma\beta\delta} \\
& \quad -\tfrac{7}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \tfrac{7}{120}\Box R + \tfrac{1}{12}R_{;\alpha\beta}{}^{\beta\alpha}) \\
& +g^{\mu\alpha}g^{\nu\beta}(\tfrac{29}{90}R^{\gamma\delta}R_{\gamma\alpha\delta\beta} + \tfrac{127}{1080}R^{\gamma\delta\zeta}{}_{\alpha}R_{\gamma\delta\zeta\beta} \\
& \quad -\tfrac{49}{540}R^{\gamma\delta\zeta}{}_{\alpha}R_{\gamma\zeta\delta\beta} + \tfrac{17}{90}R_{\alpha\gamma}R^{\gamma}_{\beta} \\
& \quad +\tfrac{1}{60}(R_{\alpha\gamma;}{}^{\gamma}{}_{\beta} + R_{\beta\gamma;}{}^{\gamma}{}_{\alpha}) \\
& \quad +\tfrac{1}{24}(R_{\alpha\gamma;\beta}{}^{\gamma} + R_{\beta\gamma;\alpha}{}^{\gamma}) \\
& \quad -\tfrac{7}{120}R_{;\alpha\beta} + \tfrac{1}{18}R\,R_{\alpha\beta} - \tfrac{11}{60}\Box R_{\alpha\beta})].
\end{aligned} \tag{6.97}
$$

Although the expressions for the divergent part of $\left[G^{(H)}_{\gamma\beta';\rho\tau'}\right]$ and $\langle T^{\mu\nu}\rangle^{\rm div}$ may seem cumbersome, a more careful inspection shows that they result from a very large number of terms: by looking at the coincidence limits eqs. (6.82)-(6.89), it is easy to see that each divergent part of $\left[G^{(H)}_{\gamma\beta';\rho\tau'}\right]$ is obtained from the sum of eighty contractions variously involving the metric tensor, the Ricci and Riemann tensor, and that $\langle T^{\mu\nu}\rangle^{\rm div}$ is obtained by summing thirty of these objects; therefore expression (6.97) for $\langle T^{\mu\nu}\rangle^{\rm div}$ is obtained from a careful handling of more than two thousand terms. For this purpose, a program has been written in FORM, which is a symbolic manipulation system whose original author is Jos Vermaseren at NIKHEF (see Heck [21], Vermaseren, Kaneko, Kuipers, Ruijl, Tentyukov, Ueda and Vollinga [33] and the courses [1]).

Conclusions

In this work a powerful formalism for gauge field theories has been introduced and has been used to obtain a manifesly covariant quantization of such theories, even in curved space-time; then special attention has been paid to the evaluation of propagators for scalar field theory and Maxwell's theory and, through the point-splitting method, the stress-energy tensor for Maxwell's theory has been derived in terms of second derivatives of the Hadamard Green function of the electromagnetic field. Last, an original computation has been presented: a concise, explicit formula for the divergent part of the stress-energy tensor: it was obtained from a careful handling of more than two thousand terms; for this purpose, a program has been written in FORM, which is a symbolic manipulation system whose original author is Jos Vermaseren at NIKHEF.

The results obtained are interesting in the context of effective action in curved space-time, whose divergent part is essential to discuss renormalization group equations for the Newton constant and the cosmological constant; moreover, they can be used to obtain a proper source in the semiclassical gravitational field equations, when doing a back reaction problem, for every background gravitational field.

It would be of great interest to further generalize these results and study the case where the gauge group is no longer Abelian, i.e., $SU(N)$-Yang-Mills theories with $N \neq 1$, or the dynamical fields are no longer bosonic, i.e., fermionic and supersymmetric field theories. Nevertheless, in the author's opinion, the most fascinating element of investigation would be to frame the present work in a more general context of quantum field theory in curved space-time: it is a realm where the idea of particles as unitary irreducible representations of the Poincaré group (cf. the work in refs. [15], [16]) ceases to exist (together with the possibility to place them in the momentum space), since the isometry group for the dynamical gravitational field is the diffeomorphism group; progress can be made in this direction deepening the understanding of the infinite-dimensional manifolds, which arise quite naturally when dealing with quantum field theory and its interaction with gravitation.

Appendix A

Superanalysis

A.1 Supernumbers

Here the basic ideas behind supernumbers and superanalysis will be introduced; we refer to [10], [11] for a detailed discussion and for the introduction of supervector spaces, dual supervector space, supermanifolds and super Hilbert spaces.

Let ζ^a, $a = 1, ..., N$, be a set of generators for an algebra, which anticommute:

$$\zeta^a \zeta^b = -\zeta^b \zeta^a, \qquad (\zeta^a)^2 = 0, \qquad \forall a, b. \tag{A.1}$$

The algebra is called a *Grassmann algebra* and denoted by Λ_N. In this work the formal limit $N \to \infty$ is always taken, and elements of the algebra, when expressed in terms of the ζ^a, are formal series.

The elements of Λ_∞ are called *supernumbers*. Every supernumber can be expressed in the form

$$z = z_B + z_S, \tag{A.2}$$

where z_B is an ordinary complex number and

$$z_S = \sum_{n=1}^{\infty} \frac{1}{n!} c_{a_1 \ldots a_n} \zeta^{a_n} ... \zeta^{a_1} \tag{A.3}$$

the c's also being complex numbers. The c's are completely antisymmetric in their indices, and summation over repeated indices is to be understood, z_B is called the body and z_S the soul of z. If $\zeta^a z = 0$ for all a, then $z = 0$.

Every analytic function on the complex numbers may be extended to a supernumber-valued function on Λ_∞ by the formal series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_B) z_S^n. \tag{A.4}$$

Let $\pi : \Lambda_\infty \to \mathcal{C}$ be the mapping that replaces each supernumber by its body, and let z_0 be a singular point of $f\big|_{\pi(\Lambda_\infty)} = \mathcal{C}$. Then every element of $\pi^{-1}(z_0)$ is a singular point of f.

One may consider matrices whose elements are supernumbers. The body or a matrix is then defined as the ordinary matrix obtained by replacing each element with its body. The soul of the matrix is the remainder. A square matrix has an inverse if and only if its body is nonsingular. The inverse is unique and is expressible as

$$M^{-1} = M_B^{-1} - M_B^{-1} M_S M_B^{-1} + M_B^{-1} M_S M_B^{-1} M_S M_B^{-1} - ... \tag{A.5}$$

Every supernumber may be split into its even and odd parts:

$$z = u + v,$$
$$u = z_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{a_1 ... a_{2n}} \zeta^{a_{2n}} ... \zeta^{a_1}, \tag{A.6}$$
$$v = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{a_1 ... a_{2n+1}} \zeta^{a_{2n+1}} ... \zeta^{a_1}. \tag{A.7}$$

Supernumbers that are either purely odd or purely even are called *pure*. Odd supernumbers anticommute among themselves and are called *a-numbers*. Even supernumbers commute with everything and are called *c-numbers*. The set of all c-numbers is a commutative sub-algebra of Λ_∞, denoted by $\mathcal{C}_c$. The set of all a-numbers is denoted by $\mathcal{C}_a$; it is not a subalgebra. The square of every a-number vanishes.

A.2 Superanalytic Functions

Let $f : \mathcal{C}_a \to \Lambda_\infty$ be a supernumber-valued function on $\mathcal{C}_a$. f is called *superanalytic* if it satisfies the following condition: Let v be an arbitrary element of $\mathcal{C}_a$. Let v be given an arbitrary infinitesimal a-number displacement dv. Then its image $f(v)$ in Λ_∞ suffers a displacement which, for all dv, takes the form

$$df(v) = dv \left[\frac{\overrightarrow{d}}{dv} f(v) \right] = \left[f(v) \frac{\overleftarrow{d}}{dv} \right] dv, \tag{A.8}$$

where the coefficients $\overrightarrow{d}/dv f(v)$ and $f(v)\overleftarrow{d}/dv$ are independent of dv and depend (at most) only on v. These coefficients are called respectively the left and right derivatives of f with respect to v.

