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Combinatorial Aspects and Renormalization

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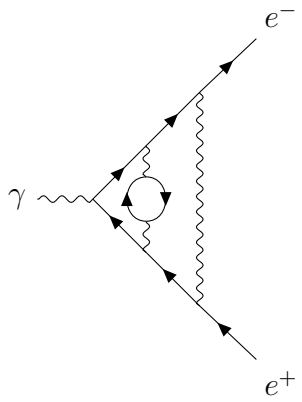
"The effort to understand the universe is one of the very few things which lifts human life a little above the level of farce and gives it some of the grace of tragedy."

Steven Weinberg [Wq8]

Preface



Abstract

We study the Connes-Kreimer approach to renormalization for the self-interacting scalar theory. We give a brief, self-contained account of perturbative quantum field theory and a detailed description of the BPHZ renormalization procedure for the ϕ^3 theory. We then introduce Hopf algebras and we show that the underlying structure of the Feynman graphs in the theory can be described by a Hopf algebra. We finally arrive at the major result in the Connes-Kreimer theory that the solution to the Birkhoff factorization for the Hopf algebra of Feynman graphs is given exactly by the BPHZ procedure.

Sommario

Studiamo l'approccio di Connes-Kreimer alla rinormalizzazione nel caso di una teoria scalare auto-interagente. Diamo una breve, autosufficiente descrizione della teoria quantistica dei campi perturbativa e una descrizione dettagliata della procedura BPHZ di rinormalizzazione per la teoria ϕ^3. Dopodiché introduciamo le algebre di Hopf e mostriamo come la struttura soggiacente ai diagrammi di Feynman della teoria possa essere descritta da una Hopf algebra. Infine giungiamo al risultato principale della teoria di Connes-Kreimer, ovvero che la soluzione alla fattorizzazione di Birkhoff nel caso della Hopf algebra dei diagrammi di Feynman è data esattamente dalla procedura BPHZ.

Acknowledgements

This thesis is the summa of my journey as student.
Despite being regarded as a personal achievement, it is sometimes forgotten this is built up of many little contributions collected along the road, I want to try to acknowledge at least partially, overcoming my shyness.

I would like to express my gratitude to my supervisors F. D'Andrea, P. Vitale and M. Arzano for believing in me and for their help in this work. I would like to thank many people of the theoretical physics section, particularly F. Lizzi and P. Vitale, for the beautiful and instructive conversations and for making the students feel part of a great big familiy. Very special thanks to Nicola, without him I could have been lost so many times. I want to thank my adventure companions Lorenzo, Annarita, Aurora, Mirko, Vincenzo amongst the others, for making the life in department a cheerful time even for a personality like mine. Special thanks also to all QBR for bearing my blabbering in front of a beer. My deepest thankfulness goes to my grandfather (mio nonno), my brother and my family for their support and affection, to Agata and Noi for the patience and the love.

Finally, thank you to my parents for the countless coffees.

Sincerely yours,
Salvatore

Contents

Chapter 0

Introduction

Despite in a non-relativistic (i.e. Galilean) theory the concept of field may however be useful and elegant, it is only in the regime of the speed of light where it becomes essential in order to deal with the locality of interactions. On the other hand particles, which played a substantial role in the framework of Newtonian mechanics, become too much of a histrionic character in such an extremely broader scenario providing for features as creation and annihilation of identical components, where there is no clue of what the structure of spacetime should be below a certain lenghts scale, when energies can go far beyond the nowadays possibilities of experimentation... so that we have bit of some hints of what and not only which could be the fundamental constituents of nature: this is when (special) relativity meets quantum mechanics.
Nonetheless, in the last century human resourcefulness and initiative guided us in this somewhat bizarre world and the work of many, driven by needful curiosity (and skillful mathematics), has lead us to a truly beautiful description for this comprehensive-seeking theory of physics.
Driven by the *"shut up and calculate!"* motto, quantum physicists have been able to overcome the difficulties in the conceptual understanding of quantum mechanics relying on the consistence of its mathematics, despite the operations of translation has not always been an easy task. But troubles seem even to multiply in the context of quantum field theory, with inconsistences and infinites arising everywhere!
As before, people returned back to their mathematical foothold and experimental handhold, and began to develop techniques and manipulations whose results fit surprisingly well with experiments (or vice versa?). Of these, renormalization represents perhaps the major instance, getting rid of cumbersone divergences in the computations and maybe revealing the path to some deep truth of nature we still struggle to appreciate. Once again, physics became source of inspiration for mathematics. And analogously, mathematics permeate our understanding of physical reality.

Outline

This thesis is based on the Connes-Kreimer theory, a wide framework of mathematical physics in the context of Hopf algebras and more generally of noncommutative geometry. The main interest is the perturbative regime of quantum field theory, possibly the most fruitful resource of results in the whole theoretical physics, but somehow lacking a bit of mathematical strictness.
Their approach heads directly towards the Feynman diagrams of a theory, which we will define and interpret in a graph-theoretical manner. Accordingly, we shall frequently use the terms diagram and graphs interchangeably. Scalar fields will accompany us throughout in order to reconnect with [C] and [CM]. We will focus on self-interacting theory, specifically ϕ^3, in order to keep nontriviality without losing too much generality. Despite not describing a fundamental theory, this is a simple toy model which lets us avoiding to give up primary insights. Nevertheless, self-interacting scalar theories still retain importance as effective field theories and in the context of condensed matter physics. We will mainly treat the theory's primary result until now, understanding the renormalization procedure on the basis of the combinatorics present in different physical constructions.

This work is articulated as follows:

Chapter 1 is a summary of perturbative quantum field theory. It serves both as background and backbone for the chapters which follow, so we sought for the bare minimum, without involving too much into details which we shall not need. Here the language is the usual informal physicists' one, introducing the instruments used in the subsequent computations in an intuitive manner, with some little clarification where required.

Chapter 2 is a presentation of renormalization. Since it is a central topic both in quantum field theory and in the Connes-Kreimer theory, we aimed at completeness when clarifying the somehow odd physical ideas behind these manipulations. We deal essentially with our toy model, implementing the techniques directly. This allowed us to work out examples from basic graphs to more general constructions. Furthermore, this is supplied by explicit calculations in the appendix.

Chapter 3 is a mathematical interlude around the concept of Hopf algebra. This provides prerequisites to the formalism of Connes-Kreimer, but also plays the role of preparation to chapter 4. The main definitions are piled up in a strict

vocabulary. Afterwards we preliminarly investigate some useful ideas.

Chapter 4 is the recollection of probably the major result within the Connes-Kreimer theory. We begin to shore up some of the previous objects of chapter 1 and 2 in a generalized, mathematical formalism. Then we make use of the structures of chapter 3 to get a precise algebraic interpretation of the renormalization procedure, giving the demonstrations of some theorems and reworking the examples of chapter 2 in this new context. Finally we sketch some other results of this framework.

Further details can be found in the bibliography and therein references. For instance, some curious historical notes can be found throughout [Wei].

Chapter 1

Quantum Field Theory

"The theory of quantum electrodynamics describes nature as absurd from the point of view of common sense. And it fully agrees with experiment. So I hope you can accept nature as She is - absurd. "

Richard P. Feynman [Fq8]

1.1 Introduction

We begin by trying to do the minimum injustice in the purpose of outlining the main aspects of quantum field theory.
We shall take as granted some special relativity covariant formalism and we will not delay much with basic quantum mechanics. As usual in this context we will work in natural units, in which the Planck constant and the speed of light are defined by:

$$\hbar = c = 1$$

This choice allows us to pick a single dimensionful quantity, namely the mass, and to refer to it simply as *dimension* throughout the sequel.

In this section we will give a brief account of what a classical field is, focusing on the Lagrangian description for a scalar theory and then giving an idea of how canonical quantization is performed.
After that, we look at self-interacting theories by means of perturbation theory approach. The checkpoint is to reach Wick's theorem, which encodes the combinatorics of perturbative quantum field theory and then we give a first

heuristical derivation of Feynman diagrams. After showing some basic example along the road, we also write down the Feynman rules.
Finally we introduce Feynman's path integrals formalism and the effective action which will turn out useful in the next sections.

We gliss over some mathematical technicalities which are sometimes difficult to precisely define and would drift us apart from the purpose of a summary. In general the derivation shall follow classical textbooks such as [PS] and [Wei]. Some details using path integrals language can also be found in [Ram] and [Sre]. For further overall mathematical details we refer to [CM].

1.2 Free Theory

1.2.1 Scalar Fields

A (classical) field $\phi(x)$ is a section of an opportune Hermitian vector bundle on a (pseudo-)Riemannian manifold. When this manifold is the space of configurations of a certain system, one can then define the Lagrangian (density), i.e. a smooth scalar function of these fields:

$$\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}_0 + \mathcal{L}_{int} \tag{1.1}$$

where we split the free term and the interaction term beforehand.
We won't explicitly handle much geometry since we shall only work with scalar theory in $d-$dimensional Minkowski space with signature $(+, -, \dots, -)$. An element of this space is written as x^μ, where $\mu = 0, \dots, d-1$, but at times we may also make use of the notation $x^\mu = (t, \mathbf{x})$.
The interaction term $\mathcal{L}_{int}$ is assumed to be a polynomial in the fields. Usually it can be a potential of the form:

$$V(\phi) = \frac{g}{3!}\phi^3 \quad , \qquad V(\phi) = \frac{\lambda}{4!}\phi^4 \tag{1.2}$$

or even their sum.
Throughout our work, we will mainly use one of these two prototypical terms, one at a time. It is worth mentioning that (at least as they stand) these potentials are a bit unphysical[1]. Nonetheless self-interacting scalar theories come handy

[1] In particular any odd power of the fields renders the Hamiltonian unbounded from below, i.e. the theory has no stable minimum; meanwhile for the ϕ^4 interaction the renormalized theory turns out to be trivial (see for example [BS]), its fixed point being the trivial one.

as toy models, exhibiting most of the basic features whilst lightening the intricacies of the explicit calculations in some examples.

More fields and interaction terms can be added in the Lagrangian respecting some specific request of the theory under consideration, such as Lorentz and gauge invariance, some other symmetry, renormalizability (as we will see) and other criteria which may depend upon the field types and the space-time dimension.

The free Lagrangian of the theory $\mathcal{L}_0$ is in general made of the kinetic and (possibly) the mass terms. Our starting point is the basic Lagrangian for a (real) scalar field $\phi(x^\mu)$:

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{1.3}$$

From a Lagrangian is then defined the (classical) action functional[2]:

$$S[\phi] = \int \mathcal{L}(\phi, \partial_\mu\phi)d^dx \tag{1.4}$$

Hence, following the variational approach, we get the dynamics of the system as an extremum of the action itself. For the free scalar theory the Euler-Lagrangian equation is the Klein-Gordon equation:

$$\partial_\mu\partial^\mu\phi(x) + m^2\phi(x) = 0 \tag{1.5}$$

which is easily generalized in the presence of a potential.
Thus an action S, i.e. the datum of a Lagrangian $\mathcal{L}$ and a dimension d, is all we need to define a (classical) field theory.

At this point we want to discuss the classical system on a quantum level, thus a quantization procedure is needed. There are different approaches, whose full descriptions would involve mathematical technicalities far beyond the scope of this brief introduction. Each of them has its advantages, so we choose to start with a sketch of the most intuitive one.
To put it simple, the canonical quantization of a classical theory is the following correspondence:

Commutative algebra of functions on the phase space equipped with the Poisson bracket $\{\cdot,\cdot\}$	$\longrightarrow$	Operator algebra on an opportune Hilbert space equipped with the commutator $[\cdot,\cdot]$

[2]Here we keep track of the dimension, anyhow for all this section we could (and possibly will) simply put $d = 4$

This procedure is somehow well understood in the case where degrees of freedom are finite in number, i.e. passing from classical to quantum mechanics. Despite being ideally the very same correspondence, dealing with fields formalism is more subtle, since now we are treating an infinite number of degrees. Just in our scalar field theory one substitutes scalar functions with operator-valued distributions. At this point one expands the fields in Fourier modes:

$$\phi(x) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{1.6}$$

where $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$ is the energy of one particle.
In the classical theory, this makes generally sense with $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$ opportune functions.
What we do on a quantum level is to work out the analogy with the canonical quantization, obtaining a formal machinery which still provides us sensible results. In this regard, we now replace these functions by operators and impose the harmonic oscillator commutation rules for each mode:

$$[a_{\mathbf{p}}, a^*_{\mathbf{q}}] = (2\pi)^{d-1}\delta^{(d-1)}(\mathbf{p}-\mathbf{q}) \qquad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^*_{\mathbf{p}}, a^*_{\mathbf{q}}] = 0 \tag{1.7}$$

Altough highly heuristic, this allows us to perform the algebraic manipulations we shall need.
Indeed creation and annihilation operators suffice to construct the Fock space to represent our states as usual, defining the vacuum state $|0\rangle$ as being annihilated by all $a_{\mathbf{k}}$'s and building up each state as oscillators' towers of $a^*_{\mathbf{k}}$'s.

En passant, we also define the normal-ordering operator N which acts on products of oscillators putting the creators to the left of the annihilators, for example:

$$N(a_{\mathbf{p}} a^*_{\mathbf{k}} a_{\mathbf{q}}) \equiv a^*_{\mathbf{k}} a_{\mathbf{p}} a_{\mathbf{q}} \tag{1.8}$$

This is going to really help us in making calculations.

1.2.2 Green's Functions

Once we have a vacuum state, we can begin our computational journey by constructing the first fundamental objects in the theory, the $n-$point correlaction functions:

$$G^{(n)}(x_1, \ldots, x_n) := \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle \tag{1.9}$$

where the symbol T is the time-ordering operator, whose task is to rearrange the fields from latest to first on space-time in order to preserve causality.
Correlation functions encompass physical data as cross sections. Of particular importance for $n = 2$ we have the Feynman propagator:

$$\langle 0|T[\phi(x)\phi(y)]|0\rangle \equiv D_F(x-y) = i\int \frac{d^dp}{(2\pi)^d}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{1.10}$$

which is the Fourier transform of the Feynman propagator in the momentum space, defined as:

$$\tilde{D}_F(p) = \frac{i}{p^2-m^2+i\epsilon} \tag{1.11}$$

Observe that the small real constant $\epsilon > 0$ is inserted as a trick in order to treat poles and the limit $\epsilon \to 0$ shall be taken after the integral in (1.10) is performed. Also notice that the Feynman propagator is the Green's function[3] of the Klein-Gordon operator:

$$(\Box + m^2)D_F(x-y) = -i\delta^{(d)}(x-y) \tag{1.12}$$

This is the simplest case of the equation:

$$(\Box+m^2)\langle 0|T[\phi(x)\phi(x_1)\cdots\phi(x_n)]|0\rangle = \sum_{j=1}^{n}\langle 0|T[\phi(x_1)\cdots(-i\delta^{(d)}(x-x_j))\cdots\phi(x_n)]|0\rangle$$

which we recast in the insightful fashion:

$$(\Box+m^2)G^{(n+1)}(x,x_1,\ldots,x_n) = -i\sum_{j=1}^{n}\delta^{(d)}(x-x_j)G^{(n-1)}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n) \tag{1.13}$$

clearly showing the *correlation* amongst correlation functions.
These can be further generalized to an interacting theory by means of the equations of motion and the commutation relations.

1.3 Interactions

1.3.1 Perturbation Theory

In the quantum realm, to measure something means to interact with it. On the theoretic side, free-particle states are eigenstates of the Hamiltonian, thus one

[3]Tacitly assuming further causality conditions.

must incude nonlinear local terms in order to see interactions and scattering. These terms have to preserve the structure of the Lagrangian, namely Lorentz invariance and symmetries. A crucial role is also played by gauge symmetries, that require specific combinations of fields and coupling constants in order not to be lost. A first picking of possible candidate interaction terms is possible on the base of the sole dimensional analysis. In fact one recalls that in natural units the mass dimension is the inverse of the lenght dimension. Thus the action functional must be dimensionless and accordingly the Lagrangian dimensionality must be precisely the mass dimension d (of the space-time).

Considering the scalar theory, from the kinetic terms we find out the mass dimension of the field is $[\phi] = d/2 - 1$. Consequently the mass terms is a one-dimensional coupling as expected. For a general self-interaction term of the form $\kappa\phi^n$ the coupling has dimension $[\kappa] = d - \frac{n}{2}(d-2)$ therefore we notice that for $n > \frac{2d}{d-2}$ the coupling's dimension should be negative.

To find an exactly solvable field theory is a difficult task. The best approach generally consists in treating the interaction term as a perturbation of the free theory. Therefore one gets series expansion in the coupling constant. Usually it turns out that physically relevant models behave in a sensitive manner. For example the coupling constant in QED is small enough to render the perturbative expansion an asymptotic series, thus making the method meaningful.

Furthermore, another important aspect is renormalizability: beyond the tree-level, the perturbation series involve integrals in the momenta space of virtual particles. These integrals are usually divergent, so one imposes some sort of regulator and seeks for a consistent way to render the physics independent from this procedure. As we shall see, it turns out that negative-dimension couplings would interfere with renormalization and the corresponding interaction terms must be discarded.

Henceforth we shall adopt the conventions and notation of [PS].
The starting point is to define the analogous of the free propagator. In general we know this is the two-point correlation function:

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle \tag{1.14}$$

where $|\Omega\rangle$ is the ground state of the full, interacting theory. This is a different ground state from $|0\rangle$ of the free theory, but we will see in a moment how they can be linked.

Clearly in the latter case the two-point correlation function is just the (Feynman) propagator (1.11). Thus this retains the physical interpretation of the correlation function as the amplitude for propagation of a particle or excitation between y and x. We would be glad to relate directly these two propagators. Once this is done, one can obtain higher-order correlation functions as the generalized version of the recursive relation (1.13).
Luckily this is possible, however we just state the results without going into details, addressing to the references listed in the introduction for a complete derivation.

