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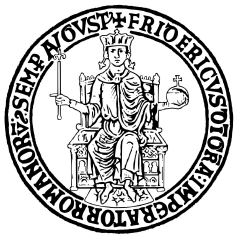

Scuola Politecnica e delle Scienze di Base

Area Didattica di Scienze Matematiche Fisiche e Naturali

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On Spontaneous Symmetry Breaking in Quantum Theory

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Chapter 1

Symmetries

1.1 Classical and quantum-mechanical view

Studying the symmetries of a given system can be a way to understand its dynamics. Our aim is to introduce the idea of symmetry and the most common formalism used to study the symmetries of a given system.

Let us introduce a classical problem which shows the distinctions between a classical description and the quantum-mechanical description: the double well potential. Consider a particle moving under the effect of a double well potential like that shown in the picture below:

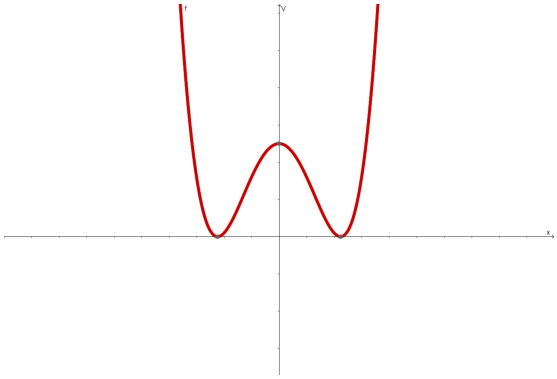

Figure 1.1: A double well potential.

In both cases, the Hamiltonian of the particle is the following:

$$H = \frac{p^2}{2m} + V(x), \qquad V = -Ax^2 + Bx^4.$$

First of all we notice that this Hamiltonian is invariant under reflection respect to

the origin of the frame of reference. We say it is symmetric under parity transformation. We would now like to discuss the classical and quantum-mechanical counterpart of this problem. The classical case is clear: the particle will be localized in one of the two well depending on the initial condition of its motion. Its final state will not be symmetric under parity transformations anymore: we say symmetry breaking occurred.

In order to describe the quantum-mechanical case, we start from our initial assumption: The Hamiltonian of the system is symmetric under parity transformation. Considering the Hamiltonian operator acting on a generic state $|\psi\rangle$, and with P representing the operator which gives the parity transformations, this means:

$$PH|\psi\rangle = HP|\psi\rangle \Rightarrow [P, H] = 0.$$

The operator P is an operator such that:

$$P\psi(x) = \psi(-x) \Rightarrow P^2\psi(x) = \psi(x) \Rightarrow P^2 = 1.$$

So we can say $P = \pm 1$, that is the eigenvalues of the Parity operator are either $P = 1$ or $P = -1$. This is an important conclusion: Every function ψ representing a quantum state has defined parity, since it has to be either even ($P = 1$) or odd ($P = -1$) under parity transformation. Coming back to our description of the double well potential, if we consider a very high central wall, we would be tempted to say the two wells are independent, so we will have a $\psi_R(x)$ or a $\psi_L(x) = \psi_R(-x)$, with clear analogy with classical case. This statement is wrong: in this case we would have a double degeneration of each energy level corresponding either to $\psi_R^n(x)$ or to $\psi_L^n(x)$, and this is not possible since we're studying a 1-dimensional system. In our discussion we ignored a fundamental issue of the quantum description of 1-dimensional systems: even with an infinitely high central wall, we know the two wells can not be considered "independent", since the probability of the particle passing from one to the other will be non-zero, even if very small. This is the tunnel effect, one of the most famous and mind-blowing consequence of quantum mechanics. In the quantum description of the system, the state of the particle will be an overlap of the two states $\psi_R^n(x)$ and $\psi_L^n(x)$:

$$\psi = A\psi_R(x) + B\psi_L(x).$$

We know, from what we said before, that a energy eigensate must have a defined parity. Considering that solutions of 1-dimensional potentials have $0, 1, 2, ..., n$ nodes respectively in the energy level $0, 1, 2, ..., n$, we want the ground state to be even, the first to be odd and so on. We can write these two states as follows:

$$\psi_0 = A\psi_R^0(x) + B\psi_L^0(x),$$

$$\psi_1 = C\psi_R^1(x) - D\psi_L^1(x).$$

So we notice that, in quantum description, the ground state of the system is still invariant under parity transformation: the symmetry breaking does not occurs due to the tunnel effect.

1.2 Mathematical Formalism for Symmetries

1.2.1 The Hamiltonian Formalism

We say a vector $|\psi\rangle$ describing the state of a quantum system is symmetric under a given transformation U if and only if $U|\psi\rangle = e^{i\phi}|\psi\rangle$. This means the action of the operator U does not change the vector $|\psi\rangle$ up to a phase factor which does not affect measured physical quantities.

In the example of the double-well potential exposed above we noticed the Hamiltonian is invariant under the symmetry transformation. In fact, as well as the sates of a given system, laws of nature can be symmetric or non-symmetric under a given transformation. In the Hamiltonian formalism the dynamics of a system is related to its Hamiltonian function, so we understand it is important to study the possible symmetries of a given Hamiltonian.

Recalling what we said in the previous section, in quantum mechanics whichever operator A is said to be symmetric under a given transformation U if and only if $[U, A] = 0$. Equivalently:

$$[U, A] = 0 \Rightarrow UA - AU = 0 \Rightarrow U^\dagger AU = A.$$

Consider a transformation U that is a symmetry of the Hamiltonian, and let $|\psi\rangle$ be an eigenstate of the Hamiltonian. We can write:

$$UH|\psi\rangle = UE|\psi\rangle = EU|\psi\rangle.$$

With E the eigenvalue of the Hamiltonian. This relation means that the state $U|\psi\rangle$ is an eigenstate of the Hamiltonian with same eigenvalue of $|\psi\rangle$. Furthermore, this means that if $|\psi\rangle$ is not symmetric under the action of the operator U, we will have a different state with same energy, so the energy level is degenerate.

Every continuous symmetry transformation is unitary, and that means that, given a transformation U there exist an operator Q such that[1]:

$$U = e^{iQ} = \sum_n \frac{(iQ)^n}{n!}.$$

And we call the operator Q the *generator* of the transformation.

As we know from Schrödinger equation, The Hamiltonian is the generator of the time-evolution operator. In fact we know:

$$U_T = e^{i\frac{H}{\hbar}t} = \sum_n \frac{(i\frac{H}{\hbar}t)^n}{n!}.$$

[1] In case of unbounded operators, Spectral theorem in necessary, but the result is unaffected.

Having defined this expression with its power series, we understand that every operator which commute with Hamiltonian commute with the time-evolution operator. Considering such an operator Q:

$$\langle\psi(t)|Q|\psi(t)\rangle = \langle\psi|U^\dagger QU|\psi\rangle = \langle\psi|Q|\psi\rangle.$$

So we understand that the expectation value of whichever operator commuting with the Hamiltonian is a constant of motion, since its expectation value does not depend on time.

Consider now continuous transformations, transformations which can be parameterised with a continuous parameter as follows:

$$U_\alpha = e^{i\alpha Q}.$$

Being a continuous transformation, we can Taylor expand this expression around $\alpha = 0$ and we have:

$$U_\alpha = 1 + i\alpha Q + o(Q^2).$$

A Global operator can be expressed as a function of a local operator:

$$Q = \int d^D x \rho(\vec{x}, t),$$

Where D is the dimension of the space we are in. For the global conservation of Q, we see that if $\rho(\vec{x}, t)$ increases in a point $\vec{x}$ of the space, then it will lower in a point $\vec{x}'$. We understand that this statement implies the existence of a *current of* ρ flowing in the space, as well as the well known case for the conservation of the electric charge and, recalling this analogy, we can write:

$$\partial_t \rho(\vec{x}, t) = -\partial_n j^n(\vec{x}, t) \qquad \Rightarrow \qquad \partial_t \rho(\vec{x}, t) + \partial_n j^n(\vec{x}, t) = \partial_\nu j^\nu(\vec{x}, t) = 0.$$

So, starting form the assumption of the conservation of the operator Q we have found a continuity relation. It is readily seen that the inverse implication is valid:

$$\begin{aligned} 0 &= \int d^D x \partial_\nu j^\nu(\vec{x}, t) = \int d^D x \partial_t \rho(\vec{x}, t) + \int d^D x \partial_n j^n(\vec{x}, t) = \\ &= \partial_t Q + \oint dS j^n(\vec{x}, t). \end{aligned}$$

Sending S to infinity and assuming $j^n(\vec{x}, t)$ vanishing quickly, the second term in the second line vanishes.