It can be shown that the general solution of eq. (A.8) is

$$f(v) = a + bv, \qquad a, b \in \Lambda_\infty. \tag{A.9}$$

That is, a superanalytic function of an a-number variable is simply a linear function. It is therefore superanalytic everywhere in $\mathcal{C}_a$ (no singularities). If the coefficient b in (A.9) is pure, then

$$\frac{\overrightarrow{d}}{dv} f(v) = (-1)^b b, \qquad f(v) \frac{\overleftarrow{d}}{dv} = b, \tag{A.10}$$

where a symbol appearing in an exponent of -1 is to be understood as taking the value 0 or 1 according as it is c-type or a-type. Superanalytic functions $f : \mathcal{C}_c \to \Lambda_\infty$ on $\mathcal{C}_c$ are defined similarly:

$$df(u) = du \left[\frac{\overrightarrow{d}}{du} f(u) \right] = \left[f(u) \frac{\overleftarrow{d}}{du} \right] du, \tag{A.11}$$

but here the similarity ends. Since du is a c-number, it follows that

$$\frac{\overrightarrow{d}}{du} f(u) = f(u) \frac{\overleftarrow{d}}{du}. \tag{A.12}$$

Then there is no need to distinguish between left and right derivatives. Moreover, the class of superanalytic functions on $\mathcal{C}_c$ is infinitely richer than the class of superanalytic functions on $\mathcal{C}_a$. The general solution of eq. (A.12) is

$$f(u) = \sum_{n=0}^{\infty} \frac{1}{n!} f_{a_1 \dots a_n}(u) \zeta^{a_n} \dots \zeta^{a_1}. \tag{A.13}$$

where the $f_{a_1 \dots a_n}$ are extensions, as in (A.4), of analytic functions of an ordinary complex variable.

A.3 Functions of Real Variables

To define real supernumbers one extends the rules of complex conjugation (denoted by *) from $\mathcal{C}$ to Λ_∞ by adding the relations

$$\zeta^{a*} = \zeta^a \quad \text{for all } a, \tag{A.14}$$

$$(z + z')^* = z^* + z'^*, \quad (zz')^* = z'^* z^* \quad \text{forall} z, z' \in \Lambda_\infty. \tag{A.15}$$

z is said to be real if $z^* = z$, imaginary if $z^* = -z$, and complex otherwise.

The sets of real elements of $\mathcal{C}_c$ and $\mathcal{C}_a$ are denoted by $\mathcal{R}_c$ and $\mathcal{R}_a$, respectively. The product of two real c-numbers is a real c-number. The product of a real c-number and a real a-number is a real a-number. The product of two real a-numbers is an imaginary bodiless c-number.

A function from $\mathcal{R}_c$ to Λ_∞ need not be the restriction to $\mathcal{R}_c$ of a superanalytic function on $\mathcal{C}_c$ to be differentiable in the sense of eq. (A.12) with du restricted to $\mathcal{R}_c$. Any C^∞ function on the real line $\mathcal{R}$ can be extended to a C^∞ function on $\mathcal{R}_c$ by the following analog of (A.4):

$$f(x) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_B) x_S^n, \quad x \in \mathcal{R}_c. \tag{A.16}$$

Differentiable functions on $\mathcal{R}_a$ are all restricted, as in eq. (A.9), to have the linear form

$$f(x) = a + bx, \quad a, b \in \Lambda_\infty, \quad x \in \mathcal{R}_a. \tag{A.17}$$

No functions on $\mathcal{R}_a$ other than those having this form will be considered in this work.

A.4 Integration

The theory of integration may be generalized from the ordinary complex plane $\mathcal{C}$ to $\mathcal{C}_c$. If f is a superanalytic function on $\mathcal{C}_c$, then line integrals $\int_C f(z)dz$, $C \subset \mathcal{C}_c$, depend only on the endpoints of the curves C and on the homotopic relations of these curves to the various singularities of f (note that the singularities all have the form $\pi^{-1}(z_0) \cap \mathcal{C}_c$, where z_0 is a singularity of f in $\mathcal{C}$). An integral

$\oint_C f(z)dz$ over a closed contour intersecting no singularities is, as in ordinary analysis, equal to $2\pi i$ times the sum of the residues at those poles that are trapped by the contour. If f has the general form (A.13), then the residues may be arbitrary supernumbers.

Integration on the real line may be similarly generalized to integrals over curves in $\mathcal{R}_c$. If f is a smooth function on $\mathcal{R}_c$ (it does not have to be analytic), then the line integral $\int_C f(x)dx, C \subset \mathcal{R}_c$, again depends only on the endpoints of the curve C.

To obtain a useful integration theory over $\mathcal{R}_a$ one cannot use measure theoretical notions and hence one proceedes purely formally. This is because line integrals of functions of the form (A.9) on $\mathcal{C}_a$, or (A.17) on $\mathcal{R}_a$, depend on their contours, not merely on the endpoints, even though there are no singularities.

Since (A.17) is linear in x, to give meaning to the symbol $\int f(x)dx$ one has only to decide what meaning to give to the symbols $\int dx$ and $\int xdx$. The integrals of all other differentiable functions are then determined by the rules

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx, \tag{A.18}$$

$$\int af(x)dx = a \int f(x)dx, \qquad a \in \Lambda_\infty. \tag{A.19}$$

In choosing the basic integrals it proves fruitful to be guided by analogy with the equation

$$\int \frac{d}{dx} f(x)dx = 0, \tag{A.20}$$

which holds for smooth functions on $\mathcal{R}_c$ satisfying $f(\pm\infty) = 0$, if the integral itself is taken between $-\infty$ and $+\infty$. If one requires (A.20) to hold also over $\mathcal{R}_a$, then one must necessarily have

$$\int dx = 0 \tag{A.21}$$

while $\int xdx$ must be set equal to some constant supenumber. These rules were first introduced by Berezin [2] who set $\int xdx = 1$. It proves to be somewhat more elegant to set

$$\int xdx = (2\pi i)^{-1/2}. \tag{A.22}$$

Equations (A.18) to (A.22) together imply the law of shifting the integration variable and the law of integration by parts:

$$\int f(x+a)dx = \int f(x)dx, \tag{A.23}$$

$$\int f(x)\frac{\overrightarrow{d}}{dx}g(x)dx = \int f(x)\frac{\overleftarrow{d}}{dx}g(x). \tag{A.24}$$

None of the integrals (A.18) to (A.24), when taken over $\mathcal{R}_a$, is to be understood as associated with a contour in $\mathcal{R}_a$. The symbols $\int ...dx$ are thus purely formal. The "volume element" dx, however, will be treated as if it were an a-number in the integrand. Hence $\int dx = -\int dxx$ and, in multiple integrals, $\int f(x,y)dxdy = -f(x,y)dydx$. Note that when one is dealing with (linear) functions of a single a-number variable, Berezin's rules appear fairly trivial. They become less so when one is dealing with (multilinear) functions of several a-number variables.

A.5 Matrices and Shifting Indices: Supertranspose

Consider a matrix whose elements are supernumbers, with the following block form

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.25}$$

where the elements of the square submatrices A(m-dimensional) and B(n-dimensional) are c-numbers and the elements of the square submatrices C and D are a-numbers. If we denote the elements of the matrix K by $\left({}_jK^i\right)$, we can estabilish the following way to shift its indices:

$${}^iK^{\sim}{}_j \equiv (-1)^{j(i+1)}{}_jK^i. \tag{A.26}$$

The matrix $K^{\sim} = \left({}^iK^{\sim}{}_j\right)$ is called the supertranspose of K. The supertranspose obeys the following standard rules:

$$(K_1K_2)^{\sim} = K_2^{\sim}K_1^{\sim}, \tag{A.27}$$

$$K^{-1\sim} = K^{\sim -1}. \tag{A.28}$$

In the theory of supervector spaces one encounters four types of matrices, one type possessing indices positioned as on the matrix K above, and three other types exemplified by matrices L, M, and N with indices positioned as follows:

$${}^jL_i, \quad {}_jM_i, \quad {}^jN^i. \tag{A.29}$$

The supertransposition rules for these matrices are

$${}_iL^{\sim j} = (-1)^{i(j+1)}\,{}^jL_i, \tag{A.30}$$

$${}_iM^{\sim}{}_j = (-1)^{i+j+ij}\,{}_jM_i, \tag{A.31}$$

$${}^iN^{\sim j} = (-1)^{ij}\,{}^jN^i. \tag{A.32}$$

These rules make it possible to write

$$K^{\sim\sim} = K, \quad L^{\sim\sim} = L, \quad M^{\sim\sim} = M, \quad N^{\sim\sim} = N, \tag{A.33}$$

$$(L_1L_2)^{\sim} = L_2^{\sim}L_1^{\sim}, \qquad (MN)^{\sim} = N^{\sim}M^{\sim}, \tag{A.34}$$

$$L^{\sim -1} = L^{-1\sim}, \quad M^{\sim -1} = M^{-1\sim}, \quad N^{\sim -1} = N^{-1\sim}. \tag{A.35}$$

Note that inversion leaves the types K and L unchanged but interchanges types M and N. Supertransposition, on the other hand, leaves the types M and N unchanged but interchanges the types K and L. Matrices of types M and N are said to be supersymmetric if $M^{\sim} = M$ and $N^{\sim} = N$. They are said to be antisupersymmetric if $M^{\sim} = -M$ and $N^{\sim} = -N$.

