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A General Approach To Find Out Wormhole Solutions

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Chapter 1

INTRODUCTION

In 1783 John Michell, an amateur astronomer, hypotized the existence of a star whose escape velocity was the speed of light. Setting the Newtonian formula for the escape velocity equal to the speed of light, c, he found that the radius of this hypotetical star was $R = 2GM/c^2$. In terms of the mass of the Sun, M_S, $R = 2.95(M/M_S)km$. However this resulting radius seemed to be unrealistically small, so the scientist lost interest about this theoretical speculation. In 1905, the "annus mirabilis", Albert Einstein published four articles on the scientific paper "Annalen der Physik" revolutionizing the concepts of space and time. In one of these articles Einstein exposed the well known formula, $E = mc^2$, and the special relativity theory which claims that time is relative to the inertial reference frame losing the concepts of absolute space and time. Indeed, he affirms the **principle of constancy of the speed of light**, that assumes the same value $c = 2.9979 \times 10^8 km/s$, indipendently from the inertial observer. Subsequently, in 1915, Einstein published the general theory of gravity which extends the special relativity theory to the accelerated reference frames, based on the **equivalence principle**, according to which the **inertial mass** m_i is equivalent to the **gravitational mass,** m_g.

Furthermore he showed that gravity is due to the presence of a mass which curves the so called **spacetime** (a four-dimensional vectorial space that is made up of the eculideian three-dimensional space and a fourth dimension represented by the time), as shown by the famous **Einstein fields equations**. Before Einstein, the most accepted theory of gravity was the Newton's one with its **universal gravitational law**, $F_g = GM_1M_2/R^2$. According to Newton's theory, time is absolute and does not depend on the observer and the gravitational force is istantaneous. Nowadays, thanks to Einstein we know that every signal transmits at a maximum velocity that is the speed of light in vacuum. The first solution of the Einstein field equations was found by K. Schwarzschild in 1916, now called **Schwarzschild metric**. This metric represents the gravitational field outside of a compact, non-rotating, discharged object and is two time singular in the (t, r, θ, ϕ) coordinates; in particular in $r = 0$ and in $r = \frac{2MG}{c^2}$. However, in 1924 A. Eddington introduced a new coordinate system in which the singularity at $r = \frac{2MG}{c^2}$ disappear (see appendix). This particular value of r is called **Schwarzschild radius**. Instead, the $r = 0$ singularity is a true singularity because it does not depend on the particular coordinate system, as we will see in the second chapter. According to the **Birkhoff theorem** [1], the Schwarzschild metric is the most general spherically symmetric vacuum solution of the Einsten fiedl equations. Later, in 1931 Chandrasekhar [2] found that a non-rotating body of electron degenerate-matter above a certain mass limit (now called **Chandrasekhar limit** corresponding to $1,44\ M_S$) has no stable solutions. In 1939, indeed Oppenheimer and others [3], using the **Pauli exclusion principle** [4], predicted that neutron stars beyond another mass limit (the **Tolman-Oppenheimer-Volkoff (TOV) limit** corresponding to $0.7M_S$) would collapse further for the same reasons predicted by Chandrasekhar and concluded that no physical law would have prevented the star from collapsing beyond its Schwarzschild radius, into a so called a **Black Hole, BH**. Later, the TOV limit was redefined first in 1996 [5] (between $1,5$ and $3,0$ solar masses) and then through the neutron star merging event, GW170817 ($2,17 M_S$) [6, 7, 8, 9, 10]. However, Oppenheimer understood also that the Schwarzschild radius was the radius of a limit surface on which time stops from the view-point of an external observer. Later, Finkelstein and Kruskal extended the analysis to the view-point of an hypothetical infalling observer into the BH (as we will see in the second chapter). They found that this limit surface acts like an unidirectional membrane, through which nothing can escape, neither the light, once passed through. However, the Schwarzschild metric is not the only solution of the Einstein field equations; infact between 1916 and 1918 Reissner and Nordstrom [11, 12] found a solution for a charged, spherical, non-rotating body. Indeed, in 1963 R. Kerr [13] found the exact solution for a rotating BH. Contemporarely to this important developments, other discoveries were done: the **pulsars** by Jocelyn Bell [14, 15] in 1967, which by 1969 were shown to be rapidly rotating neutron stars. Moreover, thanks to W. Israel [16] and others [17, 18, 19] the **no-hair theorem** emerged, stating that a stationary BH solution is completely described by the three parameters of the **Kerr-Newmann metric** [20, 21]: mass, angular momentum and electric charge . However, in the late 1960s R. Penrose [22] and S. W. Hawking used global techniques to prove that singularities appear generically, indipendently from the symmetries imposed. Other important results were done by Hawking [23] and others [24, 25, 26] in the early 1970s, which led to the formulation of BH thermodynamics. Hawking himself in 1974, using quantum field theory, showed that BHs should radiate like a black body, predicting the effect now called **Hawking radiation** [27]. Contemporarely to the black hole developments, in 1916 with L. Flamm [28] first and with A. Einstein and N. Rosen [29] later, it turns out the existence of a non-trivial topology consisting in a bridge, known as **Einstein–Rosen bridge**, connecting two areas of space-time that can be modeled as vacuum solutions of the Einstein field equations, sometimes called two different "universes" and that are now understood to be intrinsic parts of the maximally extended version of the Schwarzschild metric describing an eternal black hole with no charge and no rotation. Subsequently this non trivial structure can be seen in a space-time diagram that uses Kruskal–Szekeres coordinates. For this reason this structure is also known as **Schwarzschild wormhole**. However, in 1962, John Archibald Wheeler and Robert W. Fuller [30] published a paper showing that this type of **wormhole** is unstable if it connects two parts of the same universe, and that it will pinch off too quickly for light (or any particle moving slower than light) that falls in from one exterior region to make it to the other exterior region. Although Schwarzschild wormholes are not traversable in both directions, their existence inspired Kip Thorne and Mike Morris [31] to imagine traversable wormholes created by holding the "throat" of a Schwarzschild wormhole open with **exotic matter** (matter that has negative mass/energy density), as we will see in the chapter 3. In the last chapter, the fourth one, we will see a general approach to wormhole solutions using the Morris-Thorne metric. So let us start with BHs introducing in the next chapter the Schwarzschild metric, which is the first solution of the Einstein field equations describing a BH, as we have seen. We will analyze its properties and discuss what we mean with the concept of **singularity**. Finally we will show important theorems of Penrose and Hawking, which characterize singularities. Note that from now on I will use the $(+,-,-,-)$ signature and use the natural units, $G = 1$ and $c = 1$.

Chapter 2

SINGULAR SOLUTIONS IN METRIC THEORIES OF GRAVITY

2.1 THE GENERAL RELATIVITY THEORY: A BRIEF INTRODUCTION

General Relativity is the physical theory of gravity formulated by Einstein in 1915 and represents the best accepted theory of gravity nowaday. It is based on the above mentioned **Equivalence principle**, that establishes that the laws of physics to which the objects of a reference system in free-fall obey, for sufficiently small regions of space-time, are those of the Special Relativity. Another fundamental principle is the **principle of General Covariance**, which places all reference systems on the same level of equality. In special relativity, instead, there is a privileged class of reference systems, the inertial ones for wchich the laws of physics are covariant. In general relativity the laws of physics are required to be covariant in all the reference systems. From these principles Einstein find out his famous **gravitational field equations**:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

where $R_{\alpha\beta}$, R, $g_{\alpha\beta}$, Λ and $T_{\alpha\beta}$ are the Ricci tensor, the Ricci scalar, the metric tensor, the cosmological constant and the stress-energy tensor, respectively. This tensor equation represents a set of non-linear partial differential equations and suggests that gravity is a manifestation of the curvature of spacetime caused by matter, but also that curvature of matter is a manifestation of gravity. The two aspects are equivalent. In particular, in the presence of matter, spacetime cannot be described by a pseudo-Euclidean metric, rather by a pseudo-Riemannian variety. Even the path of light, which in vacuum is straight, will be curved in presence of a mass. The experimental verification of this consequence of general relativity was carried out by A. Eddington in 1919 and constituted proof of validity and correctness of the theory. It must also be said that general relativity has been able to predict also gravitational waves, black holes, wormholes, gravitational lensing and more. Some of these phenomena have only recently been experimentally verified. Let us present the first solution of the Einstein field equations, the Schwarzschild metric.

2.2 THE SCHWARZSCHILD SOLUTION

In 1915 Karl Schwarzschild [32] found the first exact solution of the Einstein field equations other than the trivial flat space solution and published it in January 1916 [18], a little more than a month after the publication of Einstein's theory of general relativity. The Schwarzschild solution can be written as

$$ds^2 = (1 - \tfrac{2M}{r})dt^2 - \tfrac{1}{1-\frac{2M}{r}}dr^2 - r^2(d\theta^2 + sen^2\theta d\phi^2) \tag{2.1}$$

in the coordinates (t, r, θ, ϕ). This solution is asymptotically flat, since, as r tends to infinity, it tends to the Minkowski metric in polar coordinates. This metric depends on the positive constant M, which can be interpreted as the black hole mass. The Schwarzschild solution is spherically symmetric; furthermore, it is static. As a consequence, the Schwarzschild metric is invariant for temporal translations and for time reversal $t = -t$. It is possible to show that any asymptotically flat and spherically symmetric solution of Einstein's equations in vacuum is also static, and then it is the Schwarzschild solution. This result, named the **Birkhoff theorem**, is very important; it implies, for instance, that the exterior of spherically symmetric stars is described by the Schwarzschild solution, and that spherically symmetric objects cannot be sources of gravitational radiation. The metric, in these coordinates, is singular at r = 0 and at r = 2M. However, the singularity at r = 2M depends of the coordinate choice and can be removed by changing coordinates (and then it is called **"coordinate singularity"**); the singularity r = 0, instead, is a true singularity of the metric (and is called **"curvature singularity"**). Let us consider in detail the surface r = 2M, and the other hypersurfaces r = constant. The normal of a hypersurface whose equation is $\Sigma : r - const = 0$ is :

$$n_\mu = \Sigma_{,\mu} = (0,1,0,0) \tag{2.2}$$

Let us consider a generic hypersurface. At any point of such hypersurface we can introduce a locally inertial frame and rotate it in such a way that the components of the normal vector are:

$$n^\alpha = (n^0, n^1, 0, 0) \tag{2.3.a}$$

and

$$n_\alpha n^\alpha = (n^0)^2 - (n^1)^2 \tag{2.3.b}$$

Consider a vector t^α tangent to the surface at the same point. t^α must be orthogonal to n^β:

$$n_\alpha t^\alpha = n^0 t^0 - n^1 t^1 = 0 \tag{2.4.a}$$

from which :

$$\frac{t^0}{t^1} = \frac{n^1}{n^0} \tag{2.4.b}$$

thus:

$$t^\alpha = \Lambda(n^1, n^0, a, b) \tag{2.5}$$

where Λ, a e b are constant and arbitrary. Consequently the norm of the tangent vector is:

$$t_\alpha t^\alpha = \Lambda^2[(n^1)^2 - (n^0)^2 - (a^2+b^2)] = \Lambda^2[n_\alpha n^\alpha - (a^2+b^2)] \tag{2.6}$$