The field $\phi(x)$ in the Heisenberg picture of the interacting theory can be related by a unitary transformation to the interaction picture field $\phi_I(t, \mathbf{x})$, which in the scalar case is the very same of the free field. The unitary operator which performs this trasformation is the time-evolution operator satisfying the appropriate evolution equation with the suitable Hamiltonian. It can be given an explicit form by means of the Dyson series.

Using the Gell-Mann and Low Theorem[4] one can express the ground state $|\Omega\rangle$ in terms of the free theory ground $|0\rangle$ through an adiabatic transformation:

$$\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle = \lim_{\tau\to\infty(1-i\epsilon)} \frac{\langle 0|T\left[\phi_I(x)\phi_I(y)e^{i\int_{-\tau}^{\tau}\mathcal{L}_{int}(t)dt}\right]|0\rangle}{\langle 0|T\left[e^{i\int_{-\tau}^{\tau}\mathcal{L}_{int}(t)dt}\right]|0\rangle} \tag{1.15}$$

As it stands, this is an exact formula. Its Taylor series is the starting point for perturbative calculations.

Giving up the aforementioned notation for simplicity, what we are left are the vacuum expectation values of time-ordered products of free fields:

$$\langle 0|T[\phi(x_1)\phi(x_2)\cdots\phi(x_n)]|0\rangle \tag{1.16}$$

The standard approach to evaluate these n-point correlation functions is to expand the fields in harmonic oscillators, i.e. working on the creation-annihilation operators. This turns very useful because it enables one to invoke the normal ordering N of (1.13). Thus we can also define the *contraction* of two fields as:

$$\overbrace{\phi(x)\phi(y)} := D_F(x-y) \tag{1.17}$$

[4]See [GL] for details.

The upshot for these constructions is the Wick theorem[5]:

$$T[\phi(x_1)\phi(x_2)\cdots\phi(x_m)] = N[\phi(x_1)\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}] \tag{1.18}$$

As an operator equation this is a purely combinatorial statement.
Its usefulness is manifest when the operators are inserted in the vacuum expectation value. Indeed one has $\langle 0|N(fields)|0\rangle = 0$ by definition of normal ordering (1.13), so that we must compute just the fully contracted terms. This is a tremendous semplification!

As an explicit example let's consider:

$$\begin{aligned}\langle 0|T[\phi_1\phi_2\phi_3\phi_4]|0\rangle &= \langle 0|N[\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \\ &+\overbrace{\phi_1\phi_2\phi_3\phi_4} + \phi_1\overbrace{\phi_2\phi_3}\phi_4 + \phi_1\overbrace{\phi_2\phi_3\phi_4} + \phi_1\phi_2\overbrace{\phi_3\phi_4} + \\ &+\overbrace{\phi_1}\phi_2\overbrace{\phi_3}\phi_4 + \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \overbrace{\phi_1\phi_2\phi_3\phi_4}]|0\rangle \\ &= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)\end{aligned} \tag{1.19}$$

where $\phi_m \equiv \phi(x_m)$ is a useful notation we shall use in the following. Notice that the final result is expressed in terms of free theory propagators only[6].

Armed with this powerful tool, we now go back to the Gell-Mann and Low expression for the two-point correlaction functions (1.15) and, as promised, work with the Taylor expansion of the numerator. At the $n-$th order:

$$\langle 0|T\left[\phi(x)\phi(y)\frac{i^n}{n!}\int \mathcal{L}_{int}(x_1)dx_1\cdots\int \mathcal{L}_{int}(x_n)dx_n\right]|0\rangle \tag{1.20}$$

It is important to observe the consistency of the method: the zeroth order is the same as in the free theory. One must also pay attention to the fields in the interaction term, since they are all evaluated at the same space-time point. Once we have arrived to this, all is left to do is to apply the Wick theorem. Let's focus on some specific examples. As a matter of fact the first order in the ϕ^3 theory is

[5]Formally this remains the very same expression for spinor fields too, despite some precautions in defininining an analogous contraction and the proper commutation signature due to the fermionic nature of the fields.

[6]Obviously not by chance, one may have guessed this from (1.13).

trivial by virtue of the Wick theorem. The next simplest computation is the first order in ϕ^4 theory, which gives:

$$\begin{aligned}\langle 0|T\left[\phi(x)\phi(y)\left(\frac{-i\lambda}{4!}\right)\int\phi(z)\phi(z)\phi(z)\phi(z)d^dz\right]|0\rangle = \\ = 3\left(\frac{-i\lambda}{4!}\right)D_F(x-y)\int D_F(z-z)D_F(z-z)d^dz+ \\ +12\left(\frac{-i\lambda}{4!}\right)\int D_F(x-z)D_F(y-z)D_F(z-z)d^dz\end{aligned} \tag{1.21}$$

Now let's switch to ϕ^3 theory and consider the second order:

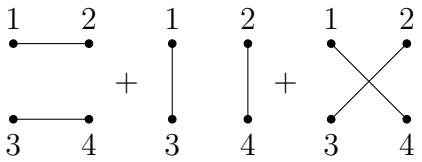
$$\begin{aligned}\langle 0|T\left[\phi(x)\phi(y)\frac{1}{2}\left(\frac{-ig}{3!}\right)\int\phi(z)\phi(z)\phi(z)d^dz\left(\frac{-ig}{3!}\right)\int\phi(w)\phi(w)\phi(w)d^dw\right]|0\rangle = \\ = 9\cdot\frac{1}{2}\left(\frac{-ig}{3!}\right)^2 D_F(x-y)\iint D_F(z-z)D_F(z-w)D_F(w-w)d^dzd^dw+ \\ +6\cdot\frac{1}{2}\left(\frac{-ig}{3!}\right)^2 D_F(x-y)\iint D_F(z-w)D_F(z-w)D_F(z-w)d^dzd^dw+ \\ +18\cdot\frac{1}{2}\left(\frac{-ig}{3!}\right)^2\iint D_F(x-z)D_F(y-z)D_F(z-w)D_F(w-w)d^dzd^dw+ \\ +9\cdot\frac{1}{2}\left(\frac{-ig}{3!}\right)^2\iint D_F(x-z)D_F(y-w)D_F(z-z)D_F(w-w)d^dzd^dw+ \\ +18\cdot\frac{1}{2}\left(\frac{-ig}{3!}\right)^2\iint D_F(x-z)D_F(y-w)D_F(z-w)D_F(z-w)d^dzd^dw+ \\ + \quad \text{the same five terms switching } (z\leftrightarrow w)\end{aligned} \tag{1.22}$$

1.3.2 Feynman Diagrams

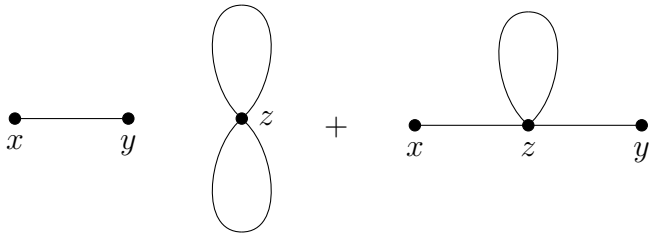
We now give a first, informal, physical introduction to the Feynman diagrams, which later on we are going to discuss in a more rigorous way. They provide a pictorial description of interaction processes. The main point is Wick's theorem: it allows to turn general correlation functions into a combination of propagators, so one is allowed to retain the intuitive interpretation of particles which propagate from a space-time point to another. The method is simple: space-time points are represented each by a vertex, propagators between points by lines joining the corresponding vertices[7].

[7]We shall define Feynman diagrams in a more general way later on.

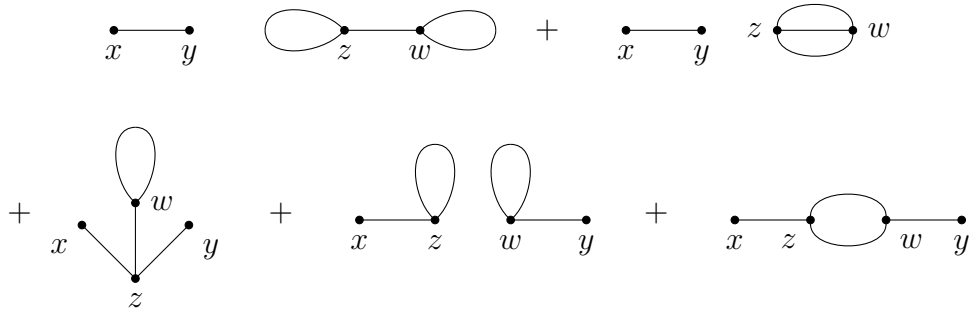
Following [PS], let's return to the example (1.19) we previously worked out in explicit computations and represent them as Feynman diagrams:

$\langle 0|T[\phi_1\phi_2\phi_3\phi_4]|0\rangle =$ 

We see that the total amplitude is the sum of all the possible (topologically) inequivalent diagrams each defined by the different way of joining all the vertices. In this sense, the combinatorial aspects manifest themselves.
In general we must keep track of *external* points x, y compared to the *internal* points. Ignoring constant factors, for example the first order in the ϕ^4 theory (1.21) is represented by two different diagrams:



Keeping track of the external points also helps out in identifying inequivalent diagrams[8]. Indeed for the ϕ^3 theory second order (1.22) one gets five contributions:



Despite we won't work with the S-matrix, it is useful in order to underpin the physical meaning of Feynman diagrams as scattering amplitudes. Indeed, the S-matrix allows us to express the cross section of an interaction process in terms of the sum of all the connected amputated[9] diagrams involved. However S-matrix elements can be calculated from the time-ordered Green's functions by means of the LSZ reduction formula[10].

[8] For the scalar theory, internal points are kind of a dummy variable.
[9] That is the largest of its 1PI subgraphs. See [PS] for a detailed discussion.
[10] See [LSZ] for details.

There is no great conceptual difference: we imagine as if the external points are "stretched" to infinity. This means that the initial and the final particle(s) are in some ideally prepared asymptotic states, joined to a vertex in the graph by means of *external lines*, the incoming ones from "the past" to a vertex, the outgoing ones from a vertex to "the future". In this sense the propagators are also called *internal lines* since they connect two vertices.

The constant factor we left behind is the product of factorials (from the Taylor expansion and combinatorics), couplings and the symmetry factor, i.e. the number of ways of interchanging vertex and lines without changing the diagram.
All of these prescriptions for associating analytic expressions with pieces of diagrams are summarized in the Feynman rules.

The Feynman rules for a self-interacting scalar theory are:

		Coordinate space	Momentum space
1.	For each propagator: (internal line)	$x \bullet\!\!-\!\!-\!\!\bullet\, y = D_F(x-y)$	$\xrightarrow[p]{} = \dfrac{i}{p^2 - m^2 + i\epsilon}$
2.	For each vertex:	$z = (-i\lambda)\int d^d z$	$= -i\lambda$
		$z = (-ig)\int d^d z$	$= -ig$
3.	For each external point:	$x \bullet\!\!-\!\!- = 1$	$x \bullet \xleftarrow[p]{} = e^{-ip\cdot x}$
	For each external line:	$z \bullet \xleftarrow[p]{} = e^{-ip\cdot z}$	$z \bullet \longleftarrow = 1$
4.	In momentum space, at each vertex, impose momentum $\delta-$conservation and integrate over each undetermined loop momentum:	$\int \dfrac{d^d p}{(2\pi)^d}$	
5.	Finally, divide by the symmetry factor in both cases.		

The symmetry factor of a graph is defined as the number of permutations of its internal vertices and internal lines which leave the external structure untouched. Besides, one must keep track of the signs assigned to the incoming and outgoing arrows. In the scalar theory, the choice of one or the other con-

vention is arbitrary, as long as it is consistent with the momentum conservation.

These rules instruct us to construct amplitudes by joining independent processes, to integrate over all points where this can occurr and then to sum each of the inequivalent amplitudes[11]. They are clearly useful to mnemonically extract the amplitude expression from diagrams.
Thus Feynman rules implicitly state that the amplitude of a disconnected graph is the product of the amplitudes of all its connected components. We define a one-particle irreducible (1PI) diagram as a connected diagram that cannot be disconnected by removing any single internal line. Then any connected graph can be seen as the concatenation of its 1PI components and the free propagators necessary to joint them.[12].

The last thing in the Gell-Man and Low formula (1.15) we must take care of is the denominator. It turns out its role is to remove the diagrams in the numerator which are disconnected from every external points[13].

At the end of the story the expression for a general $m-$points correlation functions reads:

$$\langle\Omega|T[\phi_1\cdots\phi_m]|\Omega\rangle = \text{sum of all connected diagrams with } m \text{ external vertices} \tag{1.23}$$

1.3.3 Path Integral

Another formalism for representing quantum theories is to work with (Feynman) path integrals. Formally, these are functional integrals like:

$$\langle\hat{A}\rangle \propto \int \mathcal{D}\phi\, e^{i\frac{S[\phi]}{\hbar}} A(\phi) \tag{1.24}$$

Here we find the expectation value[14] of an operator $\hat{A}$. It corresponds to the generic combination of the fields $A(\phi)$, that is found at the integrand.
The integral measure is composed of two parts. The (Feynman) measure $\mathcal{D}\phi$ is a generally ill-defined functional measure which operates as a concise way to

[11] It is worth observing that this is just the superposition principle.
[12] See [Fra] for details.
[13] That is, the different kinds of vacuum bubbles.
[14] For the free theory, this is the vacuum expectation value.

represent the infinite-dimensional integration over all possible field configurations, according to the superposition principle. We may formally write:

$$\mathcal{D}\phi \propto \prod_x d\phi(x^\mu) \tag{1.25}$$

where $x^\mu = n^\mu a$ is the discretization of the variable x on a lattice of spacing a in a finite hypervolume L^d. This is an intermediate passage that proves conceptually useful to make sense of some convergence property of these integrals, despite being rich in tricky subtleties we will ignore.

The second term is a phase containing the classical action functional of the theory under consideration. It weights the configurations and works much like for the partition function in statistical mechanics. The dimensional $\hbar$ is kept explicit in order to notice that in the limit $\hbar \to 0$ we can use the method of stationary phase to determine the classical path. In fact the stationarity condition is the variational principle from which the equations of motion are obtained.

At the end one wants to recover the infinite system by taking the thermodynamic limit $L \to \infty$ and the continuum limit $a \to 0$. It is the latter one which shows up the intrinsec ultraviolet (UV) divergences of the perturbative quantum field theory which are to be treated with renormalization theory.

This integral is defined up to an averall normalization factor, which we will later compute. Furthermore, in the measure itself is absorbed an awkward constant, which we tacitly sweep under the carpet, since it won't bother us. A rigorous mathematical treatment would be anyhow difficult. The explicit expression and complete derivation of the path integral formalism can be found in any of the references given in the introduction.

The benefit of this formalism is that it works directly with the Lagrangian, thus preserving its symmetries and manifest Lorentz invariance. This comes handy inasmuch it generalizes in a straightforward way to other, more general interacting theories. It is also useful to define operator equations as equations which hold when inserted into any path integral that has no other fields coincident with those involved in the equation.

The aim is now to reconnect with the work previously done by translating all in this new language. We begin with the path integral version of the Gell-Mann

and Low formula:

$$\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle = \lim_{\tau\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, e^{i\int_{-\tau}^{\tau}\mathcal{L}\; d^dz}\phi(x)\phi(y)}{\int \mathcal{D}\phi\, e^{i\int_{-\tau}^{\tau}\mathcal{L}\; d^dz}} \tag{1.26}$$

Notice that by making explicit the interaction term $\mathcal{L}_{int}$ we may interpret the expression as if interacting theory as well is averaged by the free theory, thus obtaining (1.15).

Next result we want to recover in this formalism is Wick's theorem.
Before all, consider the free scalar field theory S_{KG}. Therefore, according to (1.17), we define again the contraction of two fields as:

$$\overbrace{\phi(x)\phi(y)} = \frac{\int \mathcal{D}\phi\, e^{iS_{KG}}\phi(x)\phi(y)}{\int \mathcal{D}\phi\, e^{iS_{KG}}} = D_F(x-y) \tag{1.27}$$

Then all is left to do is to Taylor expand the interaction term $e^{iS_{int}}$ in the path integral, from which it follows the very same expression (1.20).
The Feynman rules are reached straightaway.

Remarking the analogy with statistical mechanics, we introduce the generating functional:

$$Z[J] := \int \mathcal{D}\phi\, e^{i\int(\mathcal{L}+J\phi)d^dx} \qquad\qquad Z_0 \equiv Z[J=0] \tag{1.28}$$

We added an external source term $J(x)\phi(x)$ to the general Lagrangian for convenience. Indeed, as it stands, this object allows us to reformulate the Gell-Man and Low formula as:

$$\langle 0|T\left[\phi(x)\phi(y)\right]|0\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right) Z[J]\Big|_{J=0} \tag{1.29}$$

where we make use of the functional derivative $\dfrac{\delta}{\delta J(x)}J(y) = \delta^{(d)}(x-y)$.

This trick is particularly useful when dealing with a free theory. If we specialize to our free scalar case with $\mathcal{L}_{KG}$ the source term is factorized and one gets a convenient expression for the generating functional:

$$Z[J] = Z_0\, e^{-\frac{1}{2}\int J(x)D_F(x-y)J(y)\, d^dx\, d^dy} \tag{1.30}$$

whose second derivative gives immediately the Feynman propagator.