It is also convenient to introduce the symbol $1_{(m,n)}$ to denote the (m,n)-dimensional unit matrix and, in block form, to write

$$1_{(m,n)} = \begin{pmatrix} 1_m & 0 \\ 0 & 1_n \end{pmatrix} \tag{A.36}$$

A.6 The Supertrace and Superdeterminant

For matrices of types K and L, which have one upper index and one lower index, one may introduce the *supertrace*:

$$\begin{aligned} \mathrm{str}K &\equiv (-1)^i{}_i K^i, \\ \mathrm{str}L &\equiv (-1)^{i\,i} L_i. \end{aligned} \tag{A.37}$$

When K or L is expressed in the block form (A.25) the supertrace becomes

$$\mathrm{str}K = \mathrm{tr}A - \mathrm{tr}B, \tag{A.38}$$

where “tr” denotes the ordinary trace.

The definitions (A.37) guarantee that the supertrace is invariant under supertransposition

$$\mathrm{str}K^{\sim} = \mathrm{str}K, \qquad \mathrm{str}L^{\sim} = \mathrm{str}L. \tag{A.39}$$

They also have the more important property of yielding the cyclic invariance laws:

$$\begin{aligned} &\mathrm{str}(K_1K_2) = \mathrm{str}(K_2K_1), \qquad \mathrm{str}(L_1L_2) = \mathrm{str}(L_2L_1), \\ &\mathrm{str}(MN) = \mathrm{str}(NM). \end{aligned} \tag{A.40}$$

The cyclic invariance laws permit one to define a *superdeterminant*, also frequently called *Berezinian*, by integrating the variational law

$$\delta(\log \mathrm{sdet}P) = \mathrm{str}(P^-1\delta P) \tag{A.41}$$

starting from the boundary condition

$$\mathrm{sdet}1_{m,n} \equiv 1. \tag{A.42}$$

Here the matrix P may be of any one of the four types. It can be verified that

$$\begin{aligned} &\mathrm{sdet}K^{\sim} = \mathrm{sdet}K, \qquad \mathrm{sdet}L^{\sim} = \mathrm{sdet}L, \\ &\mathrm{sdet}M^{\sim} = (-1)^n \mathrm{sdet}M \qquad \mathrm{sdet}N^{\sim} = (-1)^n \mathrm{sdet}N \end{aligned} \tag{A.43}$$

$$\mathrm{sdet}(P_1P_2) = \mathrm{sdet}(P_1)\mathrm{sdet}(P_2). \tag{A.44}$$

From the variational law it is easy to compute the superdeterminant in the following simple cases:

$$\mathrm{sdet}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\mathrm{det}A)(\mathrm{det}B)^{-1}, \tag{A.45}$$

$$\mathrm{sdet}\begin{pmatrix} 1_m & X \\ 0 & 1_n \end{pmatrix} = 1, \tag{A.46}$$

$$\mathrm{sdet}\begin{pmatrix} 1_m & 0 \\ Y & 1_n \end{pmatrix} = 1. \tag{A.47}$$

where “det” denotes the ordinary determinant.

A.7 Integration over $\mathcal{R}_c^m \times \mathcal{R}_a^n$

We return now to integration, specifically to multiple integration over $\mathcal{R}_c^m \times \mathcal{R}_a^n$ which, like integration over $\mathcal{R}^m$, can be defined simply by performing sequences of ordinary integrations over $\mathcal{R}_c$ and $\mathcal{R}_a$. Denote the coordinates of a point x of $\mathcal{R}_c^m \times \mathcal{R}_a^n$ by x^i. Latin indices will be understood to range over the values $-n, ..., -1, 1, ..., m$ with the negative values distinguishing the a-number coordinates. One sometimes wishes to focus on the a-number coordinates, or the c-number coordinates, by themselves. In that case Greek indices from the first part of the alphabet will be used to distinguish the a-number coordinates (e.g., x^α, x^β, x^γ, etc.) and Greek indices from the middle of the alphabet to distinguish the c-number coordinates (e.g., x^μ, x^ν, x^ρ, etc.).

The volume element in $\mathcal{R}_c^m \times \mathcal{R}_a^n$ is defined to be

$$d^{m,n}x \equiv i^{n(n-1)/2} dx^1 ... dx^m dx^{-1} ... dx^{-n}, \tag{A.48}$$

the factor $i^{n(n-1)/2}$ being included so as to make $d^{m,n}x$ formally real. A differentiable function f on $\mathcal{R}_c^m \times \mathcal{R}_a^n$ must be linear in each of the a-number coordinates separately (see eq. (A.17)). Therefore it may be expanded as a power series in the x^α terminating at the term of nth degree. This term may be expressed in the form

$$g(x^1, x^2, ..., x^m) x^{-1} ... x^{-n}, \tag{A.49}$$

g being a function of the c-number coordinates only. It is an immediate consequence of the laws (A.21) and (A.22), together with the anticommutativity of the x^α, that

$$\int f(x) d^{m,n}x = (2\pi i)^{-n/2} (-i)^{n(n-1)/2} \int g(x^1, ..., x^m) d^m x, \tag{A.50}$$

where $d^m x$ is the volume element in $\mathcal{R}_c^m$. Integration over the c-number variables proceeds just like ordinary integration.

A.8 Transformation of Variables and Gaussian Integrals

Here we report two important formulae: the first one pertains to transformations of variables, while the second one to the evaluation of Gaussian integrals.

Suppose we perform a transformation of variables in the integral of the form

$$x^i \mapsto \bar{x}^i(x);$$

then the following holds

$$\int \bar{f}(\bar{x}) d^{m,n}\bar{x} = \int f(x) J d^{m,n}x, \tag{A.51}$$

with $J \equiv \mathrm{sdet}({}_{j,}x^i)$.

Let now M be a nonsingular supersymmetric matrix of dimension (m, n), the elements ${}_iM_j$ of which have their indices in the lower position. The supersymmetry relation

$${}_iM_j = (-1)^{i+j+ij} {}_jM_i \tag{A.52}$$

implies that M has the block form

$$M = \begin{pmatrix} A & C \\ -C^{\sim} & B \end{pmatrix}, \tag{A.53}$$

where the nonsingular submatrices A and B have the symmetries

$$A^{\sim} = A, \qquad B^{\sim} = -B. \tag{A.54}$$

The antisymmetry of B implies that n is even. Suppose the elements of A are real c-numbers, the elements of B are imaginary c-numbers, and the elements of C are imaginary c-numbers. Then the quadratic form $x^i{}_i M_j\, x^j$, where the x's are the coordinates in $\mathcal{R}_c^m \times \mathcal{R}_a^n$, is a real c-number.

Consider the imaginary "Gaussian" integral

$$I \equiv \int e^{i\left(x^k{}_k M_j\, x^j\right)}. \tag{A.55}$$

The following result holds

$$I = (2\pi i)^{(m-n)/2} \left(\mathrm{sdet} M\right)^{-1/2}. \tag{A.56}$$

Appendix B

Euler-MacLaurin formula

The Euler-MacLaurin formula (see Hardy [19], Sansone [26], Schwartz [27], Wong [34]) used in chapter VI, section 3 asserts that, if $f : [0, \infty[\to \mathcal{R}$ is a function having even order derivatives which are absolutely integrable in $(0, \infty)$, then, for all $k = 1, 2, ..., \infty$,

$$\sum_{i=0}^{k} f(i) - \int_0^k f(x)dx = \frac{1}{2}[f(0) + f(k)] + \sum_{s=1}^{m-i} \frac{\tilde{B}_{2s}}{(2s)!}[f^{2s-1}(k) - f^{2s-1}(0)] + R_m(k), \tag{B.1}$$

where $\tilde{B}_{2s}$ are the Bernoulli numbers, and $R_m(k)$ is the reminder term, majorized by

$$\left|R_m(k)\right| \leq (2 - 2^{1-m}) \frac{|\tilde{B}_{2s}|}{(2m)!} \int_0^k \left|f^{2m}(x)\right| dx. \tag{B.2}$$