We have that:

- if $n_\mu n^\mu < 0$, the hypersurface is called **timelike**, and t^μ is necessarily a timelike vector.
- if $n_\mu n^\mu > 0$, the hypersurface is called **spacelike**, and t^μ can be timelike, spacelike or null.
- if $n_\mu n^\mu = 0$, the hypersurface is called **null**, and t^μ can be timelike or null.

Let us consider a point P on a surface $\Sigma = 0$. From (2.2), (2.1), in the case of an r = constant surface:

$$n_\mu n_\nu g^{\mu\nu} = g^{rr} = -(1 - \tfrac{2M}{r}) \tag{2.7}$$

thus the surfaces $r = constant$ are spacelike if $r < 2M$, null if $r = 2M$, timelike if $r > 2M$. The null hypersurface $r = 2M$, then, separates regions of spacetime where $r = const$ are timelike hypersurfaces from regions where $r = const$ are spacelike hypersurfaces; therefore, an object crossing a null hypersurface $r = const$ can never come back; for this reason, the null hypersurface $r = 2M$ is called **horizon**. The horizon r = 2M separates the spacetime in two regions:

- the region with $r > 2M$, where the $r = const.$ hypersurfaces are timelike; the r tends to infinity limit, where the metric becomes flat, is in this region, so we can consider this region as the exterior of the black hole;
- the region with $r < 2M$, where the $r = const.$ hypersurfaces are spacelike; an object which falls inside the horizon and enter in this region can only continue falling to decreasing values of r, until it reaches the curvature singularity $r = 0$; this region is then considered the interior of the BH; the name BH is due to the fact that nothing, neither objects nor signals of any kind, can escape from this region.

2.3 SINGULARITIES

A singularity of the metric is a point at which the determinant of either it or its inverse vanish. The fact that $r = 0$ is a curvature singularity can be shown by computing the curvature scalar:

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 48\frac{M^2}{r^6} \tag{2.8}$$

which diverges at $r = 0$. The fact that $r = 2M$ is a coordinate singularity is less easy to prove: the finiteness of the polynomials in the curvature tensor does not exclude, in principle, that there is a curvature singularity there. Therefore we need to study this singularity, finding the coordinate change which allows to remove it.

2.4 SPACETIME EXTENSION.

To study the structure of singularities, we must first define rigorously the concept of singularity in general relativity. This is not obvious, since a singularity does not belong to the spacetime manifold. The key notion is the **lenght of geodesics**.

definition 2.1 An n-dimensional **pseudo-Riemannian manifold** is a pair (M, g), where M is an n-dimensional differentiable manifold and g is a symmetric, nondegenerate 2-tensor field on M (called the metric). A pseudo-Riemannian manifold is said to be **Riemannian** if g has signature $(+ \ldots +)$, and is said to be **Lorentzian** if g has signature $(-, +...+)$.

definition 2.2 [33] Let M be a connected smooth manifold endowed with a smooth Riemannian metric g, i.e. g_p varies smoothly in p on M. The length of a piecewise smooth curve $\gamma : s \in [0, 1] \mapsto M$ is defined by

$$L(s) = \int_0^1 ||\gamma(s)'||\mathrm{ds} \tag{2.9}$$

where $||\gamma(s)'||=\sqrt{g_p(\gamma(s)', \gamma(s)')}$ denotes the norm of $\gamma(s)' \in T_pM$ with respect to g and T_pM is the **tangent space** in p .

definition 2.3

Let (M, g) be a Pseudo-Riemannian manifold. A curve, $\gamma : I \mapsto M$, (where $I \subseteq R$ is an interval) is a **geodesic** iff $\gamma(s)'$ is parallel along γ, that is, iff

$$\frac{D\gamma'}{ds} = \nabla_{\gamma'}\gamma' = 0 \tag{2.10}$$

where γ' is the derivative with respect to s. From (2.10) we have the so called **geodesic's equation**

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{2.11}$$

where $\Gamma^\lambda_{\mu\nu}$ is the Levi-Civita connection and $\nabla_{\gamma'}$ is the covariant derivative with respect to γ'.

definition 2.4 [34]

If the geodesic is timelike, then it's a possible world line for a freely falling particle, and its uniformly ticking parameter s (called **affine parameter**) is a multiple of the particle's proper time τ :

$$s = a\tau + b \tag{2.12}$$

In Schwarzschild spacetime, for instance, an observer falling into the BH reaches the singularity $r = 0$ in a finite amount of proper time, thus its (timelike) geodesic has a finite length and cannot be extended; this means that $r = 0$ is a singularity: whatever coordinate frame we choose, that geodesic has a finite length. Therefore, to characterize a singular behaviour we need the notion of geodesic completeness:

definition 2.5 [35]: "A spacetime is **geodesically complete** if every timelike and null geodesic can be extended to values arbitrarily large of the affine parameter. If instead the spacetime admits at least one incomplete (i.e., which cannot be extended) timelike or null geodesic, we say that it is **geodesically incomplete** and this means that there is a singularity (either a true curvature singularity or a coordinate singularity)".

In our analysis we only consider timelike or null geodesics because they represent worldlines of massive or massless particles. Let us consider the Schwarzschid metric (2.1). This is not defined at $r = 0$ and at $r = 2M$; in particular the second one $r = 2M$ is a coordinate singularity and a geodesic corresponding to an observer falling into the BH cannot be extended across the hypersurface r = 2M and such a geodesics terminates at a finite value of the affine parameter (notice that the observer arrives at r = 2M with $t = +\infty$). Howevere this singularity can be removed introducing the Kruskal frame (U, V, θ, ϕ), for which

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV - r^2(d\theta^2 + sen^2\theta d\phi^2) \tag{2.13}$$

where $r = r(U, V)$ and $UV = (1 - \frac{r}{2M})e^{\frac{r}{2M}}$ (see appendix).

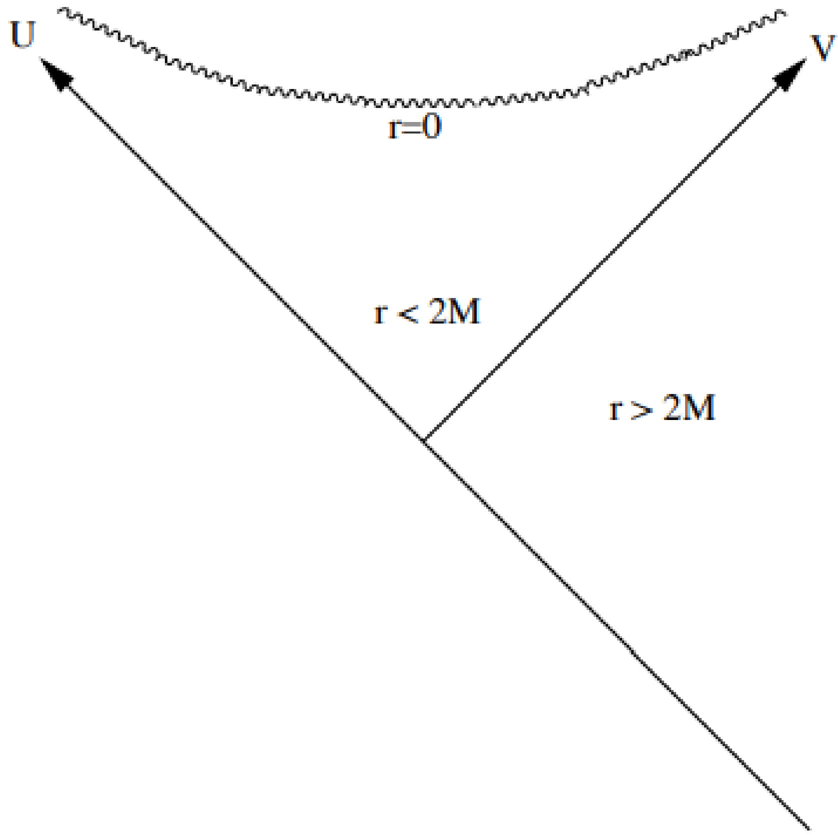

Figure 2.1 Interior and exterior of a Schwarzschild black hole in Kruskal coordinates.