1.2.2 Lagrangian Formalism: Noether's Theorem

Noether's Theorem is valid for every physical problem in which we can introduce a Lagrangian formalism. The enunciate of the Noether Theorem is the following:

Theorem 1 (Noether's Theorem)**.** *To any continuous symmetry of a local action, corresponds a current $j_\nu(\vec{x}, t)$ that is totally conserved*

We consider a set of fields as $\phi_\alpha = \phi_\alpha(\vec{x}, t)$, and define the Lagrangian density as a function of both fields and their derivatives: $\mathcal{L} = \mathcal{L}(\phi, \partial_\nu \phi)$. The action is then defined as:

$$S = \int dt d^D x \mathcal{L}.$$

Let's define the variation of the field under a transformation at a fixed point as:

$$\delta_s \phi_\alpha(\vec{x}) = \phi'_\alpha(\vec{x}) - \phi_\alpha(\vec{x}) \qquad \Rightarrow \qquad \phi'_\alpha(\vec{x}) = \phi_\alpha(\vec{x}) + \delta_s \phi_\alpha(\vec{x}),$$

And, respectively, for the lagrangian:

$$\mathcal{L}' = \mathcal{L} + \delta_s \mathcal{L}$$

Being $\mathcal{L} = \mathcal{L}(\phi, \partial_\nu \phi)$, we can then write:

$$\delta_s \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha} \delta_s \phi_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \delta_s(\partial_\nu \phi_\alpha).$$

We explicitly notice that here we use the condition of the theorem of the symmetry to be continuous, since we're considering infinitesimal variations.

Recalling the definition of action $S = \int dt d^D x \mathcal{L}$ we see that if $\delta_s \mathcal{L} = 0$ or $\delta_s \mathcal{L} = \partial_\nu K^\nu$ with K^ν a generic function, then the action will be invariant up to a constant contribute. Now we can write:

$$\delta_s \mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \delta_s \phi_\alpha \right) + \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\alpha} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \right) \right\} \delta_s \phi_\alpha.$$

Recognizing the term in the bracket, we see that if ϕ_α satisfy the equations of motion, we have:

$$\delta_s \mathcal{L} \approx \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \delta_s \phi_\alpha \right),$$

Which is a total derivative. We then have two conditions on $\delta_s \mathcal{L}$:

$$\begin{cases} \delta_s \mathcal{L} &= \partial_\nu (\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \delta_s \phi_\alpha) \\ \\ \delta_s \mathcal{L} &= \partial_\nu K^\nu \end{cases}$$

Subtracting these two conditions we have:

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \delta_s \phi_\alpha - K^\nu \right) = \alpha \partial_\nu j^\nu = 0,$$

Where we have defined *Noether current*:

$$j^\nu = \frac{1}{\alpha} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_\alpha)} \delta_s \phi_\alpha - K^\nu \right).$$

So we have proved Noether's Theorem: To any symmetry of the action corresponds a current that is locally conserved.

Starting from Noether current, we can now define *Noether charge* as :

$$\begin{aligned} Q &= \int d^D x j^t(\vec{x},t) = \int d^D x \frac{1}{\alpha}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi_\alpha)}\delta_s\phi_\alpha - K^t\right) = \\ &= \int d^D x \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi_\alpha)}\Delta_s\phi_\alpha - \frac{1}{\alpha}K^t\right). \end{aligned}$$

The Noether charge is strongly related to symmetry transformation of systems we're interested in studying. Considering the canonical momentum defined as $\pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_\alpha)}$, we know in quantum mechanic field and momentum respect the commutation rule (dimensionless):

$$[\pi_\alpha(\vec{x}), \phi_\beta(\vec{y})] = -i\delta_{\alpha\beta}\delta(\vec{x}-\vec{y}).$$

Considering the particular but common case in which $K^t = 0$ and $[\Delta_s\phi_\alpha, \phi_\beta(\vec{y})] = 0$, we have:

$$\begin{aligned} [i\alpha Q, \phi_\beta(\vec{y})] &= i\alpha \int d^D x [\pi_\alpha(\vec{x})\Delta_s\phi_\alpha(\vec{x}), \phi_\beta(\vec{y})] = i\alpha \int d^D x [\pi_\alpha(\vec{x}), \phi_\beta(\vec{y})]\Delta_s\phi_\alpha(\vec{x}) = \\ &= i\alpha \int d^D x [-i\delta_{\alpha\beta}\delta(\vec{x}-\vec{y})]\Delta_s\phi_\alpha(\vec{x}) = \alpha\Delta_s\phi_\alpha(\vec{x}) = \delta_s\phi_\alpha(\vec{x}). \end{aligned}$$

We see that the commutator of the Noether charge and the field is the variation of the field generated by the transformation. We can write:

$$\phi_\alpha(\vec{x})' = \phi_\alpha(\vec{x}) + i\alpha[Q, \phi_\alpha(\vec{x})].$$

This relation looks clearly like the relation $U_\alpha = 1 + i\alpha Q + o(Q^2)$ we have derived for continuous transformation in Hamiltonian formalism. So we recognize that the Noether charge is exactly the conserved observable we have written about in the previous chapter. Using the Baker-Campbell-Hausdorff formula, this becomes:

$$\phi_\alpha(\vec{x})' = e^{i\alpha Q}\phi_\alpha(\vec{x})e^{-i\alpha Q} + o(\alpha^2),$$

and we call Q the generator of the transformation.

1.2.3 Elements of Group Theory

Group theory is an important mathematical tool to understand the role symmetries play in physics. A set of elements is called a group if:

- The combined effect of two elements of the set is still an element of the set
- There exists the neutral element
- There exists the inverse for every element of the set

Examples of notable importance for physics are:

- The group of rotation in n dimensions $SU(n)$
- The group of rotation and reflections in n dimensions $O(n)$
- The group of rotations of a regular n-sided polygon C_n

If the elements of the group commute, the group is said to be *Abelian*Abelian. A subset H of elements of the group is said to be a *subgroup* if it has all the property previously listed. Considering $h \in H$, for each $g \in G$ we define a *coset* gH, set of gh with g fixed and $h \in H$. The set of all cosets relative to the elements g of the group G and h of the group H is called *Quotient set* $\frac{G}{H}$.

If a group can be parameterised with a continuous parameter so that a transformation of the group can be represented as $U_\alpha = e^{i\alpha_a Q_a}$, with $\alpha \in \Re$. Such group is said to be a *Lie Group*, and the operators Q_a are called the *generators* of the group. The sum of two generators of the group is itself a generator, so the set of symmetry generator is a vector space. The commutator of two generators is usually a linear combination of generators:

$$[Q_a, Q_b] = i \sum_c f_{abc} Q_c,$$

where f_{abc} are the structure constants of the group. The vector space of symmetry generators together with their structure constants define the *Lie Algebra*. A Lie Group posses a single algebra, but a single algebra is not associated to a single Lie group, since we have to take into account some discrete symmetry that cold be included in the group.
To relate these to the symmetry properties we want to study, we have to represent hhese mathematical constructs. Let's take, for example, the case of the group C_n: If we represent e regular n-side polygon in a plane with real coordinate, an element of C_n can be represented by a 2x2 matrix like:

$$\mathcal{C} = \begin{pmatrix} \cos\frac{2\pi k}{n} & \sin\frac{2\pi k}{n} \\ -\sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{pmatrix}.$$

If we represent the same n-side polygon in a complex plane, an element of the group C_n is represented by a global phase $e^{i\frac{2\pi k}{n}}$.

Chapter 2

The Spontaneous Symmetry Breaking

2.1 Singular limits: an intuitive example

Central in the discussion about spontaneous symmetry breaking is the idea of singular limit. The thermodynamic limit for a thermodynamic system is defined to take both $N \to \infty$ and $V \to \infty$ holding the quotient N/V finite. In many system this limit is singular.