By taking higher derivatives of the generating functional we can compute every Green's function of the theory:

$$G^{(n)}(x_1,\ldots,x_n) = \frac{(-i)^n}{Z[0]}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0} \tag{1.31}$$

1.3.4 Effective Action

It is very fruitful to look at some consequences of the deep connection between quantum field thery and statistical mechanics in order to draw some results. For simplicity we work with a single scalar theory and define the energy functional $E[J]$ by:

$$e^{-iE[J]} = Z[J] = \int \mathcal{D}\phi\, e^{i\int(\mathcal{L}+J\phi)d^dx} \tag{1.32}$$

This is recognizable as the amplitude $\langle\Omega|e^{-i\mathcal{H}T}|\Omega\rangle$, implying that $E[J]$ is the vacuum energy with a source. At this point, we take the functional derivative:

$$\frac{\delta}{\delta J(x)}E[J] = i\frac{\delta}{\delta J(x)}\log Z = -\frac{i}{Z}\frac{\delta}{\delta J(x)}Z = -\frac{\int\mathcal{D}\phi\, e^{i\int(\mathcal{L}+J\phi)d^dy}\,\phi(x)}{\int\mathcal{D}\phi\, e^{i\int(\mathcal{L}+J\phi)d^dy}} = -\langle\Omega|\phi(x)|\Omega\rangle_J \tag{1.33}$$

getting the vacuum expectation value in the presence of a source $J(x)$.
Following the analogy with statistical mechanics, we define the classical field:

$$\phi_{cl}(x) \equiv \langle\Omega|\phi(x)|\Omega\rangle_J \tag{1.34}$$

Then by taking the Legendre transform of $E[J]$ one defines the (quantum) effective action:

$$\Gamma[\phi_{cl}] := -E[J] - \int J(y)\phi_{cl}(y)d^dy \tag{1.35}$$

Observe that its functional derivative returns the external source, as expected:

$$\frac{\delta}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}] = -J(x) \tag{1.36}$$

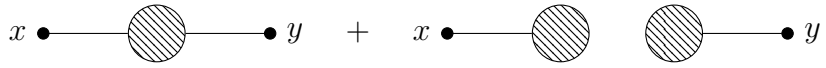
Assuming that $\Gamma[\phi_{cl}] \propto V_{eff}(\phi_{cl})$ it turns out that an extremum of the effective action is obtained by solving the corresponding equation for the effective potential $\frac{\partial}{\partial\phi_{cl}}V_{eff}(\phi_{cl}) = 0$. The solutions of this equation are translational-invariant

states (i.e. independent of x) with $J = 0$. The minima of the effective potential define exact vacuum states of the quantum theory.

Going on, we take the second derivative of $E[J]$:

$$\frac{\delta^2 E[J]}{\delta J(x)\delta J(y)} = -i\langle\phi(x)\phi(y)\rangle + i\langle\phi(x)\rangle\langle\phi(y)\rangle \equiv -i\langle\phi(x)\phi(y)\rangle_{conn} \tag{1.37}$$

This is the connected correlator (hence the notation). Indeed, diagrammatically we have:

$\langle\phi(x)\phi(y)\rangle =$ 

$\langle\phi(x)\rangle\langle\phi(y)\rangle =$ 

where the blobs stand for a sum of connected diagrams. Notice that the two disconnected terms cancel each other, as required. This is in fact true for any higher order derivative, $E[J]$ is the generating functional of connected correlation functions:

$$\frac{\delta^n E[J]}{\delta J(x_1)\cdots\delta J(x_n)} = i^{n+1}\langle\phi(x_1)\cdots\phi(x_n)\rangle_{conn} \tag{1.38}$$

There is an akin relation for the quantum effective action, highlighting its profound role in quantum field theory:

$$\frac{\delta^n \Gamma[\phi_{cl}]}{\delta\phi_{cl}(x_1)\cdots\delta\phi_{cl}(x_n)} = -i\langle\phi(x_1)\cdots\phi(x_n)\rangle_{1PI} \tag{1.39}$$

This means that the effective action is the generating functional of one-particle irreducible correlation functions.

Chapter 2

Renormalization

"This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small - not neglecting it just because it is infinitely great and you do not want it!"

P. A. M. Dirac [Dq8]

2.1 Introduction

The idea of renormalization goes back to fluid dynamics [CM], [Col] where a moving object immersed in a field acquires an additional mass due to the interaction with the sorrounding fluid.
Therefore in classical physics one can separate and measure the mass of the object and the body of fluid it drags. However in a quantum field theory it is not possible to isolate a particle from its field, thus the *bare* parameters are not physical observables. It is from this hindrance that arose bizarre albeit really fascinating skills aiming to untie this knot.

Starting from a general picture of divergences in a quantum field theory, we examine in detail the simplest example for the interacting scalar theory, the one-loop self-energy. We develop techniques to manipulate divergent integrals in order to keep infinities under control by means of some regulator.
Then we introduce local counterterms as main idea to perform renormalization by compensating divergences. After discussing some prescriptions, we focus on the combination of dimensional regularization [Wil], [tV], [BG] and minimal

subtraction scheme, both of which we will use in the following.
With this at hand we come to the description of the BPH(Z) procedure [BP], [BS], [Hep], a systematic approach to renormalize a theory in a recursive way graph by graph. In the end, solving the recursion leads to forest formula [Zim].

This section is largely based on [C], from which we retain most of the notation. Some insights are added from the other references.

2.2 Divergences

2.2.1 Regulators

We have seen (1.25) that path integrals can be regarded formally as the continuum limit of a lattice theory. The spacing a acts as a natural cut-off for the large momenta, and thereby the integrals over momenta are function of it as extremum of integration. In the limit $a \to 0$, i.e. when this cut-off is removed, these integrals appear to be divergent, despite in a definite manner depending on a itself. In this sense, the lattice spacing keeps under control divergences in the theory. It is an example of UV regulator[1].

There are many possible regulators, amongst which the more or less brute cut-offs form a large class. Another standard way is the Pauli-Villars method. To put it simple this just replaces the free scalar propagator with:

$$\frac{i}{p^2 - m^2 + i\epsilon} \longrightarrow S_F(p, m, \Lambda) = \frac{i}{p^2 - m^2 + i\epsilon} - \frac{i}{p^2 - \Lambda^2 + i\epsilon} \tag{2.1}$$

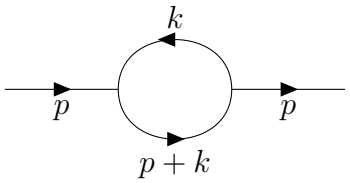
whereby one recovers the free case in the limit $\Lambda \to \infty$.
When computed with a cut-off, the integrals are not immediately divergent anymore, but the theory exhibits rather unphysical features such as loss of Poincaré invariance. However this is not a big treat, since they serve as an intermediate tool and in the end one is interested in the renormalized theory with cut-off removed. Therefore any regulator is equivalent[2] and the choice is dictated by computational convenience.

Despite divergences seem to invalidate perturbation theory, measurements give plenty of credit. Path integrals and renormalization reach the scope to construct sensible Green's functions. For their part, Green's functions encode all the physics needed to describe a quantum field theory.

[1] There also exist infrared (IR) regulators, but they won't bother us in the following.
[2] At least while working in perturbation theory.

2.2.2 Infinities

The simplest example of divergence in the ϕ^3 theory is the one-loop self energy, whose graph is:



(2.2)

Its contribution is defined to be i times the value of the graph as dictated by the Feynman rules, when the external structure is removed:

$$\Sigma_1(p^2, m^2, d) = \frac{i}{2}\frac{g^2}{(2\pi)^d}\int \frac{1}{(k^2 - m^2 + i\epsilon)}\frac{1}{((p+k)^2 - m^2 + i\epsilon)}d^d k \tag{2.3}$$

For $d \geq 4$ this integral has an overall UV[3] divergence.
One can firstly determine the asymptotic behavior by looking at the *superficial degree of divergence* D of a graph, i.e. by counting the powers of momentum k^D in the associated integral for large momentum.
For (2.3) one has $D = d - 4$, implying it is convergent only for $d \leq 3$. In particular, for $d = 4$ the power counting suggests a logarithmic divergence.

At this point we introduce the lattice regulator in order to manipulate this expression and quantify the asymptotic behavior. We add and subtract the same quantity and rewrite as:

$$\Sigma_1(p^2, m^2, d, a) = \Sigma_{1fin}(p^2, m^2, d, \mu^2, a) + \Sigma_{1div}(d, \mu^2, a) \tag{2.4}$$

Making the first term explicit, we see it is manifestly finite:

$$\Sigma_{1fin} \equiv \frac{ig^2}{2(2\pi)^d}\int \Big(\frac{1}{(k^2 - m^2 + i\epsilon)((p+k)^2 - m^2 + i\epsilon)} - \frac{1}{(k^2 - \mu^2 + i\epsilon)^2}\Big)\, d^d k \tag{2.5}$$

The second term instead contains the divergence:

$$\Sigma_{1div} \equiv \frac{ig^2}{2(2\pi)^d}\int \frac{1}{(k^2 - \mu^2 + i\epsilon)^2}\, d^d k \tag{2.6}$$

[3]Recall we are ignoring the IR behavior. Accordingly, in the following we shall make use of a fictitious mass μ to avoid problems in this regard, which can however be recognized in the limit $\mu \to 0$

It is important to observe that we not only singled out the problematic term, but we also made this independent of the external momentum p, which will turn essential in the cancellation of the divergence itself. Afterwards, one can rearrange the integral as:

$$\Sigma_{1div}(d,\mu^2,a) = \frac{g^2\mu^{d-4}}{2(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_1^{\varkappa}\left(\frac{1-y}{y}\right)^{\frac{d}{2}-1}dy \qquad \text{with } \varkappa = \frac{a^2\mu^2}{1+a^2\mu^2} \tag{2.7}$$

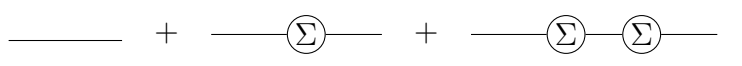
The full derivation and the computations for some value of d are is given in the appendix $A.1$.
As an example in $d=4$ the asymptotic behavior is:

$$\Sigma_1 \approx -\frac{g^2}{16\pi^2}\ln\frac{1}{a} + finite \qquad \text{as } a \to 0 \tag{2.8}$$

which is indeed a logarithmic divergence.
The superficial degree of divergence can be determined in a similar fashion for any 1PI graph with no subdivergences.

2.2.3 Counterterms

Thus far we considered Σ_1. This is the one-loop approximation of the self-energy Σ, the more general object defined as i times the sum of all 1PI diagrams with two external lines, when the external structures is removed. The self-energy gives perturbative corrections to the free propagator and as such it can be used to compute the full propagator of the interacting theory. Indeed one has the recursive[4] relation:

$\tilde{G}^{(2)}_{int}(p^2) =$  (2.9)

where the $0th-$order term is the (scalar) free theory propagator (1.11).
This is a geometric series in Σ and can be resummed. At the leading order in g one has $\Sigma = \Sigma_1 + O(g^2)$, therefore it gives:

$$\tilde{G}^{(2)}_{int}(p^2) = \frac{i}{p^2 - m^2 - \Sigma + i\epsilon} = \frac{i}{p^2 - m^2 - \Sigma_1 + i\epsilon} + O(g^2) \tag{2.10}$$

Recalling the divergent part of Σ_1 (and of the full Σ) is independent of p^2, we see

[4] In fact this is of the form: $G = G_0 + G\Sigma G$

that the self-energy represents the dynamical contribution to the mass coming from the interactions.
The actual mass m_{ph} of the particle becomes:

$$m_{ph}^2 = m^2 + \Sigma\big|_{p^2=m_{ph}^2} \tag{2.11}$$

Hence one reparametrizes the theory with the *bare mass* m_0^2 and rewrites the mass term in the Lagrangian as:

$$-m^2\frac{\phi^2}{2} \longrightarrow -m_0^2\frac{\phi^2}{2} = -m_{ph}^2\frac{\phi^2}{2} - \delta m^2\frac{\phi^2}{2} \tag{2.12}$$

This is the sum of two terms: the first one is the physical mass of the free theory and now appears in the free propagator. The second term is interpreted as a piece of the interaction Lagrangian and, as such, it gives rise to its own Feynman diagram:

 (2.13)

This is called the *mass counterterm* and is adjusted so that it exactly cancels the dynamical contribution to the particle's mass in the full propagator of the interacting theory. It is determined as a power series in g.
Thus to $O(g^2)$ the self-energy is the sum of the one-loop graph (2.2) and the counterterm, i.e. the *renormalized* self-energy:

$$\Sigma_{1R} = \lim_{a\to 0}\left(\Sigma_1(p^2, m_{ph}^2, d, a) + \delta m^2\right) = \Sigma_{1fin}(p^2, m_{ph}^2, d, \mu^2, 0) + (\delta m^2 + \Sigma_{1div}) \tag{2.14}$$

We require δm^2 to cancel the divergence in Σ_{1div} and then take the continuum limit $a \to 0$. Finally, setting $(\delta m^2 + \Sigma_{1div}) = -\Sigma_{1fin}(m_{ph}^2, m_{ph}^2, d, \mu^2)$, one gets an expression which is moreover indipendent of μ^2:

$$\Sigma_{1R} = \Sigma_1(p^2, m_{ph}^2, d) - \Sigma_1(m_{ph}^2, m_{ph}^2, d) \tag{2.15}$$

Notice that since the second term is independent of the external momentum, it is true that:

$$\frac{\partial^n \Sigma_{1R}}{\partial p^{\mu_1}\cdots\partial p^{\mu_n}} = \frac{\partial^n \Sigma_1}{\partial p^{\mu_1}\cdots\partial p^{\mu_n}} \tag{2.16}$$

This is helpful to establish the form of the counterterms needed to renormalize the theory. As a general statement, differentiating once respect to an external

momentum decreases the superficial degree of divergence by one. We see for instance:

$$\frac{\partial \Sigma_1}{\partial p^\mu} = -\frac{ig^2}{(2\pi)^d} \int \frac{(p+k)_\mu}{(k^2-m^2)((p+k)^2-m^2)^2} d^d k \tag{2.17}$$

The integral has now $D = d - 5$ and converges for $d = 4$.
Then the general idea is to lower the superficial degree enough to make the integral converge, so that we can handle it and hereupon recover Σ_{1R} from Σ_1 plus some arbitrary integration constants, to be fixed by imposing some renormalization prescription such as (2.14).
Therefore, going to $d = 4$ it is just needed to differentiate once. Besides one can utilize the equivalence:

$$\frac{\partial \Sigma_{1R}}{\partial p^2} = \frac{p^\mu}{2p^2} \frac{\partial \Sigma_{1R}}{\partial p^\mu} = \frac{p^\mu}{2p^2} \frac{\partial \Sigma_1}{\partial p^\mu} \tag{2.18}$$

and after some computations (see appendix $A.2$) we impose the condition (2.15) to get:

$$\Sigma_{1R}(p^2, m_{ph}^2, 4) = \frac{g^2}{32\pi^2} \int_0^1 \ln \left(\frac{m_{ph}^2 - p^2 x(1-x)}{m_{ph}^2 (1 - x + x^2)} \right) dx \tag{2.19}$$

and it can be worked out analytically.
This was a fairly easygoing case, since we have a single dimensionless divergence that can be reabsorbed with a simple counterterm. Moreover, it is not just a coincidence we needed to differentiate once.

If we go to $d = 6$, the one-loop self-energy has the asymptotic behavior:

$$\Sigma_1 \approx -\frac{g^2}{256\pi^3} \frac{1}{a^2} + \frac{g^2 \mu^2}{64\pi^3} \ln \frac{1}{a} + finite \qquad \text{as } a \to 0 \tag{2.20}$$

reflecting the superficial degree of divergence $D = 2$.
This is qualitatively different and is even dimensionful: clearly the single counterterm of the previous case cannot suffice.
One pursues the very same technique and differentiating Σ_1 three times is sufficient to make the integral converge in $d = 6$. Next we integrate and get three additional terms of the form: $C_0 + A_\mu p^\mu + Cp^2$. The middle term would violate the Lorentz invariance of Σ so we just discard it.
By analogy we now look at the Lagrangian for a second counterterm proportional to p^2. In the self-interacting scalar theory there is not so much of a choice, but the kinematic term accomplishes its task and provides the total counterterm:

$$\mathcal{L}_{ct} = -\delta m^2 \frac{\phi^2}{2} + \delta Z \frac{(\partial \phi)^2}{2} \tag{2.21}$$

These give rise to a revised Feynman diagram that is still represented as (2.13). Likewise the mass counterterm, one interprets this new counterterm as a redefinition in terms of a *bare field* $\phi_0 \equiv \sqrt{Z}\phi$:

$$\frac{(\partial\phi)^2}{2} \longrightarrow \frac{(\partial\phi_0)^2}{2} = Z\frac{(\partial\phi)^2}{2} = \frac{(\partial\phi)^2}{2} + \delta Z\frac{(\partial\phi)^2}{2} \tag{2.22}$$

This a wave-function renormalization that affects the propagator. In fact we reinterpret the propagator in (2.9), (2.10) as a function of the bare quantities, which in turn can be re-expressed as:

$$\tilde{G}_0^{(2)} = \frac{iZ}{p^2 - m_{ph}^2 - \Sigma_1 - \delta m^2 + \delta Z p^2 + i\epsilon} + O(g^2) = \frac{iZ}{p^2 - m_{ph}^2 - \Sigma_{1R}} + O(g^2) \tag{2.23}$$

where Σ_1 is computed in terms of the bare quantities.
Then one have to impose some prescription on the renormalized self-energy. For now, we go on with the *mass-shell* renormalization conditions:

$$\Sigma_{1R}\big|_{p^2=m_{ph}^2} = 0 \qquad \left.\frac{\partial\Sigma_{1R}}{\partial p^2}\right|_{p^2=m_{ph}^2} = 0 \tag{2.24}$$

These allow us to determine the integration constants.
What is left is to adjust the parameter Z such that the renormalized propagator is:

$$\tilde{G}^{(2)} = \frac{i}{p^2 - m_{ph}^2 + i\epsilon} + finite \qquad \text{as } p^2 \to m_{ph}^2 \tag{2.25}$$

We see there is a relation between the superficial degree of divergence and the momentum dependence of the counterterm. This is true for a general graph.
Going to a dimension $d > 6$ the superficial degree is $D > 2$.
For example, in $d = 8$ there is a quartic divergence that would need a counterterm $\propto (\Box\phi)^2$ which is not of the form of any term in the original Lagrangian, so it cannot be reabsorbed. In this case the theory is called non-renormalizable. Such occurrence can be traced back to the dimension of the coupling.
Indeed in the ϕ^3 theory one has $[g] = 3 - \frac{d}{2}$, which must be non-negative to ensure renormalizability[5].