In these coordinates (see Fig. 2.1), the BH exterior $r > 2M$ corresponds to $(U < 0, V > 0)$ and the BH interior $r < 2M$ corresponds to $(U > 0, V > 0$ with $UV < 1)$; the singularity $r = 2M$ (and $t = +\infty$) corresponds to the semiaxis $(U = 0, V > 0)$; the curvature singularity $r = 0$ corresponds to the upper branch of the hyperbole $UV = 1$. We can notice that in this new coordinate frame at the value $r = 2M$ the metric does not become singular. Furthermore we can extend this frame first considering the semiaxis

$U = 0$, $V > 0$, since the line element (2.13) is not singular there. So we are considering a new manifold M such that:

$$M \subset M_1 U M_2 \tag{2.14}$$

defined by

$$V > 0, \qquad UV < 1 \tag{2.15}$$

and where M_1 and M_2 are the $r < 2M$ and the $r > 2M$ manifold, respectively (the interior and the exterior of the BH). Therefore, in order to eliminate a coordinate singularity we have to extend the spacetime manifold. The manifold M should still be extended considering $V \leq 0$: so we can extend in this way our geodesics backward. Therefore $(-\infty < U < +\infty, -\infty < V < +\infty$, with $UV < 1)$ is the **maximal extension** (i.e., which can not be furthet extended) of the Schwarzschild spacetime. This is the usual Kruskal construction, shown in Fig. 2.2; In the Kruskal coordinates (2.13) the null worldlines with θ, ϕ constant, are straight lines at 45°, i.e. U = const. and V = const.; this can be seen easily from the line element (2.13): assuming θ, ϕ constant, if either dU = 0 or dV = 0, then ds = 0. Therefore, the light-cone can be drawn as in Minkowski spacetime:

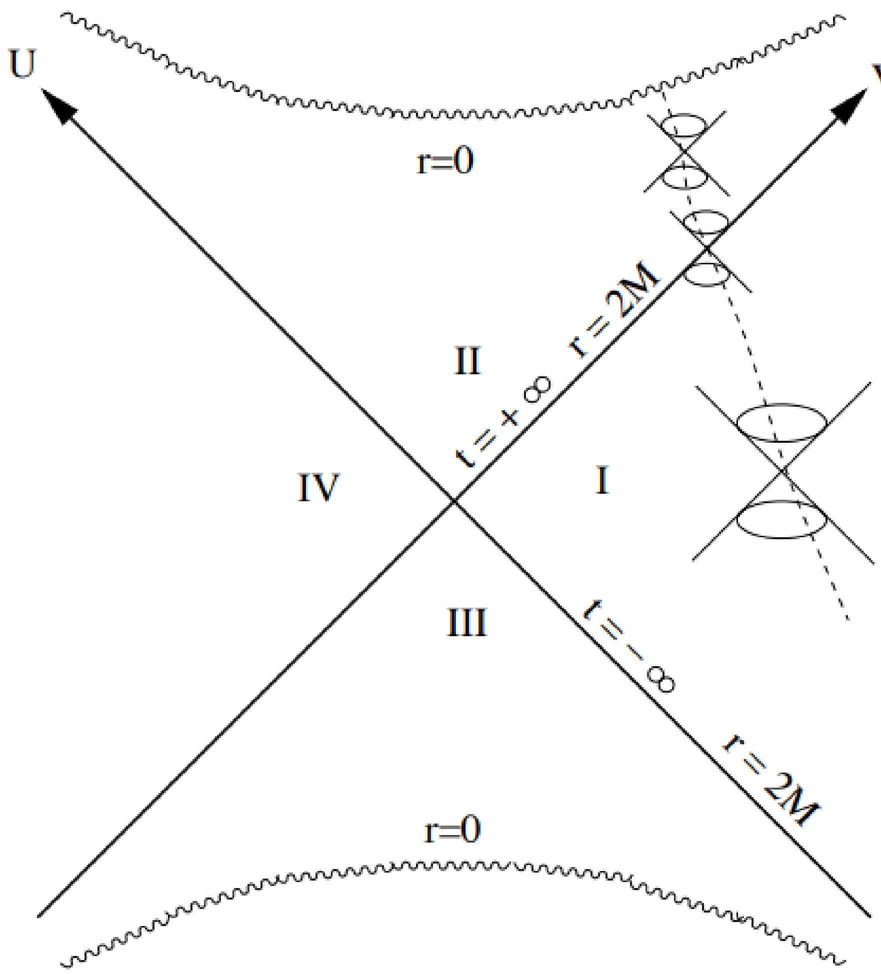

Figure 2.2: Maximal extension of the Schwarzschild spacetime in Kruskal coordinates. the dashed line represents the worldline of an observer falling into the black hole, and the wave-like curves represent the curvature singularity $r = 0$.

We can see that signals (matter and energy) can only go from sector I to sector II and not viceversa. Idem for the sectors IV and III, all signals can go only from sector III (**white hole**) to sector IV. Note that sector IV is causally disconnected from sector I. One consideration about the $r = 0$ singularity: all the geodesics of the maximally extended manifold M that reach this singularity are incomplete because they cannot be extended through $r = 0$. So this singularity, as we have already seen, is a true singularity. For instance, we can consider an observer falling inside a BH: its geodesic reachs the singularity in a finite amount of proper time and cannot be further extended. Another important consideration is that the Kruskal construction presented above describes an eternal BH because we consider also the semiaxis $U < 0$, $V = 0$ that corresponds to $t = -\infty$, while BHs are astrophysical object resulting from stellar collapse, which happen in a finite value of t. So sectors III and IV have no physical meaning in this case.

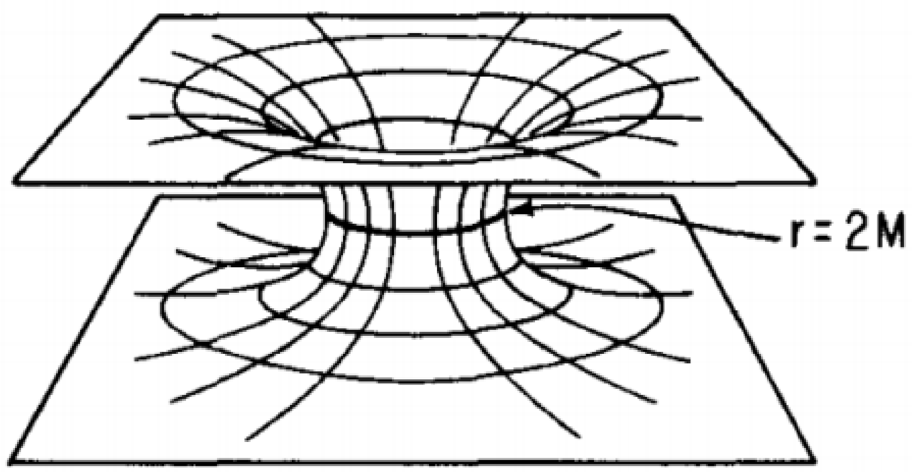

Figure (2.3) The spatial geometry of the t=0 hypersurface of Schwarzschild's space-time shows what it would look like if it were embedded in a flat space. One dimension is suppressed, that is the topology of the hypersurface is $R \times S^2$ and not $R \times S^1$. Consequently, each circle represents a 2-sphere. The portion of the surface placed above r = 2M corresponds to region I of Fig.

(2.2); similarly, the portion of the surface located below r = 2M corresponds to region IV.

2.5 NAKED SINGULARITIES AND THE COSMIC CONJECTURE CENSORSHIP

The singularity at r = 0 that occurs in spherically symmetric collapse is hidden in the sense that no signal from it can reach $\mathcal{I}^+$(future null infinity). This is not true in the Kruskal spacetime manifold since a signal from r = 0 in the white hole region can reach $\mathcal{I}^+$. This singularity is **naked** (in the sense that there is not an event horizon from which nothing can escape). Another example of a naked singularity is the $M < 0$ Schwarzschild solution:

$$ds^2 = (1 + \tfrac{|2M|}{r})dt^2 - \tfrac{1}{(1+\frac{|2M|}{r})}dr^2 - r^2 d\Omega^2 \tag{2.16}$$

This solves Einstein's equations so we have no a priori reason to exclude it. Neither of these examples is relevant to gravitational collapse because it can be shown that $M \geq 0$ for physically reasonable matter (the **positive energy theorem** [36]); moreover, a white hole is the time reverse of a black hole and both black and white holes are allowed by G.R. because of the time reversibility of Einstein equations, but white holes require very special initial conditions near the singularity, whereas black holes do not, so only black holes can occur in practice. So the possibility illustrated above (formation of a naked singularity in spherically-symmetric collapse) cannot occur. There remains the possibility that naked singularities could form in non-spherical collapse. If this were to happen the future would eventually cease to be predictable from data given on an initial spacelike hypersurface, Σ, called **Cauchy hypersurface** ; this scenario led Penrose to suggest the:

Cosmic Conjecture Censorship: "Naked singularities cannot form from gravitational collapse in an asymptotically flat spacetime that is non-singular on some initial spacelike hypersurface (Cauchy surface)".

In other words, all singularities in the universe (with the exception of a possible initial singularity) are covered by horizons. So there is no naked singularity. So the fact that we cannot say what happens to the observer as it reaches the singularity, constitutes a problem for the theory; on the other hand, such problem is not severe from an operational point of view, since no signal from the observer reaching the singularity can be sent outside the black hole: the consistency of the theory, in a certain sense, is preserved by the existence of the horizon. There is no definitive proof of this conjecture, but there are indications supporting it. Therefore, it is commonly believed that the cosmic censorship hypothesis is likely to be correct. As we mentioned above, astrophysical BHs are the product of gravitational collapse, and then are not eternal black holes. It is important to stress that, although we have discussed the entire Schwarzschild solution, only the $r > 2M$ region is directly relevant for astrophysical observations: no signal can come from the interior of the BH. Therefore, the most physically relevant properties of black holes are the properties of the exterior region. On the other hand, it is impossible to have a general understanding of the physics of black holes (and then of the behaviour of astrophysical black holes) without having at least a general idea of what's going on inside; for this reason we have briefly discussed the features of the entire solution.