To show the meaning of a limit to be singular, let's take a classical example. Consider a cylinder on a table. If we sharpen more and more the part leaning on the table, we have that it will be more and more difficult to stabilize this "pencil" we have created. The fact that it is practically impossible the keep a sharped enough pencil in equilibrium, can be expressed mathematically in term of two limits. Defining y the distance between the table and the centre of mass of the pencil, b the diameter of the pencil's tip and θ the angle between the axis of the pencil and the perpendicular to the table, we can say:

$$\lim_{b\to 0}\lim_{\theta\to 0} y > 0,$$

$$\lim_{\theta\to 0}\lim_{b\to 0} y = 0.$$

These two non-commuting limits express the fact that we can sharpen the cylinder in equilibrium keeping it in equilibrium but in this configuration a minimal perturbation to the system takes the pencil to fall on the table, breaking the rotational symmetry the pencil in equilibrium had. Spontaneous in SSB stands for this: it is not important how small the perturbation is, it is sufficient to make the system break the symmetry. Notice that the symmetric state (the pencil standing on the table) is unstable, while the broken-symmetry state is stable.

2.2 The Spontaneous Symmetry Breaking: The Heisemberg Anti-ferromagnet

Definition 1. *Spontaneous Symmetry Breaking occurs when the solutions of the equations of motion are not symmetric under a symmetry of the Hamiltonian, Lagrangian or Action.*

We would like to illustrate the mechanisms of Spontaneous Symmetry Breaking. To do so, imagine a system formed by several spins on a 3D-lattice, like in the image below:

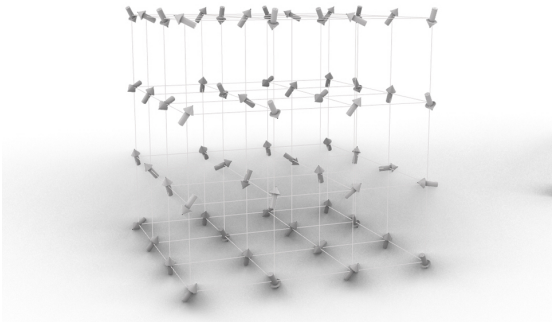

Figure 2.1: Example of spins on a 3D-lattice.

The interactions among spins is a nearest-neighbours interaction, so, from Magnetic Momenta theory, the Hamiltonian of this system can be written as:

$$H = J \sum_{i,\delta} \vec{S}_i \cdot \vec{S}_{i+\delta},$$

where i runs over each position of the lattice and δ runs over the nearest-neighbours of each spin, and we can see that this Hamiltonian is manifestly invariant under rotations around any axis for the properties of the scalar product.

As we can notice, setting the J constant to be positive, the energy of this system is minimized by a state in which nearest-neighbouring spins point in opposite directions, called *Neél state*, shown in the pictures below.

We write the above Hamiltonian as:

$$\begin{aligned} H &= J \sum_{i,\delta} S_i^x S_{i+\delta}^x + S_i^y S_{i+\delta}^y + S_i^z S_{i+\delta}^z \\ &= J \sum_{i,i+\delta} S_i^z S_{i+\delta}^z + \frac{1}{2}(S_i^+ S_{i+\delta}^- + S_i^- S_{i+\delta}^+), \end{aligned}$$

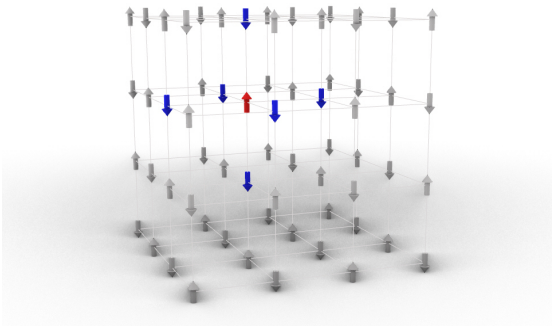

Figure 2.2: The Neél state of the Heisenberg Anti-ferromagnet.

where the $S_i^{\pm}$ are the ladder operators for S_i^z. The presence of these operator clearly shows that the Neél state is not an eigenstate of this Hamiltonian

In order to study this system further, we now introduce a largely used way to study systems like this. Let us consider a free particle in a finite volume. We Fourier-expand the momentum operator so that: $p(x) = \sum_k e^{ikx} p_k$. So we see that:

$$P_{tot} = \int p(x)dx = \sum_k \int e^{ikx} p_k = V p_{k=0},$$

and we can write, for the Hamiltonian of this system:

$$H \propto \int p^2(x)dx = V \sum_k p_k \cdot p_{-k} = \frac{1}{V} p_{tot}^2 + V \sum_{k \neq 0} p_k \cdot p_{-k}.$$

In the last part of the equation, we see that the first term represents the collective motion of the system, while the second therm represents the internal motion of the system. For our system of many spins exposed before and considering the previously described Neél state, we can identify two sublattices, the one formed by the spins pointing up and the other with the spins pointing down, having we chosen the z-axis as the direction of quantization. Each sublattice will give to the Hamiltonian a contribution we can calculate as we have done above. Considering in fact the spin eigenfunction on a single sublattice as a periodic function of the space, we have:

$$\begin{aligned} S_A &= \sum_{i \in A} \vec{S_i} = \sum_x \vec{S(x)} = \sum_x \sum_k e^{ikx} \vec{S_k} = \\ &= V \sum_k \delta(k) \vec{S_k} = V \vec{S}_{k=0}. \end{aligned}$$

So, from Lieb-Mattis' theorem, the collective part of the Hamiltonian is:

$$H = \frac{J}{N} \vec{S_A} \cdot \vec{S_B}.$$

As we already know, a Hamiltonian like thhis one can be rewritten in term of the total spin operator $\vec{S} = \vec{S_A} + \vec{S_B}$ as follows:

$$H = \frac{J}{2N}(S^2 - S_A^2 - S_B^2).$$

At this point we substitute the operators with their eigenvalues, being them manifestly commuting operators, and we then have:

$$H = \frac{J\hbar^2}{2N}[S(S+1) - S_A(S_A+1) - S_B(S_B+1)].$$

We can see that the energy of the system is minimized by a state in which the total spin is zero, while both S_A and S_B reach their maximal value of $\frac{N}{4}$.
The fact that $S = 0$ in the ground state means that $\langle S^z \rangle = \langle S^y \rangle = \langle S^x \rangle = 0$. Furthermore this means the ground state is a state with each spin aligned in the same direction, in the same or the opposite way of its nearest-neighbours depending on the sign of the constant J, but in which no direction of the space is privileged. This ground state is symmetric for rotation around any axis.

We are now introducing a arbitrarily small perturbation that breaks the rotational symmetry around $x-$ and $y-$axis, so that we write our collective Hamiltonian as follows:

$$H = \frac{J}{N}\vec{S_A} \cdot \vec{S_B} - \mu(S_A^z - S_B^z).$$

This Hamiltonian is clearly non-symmetric for rotation around $x-$ and $y-$axis, since it clearly does not commute with the operators S_A^x, S_A^y, S_B^x, S_B^y. If we take the limit for $N \to \infty$ and for $\mu \to 0$ of the difference of magnetization we have:

$$\lim_{N\to\infty} \lim_{\mu\to 0} \langle 0|(S_A^z - S_B^z)|0\rangle = 0$$

$$\lim_{\mu\to 0} \lim_{N\to\infty} \langle 0|(S_A^z - S_B^z)|0\rangle = \frac{N}{2}.$$

Here we considered the expectation value of the difference of magnetization between the two sublattices on the ground state. In the first relation, the considerations done before are still valid: the state minimizing the energy of the system will be symmetric under rotation of space, since no direction of the space is privileged. In the second case, we have that the state minimizing the energy of the system is instead a state in which spins point in opposite direction of the z-axis[1] respect to their nearest-neighbour. This is s symmetry-broken state, and we see that even the smallest perturbation is sufficient to cause the breaking. We say Spontaneous Symmetry Breking occurred.

2.3 Ordered Systems

2.3.1 The Thermodynamic Limit

From here on, we will discuss systems with some degree of translational symmetry, hence systems characterized by the repetition in space of a certain unit cell. This allows

[1] We are considering the constant J positive

us to write the state of the system as tensor product of states at different positions:

$$|\psi\rangle = \otimes_x |\psi(x)\rangle.$$

Translational invariance is sufficient to state that, in thermodynamic limit, two different symmetry-broken states of our system are orthogonal. In fact, we have:

$$\begin{aligned}\langle\psi'(x)|\psi(x)\rangle &= \langle\otimes_{x'}\psi'(x')|\otimes_x|\psi(x)\rangle = \prod_x \langle\psi'(x)|\psi(x)\rangle \\ &= \lim_{N\to\infty}(\langle\psi'(x)|\psi(x)\rangle)^N.\end{aligned}$$

We see that only if $|\psi'\rangle = e^{i\phi}|\psi\rangle$ this scalar product will be non-zero in thermodynamic limit, and it will be zero otherwise.