[5]It affects the dimension of the coefficients in the counterterms.

2.3 Renormalization Scheme

2.3.1 Prescriptions

The basic idea of renormalization is to understand divergences as a shift in the measurable quantities of the theory to some other effective value and as such they can be compensated by tuning the parameters in the Lagrangian in a proper manner. The latter is reinterpreted as written in terms of the *bare* quantities, so that *counterterms* are singled out. For the scalar theory one has:

$$\mathcal{L} = \mathcal{L}_{KG} + V(\phi) + \mathcal{L}_{ct} \tag{2.26}$$

In general, counterterms are a finite number of functions depending on the mass, the coupling and the lattice spacing.
Hence we work out the Feynman rules for the general interacting theory with $\mathcal{L}_{int} = V(\phi) + \mathcal{L}_{ct}$ and employ perturbation theory in powers of the renormalized coupling. The counterterms are expanded in infinite series and to each order they are adjusted graph by graph to cancel the divergences, so that the net effect of the interactions stays finite in the limit $a \to 0$. If this can be done, the continuum theory as constructed continues to make sense and is said to be *renormalizable*.

There are infinitely many ways to determine the counterterms for a given theory, each of these corresponding to a particular parametrization. Then one imposes some rule to resolve the ambiguity, i.e. a renormalization prescription or scheme. As a concrete example, let's consider the mass counterterm in $d = 4$.
In (2.14), (2.15) we adapted the counterterm to compensate the divergence and then we adopted the mass-shell condition to fix the overall combination. In a sense, we have split the bare mass in a precise way:

$$m_0^2 = m_{ph}^2 + \delta m_{ph}^2 = m_{ph}^2 - \Sigma_{1div} - \Sigma_{1fin}\big|_{p^2=m_{ph}^2} + O(g^4) \tag{2.27}$$

but nothing prevents us from choosing a different prescription in doing this. Indeed one has a whole family parametrized by an arbitrary constant ζ:

$$m_0^2 = m^2 + \delta m^2 = m^2 - \Sigma_{1div} - g^2\zeta + O(g^4) \tag{2.28}$$

Clearly one requires that the theory gives the same results, that is bare mass must be the same:

$$m^2 = m_{ph}^2 + g^2\zeta - \Sigma_{1fin}\big|_{p^2=m_{ph}^2} + O(g^4) \tag{2.29}$$

This translates in a certain arbitrariness in the definition of the renormalized self-energy:

$$\Sigma_{1R}^{(\zeta)}(p^2, m^2) = \Sigma_{1fin} + g^2\zeta \tag{2.30}$$

which is however physically irrelevant. What is essential is that, whenever a particular divergent graph occurs as a subgraph of a bigger graph, the ambiguity is resolved in the same way at each occurrence, since the corresponding counterterm is generated by a single term in the Lagrangian.

Amongst the possible renormalization schemes, we have seen the mass-shell (or physical) perscription (2.24) and are going to look into details the minimal subtraction (MS) scheme, which we shall use in the following.
Another example is the BPHZ scheme, also known as zero-momentum subtraction.
Given a divergent 1PI graph Γ with superficial degree $D(\Gamma) \geq 0$, let $R(\Gamma)$ be its renormalized value. The prescriptions are:

$$R(\Gamma)\big|_{p^2=0} = 0 \qquad \left.\frac{\partial^n R(\Gamma)}{\partial p^{\mu_1} \cdots \partial p^{\mu_n}}\right|_{p^2=0} = 0, \qquad n = 1, \ldots, D(\Gamma) \tag{2.31}$$

BPHZ renormalization consists in implementing these by directly subtracting off the first $D(\Gamma)$ terms in the Taylor expansion of the integrand, performing renormalization before the integration over loop momenta, thence without need of any explicit UV cut-off. Adopting this the self-energy is, at $d = 4$:

$$\Sigma_{1R}^{(BPHZ)} = \frac{ig^2}{32\pi^4} \int \left(\frac{1}{(k^2 - m^2)((p+k)^2 - m^2)} - \frac{1}{(k^2 - m^2)^2} \right) d^4k \tag{2.32}$$

while at $d = 6$:

$$\begin{aligned} \Sigma_{1R}^{(BPHZ)} = \frac{ig^2}{128\pi^6} \int \Bigg(& \frac{1}{(k^2 - m^2)((p+k)^2 - m^2)} - \frac{1}{(k^2 - m^2)^2} + \\ & + \frac{2(p \cdot k)}{(k^2 - m^2)^3} - \frac{4(p \cdot k)^2 - (k^2 - m^2)p^2}{(k^2 - m^2)^4} \Bigg) d^6k \end{aligned} \tag{2.33}$$

2.3.2 Dimensional Regularization

The BPHZ scheme (2.31) makes manifest that there is no fundamental dependence on the cut-off procedure when working in perturbation theory. Nonetheless, most of the times one is interested in a different renormalization perscription, thus needing a regulator. Rather than employing a intuitive yet cumbersome cut-off, we are now introducing *dimensional regularization*, which presents

some very convenient features we shall briefly mention afterwards.

Looking at the superficial degree of a graph one observes that going to a small enough dimension d makes the UV divergences disappear[6] Therefore the idea is to utilize the space-time dimension itself as a regulator, which is removed by taking the limit $d \to 4$. To do this one must treat d as a continuous variable.

Vector spaces of non-integer dimension do not exist as such, but at least in perturbation theory Wilson [Wil] defined a consistent integration on a complex-dimensional space. In a few words, one works on an infinite dimensional vector space where is introduced an operation which is assumed to satisfy the basic properties of an integral, i.e. linearity, an opportune scaling law and roto-translational invariance[7]. Taken as axioms, these ensure the operation is unique (up to normalization) and it reduces to the usual integration when the dimension is an integer. Furthermore the integration must split into an ordinary integral over some integer-dimensional space and a spherically symmetric integral over the remaining dimensions. It is this operation that embodies the dimensionality, forcing all vectors involved in the integrand to lie in some integer-dimensional subspace. We won't give an explicit and exhaustive formulation, however the outcome is that all the ordinary integration's features are recovered in this operation.

The upshot is we can work in a purely formal sense, assuming the usual manipulation can be worked out. Another assumption we need is this generalized version of a gaussian integral for non-integer d:

$$\int e^{k^2}\, d^d k = i\pi^{\frac{d}{2}} \tag{2.34}$$

Given the validity of this expression, one then makes use of the Schwinger representation:

$$A^{-1} = \int_0^\infty e^{-At}\, dt \tag{2.35}$$

in order to recast the one-loop self-energy (2.3) in the form:

$$\Sigma_1 = -\frac{g^2}{2(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 (m^2 - p^2 x(1-x))^{\frac{d}{2}-2}\, dx \tag{2.36}$$

This result is unique up to overall normalization. The full derivation is given

[6] For example we can look at the one-loop self-energy in $A.1.1$.

[7] In this regard it shares some analogies with the Berezin integral.

in the appendix $A.3$. The divergences now reside in the simple poles of the $\Gamma-$function for $d = 2n + 4$ whose residues are polynomials in p of degree equal to the degree of divergence.

Apart from computational advantages, dimensional regularization brings the huge benefit to retain explicitly Poincaré invariance (and also gauge symmetries). Besides it is exploitable to treat as well IR divergences in theories with massless fields. Henceforward we shall also frequently and tacitly assume we work in the euclidean metric according to the dimensional regularization procedure when performing integrals manipulations.

From (2.36) one easily gets the renormalized self-energy Σ_{1R} in $d = 4$ by adding a mass counterterm $\delta m^2(g, m^2, d)$. Let's say we impose the mass-shell conditions (2.24):

$$\Sigma_{1R}^{(ph)} = -\frac{g^2}{2(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 \left((m_{ph}^2 - p^2x(1-x))^{\frac{d}{2}-2} - (m_{ph}^2(1-x+x^2))^{\frac{d}{2}-2}\right) dx \tag{2.37}$$

Now $\Gamma(z)$ has a pole[8] at $z = 0$:

$$\Gamma(z) = \frac{1}{z} - \gamma_E + O(z) \tag{2.38}$$

so that letting $d \to 4$, at the first order we get the same result (2.19).

2.3.3 Minimal Subtraction

With dimensional regularization, one can explicitely expand Σ_1 in a power series in $d - 4$ and get the asymptotic behavior for $d \to 4$:

$$\Sigma_1 \approx \frac{2}{d-4} + \gamma_E - \ln 4\pi + \int_0^1 \ln(m_{ph}^2 - p^2x(1-x))\, dx \tag{2.39}$$

Since the divergence amounts to a simple pole at $d = 4$, we may choose a renormalization prescription in which the counterterm is defined to cancel the singularity:

$$\delta m^2 = \frac{g^2}{32\pi^2}\frac{2}{4-d} \tag{2.40}$$

In doing this, one must also allow for the dependence of the coupling dimension on d, making the mass scale explicit:

$$g \to \mu^{2-\frac{d}{2}} g \tag{2.41}$$

[8] $\gamma_E = 0.5772...$ is Euler-Mascheroni constant.

where μ is a mass parameter to render $[g] = 1$.
We also expand this term[9] and finally obtain the renormalized self-energy:

$$\Sigma_{1R}^{(MS)} = \frac{g^2}{32\pi^2} \int_0^1 (\ln \left(\frac{m^2 - p^2 x(1-x)}{4\pi\mu^2} \right) + \gamma_E)\, dx \tag{2.42}$$

This renormalization scheme is called minimal subtraction (MS). Notice that the counterterms are pure poles at the physical value of d. The parameter μ is entirely arbitrary, in fact making this a family of renormalization prescriptions. For instance, in the $\overline{MS}$ scheme one reabsorbes the universal constants in the unit of mass:

$$\Sigma_{1R}^{(\overline{MS})} = \frac{g^2}{32\pi^2} \int_0^1 \ln \left(\frac{m^2 - p^2 x(1-x)}{\bar{\mu}^2} \right) dx \qquad \bar{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E} \tag{2.43}$$

The asymptotic behavior of Σ_1 for $d \to 6$ is worked out in the same way, this time paying attention when we single out the singularity since the series is in $d - 6$. First one has:

$$\Gamma(z-1) = -\frac{1}{z} + \gamma_E - 1 + O(z) \qquad \text{as } z \to 0 \tag{2.44}$$

Notice also that the renormalized coupling is dimcnsionless in this case, so that we must insert the unit mass $\mu^{3-\frac{d}{2}}$.
Being pedantic, it may be helpful to baldly set $3 - \frac{d}{2} = z$ and rewrite Σ_1 as:

$$\Sigma_1 = -\frac{g^2}{128\pi^3} (4\pi\mu^2)^z \Gamma(z-1) \int_0^1 \xi^{1-z}\, dx \tag{2.45}$$

where we also put $\xi = (m^2 - p^2 x(1-x))$
Up to the constant factor, we simply pick the pole in the limit $z \to 0$:

$$\Sigma_1 \approx \frac{1}{z} \left(\frac{p^2}{6} - m^2 \right) + (\gamma_E - 1 - \ln 4\pi\mu^2) \left(m^2 - \frac{p^2}{6} \right) + \int_0^1 \xi \ln \xi\, dx \tag{2.46}$$

thus the minimal subtraction conditions become:

$$\delta m^2 = m^2 \left(\frac{g^2}{64\pi^3 (d-6)} \right) + O(g^4) \tag{2.47}$$

$$\delta Z = \frac{1}{6} \left(\frac{g^2}{64\pi^3 (d-6)} \right) + O(g^4) \tag{2.48}$$

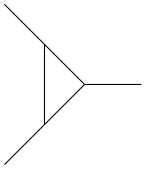
[9] Observe that: $\mu^{2-d/2} = e^{(2-d/2)\ln\mu} = 1 + (2 - d/2)\ln\mu + \frac{1}{2}(2 - d/2)^2 \ln^2 \mu + \ldots$

For the renormalized self-energy one gets:

$$\Sigma_{1R}^{(MS)} = -\frac{g^2}{128\pi^3}\int_0^1 (m^2 - p^2x(1-x))\Big(\ln\left(\frac{m^2 - p^2x(1-x)}{4\pi\mu^2}\right) + (\gamma_E - 1)\Big)\,dx \tag{2.49}$$

2.3.4 Renormalizability

Hitherto we only focused on the self-energy graph, giving a detailed description of its divergences by means of power counting and how they can be taken care of. We also introduced a set of tools and ideas that will turn useful in the following. In particular we have chosen to make use of the dimensional regularization, implementing the minimal subtraction scheme, which in the next chapter we are going to use in combination with the BPHZ procedure in order to extract divergences from any graph.
In a similar fashion one can compute the lowest order counterterm δg for the coupling from the the one-loop vertex graph:

$\mathcal{V}_1 =$  (2.50)

The full derivation is given in the appendix $A.4$.
In the minimal subtraction scheme the result for $d \to 6$ is:

$$\delta g = \mu^{3-d/2}\left(\frac{g^3}{64\pi^3(d-6)}\right) + O(g^4) \tag{2.51}$$

So far we have considered the two basic one-loop graphs in the ϕ^3 theory and worked out their respective counterterms up to g^4. We see these are generated at the lowest order in perturbation theory from the counterterm Lagrangian:

$$\mathcal{L}_{ct} = \delta Z\frac{(\partial\phi)^2}{2} - \delta m^2\frac{\phi^2}{2} - \delta g\frac{\phi^3}{3!} \tag{2.52}$$

Strictly speaking we should also include a linear counterterm $\propto \phi$ and a cosmological term. However, these are related to tadpoles and vacuum bubbles, so that we can ignore them according to (1.23).

Clearly this cannot be the end of the story, there is plenty of more general and intricate graphs in the theory which would require different counterterms. But it seems the next perturbative order would spoil the results reached up to this point, for one has to re-alter the prescriptions to include other graphs.
A crucial observation comes from considering the theory in d space-time dimensions. Looking at any 1PI graph associated to each $n-$point Green's function, this has dimensionality $\left[\tilde{G}^{(n)}_{int}\right] = n + d - \frac{nd}{2}$.

By this and the arguments below the (2.25) we can deduce that the theory is not renormalizable in $d > 6$ as a direct consequence of the negative dimension of the coupling g, since we would have divergent graphs for any n by going to a large enough perturbative order.
Morever we also see that in $d = 6$ the only divergent Green's function are those with $n = 1, 2, 3$ which we know how to renormalize using (2.52). These divergences appear to every order in g but they are the only fundamental ones in the theory, in the sense that they appear as subdivergences for other divergent graphs. In order to see how to treat renormalization of the theory order by order, another key point is to reconsider each counterterm in $\mathcal{L}_{ct}$ not a single quantity, but rather as a sum of the form:

$$\delta Z = \sum_{1PI} \delta_\Gamma Z = \delta_1 Z + \delta_2 Z + \dots \tag{2.53}$$

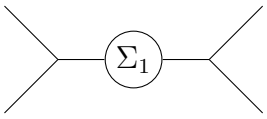
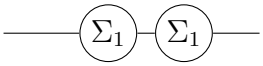
and similarly for other counterterms. Each term of the sum cancels the overall divergence in one particular graph generated by the basic interaction. Hence the expressions (2.47), (2.48) and (2.51) simply represent the first contributions. Higher contributions can be then worked out using an inductive argument, by assuming renormalizability of the theory at the lowest order and then exploiting the relations (1.13) for instance.

This idea provides a new approach on single graphs where the UV divergences are located and can be renormalized by adding specific counterterms generated from the given graph. For a renormalizable theory this can be done at every order in perturbation theory. In the graph-by-graph method one must also show these counterterms are local (i.e. polynomial in momentum) and they are tantamount only to renormalizing the parameters of the Lagrangian. We are going to see this can be done in a systematic way.

2.4 BPHZ Procedure

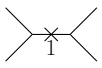
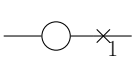
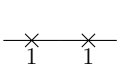
2.4.1 Subdivergences

Once we elaborated on a good technique, we are now ready to deal with more general, divergent graphs for the ϕ^3 theory in $d = 6$.
Divergences come from looped lines when the involved loop momenta get large and they are always confined in 1PI (sub)graphs. Hence there is no loss of generality in considering as subgraphs the familiar one-loop self-energy (2.2) and one-loop vertex (2.50), which we know how to treat. If a graph diverges when all its looped momenta get large, it is said to have an overall divergence. Besides a graph can also possess *subdivergences*, which involve smaller subgraphs (of which they are the overall divergence). In fact a general graph may have no overall divergence despite being packed with subdivergences. As a warm up, consider the following graphs:

$\Gamma_1 =$  $\Gamma_3 =$  (2.54)

These are simple examples of 1PR graphs[10], since they can be made disconnected by cutting a single line. As such they cannot have an overall divergence, for some lines are not a part of any loop. In fact we have two instances of subdivergence, i.e. divergent subgraphs. We associate a specific divergence with its subgraph as determined by power-counting. Moreover it can occurr that one needs to consider the momenta behaviors individually, some getting large, the others staying finite.