2.6 SINGULARITY THEOREMS

I want to finish this chapter showing two important theorems due to Penrose (1965) and Hawking (1965-1966) which characterize singularities:

Theorem 2.6.1, R.Penrose, 1965
A spacetime which:

- admits a non-compact Cauchy hypersurface
- for which the null energy condition $T(n,n) = Ric(n) \geq 0$ holds and
- which admits a trapped surface namely a surface for which both ingoing and outgoing lightlike geodesics contract

is future null geodesically incomplete.

In 1965-66 Stephen Hawking immediately realizes that Penrose's argument works for the universe as a whole.

Theorem 2.6.2, S.W.Hawking, 1965-66
A spacetime which satisfies

- it admits a Cauchy hypersurface
- the (timelike unit) normals to the Cauchy hypersurface are expanding (universe expansion), and
- $Ric(v) \geq 0$ for every timelike vector,

is timelike geodesically past incomplete.

Like we have already said, contemporarely to the developments about BHs, other theoretical conjectures arised, wormholes, WHs. In the next chapter we will see them starting from their definition and we will see that there are various types of wormholes depending on particular properties, for instance, their traversability.

Chapter 3

WORMHOLE SOLUTIONS AND THE PROBLEM OF TOPOLOGY CHANGE

3.1 WORMHOLES: DEFINITION

Wormholes have been defined both geometrically and topologically. From a topological view-point, an intra-universe wormhole (a wormhole between two points in the same universe) is a compact region of spacetime whose boundary is topologically trivial, but whose interior is not simply connected. Formalizing this idea leads to definitions such as the following, by Matt Visser [37]:

definition3.1 If a Minkowski spacetime contains a compact region Ω and if the topology of Ω is of the form $\Omega \sim R \times \Sigma$, where Σ is a three-manifold of the nontrivial topology, whose boundary has topology of the form $\partial\Sigma \sim S^2$, and if, furthermore, the hypersurfaces Σ are all spacelike, then the region Ω contains a quasipermanent intrauniverse wormhole.

Geometrically, wormholes can be described as regions of spacetime that constrain the incremental deformation of closed surfaces. For example, in [38], a wormhole is defined informally as: " a region of spacetime containing a **world tube** (the time evolution of a closed surface) that cannot be continuously deformed to a **world line** (the time evolution of a point)". We can distinguish various types of wormholes: for example, **intra-universe** and **inter-universe** wormholes. Intra-universe wormholes are those which connect two different regions of the same universe; while inter-universe wormoholes connect two different universe. The difference between these two classes of wormholes arises only on a global level (geometry and global topology): an observer who limits himself to making local measurements, in the vicinity of the wormhole, he cannot tell if he is traveling to another universe or to a different region of the universe in which he is located. This is a trivial consequence of the fact that wormholes are solutions of the equations of Einstein field, which do not place constraints on the topology of the solutions. Another distinction is made based on the variety in which the wormhole is immersed. In fact, we speak of a **Lorentzian wormhole** if the manifold is Lorentzian (pseudo-Riemannian) or of a **Euclidean wormhole** if the manifold is Riemannian (with Euclidean metric).

3.2 THE EINSTEIN-ROSEN BRIDGE

Schwarzschild wormholes, also known as **Einstein–Rosen bridges** (named after Albert Einstein and Nathan Rosen) are connections between areas of space that can be modeled as vacuum solutions to the Einstein field equations, and that are now understood to be intrinsic parts of the maximally extended version of the Schwarzschild metric describing an eternal black hole with no charge and no rotation. As we have seen previously, the Schwarzschild metric is :

$$ds^2 = (1 - \frac{2M}{r})dt^2 - \frac{1}{(1-\frac{2M}{r})}dr^2 - r^2(d\theta^2 + sen^2\theta d\phi^2) \tag{3.1}$$

with $r > 2M$, θ from 0 to π, ϕ from 0 to 2π.

The vanishing of the determinant of the $g_{\mu\nu}$ for $\theta = 0$ is unimportant, since the corresponding (spatial) direction is not preferred. If one introduces in place of r a new variable according to the equation:

$$u^2 = r - 2M \tag{3.2}$$

one obtains for ds^2 the expression:

$$ds^2 = -4(u^2+2M)du^2 - (u^2+2M)^2(d\theta^2+sen^2\theta d\phi^2) + \frac{u^2}{u^2+2M}dt^2 \tag{3.3}$$

where $u \in (-\infty, +\infty)$ and $c = G_n = 1$.

In this new variable $u = 0$ corresponds to $r = 2M$ and, indeed, $r \in [0, 2M)$ is not considered while the asimptotically flat region $r \in [2M, +\infty)$ twice.

The four-dimensional space is described mathematically by two congruent parts or "sheets", corresponding to $u > 0$ and $u < 0$, which are joined by a hyperplane r =2m or u=0 in which g vanishes. We call such a connection between the two sheets a **bridge**. We can easily generalize the result of Einstein and Rosen considering that, without losing generality, a spherically symmetrical metric can always be written in the form:

$$ds^2 = e^{-2\phi(r)}(1 - \frac{b(r)}{r})dt^2 - \frac{dr^2}{(1-\frac{b(r)}{r})} - r^2 d\Omega^2 \tag{3.4}$$

This metric has horizons corresponding to those values of r such that $b(r_H) = r_H$. If a horizon is present, the geometry describes a black hole. We introduce the u coordinate by placing:

$$u^2 = r - r_H \tag{3.5}$$

Then (3.4) becomes:

$$ds^2 = e^{-(u^2+r_H)} \frac{r_H+u^2-b(r_H+u^2)}{r_H+u^2} dt^2 - 4\frac{r_H+u^2}{r_H+u^2-b(r_H+u^2)} u^2 du^2 - (r_H + u^2)^2 d\Omega^2 \tag{3.6}$$

The region close to u = 0 is the bridge connecting the asymptotically flat region close to $u = +\infty$ to the asymptotically flat one close to $u = -\infty$. Let's analyze the trends of the metric (3.6) for $u \approx 0$ and $u \mapsto \pm\infty$: in the first case, that is, near the throat of the wormhole:

$$ds^2 = e^{-\phi(r_H)} \frac{u^2[1-b'(r_H)]}{r_H} dt^2 - 4\frac{r_H+u^2}{1-b'(r_H)} du^2 - (r_H+u^2)^2 d\Omega^2 \tag{3.7a}$$

where with the apex we have indicated the derivative with respect to the radial coordinate r. In the second case, however, the asymptotic trend implies that $\phi \mapsto 0$ and $b \mapsto 2m$, for which the metric becomes:

$$ds^2 = \frac{r_H+u^2-2M}{r_H+u^2} dt^2 - 4\frac{r_H+u^2}{r_H+u^2-2M} u^2 du^2 - (r_H+u^2)^2 d\Omega^2 \tag{3.7b}$$

The narrowest part of the geometry is defined as the **throat**. In the Schwarzschild wormhole, the throat is located right at the event horizon. The region near the throat, however, it is called **bridge**.

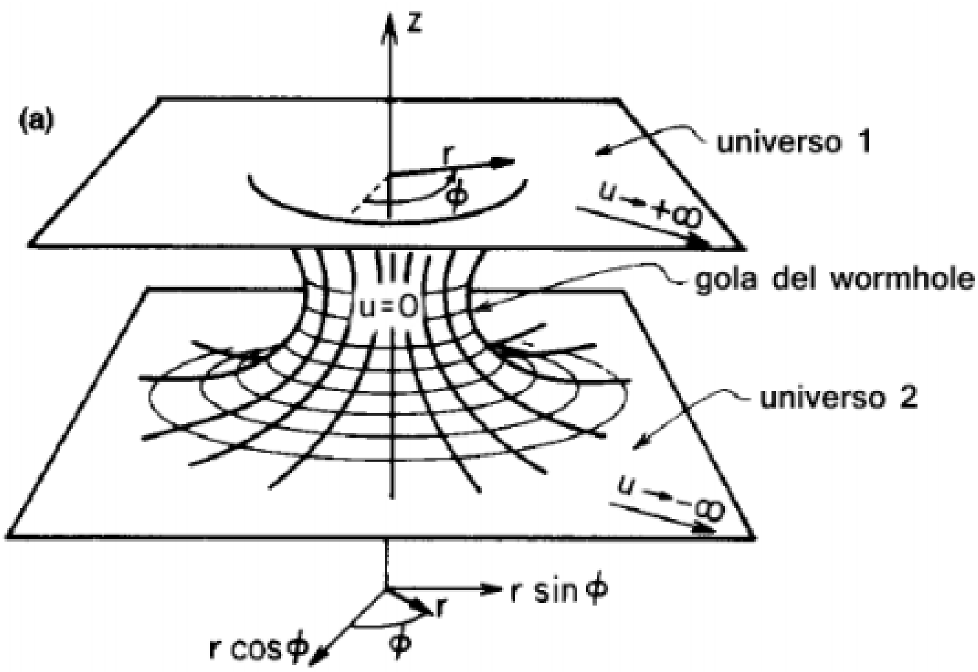

Figure 3.1(a) Wormhole inter-universe. The radial coordinate u is related to r by the relation $u = \pm\sqrt{r - r_H}$, with the + sign referring to the universe placed higher and the sign - referring to that placed further down. The graph represents the geometry of the wormhole at a given instant of time (t = cost).

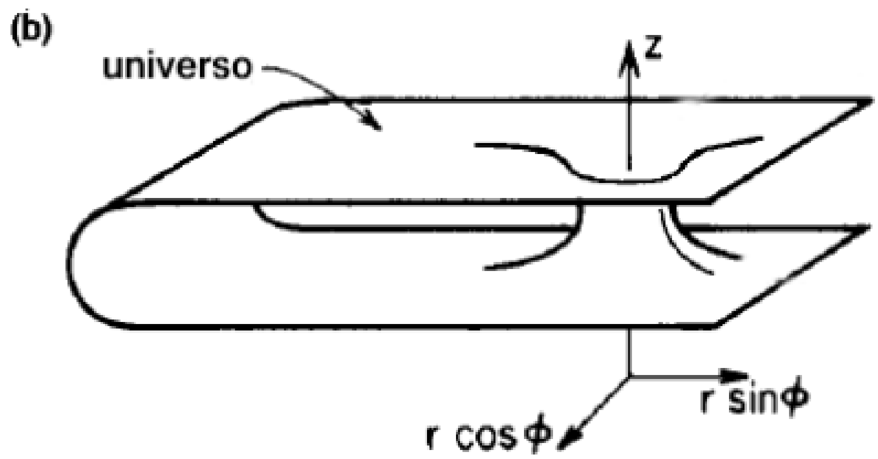

Figure 3.1(b) Intra-universe wormhole. Similarly to Fig. (3.2), the graph represents the geometry of the wormhole at a given instant of time (t = cost).