We now introduce the group theory to classify symmetry-broken states. We will refer, with notation abuse, to both the element of a group and the operator it represents with the symbol g. If we consider a state that breaks some symmetry of a given group, we will have that if g is a broken symmetry, then $g|\psi\rangle \neq |\psi\rangle$, while, if g is a symmetry of the group that is not broken by the state $|\psi\rangle$, $g|\psi\rangle = e^{i\phi}|\psi\rangle$ and the two state $g|\psi\rangle$ and $|\psi\rangle$ are equivalent. The set of all unbroken symmetry of a symmetry group G form a subgroup $H \subset G$ called *residual group*.

Recalling that two symmetry-broken states are either equivalent or orthogonal in thermodynamic limit, we now take two transformations g_1 and g_2. Being $h \in H$, if $g_1|\psi\rangle = g_2 h|\psi\rangle = e^{i\phi}g_2|\psi\rangle$, then the two vectors $g_1|\psi\rangle$ and $g_2|\psi\rangle$ are equivalent. Conversely, if this relation is not satisfied, the two vectors are two different, orthogonal, broken-symmetry states in Thermodynamic limit. Starting from an initial state $|\psi\rangle$ we can label each symmetry-broken state $|\psi'\rangle$ that is different from it for the action of given transformation g, $|\psi'\rangle = g|\psi\rangle\rangle$. Being every symmetry-broken state either orthogonal or equivalent in thermodynamic limit, this is equivalent to create a set of transformations in which no two elements satisfy the relations $g = g'h$. This is the definition of quotient set $\frac{G}{H}$ given previously.

2.3.2 The Order Parameter

We would like to distinguish among different symmetry-broken states. To do so, we would define an operator which has zero expectation value in symmetric states and has different non-zero eigenvalues in broken-symmetry states. To define such an operator we rewrite our definition of symmetric state as follows:

Definition 2. *Let U be a symmetry of a give Hamiltonian, with generator Q such that $U = e^{i\alpha Q}$ with α continuous or discrete real variable. A state $|\phi\rangle$ breaks this symmetry if and only if there exists an operator Φ such that:*

$$\langle\psi|[Q,\Phi]|\psi\rangle \neq 0.$$

The operator Φ is called *interpolating field*. Given this definition, we define:

$$O = [Q,\Phi] \qquad \text{and} \qquad \mathbf{O} = \langle\psi|[Q,\Phi]|\psi\rangle,$$

Respectively the *Order Parameter Operator* and the *Order Parameter*. We explicitly point out that O and then $\mathbf{O}$, are not necessarily Hermitian.

The order parameter operator we just defined has zero expectation value on symmetric states, but nothing in our definition enables it to distinguish among different symmetry-broken states. As we need it to do so, we want the order parameter operator to be an omomorphism between the set of its eigenvalues and the quotient set $\frac{G}{H}$. It is always possible to find such an operator, and its form is generally suggested by the physics of the system.

Let's take as example the system of many spins on a regular lattice exposed before. We'd like to write the order parameter operator for our system. We saw that a small symmetry-breaking perturbation is sufficient to make the ground state to break the translational symmetry of the Hamiltonian. We now define the *staggered magnetization* $N_i^a = (\pm 1)^i S_i^a$ where i is the index position that is even on the A-sublattice and odd on the B-sublattice. We use this staggered magnetization as interpolating field, then we have, for rotation around y-axis (dimensionless):

$$O_i = [Q_i, \Phi_j] = [S_i^y, N_j^x] = -i\delta_{ij} N_i^z.$$

For the local order parameter. The global order parameter will be:

$$O = -i \sum_{i,j} \delta_{ij} N_i^z = -i \sum_i N_i^z.$$

We see that in general, the order parameter operator as well as the interpolating field, is a local definition, from which we define the global order parameter for the system. We then modify our previous definition with the following:

$$O(x) = [Q, \Phi(x)] \qquad \text{and} \qquad \mathbf{O}(x) = \langle \psi | [Q, \Phi(x)] | \psi \rangle.$$

2.3.3 Classical State and Long Range Order

By now, we have seen that in the thermodynamic limit a small perturbation is sufficient to make a system break a symmetry of the corresponding Hamiltonian. We may suggest that the ground sate is an eigenstate of the order parameter operator, but it is seen that there is the effect of excitations at finite wave number which makes the ground state deviate from what we call the *classical state*, that is the eigenstate of the order parameter. In our examples, it would be the tensor product of the S^z eigenfunction relative to the position x on the 3-D lattice:

$$|\psi\rangle = \otimes_x S_x^z.$$

The deviation of the state of the system from the classical state are called *quantum correction* . For the much part of the real physical systems, we then don't know the exact ground state, and this is clearly a problem as we can not calculate the expectation

value of the order parameter operator for that system.
So we introduce the two-points correlation function:

$$C(x, x') = \langle \psi | O^\dagger(x) O(x') | \psi \rangle,$$

and we say:

$$C(x, x') = \begin{cases} \text{Constant} & \Rightarrow \text{long range order} \\ e^{-\frac{|x-x'|}{l}} & \Rightarrow \text{disordered} \end{cases}$$

where l is called *correlation length*. The physical meaning of the two points correlation function is clear: it is the scalar product of $O(x)|\psi\rangle$ in two different points of the space, so, if constant, it tells us the expectation value of the order parameter operator is the same in every point of the space, so the system is ordered and the symmetry we're studying is broken.

2.4 Collective Excitation: The NG-Modes

By now we have focused our attention of the collective part of the Hamiltonian. In this chapter we will expose a fundamental consequence of Nother's Theorem affecting the collective excitation of the system. The core of our discussion can be summarized with the Goldstone's Theorem :

Goldstone's Theorem 1. *If a continuous symmetry is broken in the absence of a long range-interaction leaving some translational symmetry unbroken, then there exists at least a mode in the spectrum of the Hamiltonian whose energy goes to zero as* $k \to 0$

We're going to demonstrate the Goldstone's theorem. Translational symmetry is necessary in order to express eigenvalues of the Hamiltonian as $|n, k\rangle$, where n stands for all eigenvalues of operators other than the momentum operator. These eigenstates are orthogonal and $\langle n', k'|n, k\rangle = (2\pi)^D \delta_{k,k'} \delta_{n,n'}$, with D the dimension of the space we are working in, so that we can say:

$$I = \sum_n \int \frac{d^D k}{(2\pi)^D} |n, k\rangle\langle k', n'|.$$

We can insert the identity in the definition of the order parameter and then we have:

$$\langle\psi|[Q, \Phi]|\psi\rangle = \sum_n \int \frac{d^D k}{(2\pi)^D} [\langle\psi|Q|n, k\rangle\langle k', n'|\Phi|\psi\rangle - c.c.].$$

As we have already told in chapter 1, Noether charges can be expressed as the integral in space of Noether current that, in Hamiltonian formalism, is an operator which follows the well known space- and time-translation relations. So we write:

$$\begin{aligned}
\langle\psi|[Q, \Phi]|\psi\rangle &= \sum_n \int_\Omega d^D x \int \frac{d^D k}{(2\pi)^D} [\langle\psi|j^t(x, t)|n, k\rangle\langle k', n'|\Phi|\psi\rangle - c.c.] = \\
&= \sum_n \int d^D x \int \frac{d^D k}{(2\pi)^D} [\langle\psi|e^{\frac{i}{\hbar}(Ht+\vec{P}\cdot\vec{x})} j^t(0, 0) e^{-\frac{i}{\hbar}(Ht+\vec{P}\cdot\vec{x})}|n, k\rangle\langle k', n'|\Phi|\psi\rangle - c.c.].
\end{aligned}$$

Now we call E_n the energy of the state $|n, k\rangle$ respect to the symmetry-broken state $|\psi\rangle$:

$$\langle\psi|[Q, \Phi]|\psi\rangle = \sum_n \int_\Omega d^D x \int \frac{d^D k}{(2\pi)^D} [\langle\psi|e^{\frac{i}{\hbar}\vec{P}\cdot\vec{x}} j^t(0, 0) e^{-\frac{i}{\hbar}(E_n t+\vec{P}\cdot\vec{x})}|n, k\rangle\langle k', n'|\Phi|\psi\rangle - c.c.].$$

Using the translational invariance, we have $e^{\frac{i}{\hbar}\vec{P}\cdot\vec{x}}$ acting on the state $|\psi\rangle$ has to be the identity operator. so we understand $|\psi\rangle$ is a zero-momentum state.