Appendix C

Coefficient Bivectors: Coincidence Limits

We will prove some of the coincidence limits listed in Chapter VI, i.e., eqs. (6.83), (6.84), (6.85), (6.86). Consider the recurrence relation (6.21) for the coefficient bivectors $b_{n\ \mu\nu'}$:

$$\sigma^{;\lambda} b_{n\ \mu\nu';\lambda} + n b_{n\ \mu\nu'} = \frac{1}{\sqrt{\Delta}} \left(\sqrt{\Delta} b_{n-1\ \mu\nu'} \right)_{;\lambda}{}^{\lambda} - R^{\lambda}_{\mu} b_{n-1\ \lambda\nu'}. \tag{C.1}$$

Since $b_{0\ \mu\nu'} = g_{\mu\nu'}$, for $n = 1$ the previous equation is:

$$\begin{aligned} \sigma^{;\lambda} b_{1\ \mu\nu';\lambda} + b_{1\ \mu\nu'} &= \frac{1}{\sqrt{\Delta}} \left(\sqrt{\Delta} g_{\mu\nu'} \right)_{;\lambda}{}^{\lambda} - R^{\lambda}_{\mu} g_{\lambda\nu'} \\ &= \frac{1}{\sqrt{\Delta}} \left(\sqrt{\Delta}_{;\lambda}{}^{\lambda} g_{\mu\nu'} + 2\sqrt{\Delta}_{;\lambda} g_{\mu\nu';}{}^{\lambda} + \sqrt{\Delta} g_{\mu\nu'}{}_{;\lambda}{}^{\lambda} \right) - R^{\lambda}_{\mu} g_{\lambda\nu'}. \end{aligned} \tag{C.2}$$

Taking the coincidence limit, and using (Synge [32])

$$\left[\sigma^{;\lambda}\right] = 0, \tag{C.3}$$
$$\left[\sqrt{\Delta}\right] = 1, \tag{C.4}$$
$$\left[\sqrt{\Delta}_{;\mu}\right] = 0, \tag{C.5}$$
$$\left[\sqrt{\Delta}_{;\mu\nu}\right] = \tfrac{1}{6} R_{\mu\nu}, \tag{C.6}$$
$$\left[g_{\mu\nu'}\right] = g_{\mu\nu}, \tag{C.7}$$
$$\left[g_{\mu\nu';\alpha}\right] = 0, \tag{C.8}$$
$$\left[g_{\mu\nu';\alpha\beta}\right] = -\tfrac{1}{2} R_{\mu\nu\alpha\beta}, \tag{C.9}$$

one obtains

$$\left[b_{1\ \mu\nu'}\right] = \tfrac{1}{6} R g_{\mu\nu} - R_{\mu\nu}, \tag{C.10}$$

which is precisely (6.83).

Taking the first derivative of (C.2), one obtains

$$\begin{aligned}
&\sigma^{\lambda}{}_{\rho} b_{1\ \mu\nu';\lambda} + \sigma^{\lambda} b_{1\ \mu\nu';\lambda\rho} + b_{1\ \mu\nu';\rho} = \\
&= \left(\frac{1}{\sqrt{\Delta}}\right)_{;\rho} \left(\sqrt{\Delta}_{;\lambda}{}^{\lambda} g_{\mu\nu'} + 2\sqrt{\Delta}_{;\lambda} g_{\mu\nu';}{}^{\lambda} + \sqrt{\Delta} g_{\mu\nu';\lambda}{}^{\lambda}\right) \\
&\quad + \frac{1}{\sqrt{\Delta}} \left(\sqrt{\Delta}_{;\lambda}{}^{\lambda} g_{\mu\nu'} + 2\sqrt{\Delta}_{;\lambda} g_{\mu\nu';}{}^{\lambda} + \sqrt{\Delta} g_{\mu\nu';\lambda}{}^{\lambda}\right)_{;\rho} \\
&\quad - R^{\lambda}_{\mu;\rho} g_{\lambda\nu'} - R^{\lambda}_{\mu} g_{\lambda\nu';\rho}.
\end{aligned} \tag{C.11}$$

Taking the coincidence limit, and using (C.3)-(C.9) and

$$\left[\left(\frac{1}{\sqrt{\Delta}}\right)_{;\rho}\right] = -\left[\left(\sqrt{\Delta}\right)^{-2}\right][\sqrt{\Delta}_{;\rho}] = 0, \tag{C.12}$$

$$[\sigma_{;\mu\nu}] = g_{\mu\nu}, \tag{C.13}$$

one obtains

$$2[b_{1\ \mu\nu';\rho}] = \left[\sqrt{\Delta}_{;\lambda}{}^{\lambda}{}_{\rho}\right] g_{\mu\nu} + \left[g_{\mu\nu';\lambda}{}^{\lambda}{}_{\rho}\right] - R_{\mu\nu;\rho}. \tag{C.14}$$

From

$$\left[\sqrt{\Delta}_{;\alpha\beta\gamma}\right] = \tfrac{1}{12}\left(R_{\alpha\beta;\gamma} + R_{\alpha\gamma;\beta} + R_{\beta\gamma;\alpha}\right),$$

$$\left[g_{\mu\nu';\alpha\beta\gamma}\right] = -\tfrac{1}{3}(R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\alpha\gamma;\beta}),$$

one arrives at

$$\left[\sqrt{\Delta}_{;\lambda}{}^{\lambda}{}_{\rho}\right] = \tfrac{1}{12}\left(R_{;\gamma} + 2R_{\lambda\rho;\lambda}\right), \tag{C.15}$$

$$\left[g_{\mu\nu';\lambda}{}^{\lambda}{}_{\rho}\right] = -\tfrac{1}{3} R_{\mu\nu\alpha\gamma}{}^{;\alpha}. \tag{C.16}$$

Then (C.14) reads

$$[b_{1\ \mu\nu';\rho}] = \tfrac{1}{24} g_{\mu\nu}\left(R_{;\gamma} + 2R_{\lambda\rho;\lambda}\right) - \tfrac{1}{6} R_{\mu\nu\alpha\gamma}{}^{;\alpha} - \tfrac{1}{2} R_{\mu\nu;\rho}, \tag{C.17}$$

which is precisely (6.85).

Differentiating again (C.2), one obtains

$$\begin{aligned}
&\sigma_{;}{}^{\lambda}{}_{\rho\omega} b_{1\ \mu\nu';\lambda} + \sigma_{;}{}^{\lambda}{}_{\rho} b_{1\ \mu\nu';\lambda\omega} + \sigma_{;}{}^{\lambda}{}_{\omega} b_{1\ \mu\nu';\lambda\rho} + \sigma_{;}{}^{\lambda} b\ {}_{\mu\nu';\lambda\rho\omega} + b_{1\ \mu\nu';\rho\omega} \\
&= \left(\frac{1}{\sqrt{\Delta}}\right)_{;\rho\omega} \left(\sqrt{\Delta}_{;\lambda}{}^{\lambda} g_{\mu\nu'} + 2\sqrt{\Delta}_{;\lambda} g_{\mu\nu';}{}^{\lambda} + \sqrt{\Delta} g_{\mu\nu';\lambda}{}^{\lambda}\right) \\
&\quad + \left(\frac{1}{\sqrt{\Delta}}\right)_{;\rho} \left(\sqrt{\Delta}_{;\lambda}{}^{\lambda} g_{\mu\nu'} + 2\sqrt{\Delta}_{;\lambda} g_{\mu\nu';}{}^{\lambda} + \sqrt{\Delta} g_{\mu\nu';\lambda}{}^{\lambda}\right)_{;\omega} \\
&\quad + \left(\frac{1}{\sqrt{\Delta}}\right)_{;\omega} \left(\sqrt{\Delta}_{;\lambda}{}^{\lambda} g_{\mu\nu'} + 2\sqrt{\Delta}_{;\lambda} g_{\mu\nu';}{}^{\lambda} + \sqrt{\Delta} g_{\mu\nu';\lambda}{}^{\lambda}\right)_{;\rho} \\
&\quad + \left(\frac{1}{\sqrt{\Delta}}\right) \left(\sqrt{\Delta}_{;\lambda}{}^{\lambda} g_{\mu\nu'} + 2\sqrt{\Delta}_{;\lambda} g_{\mu\nu';}{}^{\lambda} + \sqrt{\Delta} g_{\mu\nu';\lambda}{}^{\lambda}\right)_{;\rho\omega} \\
&\quad - R_{\mu}{}^{\lambda}{}_{;\rho\omega} g_{\lambda\nu'} - R_{\mu}{}^{\lambda}{}_{;\rho} g_{\lambda\nu';\omega} - R_{\mu}{}^{\lambda}{}_{;\omega} g_{\lambda\nu';\rho} - R_{\mu}{}^{\lambda} g_{\lambda\nu';\rho\omega}
\end{aligned} \tag{C.18}$$