We also note that for this bridge construction we must take m > 0, as if we have assumed m < 0, our bridge construction will fail since we require the existence of a horizon for this coordinate transformation to work. Howevere, from this description, it is clear that the main feature of Schwarzschild wormhole is the presence of an event horizon. Now an important question: are Einstein-Rosen bridges traversable? No. When we began the construction of our Schwarzschild wormhole, we started with the Schwarzschild solution which is static, with a finite throat with circumference of 2m. This is true in the region far away from the throat, since the Schwarzschild solution carries no time dependence. Can we say that it is the same for the regions close to the Schwarzschild throat? No! As we have already said in the introduction, it was argued by Fuller and Wheeler that the Schwarzschild throat is dynamic, that the throat opens and closes. This "pinch off" of the throat as they called it, happens so fast that even a particle travelling at the speed of light cannot get through the wormhole. The light will be pinched off and trapped in a region of infinite curvature when the throat closes. So they are not traversable since:

- Tidal gravitational forces at the throat are great. Traveller is killed unless wormhole's mass exceeds $10^4 M_S$ so the throat circumference will exceed 10^5 km.
- Schwarzschild wormhole is not static but dynamic. As time pass, the throat starts from zero circumference to a maximum circumference and back again to zero. This happens so fast that even light will be trapped.
- the horizon is unstable under small perturbations.

3.3 TRAVERSABLE WORMHOLES

We can ask ourselves whether or not traversable wormholes exist. Let's try to understand what then must be the properties satisfied by the metric in order to solve the problem of traversability. Considering, for simplicity, the case of spherical symmetry (and static), we expect that:

- Metric should be both spherically symmetric and static. This is just to keep everything simple.
- Solution must everywhere obey the Einstein field equations. This assumes correctness of GR.
- Solution must have a throat that connects two asymptotically flat regions of spacetime.
- No horizon, since a horizon will prevent two-way travel through the wormhole.
- Tidal gravitational forces experienced by a traveler must be bearably small.
- Traveler must be able to cross through the wormhole in a finite and reasonably small proper time.
- Physically reasonable stress-energy tensor generated by the matter and fields.
- Solution must be stable under small perturbation.
- Should be possible to assemble the wormhole, i.e. assembly should require both much less than the total mass of the universe and much less than the age of the universe.

Our construction of the wormhole should at least satisfy the first four criteria. Morris and Thorne calls this the "basic wormhole criteria". The following three criteria (from the fifth to the seventh) are called "usability criteria" since it deals with human physiological comfort. Thus we need to find a solution that will satisfy the basic wormhole criteria. We will take the simple approach of Morris and Thorne. In 1987 Morris and Thorne realized that this type of structure was possible and thus they began their analysis on traversable wormholes , considering first of all a geometry that satisfied the above requirements. They passed then to the computation of the components of the Riemann tensor and used the equations of Einstein's field to deduce what the mass-energy distribution should be. In particular, they observed that the impulse-energy tensor, near the throat, violated the so-called weak energy condition (WEC). Of fundamental importance in the study of wormholes and, in general, of Einstein's field equation are the conditions on energy.

Weak Energy Condition (WEC). "The weak energy condition states that for any timelike vector V:

$$T_{\mu\nu}V^{\nu}V^{\mu} \geq 0 \tag{3.8}$$

Physically, this implies that the weak energy condition forces the local energy density to be positive measured by any timelike observer. In terms of principal pressures,

$$WEC \iff \rho \geq 0 \qquad and \qquad \rho + \rho_j \geq 0, \qquad \forall j \tag{3.9}$$

where ρ is the energy density ".

3.4 THE MORRIS-THORNE METRIC

To simplify the discussion, Morris and Thorne considered a spherically symmetrical, time-independent, non-rotating wormhole. The variety of interest it was therefore that of a spherically symmetrical and static space-time, with two asymptotically flat regions. Let's start with the following metric, the **Morris-Thorne metric** (in this section and in the next one we will use the signature $(-,+,+,+)$, as done by Morris and Thorne in their original article):

$$ds^2 = -e^{\phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega^2 \tag{3.10}$$

where $b(r)$ and $\phi(r)$ are functions of the radial coordinate only and represent, respectively, the **shape function** and the **redshift function**. The throat of the wormhole is defined by the minimum value of the radial coordinate, $r_{min} = r_0$ given by the condition $b(r_0) = r_0$. Hence, r varies from r_0 to infinity. To be more precise, r has a non-monotone trend: decreases from $+\infty$ to r_0 when moving through the universe placed further down, increases from r_0 to $+\infty$ when you move out of the throat, towards the universe placed higher. In any static and asymptotically flat metric, including one that describes a wormhole, horizons are defined as non-singular surfaces in which $g_{00} \mapsto 0$. Then, in order to satisfy the third requirement , $\phi(r)$ is forced to have finite value $\forall r$. It is also required that the fields go to zero fast enough for $r \mapsto \infty$, in such a way as to have an **asymptotically flat space-time**:

$$\frac{b}{r} \mapsto 0 \qquad \text{and} \qquad \phi(r) \mapsto 0 \qquad \text{for } r \mapsto \infty \tag{3.11}$$

Another fundamental ingredient in wormhole physics is the **flaring-out condition**, which requires that (for more details see appendix):

$$\frac{b-b'r}{2b^2} > 0 \tag{3.12}$$

Finally, the last constraint for the function $b(r)$ is that $\frac{b(r)}{r} \leq 1$, with the valid equality only at the throat. This condition ensures that the **proper radial distance**:

$$l(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1-\frac{b(r)}{r}}} \tag{3.13}$$

has finite value and that the g_{rr} component of the metric tensor does not change sign $\forall r \geq r_0$.

3.5 THE STRESS-ENERGY TENSOR

Birkoff's theorem tells us that only one kind of vacuum, spherical, static and asymptotically flat wormhole is allowed by Einstein field equations: a (non traversable) Schwarzschild wormhole. Thus, a traversable wormhole must be threated by matter or fields with a non-zero (non-vacuum) stress-energy tensor. So let's write the field equations in matter for the metric (3.10). We will observe that the properties to be satisfied by the functions $b(r)$ and $\phi(r)$ will imply a strong constraint for the components of the momentum-energy tensor that generates the space-time curvature. Below, we report only the results obtained by Morris and Thorne in their 1987 paper, that we have already cited in the introduction. In particular, for the components of the

Einstein tensor, they obtained (using natural units, $c = 1$, $G = 1$):

$$G_{tt} = \frac{b'}{r^2} \tag{3.14}$$

$$G_{rr} = -\frac{b}{r^3} + 2(1 - \frac{b}{r})\frac{\phi'}{r} \tag{3.15}$$

$$G_{\theta\theta} = G_{\phi\phi} = (1 - \frac{b}{r})(\phi'' - \frac{b'r - b}{2r(r-b)}\phi' + \phi'^2 + \frac{\phi'}{r} - \frac{b'r - b}{2r^2(r-b)}) \tag{3.16}$$

Non-vanishing stress-energy tensor components should be the same non-vanishing components as the Einstein tensor. We denote the following:

$$T_{tt} = \rho(r) \qquad T_{rr} = -\tau(r) \qquad T_{\theta\theta} = T_{\phi\phi} = p(r) \tag{3.17}$$

where $\rho(r)$ is the total mass-energy density, $\tau(r)$ is the radial tension per unit area, and $p(r)$ is the pressure in the lateral direction (orthogonal to the radial one). Now we use,

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3.18}$$

Morris and Thorne arrived at the following field equations, as seen by static observers in an orthonormal frame :

$$b' = 8\pi r^2 \rho \tag{3.19}$$

$$\phi' = \frac{-8\pi\tau r^3 + b}{2r(r-b)} \tag{3.20}$$

$$\tau' = (\rho - \tau)\phi' - 2\frac{p+\tau}{r} \tag{3.21}$$

which can also be reversed and written in the form:

$$\rho = \frac{b'}{8\pi r^2} \tag{3.22}$$

$$\tau = \frac{\frac{b}{r} - 2(r-b)\phi'}{8\pi r^2} \tag{3.23}$$

$$p = \frac{r}{2}[(\phi - \tau)\phi' - \tau'] - \tau \tag{3.24}$$

We use the latter to write the following scalar function in terms of the geometric functions $b(r)$ and $\phi(r)$:

$$\zeta \doteq \frac{\tau - \rho}{|\rho|} = \frac{\frac{b}{r} - b' - 2(r-b)\phi'}{|b'|} \tag{3.25}$$

which we can conveniently rewrite as:

$$\zeta = \frac{2b^2}{r|b'|}\frac{b - b'r}{2b^2} - 2(r-b)\frac{\phi'}{|b'|} \tag{3.26}$$

(3.25) and (3.26), together with the finite value of ρ and consequently of b' (eq. (3.22)) and to the fact that, in the throat, $(r-b)\phi \mapsto 0$, makes the condition (3.12) be written as:

$$\zeta_0 = \frac{\tau_0 - \rho_0}{|\rho_0|} > 0 \tag{3.27}$$

at or near the wormhole throat. The constraint that follows:

$$\tau_0 > \rho_0 \tag{3.28}$$

is very strong: in the throat of the wormhole, the tension must be sufficiently large to exceed the total mass-energy density. Materials with the property $\tau > \rho > 0$ is called, **exotic**. The exotic nature of the wormhole's throat material is especially troublesome because of its implications for measurements made by an observer who moves through the throat with a radial velocity close to the speed of light, $\gamma >> 1$. Such an observer see in his frame a **negative energy density**. So clearly the WEC condition is violated by the result we obtained previously. So we may investigate whether this violation can occur or not. At least we can see some examples of observing this violation, due to quantum effects. An example of violation of energy condition is the **Casimir effect** [39].