We can then write:

$$\begin{aligned}
\langle\psi|[Q,\Phi]|\psi\rangle &= \sum_n \int_\Omega d^D x \int \frac{d^D k}{(2\pi)^D} e^{-\frac{i}{\hbar}(E_n t + \vec{P}\vec{x})}[\langle\psi|j^t(0,0)|n,k\rangle\langle k',n'|\Phi|\psi\rangle - c.c.] = \\
&= \sum_n \int_\Omega d^D x \int \frac{d^D k}{(2\pi)^D} e^{-\frac{i}{\hbar}(E_n t + \hbar k x)}[\langle\psi|j^t(0,0)|n,k\rangle\langle k',n'|\Phi|\psi\rangle - c.c.] = \\
&= \sum_n \int \frac{d^D k}{(2\pi)^D} \delta_\Omega(k) e^{-\frac{i}{\hbar}(E_n t)}[\langle\psi|j^t(0,0)|n,k\rangle\langle k',n'|\Phi|\psi\rangle - c.c.] \neq 0.
\end{aligned}$$

As we said that $|\psi\rangle$ is a symmetry-broken state. We then understand that there must exist at least one state for which the integrand is non-zero. The function $\delta_\Omega(k)$ is a peaked function of k tending towards a $\delta(k)$ with Ω approaching infinity. So we recognize that there must be at least one state, near the zero momentum state, that come from the excitation by both the Noether current operator and the Interpolating field of the broken-symmetry ground state.

As we saw in chapter 1, Noether charge is time independent and, if so it is the interpolating field, the order parameter operator is itself time independent. So, making the time derivative, this has to be equal to zero:

$$\begin{aligned}
\partial_t\langle\psi|[Q,\Phi]|\psi\rangle &= \partial_t \sum_n \int \frac{d^D k}{(2\pi)^D} \delta_\Omega(k) e^{-\frac{i}{\hbar}(E_n t)}[\langle\psi|j^t(0,0)|n,k\rangle\langle k',n'|\Phi|\psi\rangle - c.c.] = \\
&= -\sum_n \int \frac{d^D k}{(2\pi)^D} \delta_\Omega(k) \frac{i}{\hbar} E_n e^{-\frac{i}{\hbar}(E_n t)}[\langle\psi|j^t(0,0)|n,k\rangle\langle k',n'|\Phi|\psi\rangle - c.c.] = 0.
\end{aligned}$$

We now understand that the state obtained by the excitation of the ground state by both Noether's current and the interpolating field has energy that goes to zero with k approaching to zero, and then Goldstone's Theorem is demonstrated. As it is easily seen, the theorem is constructive, as it tells us that these excitation modes can be obtained by acting on the broken-symmetry ground state with either Noether current or the Interpolating field.

As demonstrated above, the theorem could suggest that for each broken symmetry there has to be a state near the ground state whose energy goes to zero with k approaching zero. This is seen to be false: the system we have described before, the Heisemberg (Anti)ferromagnet is seen to have only one NG-Mode , while two symmetries are broken. To understand why this happens, the system of the Heisemberg (Anti)ferromagnet is again a perfect example: we notice that in this case, the interpolating fields are themselves generators of broken symmetries. Calling Q_a and Q_b the generators of the broken symmetries and $j_a^t(\vec{x},t)$ and $j_b^t(\vec{x},t)$ the corresponding Noether currents, we have:

$$\begin{aligned}\langle[Q_a, j_b^t(\vec{x},t)]\rangle &= \int d^D y \langle[j_a^t(\vec{y},t), j_b^t(\vec{x},t)\rangle = \int d^D y \delta(\vec{y}-\vec{x}) \sum_c i f_{abc} \langle j_c^t(\vec{y})\rangle = \\ &= \sum_c i f_{abc} \langle j_c^t(\vec{y})\rangle = \langle[j_a^t(\vec{x},t), Q_b]\rangle,\end{aligned}$$

where f_{abc} are the structure constants of the symmetry group the symmetries we are dealing with belong. Being these two commutators equal, we see that the Noether currents and the interpolating fields relative to the two different broken symmetries excite the same NG-mode. We call such NG-modes type-B .

In order to count the NG-modes, we define the Watanabe-Brauner matrix $M_{ab} = -i\langle[Q_a, j_b^t(\vec{x},t)]\rangle$ that is a matrix with a empty principal diagonal. Furthermore, it is possible to show that type-A and type-B NG-modes have, respectively, linear and quadratic dispersion relation.

2.5 Quantum Correction

In paragraph 2.3.3 we said that the ground state of real quantum systems usually deviate from the classical states, eigenstates of the order parameter operator due to the effect of excitations at finite wave number. Our aim now is to find the energy of the ground state of the Heisemberg Anti-ferromagnet, taking into account quantum corrections. first of all we recall the Hamiltonian of the system:

$$\begin{aligned} H &= J\sum_{i,\delta}\vec{S}_i\cdot\vec{S}_{i+\delta} = \\ &= \frac{J}{2}\sum_{i,\delta}[S_i^xS_{i+\delta}^x + S_i^yS_{i+\delta}^y + S_i^zS_{i+\delta}^z], \end{aligned}$$

where again the index i runs over all the possible position of the lattice, the index δ runs over the nearest-neighbours of the i-th spin and the constant J is assumed to be positive. The factor $\frac{1}{2}$ is necessary to avoid double counting. We know that such broken-symmetry state of a system like this has all spins aligned in the same direction, and, since we have assumed the constant J to be positive, each spin points in the opposite way of its nearest-neighbours, defining two sublattices A and B. We now introduce the *rotated spin operators*, defined as follows:

$$N_{i\in A}^a = S_{i\in A}^a, \quad N_{i\in B}^x = S_{i\in B}^x, \quad N_{i\in B}^y = -S_{i\in B}^y, \quad N_{i\in B}^z = -S_{i\in B}^z.$$

We explicitly notice that these operators respect the SU(2) algebra. We can express the Hamiltonian of the system in therms of these new operators:

$$\begin{aligned} H &= \frac{J}{2}\sum_{i,\delta}N_i^xN_{i+\delta}^x - N_i^yN_{i+\delta}^y - N_i^zN_{i+\delta}^z = \\ &= \frac{J}{2}\sum_{i,\delta}[N_i^zN_{i+\delta}^z + \frac{1}{2}(N_i^+N_{i+\delta}^+ + N_i^-N_{i+\delta}^-)], \end{aligned}$$

where obviously $N_i^\pm$ are he raising and lowering operators for the N_i^z operator: $N_i^\pm = N_i^x \pm iN_i^y$. The broken-symmetry state (Neél state) is the state with maximum value of $N_{tot}^z = \sum_i N_i^z$. We explicitly notice that it is not an eigenstate of the Hamiltonian.