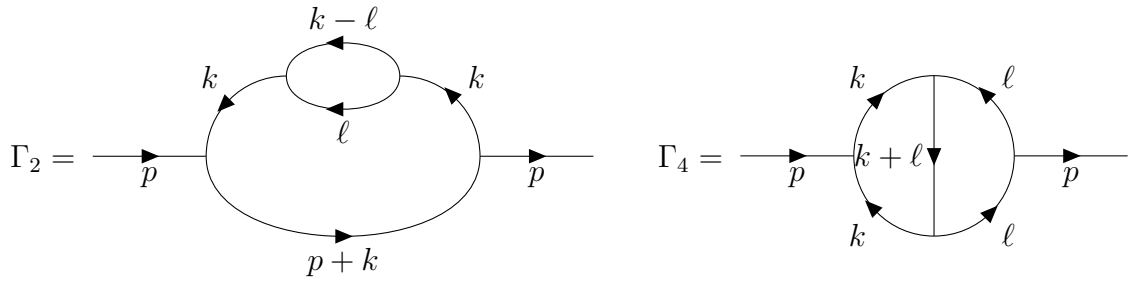
Observe that Γ_1 possesses a single subdivergence coming from the only insertion of Σ_1, while Γ_3 has three different subdivergences from the three regions where one or both the loop momenta go to infinity.
The idea to renormalize these graphs is quite intuitive: counterterms are generated replacing each occurrence of a divergent loop in a graph with its individual counterterm. In the case of any Σ_1, this is represented by a graph (2.13), so that substituting one by one we get:

$\delta_1 =$  $\delta_a =$  $\delta_b =$  $\delta_{ab} =$  (2.55)

Where a label 1 is added to recall this is the one-loop self-energy counterterm.

[10] As a remainder, 1PR stands for one particle *reducible*, the exactly opposite of a 1PI graph.

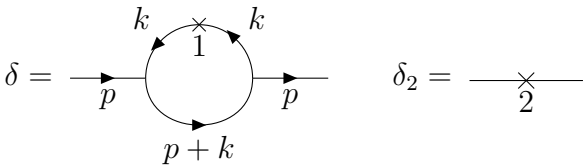
Notice how generally $\delta_a \neq \delta_b$ still contain a divergent subgraph that is then renormalized by δ_{ab}. Therefore one must add (a combination of) these counterterms to their original graphs in order to arrive at the renormalized value.
More complicated graphs may possess many different subdivergences in addition to an overall divergence:



(2.56)

The graph Γ_2 is an example of a nested subdivergence. It has two UV divergences: one is the overall divergence, when $k, \ell \to \infty$; the other is the subdivergence where $\ell \to \infty$ whilst k stays finite[11].
Instead, the graph Γ_4 presents two overlapping subdivergences, that is they share a common internal line, beyond the overall divergence $k, \ell \to \infty$.

We start by looking at the counterterms for Γ_2:



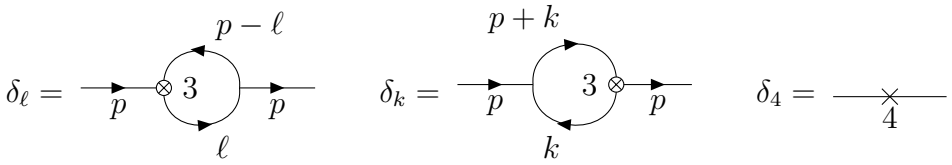
(2.57)

The label 2 means the overall counterterm is obtained by considering the contributions $\delta_2 Z$ and $\delta_2 m^2$ in the expansions of the complete counterterms (2.53).
The existence of a nested subdivergence implies a non-local term that as such cannot be removed by any local counterterm. However it disappears in the sum $\Gamma_2 + \delta$, compensated by the counterterm to the subgraph[12]. What is left is an overall divergence which is cancelled by adding the local counterterm δ_2.
Therefore the combination $\Gamma_2 + \delta + \delta_2$ gives a finite value.

[11] One could also argue we must consider the region in the space of the loop momenta where k and ℓ both go to infinity with k much more slowly, but the Weinberg's theorem [W60] tells us this does not need to be treated as a separate case.

[12] This can be computed explicitely, for example in the $d \to 6$ massless case.

For Γ_4 there are three counterterms:



(2.58)

Where this time the label 4 comes from $\delta_4 Z$ and $\delta_4 m^2$.
Like before, we first need to deal with subdivergences. Since the corresponding subgraphs overlap, in this case they must be taken care single handedly, replacing one at time and thus generating two countergraphs δ_k and δ_ℓ. Each of them corresponds to the insertion of a coupling counterterm (2.51).
The sum $\Gamma_4 + \delta_k + \delta_\ell$ is finally renormalized by and overall counterterm δ_4.

This is the pattern: given a basic graph of the theory one adds the counterterms generated from renormalizing its subdivergences and then the overall countergraph to get a finite value.
The overall divergence is still determined by power-counting. In fact one can differentiate both the graph and its counterterms respect to the external momentum to decrease the superficial degree of divergence the same way we have done before. It is moreover local, since after removing subdivergences we are left with the counterterm of a 1PI graph, which is a polynomial in its external momenta with degree equal to the degree of divergence. The trick works as well for overlapping and nested subdivergences, aside from combinatoric problems we are resolving in the following.

2.4.2 General Method

Consider a general Feynman graph Γ with L loops and N vertices and let its value be written as:

$$U(\Gamma)(p_1, \ldots, p_N) = \int I(p_1, \ldots, p_N, k_1, \ldots, k_L)\, d^d k_1 \cdots d^d k_L \tag{2.59}$$

Where the p_i's are the external momenta and the k_j's are the loop momenta.
This is the unrenormalized value of the basic graph as dictated by the Feynman rules. In constructing a sensible counterterm, one must first subtract off its subdivergences thus rendering local the overall divergence. We aim to a general method to organize subdivergences for a graph Γ, generate countergraphs from them and thereby obtaining a finite renormalized value $R(\Gamma)$:

$$R(\Gamma) = U(\Gamma) + S(\Gamma) \tag{2.60}$$

Here $S(\Gamma)$ is the subtraction procedure, that is the sum of the counterterm graphs. This is what one needs at the end of the day.
We start by looking at the case of a 1PI graph with no subdivergences, which may only possess an overall divergence. One renormalizes the graph by subtracting an overall counterterm:

$$R(\Gamma) = U(\Gamma) - T \circ U(\Gamma) \tag{2.61}$$

The operation T extracts the divergence of $U(\Gamma)$ implementing whatever renormalization prescription we choose. In the zero-momentum subtraction T takes the Taylor expansion of $U(\Gamma)$ around $p^2 = 0$ and picks out the terms up to order $D(\Gamma)$, the degree of divergence. Instead, in the minimal subtraction scheme that we shall use, T takes the Laurent expansion of $U(\Gamma)$ around the physical space-time dimension $d = d_0$ and picks up the pole terms. For $d_0 = 6$ one has[13]:

$$T \circ \Sigma_1 = \delta_1 Z p^2 - \delta_1 m^2 = \left(\frac{p^2}{6} - m^2\right)\left(\frac{g^2}{64\pi^3(d-6)}\right) \tag{2.62}$$

$$T \circ \mathcal{V}_1 = i\delta_1 g = i\mu^{3-d/2}\left(\frac{g^3}{64\pi^3(d-6)}\right) \tag{2.63}$$

Notice that the mass unit μ, in fact $U(\Gamma)$ and its pole part must have the same dimension for any d.

To renormalize a more general graph Γ we must first know how to renormalize its subdivergences.
Let $\bar{R}(\Gamma)$ be the unrenormalized value of Γ with subtractions made to cancel the subdivergences. Since the only remaining (possible) divergence is an overall divergence, again we can now make use of the operation T to define an overall counterterm by applying to $\bar{R}(\Gamma)$. One gets the improved version of (2.61) for the renormalized value of Γ:

$$R(\Gamma) = \bar{R}(\Gamma) - T \circ \bar{R}(\Gamma) \tag{2.64}$$

We have a recursive definition of $R(\Gamma)$: successive application of this expression to smaller and smaller subgraphs ultimately brings us to graphs with no subdivergences which can be renormalized with (2.61).
All that remains to do is to construct $\bar{R}(\Gamma)$, being $U(\Gamma)$ with subdivergences

[13]The notational abuse $T \circ U(\Gamma) \equiv T(\Gamma)$ may recur since it is harmless.

subtracted. Finally we define:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C_\gamma(\Gamma) \tag{2.65}$$

Observe that the sum is over all the proper subgraphs γ of Γ, rather than the divergent ones only. Hence the notation $C_\gamma(\Gamma)$ which means that we replace the subgraph γ by its overall counterterm:

$$\gamma \to C(\gamma) = \begin{cases} -T \circ \bar{R}(\gamma) & \text{if } \gamma \text{ has an overall divergence} \\ 0 & \text{if } \gamma \text{ has no overall divergence} \end{cases} \tag{2.66}$$

that we also write directly as: $C_\gamma(\Gamma) \equiv U(\Gamma)\big|_{\gamma \to C(\gamma)}$
Besides, notice that this reproduces (2.61) when Γ possesses no subdivergences, since the summation brings null contribution and $\bar{R}(\Gamma) = U(\Gamma)$

Let's see how this work with our examples (2.54).
First we apply this to the graph Γ_1, with only one subdivergence:

$$R(\Gamma_1) = \bar{R}(\Gamma_1) = U(\Gamma_1) + C_{\Sigma_1}(\Gamma_1) = U(\Gamma_1) + U(\Gamma_1)\big|_{\Sigma_1 \to -T \circ \Sigma_1} \tag{2.67}$$

the single counterterm being δ_1.
Next, consider Γ_3: this is tricky because the divergent subgraphs do not intersect. Let γ_1 and γ_2 be the self-energy bubbles and $\gamma_1 \cup \gamma_2$ their disjoint union. Again we have no overall divergence, so:

$$R(\Gamma_3) = \bar{R}(\Gamma_3) = U(\Gamma_3) + C_{\gamma_1}(\Gamma_3) + C_{\gamma_2}(\Gamma_3) + C_{\gamma_1 \cup \gamma_2}(\Gamma_3) \tag{2.68}$$

Here usually $C_{\gamma_i}(\Gamma_3) = U(\Gamma_3)\big|_{\gamma_i \to -T(\gamma_i)}$ for $i = 1, 2$ reproduce δ_a and δ_b.
The third term seems problematic. It corresponds to a subtraction for $\gamma_1 \cup \gamma_2$ when both loop momenta are large. But we must take away from it the counterterms for the regions where only one momentum is large:

$$C(\gamma_1 \cup \gamma_2) = -T \circ \Big[U(\gamma_1 \cup \gamma_2) + C(\gamma_1) U(\gamma_2) + U(\gamma_1) C(\gamma_2) \Big] \tag{2.69}$$

Now we make two observations:

$$U(\gamma_1 \cup \gamma_2) = U(\gamma_1) U(\gamma_2), \qquad T \circ C(\gamma_i) = -T \circ U(\gamma_i) \tag{2.70}$$

The first one follows from the fact that $\gamma_1 \cup \gamma_2$ is disconnected.

The second one simply means that the pole part of a pole part is itself[14].
At this point we must define the operation T when acting on a disconnected graph to act independently on its components:

$$\begin{aligned} T \circ [U(\gamma_1)U(\gamma_2)] &= \big(T \circ U(\gamma_1)\big)\big(T \circ U(\gamma_2)\big) \\ T \circ [C(\gamma_i)U(\gamma_j)] &= -\big(T \circ U(\gamma_i)\big)\big(T \circ U(\gamma_j)\big) \end{aligned} \tag{2.71}$$

Therefore we find that:

$$C(\gamma_1 \cup \gamma_2) = \big(T \circ U(\gamma_1)\big)\big(T \circ U(\gamma_2)\big) = T \circ \big[U(\gamma_1)U(\gamma_2)\big] \tag{2.72}$$

And the countegraph δ_{ab} is represented by:

$$C_{\gamma_1 \cup \gamma_2}(\Gamma_3) = U(\Gamma_3)\big|_{\gamma_1 \cup \gamma_2 \to T \circ [U(\gamma_1)U(\gamma_2)]} \tag{2.73}$$

The above procedure generalizes to an arbitrary graph. Applying to Γ_2 and Γ_4 in (2.56) the result is of the form (2.64) with:

$$\begin{aligned} \bar{R}(\Gamma_2) &= U(\Gamma_2) + U(\Gamma_2)\big|_{\Sigma_1 \to -T \circ \Sigma_1} \\ \bar{R}(\Gamma_4) &= U(\Gamma_4) + U(\Gamma_4)\big|_{\gamma_3 \to -T(\gamma_3)} + U(\Gamma_4)\big|_{\gamma_4 \to -T(\gamma_4)} \end{aligned} \tag{2.74}$$

Where γ_3 and γ_4 are the two one-loop vertex subgraphs.
These respectively represent δ, δ_ℓ and δ_k (2.58). The two overall counterterms are given by the action of T, i.e. $\delta_i \rightsquigarrow -T \circ \bar{R}(\Gamma_i)$ for $i = 2, 4$.

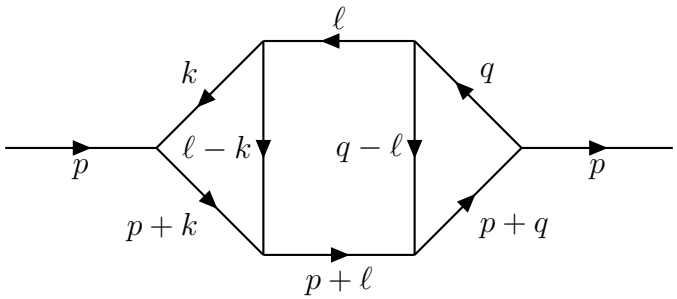
However, these are still fairly simple computations which conceal the full power of the method.

2.4.3 Forest Formula

After having set up the formalism, now we work out a general graph as guideline to reach a systematic procedure for extracting the divergences from any integral.
Consider a graph with nested and multiply overlapping divergences, the three-

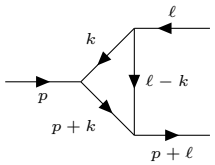
[14]Explicitely $T \circ \big[T \circ U(\gamma_i)\big] = T \circ U(\gamma_i)$ and the minus sign following from the definition (2.66).

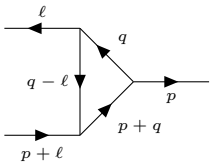
loop self-energy Γ:



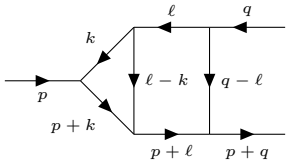
(2.75)

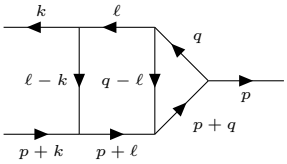
First of all, its divergent subgraphs are:

$\gamma_1 =$ 

$\gamma_2 =$ 

$\gamma_5 = \gamma_1 \cup \gamma_2$

$\gamma_3 =$ 

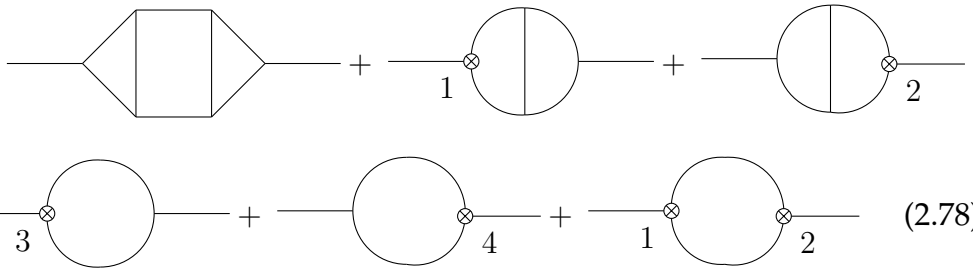
$\gamma_4 =$ 

(2.76)

They are connected 1PI graphs except for γ_5.
Following the technique, we must subtract these subdivergences according to (2.65):

$$\bar{R}(\Gamma) = \Gamma + \sum_{i=1}^{5} \Gamma\big|_{\gamma_i \to C(\gamma_i)} \tag{2.77}$$

Observe that in rewriting this expression we simply denoted the value of a (sub)graph by the (sub)graph itself. It is immediate to represent this by Feynman graphs:

$\bar{R}(\Gamma) =$ 

(2.78)

With labels specifying which subgraph has been replaced by its counterterm.

The graphs γ_1 and γ_2 have no further subdivergences. Their counterterms are the ordinary one-loop vertex's one, given in (2.50):

$$C(\gamma_i) = -T \circ U(\gamma_i) = -T \circ \mathcal{V}_1 \qquad\qquad i = 1, 2 \tag{2.79}$$

These two are the divergent subgraphs respectively of γ_3 and γ_4. Hence we have two counterterms of the more general form $-T \circ \bar{R}(\gamma_j)$:

$$C(\gamma_3) = -T \circ \left[\gamma_3 + \gamma_3|_{\gamma_1 \to -T(\gamma_1)}\right] \qquad C(\gamma_4) = -T \circ \left[\gamma_4 + \gamma_4|_{\gamma_2 \to -T(\gamma_2)}\right] \tag{2.80}$$

Finally γ_5 is a disconnected graph. Similarly to (2.72):

$$C(\gamma_5) = -T \circ \Big[\gamma_1\gamma_2 - T(\gamma_1)\gamma_2 - \gamma_1 T(\gamma_2)\Big] = T(\gamma_1)T(\gamma_2) \tag{2.81}$$

At this point we get the renormalized graph (2.64) by removing the overall divergence:

$$R(\Gamma) = \Gamma + \sum_{i=1}^{5} \Gamma\big|_{\gamma_i \to C(\gamma_i)} - T \circ \left[\Gamma + \sum_{i=1}^{5} \Gamma\big|_{\gamma_i \to C(\gamma_i)}\right] \tag{2.82}$$

The first six terms represent (2.78) and the last six form the subtraction for the overall divergence. The overall counterterm can be shown to be local and of a degree in momentum given by naive power-counting through the differentiation trick.