3.6 THE TOPOLOGICAL CENSORSHIP

Morris and Thorne's work also represents the starting point for development of the "no wormhole" theorems. The types of wormholes discussed so far had some particular characteristics that simplified the treatment. However, in general, a wormhole could be asymmetrical, with an arbitrarily long throat and with a time-dependent geometry. The analysis of such configurations is rather difficult and requires the use of global techniques. Let's see, therefore, what are the topological properties and the related theorems that characterize a completely generic wormhole. First some definitions:

- **definition 3.1 : Null Energy Condition (NEC)** This condition states that the matter energy-momentum tensor $T_{\mu\nu}$ obeys:

$$T_{\mu\nu}n^{\mu}n^{\nu} \geq 0, \tag{3.29}$$

  for any null (light-like) vector n^{μ} , i.e., for any vector satisfying $g_{\mu\nu}n^{\mu}n^{\nu} = 0$.

- **definition 3.2 : Averaged Null Energy Condition (ANEC)**

$$\int_{\Gamma} T_{\mu\nu}n^{\mu}n^{\nu}d\lambda \geq 0 \tag{3.30}$$

  where Γ is a null curve, λ is a generic affine parameterization of the curve Γ, whose corresponding tangent vector is n.

- **definition 3.3: Globally Hyperbolic Spacetime**

  A spacetime M is **globally hyperbolic** if it has a Cauchy surface Σ (a hypersurface which is met exactly once by every inextendible causal curve).

- **definition 3.4: Trivial Causal Curve**

A **trivial causal curve** is a curve which moves from the infinite past to the infinite future, lying in the asymptotically flat region.

theorem 3.1: Topological Censorship

"In any globally hyperbolic and asymptotically flat spacetime, such that every inextensible, null type geodesic satisfies the ANEC, every causal curve from the past infinity to the future infinity is deformable to a trivial causal curve."

The topological censorship theorem allows us to have a mathematically precise and general definition of traversable wormhole:

- **definition 3.5** "If an asymptotically flat space-time M has a causal curve γ that extends from the past infinity to the future infinity and such that is not deformable to a trivial causal curve, then M has a traversable wormhole and the curve γ passes through the wormhole".

The fundamental point, therefore, is that a light ray (or an observer) is able to cross the wormhole and reach the other side. Starting from this definition, we obtain the inverse of the topological censorship theorem:

theorem 3.2

Any space-time containing a traversable wormhole or (1) is not globally hyperbolic or (2) is such that there exists at least one null inextensible geodesic along which the ANEC is violated.

The inverse theorem, therefore, states that a globally hyperbolic space-time that contains a traversable wormhole that satisfies Einstein's field equations must violate the ANEC. In the next chapter, as we have already anticipated in the ontroduction, we will utilize the Morris-Thorne metric to find out two wormhole solutions.

Chapter 4

A GENERAL APPROACH TO SPHERICALLY SYMMETRIC WORMHOLES

4.1 THE RICCI CURVATURE SCALAR IN SPHERICAL SYMMETRY

As standard, the Ricci scalar can be written as

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} [\Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\beta}_{\alpha\beta} \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\beta\mu} \Gamma^{\beta}_{\nu\alpha}] \tag{4.1}$$

where

$\Gamma^{\alpha}_{\mu\nu}$ is the Christoffel symbol defined as:

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}). \tag{4.2}$$

The most general spherically symmetric metric can be written as follows :

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2 d\Omega^2 \tag{4.3}$$

and, imposing the spherical symmetry (4.3), the Ricci scalar in terms of gravitational potentials $A(r)$ and $B(r)$ reads:

$$R(t,r) = \tfrac{B[\dot{A}\dot{B} - A'^2]r^2 + A[r(\dot{B}^2 - A'B') + 2B(2A' + rA'' - r\ddot{B}] - 4A^2[B^2 - B + rB']}{r^2 A^2 B^2} \tag{4.4}$$

where the prime indicates the derivative with respect to r while the dot with respect to t. If the metric is time-independent, i.e., $A(t,r) = A(r)$, $B(t,r) = B(r)$, we have :

$$R(r) = \tfrac{A(r)[2B(r)(2A'(r) + rA''(r)) - rA'(r)B'(r)] - B(r)A'(r)^2 r^2 - 4A(r)^2(B^2 - B(r) + rB'(r))}{2r^2 A(r)^2 B(r)^2} \tag{4.5}$$

where the radial dependence of the gravitational potentials in now explicitly shown. This expression can be seen as a constraint (a lagrangian bond) for the functions $A(r)$ and $B(r)$ once a specific form of Ricci scalar is given. In particular, it reduces to a Bernoulli equation of index two, that is:

$$B'(r) + h(r)B(r) + l(r)B(r)^2 = 0, \tag{4.6}$$

with respect to the metric potential $B(r)$ [40, 41, 42]:

$$B(r)' + \frac{r^2A'(r)^2 - 4A(r)^2 - 2rA(r)[2A(r)' + rA(r)'']}{rA(r)[4A(r) + rA'(r)]}B(r) + \frac{2A(r)[2 + r^2R(r)]}{r[4A(r) + rA'(r)]}B(r)^2 = 0 \tag{4.7}$$

4.2 WORMHOLE SOLUTIONS USING THORNE-MORRIS METRIC

Now we propose to find wormhole solutions using a particular spherically symmetric metric,the Morris-Thorne metric:

$$ds^2 = e^{2\phi}dt^2 - \frac{1}{1 - \frac{b(r)}{r}}dr^2 - r^2d\Omega^2 \tag{4.8}$$

where , in this case, :

$$A(r) = e^{2\phi}$$

and

$$B(r) = \frac{1}{1 - \frac{b(r)}{r}}$$

Inserting the Thorne- Morris metric into the equation (4.7):
we have a first-order linear equation of $b(r)$:

$b'(r)(2r+r^2\phi')+b(r)(-r\phi'+2r^2\phi'^2+4r\phi'+2r^2\phi'')-2r^3\phi'^2-4r^2\phi'-2r^3\phi''+$
$r^3R(r)=0$ (4.9)

whose general solution can be expressed in the following form :

$$b(r)=\frac{\int\frac{-[2r^3\phi'^2+4r^2\phi'+2r^3\phi''+r^3R(r)]}{2r+r^2\phi'}e^{\int\frac{(-r\phi'+2r^2\phi'^2+4r\phi'+2r^2\phi'')}{2r+r^2\phi'}dr}+C}{e^{\int\frac{(-r\phi+2r^2\phi'^2+4r\phi'+2r^2\phi'')}{2r+r^2\phi'}dr}} \tag{4.10}$$

where $\phi(r)$ is the so called "redshift function" and $b(r)$ is the "shape function" of the Morris-Thorne wormhole and C is an integration constant. Chosing the redshift function as $\phi(r)=\frac{k}{r}$, where k is an arbitrary constant and substitung it into the equation (4.10) and integrating with respect to r, we obtain the following expression for the denominator D(r) of (4.10):

$$D(r)=\int\frac{\frac{k}{r}+\frac{2k^2}{r^2}}{2r-k}dr=r^{-5}(r-\tfrac{k}{2})^5e^{\frac{2k}{r}}$$

and for the numerator N(r):

$$N(r)=\int\frac{-D(r)(r^3R(r)+\frac{2k^2}{r})}{2r-k}dr=$$
$$\int\frac{-r^{-5}(r-\frac{k}{2})^5e^{\frac{2k}{r}}(\frac{2k^2}{r}+R(r)r^3)}{(2r-k)}dr+C$$

$$=-\int\frac{(r-\frac{k}{2})^4k^2e^{\frac{2k}{r}}}{r^6}dr$$

$$-\int\frac{R(r)e^{\frac{2k}{r}}(r-\frac{k}{2})^4}{2r^2}dr$$

where we can solve the first one, D(r), considering that:

$(r - \frac{k}{2})^4 = r^4 + 4r^3(-\frac{k}{2}) + 6r^2(\frac{k^2}{4}) + 4r(-\frac{k^3}{8}) + \frac{k^4}{16}$

and using the reduction formula :

$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

So we have for the shape-function, $b(r)$, the following expression:

$$b(r) = \frac{-\int \frac{e^{\frac{2k}{r}}}{2}[r^2 - 2kr + \frac{3}{2}k^2 - \frac{k^3}{2r} + \frac{k^4}{16r^2}]R(r)dr - f(r) + C}{e^{\frac{2k}{r}} r^{-5} (r - \frac{k}{2})^5} \tag{4.11}$$

where $f(r) = Ke^{2k/r}(\frac{2K^4 - 20K^3r + 78K^2r^2 - 142kr^3 + 103r^4}{64r^4})$ (4.12)

Assuming for the Ricci scalar $R(r) = \frac{1}{r^4}$ and placing the constant $k = 1$,we obtain:

$$b(r) = \frac{-f(r) + C + g(r)}{D(r)} \tag{4.13}$$

where $g(r)$ is:

$$g(r) = -\int \frac{e^{\frac{2}{r}}(r^2 - 2r + \frac{3}{2} - \frac{1}{2r} + \frac{1}{16r^2})}{2r^4} dr = \frac{e^{\frac{2}{r}}(2 - 20r + 78r^2 - 142r^3 + 103r^4)}{128r^4} = \frac{f(r)}{2} \tag{4.14}$$