Consider now a single spin-s particle. If we consider the state with maximum value of $S_z = s$ as a ground state, we can obtain every other state by acting on this state with a raising operator in the same way we do with the harmonic oscillator. In the base $|S^2, S^Z\rangle$ we have:

$$|s,s\rangle = |0\rangle \Rightarrow |s,s-n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

To express the raising and lowering operators in terms of this creation and annihilation operators we have to fix the commutation rule $[S^+, S^-] = 2\hbar S^z$. We have:

$$S^+ = \hbar\sqrt{2s}\sqrt{1+\frac{a^\dagger a}{2s}}a,$$

$$S^- = \hbar\sqrt{2s}\sqrt{1+\frac{a^\dagger a}{2s}}a^\dagger,$$
$$S^z = \hbar(s - a^\dagger a).$$

This is the so called Holstein-Primakoff transformation. In our case, we consider the Neél state as the vacuum state $|0\rangle$ and then it is possible to see that:

$$N_i^+ = \sqrt{2s}\sqrt{1+\frac{n_i}{2s}}a_i,$$
$$N_i^- = \sqrt{2s}\sqrt{1+\frac{n_i}{2s}}a_i^\dagger,$$
$$N_i^z = (s - n_i).$$

In these definitions, the square root are meant as potential series. Having we the aim of finding the energy of the ground state, the inclusion of potential series of course does not help us: since $[a_i, a_j] = \delta_{i,j} \neq 0$, it would be so difficult to make this Hamiltonian diagonal. We can overcome this problem by considering only the first term in the expansion, so that we have:

$$N_i^+ = \sqrt{2s}a_i, \qquad N_i^- = \sqrt{2s}a_i^\dagger.$$

It looks clear that the larger S is, the most this approximation is appropriate. Now we write the Hamiltonian substituting the new definition of N_i^+ and N_i^-:

$$\begin{aligned}
H &= \frac{J}{2}\sum_{i,\delta}[N_i^z N_{i+\delta}^z + \frac{1}{2}(N_i^+ N_{i+\delta}^+ + N_i^- N_{i+\delta}^-)] = \\
&= \frac{J}{2}\sum_{i,\delta}\frac{1}{2}[2N_i^z N_{i+\delta}^z + N_i^+ N_{i+\delta}^+ + N_i^- N_{i+\delta}^-] = \\
&= \frac{J}{4}\sum_{i,\delta}[2Sa_i a_{i+\delta} + 2Sa_i^\dagger a_{i+\delta}^\dagger - (S - a_i^\dagger a_i)(S - a_{i+\delta}^\dagger a_{i+\delta})] = \\
&= \frac{JS}{2}\sum_{j,\delta}(a_i a_{i+\delta} + a_i^\dagger a_{i+\delta}^\dagger) - \frac{zNS^2J}{2} - -\frac{J}{2}\sum_{j,\delta}(-Sa_{i+\delta}^\dagger a_{i+\delta} - Sa_i^\dagger a_i + a_i^\dagger a_i a_{i+\delta}^\dagger a_{i+\delta}) = \\
&= \frac{JS}{2}\sum_{j,\delta}(a_i a_{i+\delta} + a_i^\dagger a_{i+\delta}^\dagger) - \frac{zNS^2J}{2} + JS\sum_{j,\delta} a_i a_{i+\delta} = \\
&= \frac{JS}{2}\sum_{j,\delta}(a_i a_{i+\delta} + a_i^\dagger a_{i+\delta}^\dagger) - \frac{zNS^2J}{2} + JSz\sum_{j} a_i a_{i+\delta},
\end{aligned}$$

where N stands for the number of sites in the lattice and z for the number of nearest-neighbours. This Hamiltonian is not diagonal yet, then we Fourier-expand a_j, $a_j = \frac{1}{\sqrt{N}}\sum_k e^{ikj}a_k$. So we have:

$$\sum_j a_j^\dagger a_j = \frac{1}{N}\sum_{j,k,k'} e^{-ikj}a_k^\dagger \cdot e^{ik'j}a_{k'} = \sum_k a_k^\dagger a_k,$$

$$\sum_{j,\delta} a_j a_{j+\delta} = \frac{1}{N} \sum_{j,\delta,k,k'} e^{ikj} a_k^\dagger \cdot e^{ik'(j+\delta)} a_{k'} = \sum_{k,\delta} e^{-ik\delta} a_k a_{-k},$$

$$\sum_{j,\delta} a_j^\dagger a_{j+\delta}^\dagger = \frac{1}{N} \sum_{j,\delta,k,k'} e^{-ikj} a_k^\dagger \cdot e^{-ik'(j+\delta)} a_{k'}^\dagger = \sum_{k,\delta} e^{-ik\delta} a_k^\dagger a_{-k}^\dagger.$$

The Hamiltonian then becomes:

$$\begin{aligned}
H &= \frac{JS}{2} \sum_{j,\delta} (a_i a_{i+\delta} + a_i^\dagger a_{i+\delta}^\dagger) - \frac{zNS^2J}{2} + JSz \sum_j a_i a_{i+\delta} = \\
&= -\frac{zNS^2J}{2} + zJS \sum_k a_k^\dagger a_k + \frac{JS}{2} \sum_{k,\delta} e^{-ik\delta} (a_k a_{-k} + a_k^\dagger a_{-k}^\dagger) = \\
&= -\frac{zNS^2J}{2} + zJS \sum_k \{a_k^\dagger a_k + \frac{1}{2z} \sum_\delta e^{-ik\delta} (a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)\} = \\
&= -\frac{zNS^2J}{2} + zJS \sum_k \{a_k^\dagger a_k + \frac{1}{2} \gamma_k (a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)\}.
\end{aligned}$$

The presence of raising and lowering operators still hides a simple identification of the ground state. To go further, we introduce the *Bogoliubov Transformations*, introducing a second set of creation and annihilation operators b_k and $b_k^\dagger$ following the commutation rule $[b_k, b_{k'}^\dagger] = \delta_{kk'}$. So we say:

$$a_k = b_k \cosh u_k + b_{-k}^\dagger \sinh u_{-k} \qquad a_k^\dagger = b_k^\dagger \cosh u_k + b_{-k} \sinh u_{-k}$$

Where u_k are unknown but real functions of k following the relation $u_k = u_{-k}$. We now rewrite the Hamiltonian as follows:

$$
\begin{aligned}
H &= -\frac{zNS^2J}{2} + zJS\sum_k \{a_k^\dagger a_k + \frac{1}{2}\gamma_k(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)\} = \\
&= -\frac{zNS^2J}{2} + zJS\sum_k \{(\cosh u_k b_k^\dagger + \sinh u_k b_{-k})(\cosh u_k b_k + sinh u_k b_{-k}) \\
&\quad + \frac{1}{2}\gamma_k[(\cosh u_k b_k + \sinh u_k b_{-k}^\dagger)(\cosh u_k b_{-k} + \sinh u_k b_k^\dagger) \\
&\quad + (\cosh u_k b_k^\dagger + \sinh u_k b_{-k})(\cosh u_k b_{-k}^\dagger + \sinh u_k b_k)]\} = \\
&= -\frac{zNS^2J}{2} + zJS\sum_k \{\cosh^2 u_k b_k^\dagger b_k + \cosh u_k \sinh u_k b_k^\dagger b_{-k}^\dagger + \\
&\quad + \cosh u_k \sinh u_k b_k b_{-k} + \sinh^2 u_k b_{-k} b_{-k}^\dagger + \frac{1}{2}\gamma_k[\cosh^2 u_k b_k b_{-k} + \cosh u_k \sinh u_k b_k b_k^\dagger \\
&\quad + \cosh u_k \sinh u_k b_{-k}^\dagger b_{-k} + \sinh^2 u_k b_{-k}^\dagger b_k^\dagger + \cosh^2 u_k b_k^\dagger b_{-k}^\dagger + \\
&\quad + \cosh u_k \sinh u_k b_k^\dagger b_k + \cosh u_k \sinh u_k b_{-k} b_{-k}^\dagger + \sinh^2 u_k b_{-k} b_k]\} = \\
&= -\frac{zNS^2J}{2} + zJS\sum_k \{\cosh^2 u_k + \frac{1}{2}\gamma_k \cosh u_k \sinh u_k + \frac{1}{2}\gamma_k \cosh u_k \sinh u_k) b_k^\dagger b_k + \\
&\quad + \frac{1}{2}\gamma_k \cosh u_k \sinh u_k + (\cosh u_k \sinh u_k + \frac{1}{2}\gamma_k \sinh^2 u_k + \frac{1}{2}\gamma_k \cosh^2 u_k) b_k^\dagger b_{-k}^\dagger + \\
&\quad + (\cosh u_k \sinh u_k + \frac{1}{2}\gamma_k \sinh^2 u_k + \frac{1}{2}\gamma_k \cosh^2 u_k) b_k b_{-k} + \\
&\quad + (\sinh^2 u_k + \frac{1}{2}\gamma_k \cosh u_k \sinh u_k + \frac{1}{2}\gamma_k \cosh u_k \sinh u_k) b_{-k}^\dagger b_{-k} + \frac{1}{2}\gamma_k \cosh u_k \sinh u_k\} = \\
&= -\frac{zNS^2J}{2} + zJS\sum_k \{(\cosh 2u_k + \gamma_k \sinh 2u_k) b_k^\dagger b_k \\
&\quad + \frac{1}{2}(\gamma_k \cosh 2u_k + \sinh 2u_k)(b_k^\dagger b_{-k}^\dagger + b_k b_{-k}) + \sinh^2 u_k + \gamma_k \cosh u_k \sinh u_k\} = \\
&= -\frac{zNS^2J}{2} + zJS\sum_k \{\sinh^2 u_k + \frac{1}{2}\gamma_k \sinh 2u_k + (\cosh 2u_k + \gamma_k \sinh 2u_k) b_k^\dagger b_k \\
&\quad + \frac{1}{2}(\gamma_k \cosh 2u_k + \sinh 2u_k)(b_k^\dagger b_{-k}^\dagger + b_k b_{-k})\}.
\end{aligned}
$$