This expression can be recasted in a different shape defining a new operation on $\gamma \subset \Gamma$ as:

$$-T_\gamma(\Gamma) = \Gamma\big|_{\gamma \to -T(\gamma)} \tag{2.83}$$

Observe from (2.65) and the definition (2.66) that in the case of a 1PI (sub)graph with no subdivergences this is the very same of $C_\gamma(\Gamma)$, but this equality cannot continue to be true when the graph presents subdivergences, because in general $T(\gamma) \equiv T \circ U(\gamma) \neq T \circ \bar{R}(\gamma)$.
However, this turns to be exactly its usefulness. In fact, considering γ_3 and its subgraph γ_1, the (2.80) can be rewritten as:

$$C(\gamma_3) = -T \circ [\gamma_3 - T_{\gamma_1}(\gamma_3)] \tag{2.84}$$

We want to convince ourselves that the following holds:

$$C_{\gamma_3}(\Gamma) = -T_{\gamma_3} \circ [\Gamma - T_{\gamma_1}(\Gamma)] = -T_{\gamma_3} \circ [1 - T_{\gamma_1}](\Gamma) \tag{2.85}$$

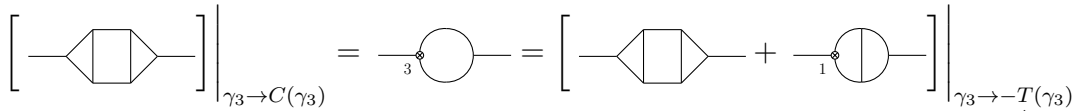
Expanding it implies the chain of equivalences:

$$\begin{aligned}\Gamma\big|_{\gamma_3\to -T(\gamma_3)+T\circ T_{\gamma_1}(\gamma_3)} &= C_{\gamma_3}(\Gamma) = -T_{\gamma_3}\circ[\Gamma - T_{\gamma_1}(\Gamma)] = \\ = \Big[\Gamma + \Gamma\big|_{\gamma_1\to -T(\gamma_1)}\Big]\Big|_{\gamma_3\to -T(\gamma_3)} &= \Gamma\big|_{\gamma_3\to -T(\gamma_3)} + \Big[\Gamma\big|_{\gamma_1\to -T(\gamma_1)}\Big]\Big|_{\gamma_3\to -T(\gamma_3)}\end{aligned} \tag{2.86}$$

This is nothing more than a request of linearity, which indeed is valid at the level of the integral $U(\Gamma)$. Moreover one has a composition law for nested subdivergences:

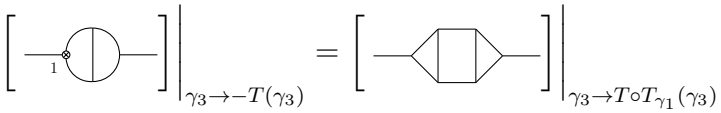
$$\begin{aligned}\Gamma\big|_{\gamma_3\to -T(\gamma_3)+T\circ T_{\gamma_1}(\gamma_3)} &= \Gamma\big|_{\gamma_3\to -T(\gamma_3)} + \Gamma\big|_{\gamma_3\to T\circ T_{\gamma_1}(\gamma_3)} \\ \Rightarrow \Gamma\big|_{\gamma_3\to T\circ T_{\gamma_1}(\gamma_3)} &= \Big[\Gamma\big|_{\gamma_1\to -T(\gamma_1)}\Big]\Big|_{\gamma_3\to -T(\gamma_3)}\end{aligned} \tag{2.87}$$

We can also see (2.4.3) directly from the viewpoint of the graphs (2.78):



(2.88)

The equation tells us that the countergraph $C_{\gamma_3}(\Gamma)$, given by the total counterterm for the subgraph γ_3, can be effectively obtained by adding the countegraph $C_{\gamma_1}(\Gamma)$ to Γ (i.e. subtracting the subdivergence) and then renormalizing the overall subdivergence.
At this point the linearity simply translates in:



(2.89)

That is the graphs formulation of (2.4.3), we now manifestly read it being the machinery for nested subdivergences.
In conclusion, substituting in (2.77) gives the subtractions:

$$\bar{R}(\Gamma) = \Gamma - T_{\gamma_1}(\Gamma) - T_{\gamma_2}(\Gamma) - T_{\gamma_3}(1-T_{\gamma_1})\Gamma - T_{\gamma_4}(1-T_{\gamma_2})\Gamma + T_{\gamma_2}T_{\gamma_1}(\Gamma) \tag{2.90}$$

And applying the T operation grants the overall counterterm.
This generalizes to arbitrary graphs. The upshot is one can explicitely solve the recursion (2.64) to find Zimmermann's *forest formula*:

$$R(\Gamma) = \sum_{F\in\mathcal{F}(\Gamma)}\prod_{\gamma\in F}[-T_\gamma](\Gamma) \tag{2.91}$$

The sum is over the set $\mathcal{F}(\Gamma)$ of all the possible forests of Γ. A forest $\mathcal{F}$ of Γ is a collection of non-overlapping divergent[15] subgraphs $\gamma \subset \Gamma$, i.e. disjoint or nested 1PI subgraphs. The empy set $\emptyset$ and the forests that do not contain the full graph Γ are called normal forests. For our graph (2.75) there are eight of them, written in set theoretical notation as:

$$\emptyset, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_1, \gamma_3\}, \{\gamma_4\}, \{\gamma_2, \gamma_4\}, \{\gamma_5\} \equiv \{\gamma_1, \gamma_2\} \tag{2.92}$$

These can be also read directly from (2.90), formally defining $-T_0 \equiv 1$.
For consistency over nested subgraphs, the T_γ operations are applied inside to outside. In fact considering the forest $\{\gamma_1, \gamma_3\}$ where $\gamma_1 \subset \gamma_3$ gives:

$$\prod_{\gamma \in \{\gamma_1, \gamma_3\}} [-T_\gamma](\Gamma) = T_{\gamma_3} T_{\gamma_1}(\Gamma) \tag{2.93}$$

The other half terms $-T \circ \bar{R}(\Gamma)$ which combine into the overall counterterm are included when considering the full forests, constructed by adding the full graph Γ to the normal forests. Hence we also identify $-T_\Gamma \equiv -T$.

The forest formula applies to individual Feynman graphs, extracting the finite part by subtracting its overall divergence and subdivergences. As we have seen in our example, it gives the same result of the recursive formula (2.64), despite we won't give the full proof for a general graph, which can be found in the reference ([C]). We are now provided with both a recursive and a non-recursive definition for the renormalization of a Feynman graph.

[15] Forests may as well be defined containing convergent graphs, for example is one is willing to change the definition of the T operation. Then the sum is simply extended to be trivial on these extra forests. However we won't need this.

Chapter 3

Hopf Algebras

載營魄抱一，能無離乎?

Tao Te Ching, chap. X

3.1 Introduction

Hopf algebras are all-encompassing objects which merge two kind of *dual* structures in a precise yet broad manner. Beyond this charming property, they also gained popularity in physics as a "generalization" for the usual description of the symmetries in a physical system, because of their nice representation theory. In fact, amongst other things, they can be seen as noncommutative deformations of both Lie groups or Lie algebras.

Clearly they are the main characters of this section. We give the algebraic concepts useful to reach the definition and list some interesting facts about them. Then we look at some standard Hopf algebras, including the definition of graded Hopf algebra which we will need later.
Afterwards we introduce some other objects over Hopf algebras to build up the content of the next section, in particular Hopf characters.

We assume familiarity with basic abstract algebra and linear algebra. The notation will be the standard one, otherwise stated. Notice that we shall make use of the tensor product $\otimes$ without any rigorous definition. This choice is made to settle things once and for all, in order not to introduce more concepts than needed, nor to use an unsatisfactory description. However, one may always think of the carefree intuitive approach looking at the tensor product as a free abelian group modulo the opportune equivalence relations. That is a quotient

space of the direct product with desired properties of distributivity and bilinearity imposed on the pairs elements "by hand".

For the Hopf algebra part we mainly followed [Maj] and [CP]. Some other insights can be found in [Kas] or the classical [Swe].

3.2 Of Bialgebras and Antipodes

3.2.1 Definitions

Let $\mathbb{K}$ be a fixed field[1].

Definition 3.2.1. An algebra is the triple (A, m, η) consisting of a vector space A over $\mathbb{K}$ with two linear maps $m : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ making the following diagrams commute:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m\otimes id} & A\otimes A \\ {\scriptstyle id\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A\otimes A & \xrightarrow{m} & A \end{array} \qquad \begin{array}{ccccc} \mathbb{K}\otimes A & \xrightarrow{\eta\otimes id} & A\otimes A & \xleftarrow{id\otimes\eta} & A\otimes\mathbb{K} \\ & \searrow^{\cong} & \downarrow{\scriptstyle m} & \swarrow_{\cong} & \\ & & A & & \end{array} \tag{3.1}$$

The first diagram tells us that the product is associative, that is written on elements as $m : a \otimes b \mapsto ab$ one has $(ab)c = a(bc) \quad \forall a, b, c \in A$
Moreover the second diagram says that the algebra has a unit element, expressed by the map $\eta(1) = 1_A$

Remark. Define the transposition map $\tau : A \otimes A \to A \otimes A$ which switches the factors $\tau(a \otimes a') = (a' \otimes a)$, hence we will simply refer to as *flip*.
The opposite algebra A^{op} is the algebra (A, m^{op}, η) with product $m^{op} = m \circ \tau$.
Then the requirement for the previously defined algebra A to be *commutative* is $A = A^{op}$, that is $m = m^{op}$

Definition 3.2.2. An algebra morphism $f : (A, m, \eta) \to (A', m', \eta')$ is a linear map $f : A \to A'$ such that:

$$m' \circ (f \otimes f) = f \circ m \qquad \text{and} \qquad f \circ \eta = \eta' \tag{3.2}$$

Remark. If A and B are two algebras over $\mathbb{K}$, their tensor product $A \otimes B$ is an algebra with product: $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, extended by linearity.

[1]This could also be made a bit more general considering a commutative unital ring k. Accordingly vector spaces and linear maps shall be replaced by k-modules and module maps and so on.

Remark. Let A be an algebra and $I \subset A$ a linear subspace. Then I is said to be a two-sided ideal if it is closed under multiplication both from left and right, i. e. $m(I \otimes A) \subset I \supset m(A \otimes I)$
The quotient vector space A/I carries a quotient algebra structure.

One of the benefits of the formulation through commutative diagrams is one may start wondering what happens by sistematically reversing all the arrows. The upshot is in a certain sense the *dual* notion.

Definition 3.2.3. A coalgebra is the triple (C, Δ, ε) consisting of a vector space C over $\mathbb{K}$ with two linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{K}$ making the following diagrams commute:

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{\Delta \otimes id} & C \otimes C \\ {\scriptstyle id \otimes \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ C \otimes C & \xleftarrow{\Delta} & C \end{array} \qquad \begin{array}{ccccc} \mathbb{K} \otimes C & \xleftarrow{\varepsilon \otimes id} & C \otimes C & \xrightarrow{id \otimes \varepsilon} & C \otimes \mathbb{K} \\ & {\scriptstyle \cong}\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow{\scriptstyle \cong} & \\ & & C & & \end{array} \tag{3.3}$$

The coproduct Δ is coassociative, i.e. it (co)satisfies the relation:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$

The counit ε is thus defined on any element: $(\varepsilon \otimes id) \circ \Delta(c) = c = (id \otimes \varepsilon) \circ \Delta(c)$

One can expand the coproduct of an element as a linear combination in $C \otimes C$. In writing these, there exists a well established and useful convention.

Definition 3.2.4. Sweedler's (sigma) notation:

$$\Delta(c) = \sum_i c_{i(1)} \otimes c_{i(2)} \equiv \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

The choice of the elements $c_{i(j)} \in C$ is far from unique. The convention consists in dropping out the first index i. When there is no room for confusion, we shall omit the summation symbol too.
Moreover we see there is no order ambiguity for a further coproduct[2]:

$$c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} \equiv c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$$

where we made use of coassociativity in the first equation.
Therefore this notation can be extended to any iterated coproduct, which we indicate as $\Delta^n : C \to C^{\otimes n+1}$

[2]Majid points out: *"Physicists should think of c as being like a probability density function: its total probability mass $\varepsilon(c)$ is being shared out among different spaces."*

Remark. In an analogous way to the case for algebras, if we set $\Delta_{op} = \tau \circ \Delta$ with τ being the flip, one obtains the opposite coalgebra $(C, \Delta_{op}, \varepsilon)$ denotd by C_{op}. The coalgebra C is then said to be cocommutative if $C = C_{op}$, that is $\Delta = \Delta_{op}$. An example of cocommutative coalgebra is the field $\mathbb{K}$ regarded as a vector space over itself, with natural maps: $\varepsilon = id_{\mathbb{K}}$ and $\Delta : \mathbb{K} \to \mathbb{K} \otimes \mathbb{K}$

Definition 3.2.5. A coalgebra morphism $f : (C, \Delta, \varepsilon) \to (C', \Delta', \varepsilon')$ is a linear map $f : C \to C'$ such that:

$$(f \otimes f) \circ \Delta = \Delta' \circ f \qquad \text{and} \qquad \varepsilon' \circ f = \varepsilon \tag{3.4}$$

Remark. If C and D are two algebras over $\mathbb{K}$, their tensor product $C \otimes D$ is an algebra with product: $\Delta(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}$

Remark. A coideal is a subspace of a coalgebra $J \subset C$ such that:

$$\Delta(J) \subset J \otimes C + C \otimes J \qquad \text{and} \qquad \varepsilon(J) = 0 \tag{3.5}$$

The quotient vector space C/J carries a quotient coalgebra structure.

Nothing prevents a linear space to be endowed with both an algebra and coalgebra structure, respecting some conditions.

Definition 3.2.6. A bialgebra is the quintuple $(H, m, \eta, \Delta, \varepsilon)$ consisting of a vector space H over $\mathbb{K}$ carrying the two structures of an algebra (H, m, η) and a coalgebra (H, Δ, ε) with one of the two *dual* conditions holding:

- Δ, ε are algebra morphisms, i.e.: $\Delta(ab) = \Delta(a)\Delta(b)$ and $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$;
- m, η are coalgebra morphisms, i.e.: $\Delta(a)\Delta(b) = \Delta(ab)$, which is the same as before, and the definitory $\eta(x_{(1)}) \otimes \eta(x_{(2)}) = \Delta(\eta(x))$.

These provides specular interpretations of bialgebra.
In terms of commuting diagrams, the first condition for instance is given by:

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{m} & H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \Delta\otimes\Delta}\downarrow & & & & \uparrow{\scriptstyle m\otimes m} \\ H \otimes H \otimes H \otimes H & & \xrightarrow{id\otimes\tau\otimes id} & & H \otimes H \otimes H \otimes H \end{array} \tag{3.6}$$

We shall need another important map which underpins a special class of bialgebras.

Definition 3.2.7. A Hopf algebra is a bialgebra H with a linear map $S : H \to H$ obeying:

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta \tag{3.7}$$

This map S is called an *antipode* and makes the following diagrams commute:

$$\begin{array}{ccccc} H & \xrightarrow{\varepsilon} & \mathbb{K} & \xrightarrow{\eta} & H \\ {\scriptstyle\Delta}\downarrow & & & & \uparrow{\scriptstyle m} \\ H \otimes H & & \xrightarrow[id\otimes S]{S\otimes id} & & H \otimes H \end{array}$$

The antipode plays a similar role to the inverse map in the group structure, although it does not satisfy $S^2 = id$ in the general case.[3] Being such a prominent object, let's summarize some important features.

Proposition 3.1. *Let S be the antipode of a Hopf algebra H. Then:*

- *If S' is another antipode of H, then $S' = S$;*
- *S is an algebra antimorphism, that is:*

$$S(hg) = S(g)S(h) \qquad \textit{and} \qquad S(1) = 1$$

- *S is a coalgebra antimorphism, that is:*

$$(S \otimes S) \circ \Delta(h) = \tau \circ \Delta \circ S(h) \qquad \textit{and} \qquad \varepsilon \circ S(h) = \varepsilon(h)$$

Proof. We prove the first point, the others follow by axioms rearranging the terms in a similar way.
By the counit axiom we have:

$$S'(h) = m \circ (S' \otimes \{\eta \circ \epsilon\}) \circ \Delta(h)$$

Then we insert the definition of the antipode S:

$$= m \circ (S' \otimes \{m \circ (id \otimes S) \circ \Delta\}) \circ \Delta(h)$$

but using the tensor product property this becomes:

$$= m \circ (S' \otimes \{m \circ (id \otimes S)\}) \circ (id \otimes \Delta) \circ \Delta(h)$$

and at this point we can use the associativity of m and the coassociativity of Δ:

$$= m \circ (\{m \circ (S' \otimes id)\} \otimes S) \circ (\Delta \otimes id) \circ \Delta(h)$$

[3] In infinite dimension it is not even true the existence of an inverse linear map S^{-1}

By reversing the process we first get the tensor product:

$$= m \circ (\{m \circ (S' \otimes id) \circ \Delta\} \otimes S) \circ \Delta(h)$$

thus reabsorbing S', which by hypotesis is an anitpode, we have:

$$= m \circ (\{\eta \circ \epsilon\} \otimes S) \circ \Delta(h) = S(h)$$

where thesis is obtained by the counit axiom. □

One can add some more insights on the significance of the antipode.

Definition 3.2.8. Given an algebra (A, m, η) and a coalgebra (C, Δ, ε) on a field $\mathbb{K}$ and two linear maps $f, g : C \to A$, the convolution product is defined as the composition:

$$f * g = m \circ (f \otimes g) \circ \Delta \tag{3.8}$$

Remark. The convolution product is a bilinear map on the vector space $Hom(C, A)$ of the linear maps from C to A.
Actually this makes the triple $(Hom(C, A), *, \eta \circ \varepsilon)$ into an algebra, called the convolution algebra.