So, definitively, we have for the shape function, $b(r)$, the following expression:

$$b(r) = \frac{C - \frac{e^{\frac{2}{r}}(2-20r+78r^2-142r^3+103r^4)}{128r^4}}{r^{-5}(r-\frac{1}{2})^5 e^{\frac{2}{r}}} \tag{4.15}$$

Now we can notice that $\frac{b(r)}{r}$ goes to 0 if r goes to infinity and , so :

$$\frac{1}{1-\frac{b(r)}{r}}$$

goes to 1 if r goes to infinity. So this metric respects the asymptotic flatness condition. If we consider the symplest spherically symmetric metric:

$$ds^2 = (1 - \frac{\tilde{b}(r)}{r})dt^2 - \frac{dr^2}{(1-\frac{\tilde{b}(r)}{r})} - r^2 d\Omega \tag{4.16}$$

and put it into the equation (4.7), where , in this case, $\tilde{A}(r) = 1 - \frac{\tilde{b}(r)}{r}$ and $\tilde{B}(r) = \frac{1}{\tilde{A}(r)}$, then we obtain the following equation for $\tilde{b}(r)$:

$$\tilde{b}(r)'' + \frac{2\tilde{b}(r)'}{r} + rR(r) = 0 \tag{4.17}$$

which can be resoluted with the Lagrange's method of variation of arbitrary constants, whose solution in function of the Ricci scalar $R(r)$ is:

$$\tilde{b}(r) = C_1 + \frac{C_2}{r} - \frac{\int r^2 R(r)dr}{2} - \frac{\int r^4 R(r)dr}{2} \tag{4.18}$$

Assuming for the Ricci scalar ,$R(r) = \frac{1}{r^4}$ we have :

$$\tilde{b}(r) = C_1 + \frac{C_2}{r} + \frac{1}{2r} - \frac{1}{2} \tag{4.19}$$

And so:

$$\frac{\tilde{b}(r)}{r} = \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{1}{2r^2} - \frac{1}{2r} \tag{4.20}$$

and if r goes to infinity then $\frac{\tilde{b}(r)}{r}$ goes to 0 and this metric respects the asymptotic flatness condition too. We can recognize that the second kind of metric , the metric (4.19) , is a Reissner–Nordström-like metric, for an opportune choice of the arbitrary constants C_2 and C_1, even if there is no charge. In fact the Reissner–Nordström metric is:

$$ds^2 = (1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2})dt^2 - (1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2})^{-1}dr^2 - r^2 d\Omega^2, \tag{4.21}$$

which is a static solution to the Einstein field equations, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body of mass M. In this metric r_S is the Schwarzschild radius, $r_S = \frac{2GM}{c^2}$, while $r_Q^2 = \frac{Q^2 G}{4\pi\epsilon_0 c^4}$ and Q is the charge of the body. Howevere, we can notice that for a particular choice of the constant C_2 ($C_2 = -\frac{1}{2}$) we obtain a Schwarzschild-like metric. So this is a non-traversable wormhole because presents an event horizon. If we consider the equation (4.11) and assume the constant $K = r_0/2$ and for the Ricci scalar, $R(r) = g/(4r^4)$, where g is a new constant, we obtain:

$$\phi(r) = \frac{b(r)}{r} = \frac{C_1 e^{-r_0/r} r^4}{(4r-r_0)^5} - \frac{[(2r_0^2+g)(r_0^4-20rr_0^3+156r_0^2r^2-568r_0r^3+824r^4)]}{((2r_0)(4r-r_0)^5)} \tag{4.22}$$

And, if we assume $g = -4r_0^2$, we obtain the following solution :

$$\phi(r) = -\frac{r_0^2(r_0^4-20rr_0^3+156r^2r_0^2-568r^3r_0+824r^4}{r_0(4r-r_0)^5} + \frac{C_1 e^{-\frac{r_0}{r}} r^4}{(4r-r_0)^5} \tag{4.23}$$

We immediately notice how (4.23) satisfies the asymptotic condition: tends to zero for $r \mapsto \infty$. We have therefore found a class of functions $\phi(r)$ that satisfy the geometric properties necessary for the traversability of the wormhole.

4.3 THE FLARING-OUT CONDITION

The flaring-out condition plays a fundamental role in wormhole physics. Its validity, in fact, implies the violation of the conditions on energy, as we have could see in the third chapter and, in particular, from (3.27). Let's see what happens when it is applied to the metric under consideration, that is (4.23). First, we observe that the flaring-out condition, in this case, becomes:

$$\frac{\phi'(r)}{2\phi(r)^2} < 0 \tag{4.24}$$

being $\phi(r) = \frac{b(r)}{r}$. So, the flaring-out condition is violated if:

$$\frac{\phi(r)'}{2\phi(r)^2} \geq 0 \tag{4.25}$$

Taking into account the solution (4.23) and its derivative:

$$\phi(r)' = \frac{1}{r_0(r_0-4r)^6}[(r_0 - 4r)[-4c_1 r_0 r^3 e^{-\frac{r_0}{r}} - c_1 r_0^2 r^2 e^{-\frac{r_0}{r}} + r_0^2(-20r_0^3$$

$$+312r_0^2 r - 1704 r_0 r^2 + 3296 r^3)] - 20 c_1 r_0 r^4 e^{-\frac{r_0}{r}} + 20 r_0^2 (r_0^4 - 20 r_0^3 r + 156 r_0^2 r^2 - 568 r_0 r^3 + 824 r^4)] \tag{4.26}$$

So, the (4.25) becomes:

$$\frac{r_0}{r^2}\frac{e^{\frac{r_0}{r}}}{r_0^2+4r^2}[(r_0 - 4r)(-20r_0^3 + 312r_0^2 r - 1704 r_0 r^2 + 3296 r^3) + 20(r_0^4 - 20 r_0^3 r + 156 r_0^2 r^2 - 568 r_0 r^3 + 824 r^4)] \geq c_1 \tag{4.27}$$

It has been demonstrated [43] that (4.23) is a solution only in a particulare extended gravity theory, the f(R)-gravity. It is not a solution in the GR-case. Moreover, it's possible to see that this solution satisfy the conditions on energy.

Chapter 5

DISCUSSION AND CONCLUSIONS

In this thesis work we discussed a general approach to wormhole solutions. Our starting point was a lagrangian constraint (eq. (4.5)) for the Ricci scalar $R(r)$ as a function of the gravitational potentials $A(r)$ and $B(r)$ and their corresponding derivatives, which has led to a Bernoulli equation (see (4.6) and (4.7)). It is important to stress that this lagrangian constraint yields different classes of solutions that in general are not solutions also for the Einstein fields equation. Subsequently, we used a spherically symmetric static metric, the Morris-Thorne metric, whit the aim to select two general classes of wormhole solutions, choosing a particular function of the Ricci scalar ($R = R_{\mu\nu}R^{\mu\nu}$), namely $R(r) = \frac{1}{r^4}$. We physically expect this subcase to occur for particular choices of the starting action. The first class of wormholes, represented by eq. (4.23), respects the conditions of energy [43] and violate the flaring-out condition for a specific choice of the arbitrary integration constant. One can check the physical relevance of such potential by means of astrophysical data. It is worth noticing that a general case is treated in this work, where the scalar curvature does not vanish identically, unlike the well known Schwarzschild-deSitter case. The solution (4.23), as mentioned above, is a solution in the f(R)-gravity context, where f(R) is a generic function of the Ricci scalar. In the cosmological framework, however, the f(R)-gravity manifests an effective energy-momentum tensor of the gravitational field which can play the role of Dark Energy. Thanks to astrometric and cosmological observations an exponentially accelerated universe emerges, which is supposed to include 4 % of baryonic matter, 20 % of dark matter and 76 % of dark energy, with exotic properties. Several attempts to explain the origin and nature of dark energy and dark matter were performed, as well as, the introduction of the cosmological constant in the field equations. Nevertheless, it has been observed that its value is enormously smaller than the vacuum energy predicted by field theory in curved spacetimes. Otherwise we can think of dark energy and dark matter as unknown forms of energy and matter which escape direct detection and do not thicken like ordinary matter. Moreover, this dark energy must have negative pressure (and therefore exotic), as it is possible to infer from Friedmann's equations, $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$, to satisfy the accelerated expansion. So this implies the violation of the **strong energy condition**, $\rho + 3P \geq 0$. Another explanation for dark energy is the so called **quintessence**, according to which the universe expansion should be driven by a scalar field. However, this does not provide a satisfactory solution to the problem. So an alternative to dark energy can be found in the context of extended theories of gravity. In particular, it has been observed [44, 45] that, for a f(R)-gravity theory, it is possible to obtain a positive energy density. So the cosmology described by f(R)-theory can reproduce an accelerated expansion without having to resort to dark energy. Subsequently the second solution found is a Reissner–Nordström-like solution even if there is no charge and are a class of solutions also in the General Relativity case for a particular choice of the integration constant C_2. Therefore this kind of solution presents an event horizon and cannot be used to describe stable wormholes. Schwarzschild's wormholes are theoretically plausible but also if they exist, they will probably never be crossed, even by a very advanced civilization. On the other hand Morris-Thorne wormholes are certainly traversable and, therefore, if it were possible to find scientific evidences confirming the existence of the material that makes them, an advanced civilization could exploit them to carry out interstellar (and temporal) travel, covering such huge distances in a short time. Regarding the above mentioned Morris-Thorne wormhole with exotic matter, the Casimir effect shows that quantum field theory allows the energy density in certain regions of space to be negative relatively to the ordinary vacuum energy, and it has been shown theoretically that quantum field theory allows states where energy can be arbitrarily negative at a given point. Many physicists, such as Stephen Hawking, Kip Thorne [46] and others, argued that such effects might make it possible to stabilize a traversable wormhole. In the Casimir effect, in the region between two conducting plates held parallel at a very small separation d, there is a negative energy density: $\epsilon = -\frac{\pi^2 \hbar}{720 d^4}$, between the plates. Finally the flaring-out condition, which in General Relativity implied the above energy constraint, is violated for the first metric (see equation (4.31)), the traversable wormhole's one. Therefore in the f(R)-gravity context it is no longer the exotic matter to support these structures and make them traversable but they are higher order terms in the curvature scalar. We have suitably modified only the geometric part of the field equations of Einstein, planting unchanged that of matter (standard).