Taking the coincidence limit, and using

$$[\sigma_{;\alpha\beta\gamma}] = 0 \tag{C.19}$$
$$\left[\Delta^{-1/2}_{;\alpha\beta}\right] = \left[(-\Delta^{1/2}_{;\alpha\beta}/\Delta) + (2\Delta^{1/2}_{;\alpha}\Delta^{1/2}_{;\beta}/\Delta^{3/2})\right] = -\tfrac{1}{6}R_{\alpha\beta}, \tag{C.20}$$

one arrives at

$$\begin{aligned} &2[b_{1\ \mu\nu';\rho\omega}] + [b_{1\ \mu\nu';\omega\rho}] \\ &= -\tfrac{1}{36}Rg_{\mu\nu}R_{\rho\omega} + \left[\Delta^{1/2}{}_{;\lambda}{}^{\lambda}{}_{\rho\omega}\right]g_{\mu\nu} - \tfrac{1}{12}RR_{\mu\nu\rho\omega} \\ &\quad -\tfrac{1}{6}R^{\lambda}_{\rho}R_{\mu\nu\lambda\omega} - \tfrac{1}{6}R^{\lambda}_{\omega}R_{\mu\nu\lambda\rho} + [g_{\mu\nu';\lambda}{}^{\lambda}{}_{\rho\omega}] \\ &\quad -R_{\mu\nu;\rho\omega} + \tfrac{1}{2}R_{\mu}{}^{\lambda}R_{\lambda\nu\rho\omega}. \end{aligned} \tag{C.21}$$

Now we can exploit the close relation between the commutator of covariant derivatives acting on a (co -)vector and the Riemann tensor:

$$b_{1\ \mu\nu';\rho\omega} - b_{1\ \mu\nu';\omega\rho} = -R_{\mu}{}^{\sigma}{}_{\rho\omega}b_{1\ \sigma\nu'}; \tag{C.22}$$

hence, taking the limit, we have

$$[b_{1\ \mu\nu';\rho\omega}] - [b_{1\ \mu\nu';\omega\rho}] = -R_{\mu}{}^{\sigma}{}_{\rho\omega}[b_{1\ \sigma\nu'}]. \tag{C.23}$$

Inserting this equation in (C.21), one obtains

$$\begin{aligned} 3[b_{1\ \mu\nu';\rho\omega}] &= -R_{\mu}{}^{\sigma}{}_{\rho\omega}[b_{1\ \sigma\nu'}] \\ &\quad -\tfrac{1}{36}Rg_{\mu\nu}R_{\rho\omega} + \left[\Delta^{1/2}{}_{;\lambda}{}^{\lambda}{}_{\rho\omega}\right]g_{\mu\nu} - \tfrac{1}{12}RR_{\mu\nu\rho\omega} \\ &\quad -\tfrac{1}{6}R^{\lambda}_{\rho}R_{\mu\nu\lambda\omega} - \tfrac{1}{6}R^{\lambda}_{\omega}R_{\mu\nu\lambda\rho} + [g_{\mu\nu';\lambda}{}^{\lambda}{}_{\rho\omega}] \\ &\quad -R_{\mu\nu;\rho\omega} + \tfrac{1}{2}R_{\mu}{}^{\lambda}R_{\lambda\nu\rho\omega}, \end{aligned} \tag{C.24}$$

which is precisely (6.86).

For $n = 2$, eq. (6.21) is:

$$\begin{aligned} &\sigma^{;\lambda}b_{2\ \mu\nu';\lambda} + 2b_{2\ \mu\nu'} \\ &= \frac{1}{\sqrt{\Delta}}\left(\sqrt{\Delta}_{;\lambda}{}^{\lambda}b_{1\ \mu\nu'} + 2\sqrt{\Delta}_{;\lambda}b_{1\ \mu\nu';}{}^{\lambda} + \sqrt{\Delta}b_{1\ \mu\nu';\lambda}{}^{\lambda}\right) - R^{\lambda}_{\mu}b_{1\ \lambda\nu'}. \end{aligned} \tag{C.25}$$

Taking the coincidence limit, one obtains

$$\begin{aligned} 2[b_{2\ \mu\nu'}] &= \tfrac{1}{6}R[b_{1\ \mu\nu'}] + [b_{1\ \mu\nu'}]_{;\lambda}{}^{\lambda} - R^{\lambda}_{\mu}[b_{1\ \lambda\nu'}] \\ &= (\tfrac{1}{36}R^2g_{\mu\nu} - \tfrac{1}{6}RR_{\mu\nu}) + \left\{\tfrac{1}{3}g_{\mu\nu}\left(-\tfrac{1}{36}R^2 + \left[\sqrt{\Delta}^{1/2}{}_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}\right]\right)\right. \\ &\quad \left. -\tfrac{1}{3}\Box R_{\mu\nu} + \tfrac{1}{3}\left[g_{\mu\nu';\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}\right]\right\} + (-\tfrac{1}{6}RR_{\mu\nu} + R^{\lambda}_{\mu}R_{\lambda\nu}). \end{aligned} \tag{C.26}$$

Using (6.87), (6.88), one arrives at

$$\left[\sqrt{\Delta}^{1/2}{}_{;\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}\right] = \tfrac{1}{5}\Box R + \tfrac{1}{36}R^2 - \tfrac{1}{30}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{30}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \tag{C.27}$$
$$\left[g_{\mu\nu';\alpha}{}^{\alpha}{}_{\beta}{}^{\beta}\right] = -\tfrac{1}{2}R^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\nu}; \tag{C.28}$$

Hence (C.26) reads

$$2[b_{2\ \mu\nu'}] = R^{\lambda}_{\mu}R_{\lambda\nu} - \frac{1}{3}RR_{\mu\nu} - \frac{1}{3}\Box R_{\mu\nu} - \frac{1}{6}R^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\nu}$$
$$+(\frac{1}{36}R^2 + \frac{1}{15}\Box R - \frac{1}{90}R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{90}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta})g_{\mu\nu}, \qquad (C.29)$$

which is precisely (6.84).

Appendix D

World Function: Coincidence Limits

D.1 General equations satisfied by the covariant derivatives of the world function

Following Synge [31], we will show a method to calculate the coincidence limits of the covariant derivatives of the world function, up to order six. First, one needs to recall the fundamental equations satisfied by the world function:

$$\sigma_{;\mu}\sigma^{;\mu} = 2\sigma, \tag{D.1}$$
$$\sigma_{;\mu\nu\rho} - \sigma_{;\mu\rho\nu} = -R_{\mu\lambda\nu\rho}\sigma^{;\lambda}. \tag{D.2}$$

Differentiating the first equation, and indicating with a subscript that a certain index is free, one obtains

$$\sigma_{;\alpha_1\mu}\sigma^{;\mu} = \sigma_{;\alpha_1}, \tag{D.3}$$
$$\sigma_{;\alpha_1\mu\alpha_2}\sigma^{;\mu} + \sigma_{;\alpha_1\mu}\sigma^{;\mu}{}_{\alpha_2} = \sigma_{;\alpha_1\alpha_2}, \tag{D.4}$$
$$\sigma_{;\alpha_1\mu\alpha_2\alpha_3}\sigma^{;\mu} + \sigma_{;\alpha_1\mu\alpha_2}\sigma^{;\mu}{}_{\alpha_3} + \sigma_{;\alpha_1\mu\alpha_3}\sigma^{;\mu}{}_{\alpha_2} + \sigma_{;\alpha_1\mu}\sigma^{;\mu}{}_{\alpha_2\alpha_3}$$
$$= \sigma_{;\alpha_1\alpha_2\alpha_3}, \tag{D.5}$$

while differentiating the second one, one obtains

$$\sigma_{;\alpha_1\alpha_2\alpha_3\alpha_4} - \sigma_{;\alpha_1\alpha_3\alpha_2\alpha_4} = -R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_4}\sigma^{;\lambda} - R_{\alpha_1\lambda\alpha_2\alpha_3}\sigma^{;\lambda}{}_{\alpha_4}, \tag{D.6}$$

$$\begin{aligned}&\sigma_{;\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} - \sigma_{;\alpha_1\alpha_3\alpha_2\alpha_4\alpha_5}\\ &\quad= -R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_4\alpha_5}\sigma^{;\lambda} - R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_4}\sigma^{;\lambda}{}_{\alpha_5}\\ &\quad-R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_5}\sigma^{;\lambda}{}_{\alpha_4} - R_{\alpha_1\lambda\alpha_2\alpha_3}\sigma^{;\lambda}{}_{\alpha_4\alpha_5},\end{aligned} \tag{D.7}$$

$$\begin{aligned}&\sigma_{;\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} - \sigma_{;\alpha_1\alpha_3\alpha_2\alpha_4\alpha_5\alpha_6}\\ &\quad= -R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_4\alpha_5\alpha_6}\sigma^{;\lambda} - R_{\alpha_1\lambda\alpha_3\alpha_2;\alpha_4\alpha_5}\sigma^{;\lambda}{}_{\alpha_6}\\ &\quad-R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_4\alpha_6}\sigma^{;\lambda}{}_{\alpha_5} - R_{\alpha_1\lambda\alpha_3\alpha_2;\alpha_4}\sigma^{;\lambda}{}_{\alpha_5\alpha_6}\\ &\quad-R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_5\alpha_6}\sigma^{;\lambda}{}_{\alpha_4} - R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_5}\sigma^{;\lambda}{}_{\alpha_4\alpha_6}\\ &\quad-R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_6}\sigma^{;\lambda}{}_{\alpha_4\alpha_5} - R_{\alpha_1\lambda\alpha_2\alpha_3}\sigma^{;\lambda}{}_{\alpha_4\alpha_5\alpha_6}.\end{aligned} \tag{D.8}$$