Of course, this Hamiltonian is still not diagonal, but it can be diagonalized setting:

$$\gamma_k \cosh 2u_k + \sinh 2u_k = 0.$$

Recalling fundamental relations for hyperbolic sine and cosine $\cosh^2 x - \sinh^2 x = 1$, we have:

$$\sinh 2u_k = -\frac{\gamma_k}{\sqrt{\gamma_k^2 - 1}} \qquad \cosh 2u_k = -\frac{1}{\sqrt{\gamma_k^2 - 1}}.$$

Bearing in mind the relation previously reported, we have:

$$\cosh 2u_k = 2\sinh 2u_k \cosh 2u_k = 2\cosh^2 2u_k - 1 = \cosh^2 2u_k + \sinh^2 2u_k \Rightarrow$$

$$\Rightarrow \cosh^2 2u_k = \frac{1}{2} + \frac{1}{2\sqrt{1-\gamma_k^2}} \qquad \sinh^2 2u_k = -\frac{1}{2} + \frac{1}{2\sqrt{1-\gamma_k^2}}.$$

Finally we can write our Hamiltonian in a manifestly diagonal form:

$$\begin{aligned}
H &= -\frac{zNS^2J}{2} + zJS\sum_k \{\sinh^2 u_k + \gamma_k \cosh u_k \sinh u_k + (\cosh 2u_k + \gamma_k \sinh 2u_k) b_k^\dagger b_k\} \\
&= -\frac{zNS^2J}{2} + zJS\sum_k \left\{ -\frac{1}{2} + \frac{1}{2\sqrt{1-\gamma_k^2}} - \frac{\gamma_k^2}{2\sqrt{1-\gamma_k^2}} + \left(\frac{1}{\sqrt{1-\gamma_k^2}} - \frac{\gamma_k^2}{\sqrt{1-\gamma_k^2}} \right) b_k^\dagger b_k \right\} = \\
&= -\frac{zNS^2J}{2} + zJS\sum_k \left[-\frac{1}{2} + \sqrt{1-\gamma_k^2}(b_k^\dagger b_k + \frac{1}{2}) \right] = \\
&= -\frac{zNS^2J}{2} - \frac{zJS}{2}\sum_k (1 - \sqrt{1-\gamma_k^2}) + zJS\sum_k \sqrt{1-\gamma_k^2} b_k^\dagger b_k.
\end{aligned}$$

In this expression we can see the first two terms of represent the energy of the ground sate, while the third term has non-zero expectation value in the excited states: it is the energy associated with excitation modes with non-zero wave number, namely the NG-modes .

As the system is not in the classical sate, we're interested in the correction to the order parameter. To do so, we calculate the mean value of the staggered magnetization expressed in the fort $N_i^z = (S - n_i)$. We have:

$$\langle 0| \frac{1}{N} \sum_i N_i^z |0\rangle = S - \frac{1}{N} \sum_i \langle 0| a_i^\dagger a_i |0\rangle.$$

The state $|0\rangle$ refers to the ground state of the Bogoliubov' raising and lowering operator, so we have to do the decomposition:

$$\begin{aligned}
a_k^\dagger a_k &= (\cosh u_k b_k^\dagger + \sinh u_k b_{-k})(\cosh u_k b_k + \sinh u_k b_{-k}^\dagger) \\
&= \cosh^2 u_k b_k^\dagger b_k + \cosh u_k \sinh u_k b_k^\dagger b_{-k}^\dagger + \cosh u_k \sinh u_k b_k b_{-k} + \sinh^2 u_k b_{-k} b_{-k}^\dagger.
\end{aligned}$$

We see that only the last term in this sum has non-zero expectation value on the ground state. so we write

$$\begin{aligned}
\langle 0| \frac{1}{N} \sum_i N_i^z |0\rangle &= S - \frac{1}{N} \sum_i \langle 0| a_i^\dagger a_i |0\rangle = S - \frac{1}{N} \sum_k \sinh^2 u_k \\
&= S + \frac{1}{2} - \int \frac{d^D k}{(2\pi)^D} \frac{1}{2\sqrt{1-\gamma_k^2}}.
\end{aligned}$$

This approximation method could seams to be far too approximate, but it is seen to give good results even compared with more advanced approximations.

2.6 How SSB occurs: The Landau-Ginzburg Theory

Mermin–Wagner–Hohenberg–Coleman' theorem tells us that in some precises temperature conditions, spontaneous symmetry breaking is not possible: there is an upper and a lower bound for this phenomenon to happen. We are now interested in studying how the transition from a symmetric state to a symmetry-broken state actually happen. We know we can distinguish a symmetry-broken state by a symmetric state by order parameter operator's expectation value. We then recognize two different ways the order parameter can go to zero to a non-zero value in a transition from a symmetric state to a symmetry-broken state (or vice versa):

- Phase transitions in which order parameter jumps discontinuously from zero to a non-zero value are called **First Order** phase transitions.
- Phase transitions in which order parameter goes continuously from zero to a non-zero value are called **Second Order** phase transitions.

We know from statistical thermodynamic that every thermodynamic quantity can be written in term of the Free Energy. The central idea of what is called *Landau-Ginzburg Theory* for phase transitions, is that the Free Energy can always be expressed as a functional of the order parameter. In first order phase transition, the order parameter is small near zero as the transition is continuous. Refering to the order parameter as o We can then Taylor-expand the Free Energy for o near to zero, thus close to what we call the *Critical Temperature*:

$$F[o,T] = F[0,T] + \frac{1}{2}r(T)o^2 + \frac{1}{4}u(T)o^4,$$

where we assumed the Free Energy Functional to be symmetric for $o \to -o$. This commonly occurring assumption makes the odd powers vanish. The value of the order parameter for a given temperature T is obtained minimizing the Free Energy for that temperature. Furthermore, the Free Energy has to be bounded from below: in fact, we know that for any fixed temperature, the order parameter has to assume a finite value. Thus we have that the coefficient of the highest power in the expansion ($u(T)$ in our case) has to be positive.

We can now identify two different cases:

- For $r(T) > 0$, we have the $o(T)$ for which $F[o,T]$ is minimum is given by:

  $$|r(T)|o + u(T)o^3 = 0,$$

  which means $o = 0$
- For $r(T) < 0$, we have:

  $$-|r(T)|o + u(T)o^3 = 0,$$

which give us, for the position of the minimum:

$$o = \pm\sqrt{\frac{|r(T)|}{u(T)}}.$$

So we can see the position of the minimum of the Free Energy is shifted from zero to a non-zero value of the order parameter. This shifting is smooth, so this process is a continuous phase transition.

So, while the system goes through a phase transition, $r(T)$ changes his sign, passing, of course through the zero: we can then taylor expand $r(T)$ to the first order near the critical temperature, obtaining:

$$r(t) \approx r_0 \frac{T - T_c}{T_c} \equiv r_0 t.$$

Furthermore, being $u(T)$ always positive we can Taylor expand it to the zero order and then assuming it to be constant near T_c. From the condition for the minimum of the Free Energy, we then have:

$$o = \pm\sqrt{\frac{|r(T)|}{u(T)}} \quad \Rightarrow \quad o(T) \propto \sqrt{(T - T_c)}.$$

We would now like to see how we can determine a *Landau functional* in a practical case. We take the example of the Heisemberg Anti-Ferromagnet again. we know the Hamiltonian of such a physical system is:

$$J \sum_{i,\delta} \vec{S_i} \cdot \vec{S_\delta},$$

where, as always, the summation on i is on each spin of the ferromagnet, while the one on δ is on the nearest-neighbours.
We already showed, assuming the z axis as the direction of polarization, the order parameter operator of such a system. For the breaking of rotational symmetry around y-axis, it is:

$$O = \sum_{i,j} [S_i^y, N_j^x] = \sum_{i,j} \delta_{ij} (-1)^j S_j^z = \sum_i (-1)^i S_i^z,$$

while the local order parameter operator is

$$O_i = (-1)^i S_i.$$

Note that this order parameter is independent from the position both in symmetric and non-symmetric states.