Remark. When $A = C = H$ is a Hopf algebra, the antipode $S \in End(H)$ is the inverse of the identity map for the convolution product:

$$S * id = id * S = \eta \circ \varepsilon \tag{3.9}$$

Remark. Given a bialgebra H, we can flip the underlying structures to get the three bialgebras H_{op}, H^{op} and H^{op}_{op} . Thus H is said to be commutative or cocommutative if it such as an algebra or a coalgebra respectively. In both cases the antipode simply becomes a morphism of algebras and of coalgebras. Furthermore if H owns a Hopf algebra structure with invertible antipode S, one gets three new Hopf algebras with antipodes respectively S^{-1}, S^{-1} and S

Definition 3.2.9. A morphism of Hopf algebras is both an algebra morphism and a coalgebra morphism.

Remark. Actually this is the same definition for a bialgebra morphism. Indeed, given a bialgebra morphism between two Hopf algebras $f : H \to H'$ a further condition should be the compatibility with antipodes $S' \circ f = f \circ S$ but this is automatically satisfied.

Remark. Combining the results, an invertible antipode $S : H \to H$ is an anti-automorphism of the Hopf algebra H , i.e. regarded as a map $S : H \to H^{op}_{op}$ it is an isomorphism of Hopf algebras. Accordingly, S^2 is an automorphism and if moreover H is commutative or cocommutative it can be shown one has $S^2 = id$

Remark. If H and H' are two bialgebras over $\mathbb{K}$, their tensor product $H \otimes H'$ brings the tensor product structures of algebra and coalgebra and is therefore a bialgebra. Besides, if they are equipped with two antipodes S and S' , then $(H \otimes H', S \otimes S')$ is a Hopf algebra.

Remark. A Hopf ideal is two-sided ideal of a Hopf algebra $I \subset H$ such that:

$$\Delta(I) \subset I \otimes H + H \otimes I, \qquad \varepsilon(I) = 0, \qquad S(I) \subset I \tag{3.10}$$

As usual the quotient vector space H/I inherits the Hopf algebra structure.

Now time has come to keep the promise and go thoroughly the question of duality. We start noticing the nice fact that reversing all the arrows in the definition of the Hopf algebra one obatins the very same definition of Hopf algebra, which is generally a different one, although clearly related in some way to the underlying vector space structure. This is no surprise, since we guessed the notion of coalgebra by analoguously reversing the diagram for an algebra and then we glued both of them to get a bialgebra. However there seems to be no contraindication in beginning this process from a coalgebra. It is a feature of several algebraic constructions concerning the reversal of tensor product. Still we want to clarify how this equivalence works: the core of the duality is the reversing of arrows and, needless to say, it can be given a precise meaning in terms of dual spaces.

Definition 3.2.10. Let V be a vector space over $\mathbb{K}$. The set of all $\mathbb{K}$-linear maps $\varphi : V \to \mathbb{K}$ forms a vector space equipped with pointwise addition and scalar multiplication, the dual vector space V^*.

Remark. Recall that by setting $x[\varphi] = \varphi(x) \quad \forall x \in V, \forall \varphi \in V^*$ one induces a dual pairing, i.e. a bilinear map $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{K}$.
The pairing implies the inclusion $V \subset (V^*)^*$. Moreover it extends to tensor products pairwise[4], so that for any two vector spaces $V_1^* \otimes V_2^* \subset (V_1 \otimes V_2)^*$. This inclusions are actually isomoprhisms in finite dimension, in fact one can choose two bases and show that the dual pairing is non-degenerate.

Remark. Given a coalgebra (C, Δ, ε), the pairing $\langle \cdot, \cdot \rangle = ev$ defines adjoint maps:

$$\Delta^* : C^* \otimes C^* \to C^* \qquad \varepsilon^* : \mathbb{K} \to C^*$$
$$\langle \Delta^*(\phi \otimes \psi), x \rangle = \langle \phi \otimes \psi, \Delta(x) \rangle \qquad \langle \varepsilon^*(1), x \rangle = \langle 1_{C^*}, \varepsilon(x) \rangle$$

[4]That is $\langle \phi \otimes \psi, x \otimes y \rangle = \langle \phi, x \rangle \langle \psi, y \rangle$

The coalgebra axioms make the triple $(C^*, \Delta^*, \varepsilon^*)$ into an algebra.
The converse is not true in the general case[5], because the dual map m^* might not take values in the proper subspace. Nevertheless it also holds in finite dimension that for an algebra (A, m, η) the similar pairing gives rise to a dual coalgebra (A^*, m^*, η^*)

Therefore the idea of reversing the arrows in the diagrams can be understood as linear dualization. When finite-dimensional algebra and coalgebra structures on the same vector space are assembled into a bialgebra, we have the following:

Proposition 3.2. *Let $(H, m, \eta, \Delta, \varepsilon)$ be a finite-dimensional bialgebra over $\mathbb{K}$ and let $\langle \cdot, \cdot \rangle : H^* \times H \to \mathbb{K}$ be the dual pairing defined by the evaluation map ev and extended to tensor products pairwise. Then on the dual space is carried the bialgebra structure $(H^*, \Delta^*, \varepsilon^*, m^*, \eta^*)$ specified $\forall \phi, \psi \in H^*, \forall x, y \in H$ by:*

$$\langle \Delta^*(\phi \otimes \psi), x \rangle = \langle \phi \otimes \psi, \Delta(x) \rangle \qquad \langle \varepsilon^*(1), x \rangle = \langle 1_{H^*}, \varepsilon(x) \rangle$$
$$\langle m^*(\phi), x \otimes y \rangle = \langle \phi, m(h \otimes g) \rangle \qquad \langle \eta^*(\phi), 1_H \rangle = \langle \phi, \eta(1) \rangle$$

Proof. In the previous observation we already give a pictorial explanation of how associativity and coassociativity translate into one another. What's left is to verify the structure by the axioms of H, which is just a matter of calculation. We have from the definitions:

$$\langle m^* \circ \Delta^*(\phi \otimes \psi), x \otimes y \rangle = \langle \Delta^*(\phi \otimes \psi), m(x \otimes y) \rangle = \langle \phi \otimes \psi, \Delta \circ m(x \otimes y) \rangle =$$
H is a bialgebra, so we use the fact: $\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$
$$= \langle \phi \otimes \psi, (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta(x) \otimes \Delta(y)) \rangle =$$
Recalling we are in finite dimension and with some slight abuse of notation:
$$= \langle (id \otimes \tau \otimes id) \circ (m^*(\phi) \otimes m^*(\psi)), \Delta(x) \otimes \Delta(y) \rangle =$$
And finally, by the multilinearity of the tensor product and the pairing:
$$= \langle (\Delta^* \otimes \Delta^*) \circ (id \otimes \tau \otimes id) \circ (m^*(\phi) \otimes m^*(\psi)), x \otimes y \rangle$$

Which asserts exactly the compatibility condition between product and coprod-

[5]For the coalgebra it is essential that we are able to define the multiplication on the dual space by restricting the domain of the map Δ^*.

uct in H^*. Similarly, for unit and counit we have:

$$\langle m^* \circ \varepsilon^* \circ \eta^* \circ \Delta^*(\phi \otimes \psi), x \otimes y \rangle = \langle \varepsilon^* \circ \eta^* \circ \Delta^*(\phi \otimes \psi), m(x \otimes y) \rangle =$$

After omitting some obvious passages, we make use of the bialgebra structure on H, in particular we make us of: $\Delta \circ \eta \circ \varepsilon \circ m = (\eta \otimes \eta) \circ (\varepsilon \otimes \varepsilon)$ to obtain:

$$= \langle (\phi \otimes \psi), \Delta \circ \eta \circ \varepsilon \circ m(x \otimes y) \rangle = \langle \phi \otimes \psi, (\eta \otimes \eta) \circ (\varepsilon(x) \otimes \varepsilon(y)) \rangle =$$

Again, relying on the finite dimension, by multilinearity one gets:

$$= \langle \eta^*(\phi) \otimes \eta^*(\psi), \varepsilon(x) \otimes \varepsilon(y) \rangle = \langle (\varepsilon^* \otimes \varepsilon^*) \circ (\eta^*(\phi) \otimes \eta^*(\psi)), x \otimes y \rangle$$

That is the second assertion we needed. Thus H^* is a bialgebra. □

The infinite-dimensional case is more tricky for it carries the usual issues present in the correct definition of dual space. One way to proceed is to restrict to a subalgebra $H^\circ \subset H^*$ with the right properties and define this as the dual. However, in some case it may result somehow clumsy to extricate in such subtle task. Thus a different approach is to abstract the finite-dimensional pairing's feature, i.e. to compensate for the infinite dimension by assuming the non-degeneracy.

Definition 3.2.11. Two bialgebras H_1, H_2 over $\mathbb{K}$ are dually paired if the bilinear map $\langle \cdot, \cdot \rangle : H_1 \times H_2 \to \mathbb{K}$ is a pairing, that is with obvious notation:

$$\langle m_1(x_1 \otimes y_1), x_2 \rangle = \langle x_1 \otimes y_1, \Delta_2(x_2) \rangle \qquad \langle \eta_1(1), x_2 \rangle = \langle 1_{H_1}, \varepsilon_2(x_2) \rangle$$
$$\langle \Delta_1(x_1), x_2 \otimes y_2 \rangle = \langle x_1, m_2(x_2 \otimes y_2) \rangle \qquad \langle \varepsilon_1(x_1), 1_{H_2} \rangle = \langle x_1, \eta_2(1) \rangle$$

If H_1, H_2 are equipped with the additional structure of Hopf algebra, the condition for the antipodes is:

$$\langle S_1(x_1), x_2 \rangle = \langle x_1, S_2(x_2) \rangle \tag{3.11}$$

Futhermore they are said to be a *strictly dual pair* if the pairing is non-degenerate, i.e. for any element $y \in H_2$ such that $\langle x, y \rangle = 0 \quad \forall x \in H_1$ then it implies $y = 0$, and vice versa.

Remark. Clearly this is consistent in finite dimension, where the pairing means that H_1 and H_2 are dual each other.
It is important to note that one can always make a pairing non-degenerate by quotienting with opportune (bi)ideals in order to identify all the null elements. The resulting sub-bialgebras are then strictly dual pair.

Remark. In the case of H being a finite-dimensional Hopf algebra with antipode S, then the bialgebra H^* is a Hopf algebra with antipode S^*, following by the definitions and then the property of being unique.
Another nice feature one can check is that the dual of a commutative Hopf algebra is cocommutative, and vice versa.

3.2.2 Some Examples

Stated the basic concepts, we now give a review of standard Hopf algebras. Along the road we also exploit this section to introduce some other definitions we shall use in the following.

Definition 3.2.12. Let X be any set. A reduced[6] word in X of length n is any expression of the form: $x_1^{\pm 1}x_2^{\pm 1}\cdots x_n^{\pm 1}$ where $x_1,\dots,x_n \in X$ and $x_i \neq x_{i+1}$
When X is (in corrispondence with) a subset of a group, any element of this group can be represented by a word in X and in particular the identity is represented by the empty word $\emptyset$ which is the unique word of lenght zero.

Remark. The vector space $\mathbb{K}\langle X\rangle$ with basis the set of all words in X can be provided an algebra structure wherein multiplication is defined as concatenation of words:

$$(x_{i_1}\cdots x_{i_p})(x_{i_{p+1}}\cdots x_{i_n}) = x_{i_1}\cdots x_{i_p}x_{i_{p+1}}\cdots x_{i_n} \tag{3.12}$$

This is called the free algebra with basis X.

Group Hopf algebra $\mathbb{K}G$

Let $(G,\cdot)$ be a finite group with identity e. The group Hopf algebra $\mathbb{K}G$ is the free algebra with basis G obtained extending linearly the structure maps:

$$m = \cdot \quad \eta(1) = e \quad \varepsilon(g) = 1 \quad \Delta(g) = g\otimes g \quad S(g) = g^{-1} \tag{3.13}$$

$\mathbb{K}G$ is always cocommutative and is also commutative only if G is abelian.

Group function Hopf algebra $\mathbb{K}(G)$

Let G be a finite group with identity e. The group function Hopf algebra $\mathbb{K}(G)$ is the algebra of functions on G with values in $\mathbb{K}$ and pointwise product: $(\varphi\psi)(g) = \varphi(g)\psi(g) \quad \forall g\in G, \forall\varphi,\psi\in\mathbb{K}(G)$ supplemented by the structures[7]:

$$\Delta(\phi)(g_1,g_2) = \phi(g_1g_2) \qquad \varepsilon(\phi) = \phi(e) \qquad S(\phi)(g) = \phi(g^{-1}) \tag{3.14}$$

[6] There is no loss of generality since any word can be put in this form by successive simplification of terms like xx^{-1} and $x^{-1}x$

[7] There is the identification: $\mathbb{K}(G\times G)\cong\mathbb{K}(G)\otimes\mathbb{K}(G)$

$\mathbb{K}(G)$ is always commutative and is also cocommutative only if G is abelian.

Remark. One may see some similarity between these two Hopf algebras and there is no coincidence. With the pairing given by the evaluation map, $\mathbb{K}(G)$ and $\mathbb{K}G$ are a strictly dual pair. This has some significant consequence: if G is a finite abelian group, the isomorphism induced by this duality entails Pontryagin's duality. Moreover, when G is not abelian, one can still make sense of this picture in the context of noncommutative geometry.

Graded Hopf algebra

Let $(H, m, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra which is also a graded algebra, i.e. there exist a family $\{H_k\}_{k\in\mathbb{N}}$ of subspaces of H such that:

$$H = \bigoplus_{k\in\mathbb{N}} H_k \qquad \text{with} \qquad H_i \cdot H_j \subset H_{i+j} \quad \forall i, j \in \mathbb{N} \tag{3.15}$$

This is furthermore said to be a graded Hopf algebra if the structure maps preserve the grading:

$$\begin{gathered} m(H_i \otimes H_j) \subset H_{i+j} \qquad \Delta(H_k) \subset \oplus_{i+j=k} H_i \otimes H_j \\ \eta(\mathbb{K}) \subset H_0 \qquad S(H_k) = H_k \qquad \varepsilon(H_k) = 0 \quad \forall k > 0 \\ \text{and the flip is graded:} \quad \tau(h \otimes h') = (-1)^{ij} h' \otimes h \quad \forall h \in H_i, \forall h' \in H_j \end{gathered}$$

Tensor Hopf algebra $T(A)$

Let (A, m, η) be an algebra on $\mathbb{K}$. The noncommutative algebra generated by 1 and linear combinations of finite tensor products of elements of A forms the tensor algebra:

$$T(A) = \bigoplus_{n\geq 0} A^{\otimes n} \qquad A^0 \equiv \mathbb{K} \tag{3.16}$$

This has a further Hopf algebra structure with:

$$\Delta(a) = a \otimes 1 + 1 \otimes a \qquad \varepsilon(a) = 0 \qquad S(a) = -a$$

A multitude of Hopf algebras can be constructed from tensor and graded Hopf algebra by quotienting with an opportune Hopf ideal. Some standard examples are the symmetric algebra, the exterior algebra or the following.

Universal enveloping Hopf algebra $U(\mathfrak{g})$

For any Lie algebra $(\mathfrak{g}, [,])$, the universal enveloping Hopf algebra $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ modulo the Hopf ideal generated by the relations:

$$x \otimes y - y \otimes x = [x, y] \tag{3.17}$$

Note that $\Delta(\mathfrak{g})$ is in the symmetric subalgebra of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, therefore $U(\mathfrak{g})$ is always cocommutative. It is also commutative if $\mathfrak{g}$ is abelian.

Remark. As a corollary of the Poincaré-Birkhoff-Witt, a Lie algebra $\mathfrak{g}$ over a field is canonically injected into its universal enveloping algebra. Moreover $\mathfrak{g}$ generates $T(\mathfrak{g})$ and consequently $U(\mathfrak{g})$ itself. Thus it suffices to impose the same coalgebra and antipode structure on $\mathfrak{g}$:

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \varepsilon(x) = 0 \qquad S(x) = -x \qquad \forall x \in \mathfrak{g} \tag{3.18}$$

This is tantamount to say that $\mathfrak{g}$ is the set of the primitive[8] elements of $U(\mathfrak{g})$

Definition 3.2.13. Let H be a general Hopf algebra with coproduct Δ. An element $x \in H$ is called:

- $grouplike$, if: $\Delta(x) = x \otimes x$
- $primitive$, if: $\Delta(x) = x \otimes 1 + 1 \otimes x$

Remark. We have seen that the set of grouplike elements is a group with inverse given by the antipode, whilst the set of primitive elements equipped with the bracket is a Lie algebra. Moreover they are also commutative or cocommutative as Hopf algebras. It may be argued that Hopf algebras are to be considered as the generalization of these structures. It's worth to briefly highlight that indeed the (sub)categories of finite groups, compact topological groups, compact Lie groups and affine algebraic groups functorially embed in a subcategory of the category of Hopf algebras[9]. Specifically, it can be shown that any commutative or cocommutative Hopf algebra in a certain sense is the same as one of the examples given.

Finally, where the magic happens. Let's give a standard example of noncommutative nor cocommutative Hopf algebra, i.e. the Hopf algebras of what are also known as quantum groups.

[8] In the sense of the subsequent definition.

[9] See [Abe] for details.