For the sake of readability, the following notation will be introduced: when an index is not a summation index, it will be indicated with a number, e.g.

$$\sigma_{;\alpha_1\alpha_2\lambda\alpha_3\alpha_4\alpha_5} \equiv \sigma_{;12\lambda345},$$

and the Riemann tensor, with its derivatives, will be indicated through the sequence of its indices between parentheses, e.g.

$$R_{\alpha_1\lambda\alpha_2\alpha_3;\alpha_6} \equiv (1, \lambda, 2, 3; 6).$$

Then the equations which result from differentiations of (D.1) can be put in the form

$$\sigma_{;123\ldots l} = \sum \sigma_{;1\lambda a_2 a_3 \ldots a_k} \sigma^{;\lambda}{}_{a_{k+1}\ldots a_l} \tag{D.9}$$

where the summation extends over all expressions for which $a_2, a_3, ..., a_k$ is a set selected in order from $2, ..., l$ and $a_{k+1}, ..., a_l$ is the remainder of the set $2, ..., l$ in order, and k takes all values from 2 to l. The equations which follow from differentiations on (D.2) take the form

$$\sigma_{;123\ldots l} - \sigma_{;132\ldots l} = -\sum (1, \lambda, 2, 3, b_4, ..., b_k) \sigma^{;\lambda}{}_{b_{k+1}\ldots b_l}, \tag{D.10}$$

where the summation extends over all expressions for which $b_4, ..., b_k$ is a set selected in order from $4, ..., l$ and $b_{k+1}, ..., b_l$ is the remainder of the set $4, ..., l$ in order, and k takes all values from 4 to l.

D.2 Coincidence limits of the covariant derivatives

Dividing (D.3) by σ, one obtains

$$\sigma_{;1\mu} t^{\mu} = t_1, \tag{D.11}$$

where t is the unit tangent vector along the geodesic; then the coincidence limit reads

$$[\sigma_{;12}] = g_{12}. \tag{D.12}$$

From (D.4) and (D.6) one has

$$[\sigma_{;123}] + [\sigma_{;132}] = 0 = [\sigma_{;123}] - [\sigma_{;132}], \tag{D.13}$$

which implies

$$[\sigma_{;123}] = 0. \tag{D.14}$$

For the next order, by (D.5) and (D.7) one has

$$[\sigma_{;1234}] + [\sigma_{;1423}] + [\sigma_{;1324}] = 0, \tag{D.15}$$

$$[\sigma_{;1234}] - [\sigma_{;1323}] = -(1, 4, 2, 3). \tag{D.16}$$

By (D.16) we convert (D.15) into

$$[\sigma_{;1243}] + 2[\sigma_{;1234}] + (1, 3, 2, 4) + (1, 4, 2, 3) = 0; \tag{D.17}$$

Interchanging 3 and 4 and subtracting, we get

$$[\sigma_{;1234}] = [\sigma_{;1243}], \tag{D.18}$$

and thus (D.17) gives

$$[\sigma_{;1234}] = -\tfrac{1}{3}((1,3,2,4) + (1,4,2,3)), \tag{D.19}$$

or

$$[\sigma_{;1234}] = -\tfrac{1}{3}\mathcal{P}_{3,4}(1,3,2,4), \tag{D.20}$$

where $\mathcal{P}_{3,4}$ denotes the sum of expressions obtained by permutation of 3, 4.

For the fifth order, by (D.9) and (D.10),

$$[\sigma_{;15234}] + [\sigma_{;14235}] + [\sigma_{;13245}] + [\sigma_{;12345}] = 0, \tag{D.21}$$

$$[\sigma_{;12345}] - [\sigma_{;13245}] = -(1,4,2,3;5) - (1,5,2,3;4). \tag{D.22}$$

By (D.22) we convert (D.21) into

$$\begin{aligned}&[\sigma_{;12534}] + [\sigma_{;12435}] + 2[\sigma_{;12345}]\\ &\quad +(1,4,2,3;5) + (1,5,2,3;4) + (1,3,2,4;5)\\ &\quad +(1,5,2,4;3) + (1,3,2,5;4) + (1,4,2,5;3) = 0.\end{aligned} \tag{D.23}$$

Interchanging 4 and 5, and subtracting, we get

$$[\sigma_{;12345}] = [\sigma_{;12354}], \tag{D.24}$$

interchanging 3 and 4 in (D.23), subtracting, and using (D.24), we get

$$[\sigma_{;12345}] = [\sigma_{;12435}]; \tag{D.25}$$

thus (D.23) becomes

$$\begin{aligned}[\sigma_{;12345}] = -\tfrac{1}{4}(&(1,3,2,4;5) + (1,3,2,5;4) + (1,4,2,3;5)\\ &+(1,4,2,5;3) + (1,5,2,3;4) + (1,5,2,4;3)).\end{aligned} \tag{D.26}$$

or

$$[\sigma_{;12345}] = -\tfrac{1}{4}\mathcal{P}_{3,4,5}(1,3,2,4;5). \tag{D.27}$$

Fo the sixth order we have, by (D.9),

$$\begin{aligned}&[\sigma_{;162345}] + [\sigma_{;152346}] + [\sigma_{;142356}]\\ &\quad +[\sigma_{;132456}] + [\sigma_{;123456}] + [H_{123456}] = 0,\end{aligned} \tag{D.28}$$

where

$$\begin{aligned}&[H_{123456}]\\ &\quad = [\sigma_{;1\lambda 23}][\sigma^{;\lambda}{}_{456}] + [\sigma_{;1\lambda 24}][\sigma^{;\lambda}{}_{356}] + [\sigma_{;1\lambda 25}][\sigma^{;\lambda}{}_{346}] + [\sigma_{;1\lambda 26}][\sigma^{;\lambda}{}_{345}]\\ &\quad +[\sigma_{;1\lambda 34}][\sigma^{;\lambda}{}_{256}] + [\sigma_{;1\lambda 35}][\sigma^{;\lambda}{}_{246}] + [\sigma_{;1\lambda 36}][\sigma^{;\lambda}{}_{245}]\\ &\quad +[\sigma_{;1\lambda 45}][\sigma^{;\lambda}{}_{236}] + [\sigma_{;1\lambda 46}][\sigma^{;\lambda}{}_{235}]\\ &\quad +[\sigma_{;1\lambda 56}][\sigma^{;\lambda}{}_{234}]\end{aligned} \tag{D.29}$$

is a tensor whose value is known by (D.20); from (D.10) we have

$$\begin{aligned}&[\sigma_{;123456}] - [\sigma_{;132456}] \\ &\quad = -(1,4,2,3;5,6) - (1,5,2,3;4,6) \\ &\quad -(1,6,2,3;4,5) - (1,\lambda,2,3)[\sigma^{;\lambda}{}_{4,5,6}]. \end{aligned} \tag{D.30}$$

By means of this equation we convert (D.28) into

$$\begin{aligned}&[\sigma_{;126345}] + [\sigma_{;125346}] + [\sigma_{;124356}] + 2[\sigma_{;123456}] + [H_{123456}] \\ &\quad +(1,4,2,3;5,6) + (1,5,2,3;4,6) + (1,6,2,3;4,5) + (1,\lambda,2,3)\sigma^{;\lambda}{}_{456} \\ &\quad +(1,3,2,4;5,6) + (1,5,2,4;3,6) + (1,6,2,4;3,5) + (1,\lambda,2,4)\sigma^{;\lambda}{}_{356} \\ &\quad +(1,3,2,5;4,6) + (1,4,2,5;3,6) + (1,6,2,5;3,4) + (1,\lambda,2,5)\sigma^{;\lambda}{}_{346} \\ &\quad +(1,3,2,6;4,5) + (1,4,2,6:3,5) + (1,5,2,3;6,4) + (1,\lambda,2,6)\sigma^{;\lambda}{}_{345} \\ &\quad = 0. \end{aligned} \tag{D.31}$$

If we employ the symbol I to denote the operation of interchanging numerals, so that $I(3,4)$, for example, denotes an interchange of 3 and 4, the previous equation may be written

$$\begin{aligned}&\{2 + I(3,4) + I(4,5)I(3,4) + I(5,6)I(4,5)I(3,4)\}[\sigma_{;123456}] \\ &\quad +\{1 + I(3,4) + I(4,5)I(3,4) + I(5,6)I(4,5)I(3,4)\}[L_{123456}] \\ &\quad +[H_{123456}] = 0, \end{aligned} \tag{D.32}$$

where $[L_{123456}]$ denotes the second line of (D.31).