We know from previous sections that the ground symmetry-broken state is not the Neél state , that is the state with each spin pointing at different sides on the same direction, which is not an energy eigenstate. We have seen that the state occurring in real

quantum systems is described by a perturbation of such state, and the perturbations to this collective organization of spins are what we called Quantum Corrections .

Thus, we now write the z component of spin operator operator as its eigenvalues to which we add the perturbation due to quantum corrections as follow:

$$S_i = (-1)^i m + \delta S_i,$$

where we indicated δS_i the perturbation around the eigenstate of S_i^z. With this new definition the Hamiltonian of the system becomes:

$$\begin{aligned} H &= -J\sum_{i,\delta}[(-1)^i m + \delta S_i][(-1)^{i+1} m + \delta S_{i+\delta}] \\ &= JNzm^2 + J\sum_{i,\delta}(-1)^i m \cdot \delta S_{i+\delta} + J\sum_{i,\delta}(-1)^{i+1} m \cdot \delta S_i + J\sum_{i,\delta}\delta S_i \cdot \delta S_{i+\delta}. \end{aligned}$$

Noting that second and third summations are in fact the same summation, this becomes:

$$H = -JNzm^2 - 2Jz\sum_i (-1)^i m \cdot \delta S_i + J\sum_{i,\delta}\delta S_i \cdot \delta S_{i+\delta}.$$

As we can see, the third term on the right side of the above equation is of second order in the variation, so we can neglect it. The Hamiltonian then becomes:

$$H = -JNzm^2 - 2Jz\sum_i (-1)^i m \cdot \delta S_i.$$

To go further, let's write δS_i in term of the spin operator as follow:

$$S_i = (-1)^i m + \delta S_i \quad \Rightarrow \quad \delta S_i = S_i - (-1)^i m.$$

This leads to the following expression for the Hamiltonian of the system:

$$\begin{aligned} H &= -JNzm^2 + 2JNzm^2 - 2Jz\sum (-1)^i m \cdot S_i \\ &= JNzm^2 - 2Jz\sum (-1)^i m \cdot S_i. \end{aligned}$$

Our aim is that of finding the Landau Functional for this system. To do so, we now calculate the partition function starting from the Hamiltonian we have just found and then we will write the free energy of the system as a function of the order parameter:

$$\begin{aligned} Z &= Tre^{-\beta H} = Tre^{-\beta JNzm^2 - 2J\beta z\sum(-1)^i m\cdot S_i} = \\ &= \sum_{S_1=\pm 1/2}\sum_{S_2=\pm 1/2}\dots\sum_{S_n=\pm 1/2} e^{-\beta JNzm^2}\cdot e^{-2J\beta z\sum(-1)^i m\cdot S_i} \\ &= e^{-\beta JNzm^2}\sum_{S_1=\pm 1/2}\sum_{S_2=\pm 1/2}\dots\sum_{S_n=\pm 1/2} e^{2J\beta zm^2 S_1} e^{-2J\beta zm^2 S_2} e^{2J\beta zm^2 S_3}\dots e^{-2J(-1)^N \beta zm^2 S_N} \\ &= e^{-\beta JNzm^2}[2\cosh(J\beta zm^2)]^N. \end{aligned}$$

This expression of the partition function leads to the following expression for the free energy:

$$F = -\beta^{-}1 \log Z = -JNzm^2 - \beta^{-1} \log \left[2\cosh(J\beta z m^2)\right]^N.$$

We now taylor expand the free energy for small values of m we have:

$$\frac{F}{N} = -\frac{\log 2}{\beta} + 2Jz\left[1 - \frac{Jz\beta}{2}\right] m^2.$$

We see that for high temperature, the free energy is minimized by $m = 0$, so that we do not have ordered state so we know spin rotational symmetry is unbroken. With temperature lowering, the coefficient of the quadratic term decreases, going trough zero at $Jz\beta = 2$. For what we have seen, now the free energy is minimized by a non-zero valued order parameter operator.

2.7 Conclusions

We have exposed the main features of Spontaneous Symmetry Breaking, using the example of the Heisemberg Anti-ferromagnet and then we showed Landau-Ginzburg Theory's main concepts in order to give a practical idea of how this phenomenon occurs in thermodynamic systems. However, Spontaneous Symmetry Breaking plays an important role in many aspects of nowadays' physics, and give a reasonable solutions for unsolved problems.

During the course of *Elements of Quantum Theory*, one of the "problems" of the theory was that of the delocalization of the wave function for macroscopic systems or, for what we said in this thesis, systems in the thermodynamic limit: the fact that the width of the wave packet relative to a particle increase with time does not match the evidence of everyday life. Things in our everyday experience does not delocalize, even if they are made of particles described by quantum theory. One of the possible solution of this conundrum is given by the Spontaneous Symmetry Breaking. Considering a solid body made of atoms oscillating around an equilibrium position under a nearest-neighbours interaction, we have the following Hamiltonian for this system:

$$H = \sum_{x,\delta} \frac{P(x)^2}{2m} + \frac{1}{2} m\omega^2 [X(x) - X(x-\Delta)]^2,$$

where, as said in the whole thesis before, x runs over the atoms' positions and δ over the nearest-neighbours of each atom. Δ is the shifting by the equilibrium position of a given atom and $P(x)$ and $X(x)$ follow the commutation rule (adimensional): $[X(x), P(x')] = i\delta(x'-x)$. Studying this system in the same way we did for Heisemberg Anti-ferromagnet, we have:

$$H_{coll} \propto \frac{P_{CoM}^2}{2Nm},$$

with N the number of particles and P_{CoM} is the total momentum operator, or equivalently the momentum of the centre of mass. This Hamiltonian is symmetric under translation on the x-axis and the ground state is the eigenstate relative to the eigenvalue $P = 0$, which is a constant function spread all over the space. If we introduce a perturbation which breaks the translational symmetry we have:

$$H_{coll} \propto \frac{P_{tot}^2}{2Nm} + \mu X(x)_{CoM}^2.$$

This is the Hamiltonian of a harmonic oscillator with pulse $\omega = \sqrt{\frac{2\mu}{mN}}$, which ground state (and relative energy) is well known:

$$|\psi\rangle_0 = \left(\frac{2mN\mu}{\pi^2\hbar^2}\right)^{\frac{3}{8}} e^{-\sqrt{\frac{mN\mu}{2\hbar^2}}x^2}.$$

The width of the gaussian is then $\sigma^2 = \frac{\hbar}{\sqrt{2mN\mu}}$, and we see:

$$\lim_{N\to\infty} \lim_{\mu\to 0} |\psi\rangle_0 = const.$$

$$\lim_{\mu \to 0} \lim_{N \to \infty} |\psi\rangle_0 = \delta(x),$$

and, as we have understood, this means that in the thermodynamic limit, the smallest perturbation is sufficient to completely localize the wave function.

When speaking about the Spontaneous Symmetry Breaking however, it is a must to mention one of the most important result of modern physics. In 2012, the experiments ATLAS and CMS measured for the first time a clear evidence of the existence of the *Higgs' Boson*. Unfortunately, the writer of this thesis has not the right knowledge to treat this issue in detail (by now). What we can say is that, at the beginning, the theory of *Standard Model* predicted that the bosons mediating fundamental interactions had to be massless. Standard Model is currently the most important physical theory explaining fundamental interactions of particles, and is based on the gauge symmetry group:

$$SU(2) \times SU(3) \times U(1).$$

The prediction for the bosons mediating fundamental interactions to be massless was a tremendous issue at the time: in fact, several experiments showed that these particles had defined masses. In 1964 Peter Higgs theorised the existence of a new boson, called in the following years Higgs' Boson, which mediate the interaction between particles and the *Higgs' Field.* The *Higg's mechanism* generalizes the pattern of symmetry breaking to the case of gauge symmetries. In this case the Goldstone's Theorem does not hold and this enables particles to acquire mass.

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