It is important to point out right away how wormholes, unlike black holes, are something purely theoretical: to this date one has never been observed and there is no scientific evidence to suggest its existence. However at the same time, there is no theoretical reason nor experimental evidence that these objects cannot exist , therefore it makes sense to talk about them and hypothesize that they can be used to carry out interstellar travel. Ultimately the method described in this thesis work is a general approach to wormhole and black hole solutions of the Einstein field equations and does not depend of the particular class of the field equation; in fact we have found two solutions: the first in f(R)-gravity and the second in the General Relativity case. We have found static solution because we have eliminated the time-dependence in the constraint (4.5).

Chapter 6

APPENDIX

6.1 Kruskal and Eddington-Finkelstein Coordinates

The Kruskal coordinates are defined as follows. Let us consider the $r > 2M$ manifold, and define

$$u \doteq t - r_*, \qquad v \doteq t + r_* \tag{6.1}$$

with :

$$r_* \doteq r + 2M ln(\tfrac{r}{2M} - 1) \tag{6.2}$$

(also known as **Regge-Wheeler range**) which tends to $-\infty$ as r tends to 2M. Notice that $r > 2M$ corresponds to

$$-\infty < u < +\infty \qquad -\infty < v < +\infty \tag{6.3}$$

and the limit $r \longmapsto 2M, \quad t \longmapsto +\infty$ corresponds to $u \longmapsto +\infty$, with finite v.

Since

$$\frac{dr_*}{dr} = \frac{1}{1-\frac{2M}{r}} \tag{6.4}$$

the metric is:

$$ds^2 = (1-\frac{2M}{r})(dt^2 - dr_*^2) - r^2(d\theta^2 + sen^2\theta d\phi^2) \tag{6.5}$$

$$ds^2 = (1-\frac{2M}{r})dudv - r^2 d\Omega \tag{6.6}$$

We have :

$$\frac{r_* - r}{2M} = ln(\frac{r}{2M} - 1) \tag{6.7}$$

from which :

$$1 - \frac{2M}{r} = \frac{2M}{r} e^{\frac{r_* - r}{2M}} = \tag{6.8}$$

$$= (\frac{2M}{r}) e^{\frac{-r}{2M}} e^{\frac{v-u}{4M}} \tag{6.9}$$

Thus, defining:

$$U \doteq -e^{\frac{-u}{4M}} \qquad V \doteq e^{\frac{v}{4M}} \tag{6.10}$$

we have the Kruskal metric:

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} dUdV - r^2(d\theta^2 + sen^2\phi^2) \tag{6.11}$$

with $U < 0$, $V > 0$.

Let us now consider the manifold $0 < r < 2M$, and define u, v as in (6.12),

$$u \doteq t - r_*, \qquad v \doteq t + r_* \tag{6.13}$$

but with a new coordinate sistem:

$$r_* \doteq r + 2M ln(1 - \frac{r}{2M}) \tag{6.14}$$

which is always negative, tends to zero as $r \longmapsto 0$, and to ∞ as r $\longmapsto \infty$. Differentiating this new coordinates we get the same expression as (6.4),

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{2M}{r}} \tag{6.15}$$

then the metric in u, v is still given by:

$$ds^2 = (1 - \tfrac{2M}{r})dudv - r^2 d\Omega \tag{6.16}$$

but :

$$1 - \tfrac{2M}{r} = -\tfrac{2M}{r} e^{\frac{-r}{2M}} e^{\frac{v-u}{4M}} \tag{6.17}$$

thus, defining:

$$U \doteq +e^{\frac{-u}{4M}}, \qquad V \doteq +e^{\frac{v}{4M}} \tag{6.18}$$

we have the same metric of eq. (6.11),

$$ds^2 = \tfrac{32M^3}{r} e^{-\frac{r}{2M}} dUdV - r^2(d\theta^2 + sen^2\phi^2) \tag{6.19}$$

but with $U > 0$,$V > 0$.

This metric, with $V > 0$ and extended to $-\infty < U < +\infty$, describes then the exterior and the interior of the BH, as anticipated.

Actually, there is a simpler coordinate system that covers the region I and II: the **Eddington-Finkelstein (EF) coordinates**:

$$(v, r, \theta, \phi) \qquad -\infty < v < +\infty, \qquad 0 < r < +\infty \tag{6.20}$$

these coordinate system is the **ingoing Eddington-Finkelstein coordinate system**. The two definitions (6.2), (6.15) can be put together as

$$r_* = r + 2Mln|\tfrac{r}{2M} - 1| \tag{6.21}$$

and $v = t + r_*$.

Then, since

$$dt^2 - dr_*^2 = dv^2 - 2dvdr_* = dv^2 - 2dvdr\tfrac{dr_*}{dr} = dv^2 - \tfrac{2dvdr}{(1-\frac{2M}{r})} \tag{6.22}$$

the metric in the EF coordinates is

$$ds^2 = (1 - \tfrac{2M}{r})dv^2 2dvdr - r^2 d\Omega \tag{6.23}$$

This metric covers both the interior and the exterior of the BH, i.e. the sectors I and II of the Kruskal construction, and is not singular at the horizon, which is simply $r = 2M$. Notice that on the horizon v is finite because $t \mapsto +\infty$ and $r \mapsto \infty$, while u is instead divergent.

6.2 The Flaring-Out Condition

To construct an embedding diagram of the wormhole one considers the geometry of a t = const. slice. Using the spherical symmetry, we can set $\theta = \frac{\pi}{2}$ (an "equatorial" slice). The metric on the resulting two-surface is

$$ds^2 = \frac{dr^2}{(1-\frac{b(r)}{r})}+r^2d\phi^2 \tag{6.24}$$

The three-dimensional Euclidean embedding space metric can be written as:

$$ds^2 = dz^2+dr^2+r^2d\phi^2 \tag{6.25}$$

Since the embedded surface is axially symmetric, it can be described by z = z(r), sometimes called the "lift function". The metric on the embedded surface can then be expressed as:

$$ds^2 = [1+(\frac{dz}{dr})^2]dr^2+r^2d\phi^2 \tag{6.26}$$

Equation (6.26) will be the same as Eq. (6.25) if we identify the r, ϕ coordinates of the embedding space with those of the wormhole spacetime, and also require:

$$\frac{dz}{dr} = \pm(\frac{r}{b(r)} - 1)^{-1/2} \tag{6.27}$$

A graph of z(r) yields the characteristic wormhole picture, like in figure 6.1. For the space to be asymptotically flat far from the throat, Morris and Thorne require that $dz/dr \mapsto 0$ as $l \mapsto \pm\infty$, i.e., $b/r \mapsto 0$ as $l \mapsto \pm\infty$, where l is the proper lenght, $l(r) = \pm \int_{b_0}^{r} \frac{dr}{(1-\frac{b(r)}{r})^{1/2}}$. In order for this condition to be satisfied, the wormhole must flare outward near the throat, i.e.,

$$\frac{d^2r(z)}{dz^2} > 0 \tag{6.28}$$

at or near the throat. Therefore:

$$\frac{d^2r(z)}{dz^2} = \frac{b-b'r}{2b^2} > 0 \tag{6.29}$$

at or near the throat, $r = b = b_0$, where the prime denotes differentiation with respect to r. The last equation condition in geometry also represent the minimality of the wormhole throat.

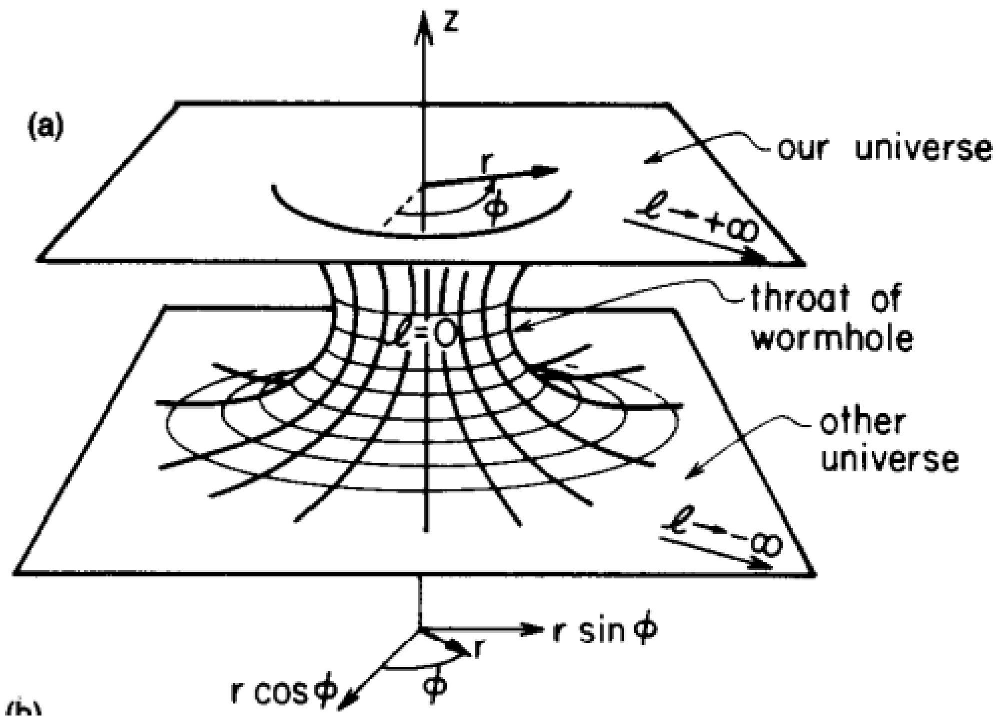

Figure 6.1 : the embedded shape of the wormhole

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