Operating with$\{1 - I(5,6)\}$ on (D.31), and using (D.18), we get

$$\begin{aligned}&\{2 + I(3,4)\}\{1 - I(5,6)\}[\sigma_{;123456}] \\ &\quad +\{1 + I(3,4)\}\{1 - I(5,6)\}(1,3,2,4;5,6) = 0. \end{aligned} \tag{D.33}$$

Since

$$I(3,4)I(3,4) = 1,$$

we have

$$\begin{aligned}\{2 - I(3,4)\}\{2 + I(3,4)\} &= 3 \\ \{2 - I(3,4)\}\{1 + I(3,4)\} &= 1 + I(3,4) \end{aligned} \tag{D.34}$$

and therefore operation on (D.33) with $\{2 - I(3,4)\}$ gives

$$\{1 - I(5,6)\}[\sigma_{;123456}] = -\tfrac{1}{3}\{1 + I(3,4)\}\{1 - I(5,6)\}(1,3,2,4;5,6) \tag{D.35}$$

Operation on (D.31) with $\{1 - I(4,5)\}$ gives

$$\begin{aligned}&\{1 - I(4,5)\}[\sigma_{;126345}] + 2\{1 - I(4,5)\}[\sigma_{;123456}] + \{1 - I(4,5)\}[H_{123456}] \\ &\quad +\{1 + I(3,6)\}\{1 - I(4,5)\}(1,3,2,6;4,5) \\ &\quad +\{1 - I(4,5)\}(1,\lambda,2,3)[\sigma^{;\lambda}{}_{456}] = 0, \end{aligned} \tag{D.36}$$

in which the first term may be evaluated by applying to (D.35) the substitution

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 6 & 3 & 4 & 5 \end{matrix}$$

while the term in $[H]$ is, by (D.29)

$$\{1 - I(4,5)\}[\sigma_{;1\lambda 23}][\sigma^{;\lambda}{}_{456}]. \tag{D.37}$$

Thus (D.36) may be written

$$\begin{aligned}\{1 - I(4,5)\}[\sigma_{;123456}] &= -\tfrac{1}{3}\{1 + I(3,6)\}\{1 - I(4,5)\}(1,3,2,6;4,5)\\ &-\tfrac{1}{2}\{1 - I(4,5)\}((1,\lambda,2,3) + [\sigma_{;1\lambda 23}][\sigma^{;\lambda}{}_{456}].\end{aligned} \tag{D.38}$$

Operating on (D.31) with $\{1 - I(3,4)\}$ gives

$$\begin{aligned}&\{1 + I(5,6)\}\{1 - I(3,4)\}[\sigma_{;125346}] + \{1 - I(3,4)\}[\sigma_{;123456}]\\ &\quad+\{1 - I(3,4)\}[H_{123456}]\\ &\quad+\{1 + I(5,6)\}\{1 - I(3,4)\}(1,5,2,6;3,4)\\ &\quad+\{1 + I(5,6)\}\{1 - I(3,4)\}(1,\lambda,2,5)[\sigma^{;\lambda}{}_{346}] = 0.\end{aligned} \tag{D.39}$$

The first term may be evaluated by applying to (D.38) the substitution

$$\begin{matrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 2 & 5 & 3 & 4 & 6\end{matrix}$$

and it becomes

$$\begin{aligned}&-\tfrac{1}{3}\{1 + I(5,6)\}\{1 + I(5,6)\}\{1 - I(3,4)\}(1,5,2,6;3,4)\\ &\quad-\tfrac{1}{2}\{1 + I(5,6)\}\{1 - I(3,4)\}((1,\lambda,2,5) + [\sigma_{;1\lambda 25}])[\sigma^{;\lambda}{}_{346}],\end{aligned} \tag{D.40}$$

in which the first term can be simplified, since

$$\{1 + I(5,6)\}\{1 + I(5,6)\} = 2\{1 + I(5,6)\}. \tag{D.41}$$

The $[H]$ term in (D.39) is, by (D.29)

$$\{1 + I(5,6)\}\{1 - I(3,4)\}[\sigma_{;1\lambda 25}][\sigma^{;\lambda}{}_{346}]. \tag{D.42}$$

Thus (D.39) may be written

$$\begin{aligned}\{1 - I(3,4)\}[\sigma_{;123456}] &= -\tfrac{1}{3}\{1 + I(5,6)\}\{1 - I(3,4)\}(1,5,2,6;3,4)\\ &-\tfrac{1}{2}\{1 + I(5,6)\}\{1 - I(3,4)\}((1,\lambda,2,5) + [\sigma_{;1\lambda 25}])[\sigma^{;\lambda}{}_{346}].\end{aligned} \tag{D.43}$$

We are now to apply (D.35), (D.38), (D.43) to the solution of (D.31) or (D.32). The first term in (D.32) is

$$\begin{aligned}&\{2 + (l + I(4,5) + I(5,6)I(4,5))I(3,4)\}[\sigma_{;123456}]\\ &\quad= \{3 + (1 + I(5,6))I(4,5)\}[\sigma_{;123456}]\\ &\quad-\{1 + I(4,5) + I(5,6)I(4,5)\}\{1 - I(3,4)\}[\sigma_{;123456}];\end{aligned} \tag{D.44}$$

but

$$\begin{aligned}&\{3 + (1 + I(5,6))I(4,5)\}[\sigma_{;123456}]\\ &= \{4 + I(5,6)\}[\sigma_{;123456}] - \{1 + I(5,6)\}\{1 - I(4,5)\}[\sigma_{;123456}],\end{aligned} \tag{D.45}$$

and

$$\{4+I(5,6)\}[\sigma_{;123456}] = 5[\sigma_{;123456}] - \{1-I(5,6)\}[\sigma_{;123456}].$$

Adding these three equations , and using (D.35), (D.38), (D.43), we find that the first term of (D.32) is

$$\begin{aligned}
&5[\sigma_{;123456}] + \tfrac{1}{3}\{1+I(3,4)\}\{1-I(5,6)\}(1,3,2,4;5,6)\\
&\quad+\tfrac{1}{3}\{1+I(5,6)\}\{1+I(3,6)\}\{1-I(4,6)\}(1,3,2,6;4,5)\\
&\quad+\tfrac{1}{2}\{1+I(5,6)\}\{1-I(4,5)\}((1,\lambda,2,3)+\sigma_{;1\lambda 23})\sigma^{;\lambda}{}_{456}\\
&\quad+\tfrac{1}{3}\{1+I(4,5)+I(5,6)I(4,5)\}\{1+I(5,6)\}\{1-I(3,4)\}(1,5,2,3;6,4)\\
&\quad+\tfrac{1}{2}\{1+I(4,5)+I(5,6)I(4,5)\}\{1+I(5,6)\}\{1-I(3,4)\}\bullet\\
&\qquad\bullet((1,\lambda,2,5)+\sigma_{;1\lambda 25})\sigma^{;\lambda}{}_{346}.
\end{aligned} \tag{D.46}$$

Substituting this expression for the first term in (D.32), we get

$$\begin{aligned}
&5[\sigma_{;123456}]\\
&\quad= -(1,4,2,3;5,6)-(1,5,2,3;4,6)-(1,6,2,3;4,5)-(1,\lambda,2,3)[\sigma^{;\lambda}{}_{456}]\\
&\quad-(1,3,2,4;5,6)-(1,5,2,4;3,6)-(1,6,2,4;3,5)-(1,\lambda,2,4)[\sigma^{;\lambda}{}_{356}]\\
&\quad-(1,3,2,5;4,6)-(1,4,2,5;3,6)-(1,6,2,5;3,4)-(1,\lambda,2,5)[\sigma^{;\lambda}{}_{346}]\\
&\quad-(1,3,2,6;4,5)-(1,4,2,6;3,5)-(1,5,2,6;3,4)-(1,\lambda,2,6)[\sigma^{;\lambda}{}_{345}]\\
&\quad-\tfrac{1}{3}((1,3,2,4;5,6)-(1,3,2,4;6,5)+(1,3,2,6;4,5)-(1,3,2,6;5,4)\\
&\qquad+(1,4,2,3;5,6)-(1,4,2,3;6,5)+(1,6,2,3;4,5)-(1,6,2,3;5,4)\\
&\qquad+(1,3,2,5;4,6)-(1,3,2,5;6,4)+(1,5,2,6;3,4)-(1,5,2,6;4,3)\\
&\qquad+(1,5,2,3;4,6)-(1,5,2,3;6,4)+(1,6,2,5;3,4)-(1,6,2,5;4,3)\\
&\qquad+(1,4,2,6;3,5)-(1,4,2,6;5,3)+(1,4,2,5;3,6)-(1,4,2,5;6,3)\\
&\qquad+(1,6,2,4;3,5)-(1,6,2,4;5,3)+(1,5,2,4;3,6)-(1,5,2,4;6,3))\\
&\quad-\tfrac{1}{2}((1,\lambda,2,3)+[\sigma_{;1\lambda 23}])([\sigma^{;\lambda}{}_{456}]-[\sigma^{;\lambda}{}_{546}]+[\sigma^{;\lambda}{}_{465}]-[\sigma^{;\lambda}{}_{645}])\\
&\quad-\tfrac{1}{2}((1,\lambda,2,4)+[\sigma_{;1\lambda 24}])([\sigma^{;\lambda}{}_{356}]-[\sigma^{;\lambda}{}_{536}]+[\sigma^{;\lambda}{}_{365}]-[\sigma^{;\lambda}{}_{356}])\\
&\quad-\tfrac{1}{2}((1,\lambda,2,5)+[\sigma_{;1\lambda 25}])([\sigma^{;\lambda}{}_{346}]-[\sigma^{;\lambda}{}_{436}]+[\sigma^{;\lambda}{}_{364}]-[\sigma^{;\lambda}{}_{634}])\\
&\quad-\tfrac{1}{2}((1,\lambda,2,6)+[\sigma_{;1\lambda 26}])([\sigma^{;\lambda}{}_{345}]-[\sigma^{;\lambda}{}_{435}]+[\sigma^{;\lambda}{}_{354}]-[\sigma^{;\lambda}{}_{534}])\\
&\quad-[H_{123456}].
\end{aligned} \tag{D.47}$$

The last term in the first line, the terms in the eleventh line, and the first term in $-[H_{123456}]$ (see eq. (D.29)), together make up (by use (D.18) and (D.20))

$$\begin{aligned}
&\{(1,\lambda,2,3)+[\sigma_{;1\lambda 23}]\}\{-2[\sigma^{;\lambda}{}_{456}]+\tfrac{1}{2}([\sigma^{;\lambda}{}_{546}]+[\sigma^{;\lambda}{}_{645}]\\
&\quad=\{-(2,3,\lambda,1)-\tfrac{1}{3}((1,2,\lambda,3)+(1,3,\lambda,2))\}\bullet\\
&\qquad\bullet\{\tfrac{2}{3}g^{\lambda\mu}((\mu,5,4,6)+(\mu,6,4,5))-\tfrac{1}{6}g^{\lambda\mu}((\mu,6,5,4)+(\mu,5,6,4))\}\\
&\quad=-\tfrac{5}{6}g^{\lambda\mu}\{(2,3,\lambda,1)+\tfrac{1}{3}(1,2,\lambda,3)+\tfrac{1}{3}(1,3,\lambda,2)\}\bullet\\
&\qquad\bullet\{(\mu,5,4,6)+(\mu,6,4,5)\}\\
&\quad=-\tfrac{5}{6}g^{\lambda\mu}\{(2,3,\lambda,1)+\tfrac{1}{3}(1,2,\lambda,3)+\tfrac{1}{3}(1,3,\lambda,2)\}\bullet\\
&\qquad\bullet\{(4,5,\mu,6)+(4,6,\mu,5)\}.
\end{aligned} \tag{D.48}$$

But

$$(2,3,\lambda,1)+(2,\lambda,1,3)+(2,1,3,\lambda)=0$$

and therefore

$$(2,3,\lambda,1)=-(2,\lambda,1,3)-(2,1,3,\lambda), \tag{D.49}$$

so that (D.48) is equal to

$$-\tfrac{5}{9}g^{\lambda\mu}\{2(1,3,\lambda,2)-(1,2,\lambda,3)\}\{(4,5,\mu,6)+(4,6,\mu,5)\}. \tag{D.50}$$

The first of the remaining terms in $-[H_{123456}]$ to which are not used up in forming the previous expression and the three expressions similar to it, is (see (D.29)) equal to

$$-\tfrac{1}{9}g^{\lambda\mu}\{(1,3,\lambda,4)+(1,4,\lambda,3)\}\{(2,3,\mu,6)+(2,6,\mu,3)\}. \tag{D.51}$$

Thus we see that (D.47) may be written

$$\begin{aligned}
&[\sigma_{;123456}]\\
&\quad=-\tfrac{4}{15}\{(1,3,2,4;5,6)+(1,3,2,5;4,6)+(1,3,2,6;4,5)\\
&\qquad+(1,4,2,3;5,6)+(1,4,2,5;3,6)+(1,4,2,6;3,5)\\
&\qquad+(1,5,2,3;4,6)+(1,5,2,4;3,6)+(1,5,2,6;3,4)\\
&\qquad+(1,6,2,3;4,5)+(1,6,2,4;3,5)+(1,6,2,5;3,4)\}\\
&\qquad-\tfrac{1}{15}\{(1,3,2,4;6,5)+(1,3,2,5;6,4)+(1,3,2,6;5,4)\\
&\qquad+(1,4,2,3;6,5)+(1,4,2,5;6,3)+(1,4,2,6;5,3)\\
&\qquad+(1,5,2,3;6,4)+(1,5,2,4;6,3)+(1,5,2,6;4,3)\\
&\qquad+(1,6,2,3;5,4)+(1,6,2,4;5,3)+(1,6,2,5;4,3)\}\\
&\qquad-\tfrac{1}{9}g^{\lambda\mu}\sum_{i=1}^{4}A^{(i)}_{\lambda\mu\ 123456}\\
&\qquad-\tfrac{1}{45}g^{\lambda\mu}\sum_{j=1}^{6}B^{(i)}_{\lambda\mu\ 123456},
\end{aligned}\tag{D.52}$$

where

$$\begin{aligned}
A^{(1)}_{\lambda\mu\ 123456}&=(2(1,3,\lambda,2)-(1,2,\lambda,3))((4,5,\mu,6)+(4,6,\mu,5)),\\
A^{(2)}_{\lambda\mu\ 123456}&=(2(1,4,\lambda,2)-(1,2,\lambda,4))((3,5,\mu,6)+(3,6,\mu,5)),\\
A^{(3)}_{\lambda\mu\ 123456}&=(2(1,5,\lambda,2)-(1,2,\lambda,5))((3,4,\mu,6)+(3,6,\mu,4)),\\
A^{(4)}_{\lambda\mu\ 123456}&=(2(1,6,\lambda,2)-(1,2,\lambda,6))((3,4,\mu,5)+(3,5,\mu,4)),\\
B^{(1)}_{\lambda\mu\ 123456}&=((1,3,\lambda,4)+(1,4,\lambda,3))((2,5,\mu,6)+(2,6,\mu,5)),\\
B^{(2)}_{\lambda\mu\ 123456}&=((1,3,\lambda,5)+(1,5,\lambda,3))((2,4,\mu,6)+(2,6,\mu,4)),\\
B^{(3)}_{\lambda\mu\ 123456}&=((1,3,\lambda,6)+(1,6,\lambda,3))((2,4,\mu,5)+(2,5,\mu,4)),\\
B^{(4)}_{\lambda\mu\ 123456}&=((1,4,\lambda,5)+(1,5,\lambda,4))((2,3,\mu,6)+(2,6,\mu,3)),\\
B^{(5)}_{\lambda\mu\ 123456}&=((1,4,\lambda,6)+(1,6,\lambda,4))((2,3,\mu,5)+(2,5,\mu,3)),\\
B^{(6)}_{\lambda\mu\ 123456}&=((1,5,\lambda,6)+(1,6,\lambda,5))((2,3,\mu,4)+(2,4,\mu,3)).
\end{aligned}$$

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