Weyl Hopf algebra $U_q(b_+)$

Let $U_q(b_+)$ be the free algebra on the vector space $\mathbb{K}$ generated by the elements $1, X, g, g^{-1}$ with the relations:

$$gg^{-1} = 1 = g^{-1}g \qquad gX = qXg \tag{3.19}$$

where $q \in \mathbb{K}$ is a fixed invertible element. It has the Hopf algebra structure:

$$\begin{array}{lll} \Delta(X) = X \otimes 1 + g \otimes X & \varepsilon(X) = 0 & S(X) = -g^{-1}X \\ \Delta(g) = g \otimes g & \varepsilon(g) = 1 & S(g) = g^{-1} \\ \Delta(g^{-1}) = g^{-1} \otimes g^{-1} & \varepsilon(g^{-1}) = 1 & S(g^{-1}) = g \end{array}$$

Remark. The Hopf algebra $U_q(b_+)$ is clearly noncommutative nor cocommutative. Notice that we have: $S^2(X) = S(-g^{-1}X) = -S(X)S(g^{-1}) = g^{-1}Xg = q^{-1}X$ therefore this is an example where $S^2 \neq id$ as expected. The parameter q works as a deformation of the commutative case $q = 1$. The same happens for the sub-bialgebra generated by $1, X, g$. This is also known as quantum plane since in the same spirit we see it is a deformation of the affine plane.

Remark. The Hopf algebra $U_q(b_+)$ is dually paired with itself, the pairing given by:

$$\langle g, g \rangle = q \qquad \langle X, X \rangle = 1 \qquad \langle X, g \rangle = \langle g, X \rangle = 0$$

3.3 Characters

3.3.1 Hopf Characters

Now we give a name to a special class of morphisms.
Let $(H, m_H, \eta_H, \Delta, \varepsilon, S)$ be a graded (commutative) Hopf algebra over $\mathbb{K}$ with grading $\{H_k\}_{k\in\mathbb{N}}$ such that $H_0 \simeq \mathbb{K}$ and let (A, m_A, η_A) be a commutative $\mathbb{K}$-algebra.

Definition 3.3.1. A Hopf (algebra) character is an algebra morphism $\phi : H \to A$ such that $\phi(1_H) = 1_A$

In other words, a character is a linear map which preserves the structure of algebra and moreover the neutral element of multiplication. Given this unity element $1_A \in A$ one can define a unity map $\eta_A : \mathbb{K} \to A$ for any character ϕ as

$\eta_A := \phi \circ \eta_H$, so that we have $\eta_A(1) = 1_A$

Endowed with a convolution product (3.8), the characters form a group. Indeed one can easily show the following:

Proposition 3.3. *The set $G_H(A) = Hom_{\mathcal{A}_{\mathbb{K}}}(H, A)$ of all Hopf characters, i. e. algebra morphisms from H to A which preserve the unit, forms a group with respect the structure provided for any two characters $\phi, \psi : H \to A$ by:*

$$\phi * \psi = m_A \circ (\phi \otimes \psi) \circ \Delta$$

With neutral element $e := \eta_A \circ \varepsilon$.
Finally, the inverse of any $\phi \in G_H(A)$ is given by $\hat{\phi} := \phi \circ S$

This enable us to translate some of the machinery into groups of functions.
Moreover, recall for a graded Hopf algebra (3.15) any homogeneous element is in the kernel of the counit, i.e. one has $\varepsilon(H_k) = 0$ for any $k > 0$. Keeping this in mind, we introduce some other concepts which we are going to use.

Definition 3.3.2. A linear map $\mathcal{R} : A \to A$ is said to be a Rota-Baxter operator if:

$$\mathcal{R}[ab] + \mathcal{R}[a]\mathcal{R}[b] = \mathcal{R}[\mathcal{R}[a]b + a\mathcal{R}[b]] \qquad \forall a, b \in A \tag{3.20}$$

This map satisfies an inhomogeneous version of anti-multiplicativity.
We use a Rota-Baxter operator to construct generalized antipodes.
For a character $\phi \in G_H(A)$ we define a linear map $S^{\phi}_{\mathcal{R}} : H \to A$ recursively as:

$$S^{\phi}_{\mathcal{R}}(1_H) := 1_A \qquad S^{\phi}_{\mathcal{R}}(h) = -\mathcal{R}[(S^{\phi}_{\mathcal{R}} * (\phi \circ \mathcal{P}))(h)] \qquad \forall h \in ker(\varepsilon) \tag{3.21}$$

where the map $\mathcal{P} = id_H - \eta \circ \varepsilon : H \to ker(\varepsilon)$ is a projector which takes care of some extra terms getting out from the coproduct.

Notice that beacuse of the commutativity of the product, it follows from the Rota-Baxter property (3.20) that this map respects the characters group operations:

$$S^{\phi}_{\mathcal{R}}(hg) = S^{\phi}_{\mathcal{R}}(h) S^{\phi}_{\mathcal{R}}(g) \qquad S^{\phi}_{\mathcal{R}}(h^{-1}) = S^{\phi}_{\mathcal{R}} \circ S(h) \tag{3.22}$$

To make contact with the antipode as anticipated, consider the convolution algebra $End(H)$ and the identity map on H both as character and Rota-Baxter operator. Then the antipode satisfies the recursive relation:

$$S(h) = -(S * \mathcal{P})(h) \qquad \forall h \in ker(\varepsilon) \tag{3.23}$$

Now it is important to observe that since a character ϕ preserves the unity, it holds that:

$$S^{\phi}_{\mathcal{R}} * \phi = S^{\phi}_{\mathcal{R}} + S^{\phi}_{\mathcal{R}} * (\phi \circ \mathcal{P}) \tag{3.24}$$

At this point with a little fantasy we may already get a major hint[10] just rewriting as:

$$\phi_{\mathcal{R}}(x) = (id_A - \mathcal{R})\bar{\phi}(x) \qquad\qquad x \in ker(\varepsilon) \tag{3.25}$$

where we posed $\phi_{\mathcal{R}} \equiv S^{\phi}_{\mathcal{R}} * \phi$ and $\bar{\phi} \equiv S^{\phi}_{\mathcal{R}} * (\phi \circ \mathcal{P})$, and used the definition (3.21) in the form $-\mathcal{R}[\bar{\phi}(x)] = S^{\phi}_{\mathcal{R}}(x)$.

If we identify these terms respectively with the renormalized Feynman rules, the Bogoliubov map and the counterterms we see that one gets exactly the same expression (2.64) that we previously derived in the BPHZ procedure.
This is exciting, but let's not rush and go on by order.

3.3.2 Birkhoff Factorization

As a general statement, the Birkhoff factorization is used for the solution to a particular class of problems in the ambit of the Riemann-Hilbert correspondence[11].
Our formulation is aimed to the much more modest purpose to lay a bridge between renormalization procedure and the setting of Hopf algebras, as we will se in the next chapter.

Let $C = \partial D$ be the boundary of a disk D on the complex plane $\mathbb{C}$ with center in $z = 0$. Consider the Riemann sphere $S^2 \simeq \mathbb{P}^1(\mathbb{C})$ and take the quotient $\mathbb{P}^1(\mathbb{C}) \setminus C$. We denote by $C_{\pm}$ the two components of this quotient space, such that $0 \in C_+$ and $\infty \in C_-$. Finally let $G(\mathbb{C})$ be a connected complex Lie group.

Definition 3.3.3. A smooth loop $\gamma : C \to G(\mathbb{C})$ admits a Birkhoff factorization if it can be written as a product

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z) \qquad\qquad \forall z \in C \tag{3.26}$$

with $\gamma_{\pm} : C_{\pm} \to G(\mathbb{C})$ two holomorphic functions such that $\gamma_-(\infty) = 1$
Observe that $\gamma_{\pm}$ are defined on C by means of a limit procedure.

[10] As a matter of fact this is a spoiler.
[11] See [CM] for details.

We consider the case when D is an infinitesimal disk around $z = 0$ and C is an infinitesimal loop. Assume the Lie group is $G(\mathbb{C}) = G_H(\mathbb{C}) \equiv Hom_{\mathcal{A}_\mathbb{C}}(H, \mathbb{C})$, i.e. the set of $\mathcal{A}_\mathbb{C}$-algebra morphisms from H to $\mathbb{C}$, where H is a commutative Hopf algebra over $\mathbb{C}$.

Let denote by $\mathcal{K}$ the field of convergent Laurent series, by $\mathcal{O}$ the ring of convergent power series and by $\mathcal{Q}$ the unital ring of inverse power series.
Regarded as commutative algebras over $\mathbb{C}$, we can consider the corresponding groups of characters:

$$G_H(\mathcal{K}) = Hom_{\mathcal{A}_\mathbb{C}}(H, \mathcal{K}) \qquad G_H(\mathcal{O}) = Hom_{\mathcal{A}_\mathbb{C}}(H, \mathcal{O}) \qquad G_H(\mathcal{Q}) = Hom_{\mathcal{A}_\mathbb{C}}(H, \mathcal{Q})$$

The elements in $G_H(\mathcal{K})$ may be seen as loops $\gamma(z)$ on an infinitesimal circle around $z = 0$ whereas the elements in $G_H(\mathcal{O})$ as loops $\gamma(z)$ which admit a holomorphic extension to $z = 0$.

Last, for an element $\phi \in G_H(\mathcal{Q})$ one shall impose the normalization $\gamma_-(\infty) = 1$. This can be done by means of the augmentation map ε_- of $G_H(\mathcal{Q})$ seen as a group ring over $\mathcal{Q}$, i.e. a map such that $\varepsilon_-(\phi) = 1$ which reduces to the identity map on the ring. Hence we obtain the corresponding condition:

$$\varepsilon_- \circ \phi = \varepsilon \tag{3.27}$$

where ε_- is the augmentation in $\mathcal{Q}$ and ε is the counit in H

Using this same notation, one has the restatement:

Definition 3.3.4. Let S be the antipode of H. An element $\phi \in G_H(K)$ admits a Birkhoff factorization if it can be written as a product

$$\phi = (\phi_- \circ S) * \phi_+ \tag{3.28}$$

where $\phi_+ \in G_H(\mathcal{O})$, $\phi_- \in G_H(\mathcal{Q})$ and such that $\varepsilon_- \circ \phi_- = \varepsilon$

It is this form of the Birkhoff factorization that we will use in the next chapter.

Chapter 4

Connes-Kreimer Theory

"Es ist dafür gesorgt, daß die Bäume nicht in den Himmel wachsen."

J. W. von Goethe [Gq8]

4.1 Introduction

The works [K98], [K99], [CK] of Kreimer and Connes exhibited that the combinatorics of perturbative quantum field theory hides a beautiful underlying structure of Hopf algebra, whose elements are Feynman graphs.
In this framework, Feynman rules associated to a graph are constructed as characters in the algebra of the meromorphic functions. This as usual are generally divergent integrals needing regularization which afterwards require renormalization. The Connes-Kreimer theory gives a clear mathematical understanding of the BPHZ procedure in terms of the Birkhoff factorization in a particular complex Lie group associated to the aforementioned Hopf algebra.

This final section employs the Hopf algebraic formalism we introduced to provide a reinterpretation of the manipulations performed in the physics of renormalization.
In order to work out our objects we give new precise definitions to Feynman diagrams and related concepts, forming an algebra.
Then is showed this can be equipped with a coalgebra structure and moreover an antipode, which encode the counterterms to a graph.

After that we show a general graded Hopf algebra admits a Birkhoff factorization and the recursive formula of BPHZ procedure is the exact solution in the case of the Hopf algebra of Feynman graphs.
In the end we briefly mention some other arguments, which are unfortunately beyond the scope of this script.

We shall mainly refer to [CM], [Kre] and in general the references of Kreimer and Connes.

4.2 General Definitions

4.2.1 Feynman Graphs

We are now prepared to implement the structures and tools previously defined, translating Feynman diagrams and the renormalization techniques into an algebraic language.

Let's generalize some quantum field theory concepts adopting a complementary yet informal point of view. Some further clarifications are given in the appendix B.

Definition 4.2.1. A Feynman graph Γ consists of a finite set of labelled vertices $\Gamma^{(0)}$ and a finite set of labelled (oriented) edges $\Gamma^{(1)}$, also called lines[1]. Each edge $e \in \Gamma^{(1)}$ is assigned to one or two vertices: if $|e \cap \Gamma^{(0)}| = 1$ then it is an external edge, oterwhise we have an internal edge when $|e \cap \Gamma^{(0)}| = 2$. This splits the set $\Gamma^{(1)}$ in two complementary subset $\Gamma^{(1)}_{ext}$ and $\Gamma^{(1)}_{int}$.

Labels mean Feynman graphs are endowed with maps[2] that assign analyitical informations and quantities to its edges and vertices, depending upon the theory under consideration.

Definition 4.2.2. A Feynman subgraph $\gamma \subset \Gamma$ consists of a finite subset of vertices $\gamma^{(0)} \subset \Gamma^{(0)}$, a finite subset of edges $\gamma^{(1)} \subset \Gamma^{(1)}$ and the restriction in the obvious way of the labelling maps aforementioned.

It is worth noticing edges in $\gamma^{(1)}$ are again assigned to the same vertices, if present in $\gamma^{(0)}$: in principle the internal and external lines of γ can be totally different from those of Γ.

[1] Since we are mainly dealing with scalar fields, the orientation won't bother us and we shall use the two terms interchangeably.

[2] These maps are left implicit in order to avoid irrelevant overcomplications.

Definition 4.2.3. A connected Feynman graph Γ is a one-particle irreducible (1PI) graph if the following holds:

1. Γ is not a tree (such as the free propagator or the simple vertex graphs).
2. Γ cannot be disconnected by cutting a single (internal) edge.

As a matter of fact, all Feynman graphs (and subgraphs) are made up of 1PI subgraphs. Furthermore, a graph can be characterized by the number of its external lines $|\Gamma^{(1)}_{ext}|$:

- A vacuum bubble has no external edges $|\Gamma^{(1)}_{ext}| = 0$ whilst for tadpole graphs $|\Gamma^{(1)}_{ext}| = 1$. We shall discard these graphs by the considerations at the end of (1.23).
- Propagators and self-energy graphs have $|\Gamma^{(1)}_{ext}| = 2$.
- General graphs with $|\Gamma^{(1)}_{ext}| \geq 3$ are called vertex graphs.

Feynman rules (1.23) allow us to obtain the unrenormalized value $V(\Gamma)(p_1, \ldots, p_N)$ associated to a graph, which in general is a multiple integrals with the following features:

$$V(\Gamma) = \prod V(\Gamma_c) \qquad\qquad V(\Gamma) \propto U(\Gamma) \tag{4.1}$$

The former expression means this value factorizes as a product over the connected components of the graph. The latter says it is proportional to the general form (2.59) we rewrite here for convenience:

$$U(\Gamma)(p_1, \ldots, p_N) = \int I(p_1, \ldots, p_N, k_1, \ldots, k_L)\, d^d k_1 \cdots d^d k_L \tag{4.2}$$

The proportionality factor is a combination of external propagators and a delta function to impose momentum conservation, hence we refer at it as the unrenormalized value of the graph Γ with assigned external momenta $(p_1, \ldots, p_N)$. Then one can write Green's functions as:

$$G^{(n)}(x_1, \ldots, x_n) = \sum_{\Gamma} \int \frac{V(\Gamma)(p_1, \ldots, p_n)}{Sym(\Gamma)} e^{i(x_1 \cdot p_1 + \cdots + x_n \cdot p_n)} \prod_j \frac{dp_j}{(2\pi)^d} \tag{4.3}$$

where the symmetry factor $Sym(\Gamma)$ is defined as the cardinality of the group $Aut(\Gamma)$ consisting in the automorphisms of the graph Γ.

From the discourse in (1.38), we now see the connected Green's functions are simply given by restricting the sum over the connected graphs Γ. Moreover we know that in order to work with the effective action we are really interested in just the 1PI graphs of the theory. The contribution of a 1PI graph to the effective action is of the form:

$$U(\Gamma)(\phi) = \frac{1}{N!}\int_{\sum p_j=0} \tilde{\phi}(p_1)\cdots\tilde{\phi}(p_N)U(\Gamma)(p_1,\ldots,p_N)\prod_j \frac{dp_j}{(2\pi)^d} \tag{4.4}$$

where $\tilde{\phi}$ denotes the Fourier transform of ϕ.
One understands the formula as a pairing between fields and the distribution $U(\Gamma)$ which is build up as a smooth function of the external momenta. Thus we arrive at the following:

Theorem 4.1. *The effective action is given by the formal series:*

$$S_{eff}[\phi] = S[\phi] - \sum_{\Gamma\in 1PI} \frac{U(\Gamma)(\phi)}{Sym(\Gamma)}$$

Proof. The proof basically follows from the Feynman rules applied to 1PI graphs, which allow to rewrite the path integral only on tree graphs weighted by S_{eff}. Details can be found in [CM] th 1.5 □

This result justifies the somehow heuristic derivation of (1.39).

In defining dimensional regularization and minimal subtraction scheme, we glibly assumed that $U(\Gamma)$ keeps on making sense when the space-time dimension d is extended to a complex number $d-z$ and furthermore that the resulting $U^z(\Gamma)$ admits a Laurent expansion for $z=0$.
One could circumvent the problems by straightly working with the BPHZ scheme for instance, that only requires $U(\Gamma)$ can be expanded in a Taylor series.
However there is no need in doing so, by virtue of the following[3]:

Theorem 4.2. *The Taylor coefficients at $p=0$ of $U^z(\Gamma)(p_1,\ldots,p_N)$ admit a meromorphic continuation to the whole complex plane $z\in\mathbb{C}$.*

Thus we can define minimal subtraction (2.40), (2.62) as the removal of the polar part $T(L(z))$ from a general Laurent series $L(z)$. Indeed the theorem allows us to apply this to the Taylor expansion of $U^z(\Gamma)(p_1,\ldots,p_N)$ and to work out the machinery.

[3]For the proof see [CM] th 1.9.

Beyond analytical quantities, one can also assign some interesting integers to a graph Γ. We have already seen that one gives names to particular graphs relying on the number N of external lines. Moreover, another important integer is the number L of independent momenta loops in a graph[4], which encodes the perturbative order. In particular one is able to express the superficial degree of divergence in a self-interacting scalar theory as:

$$D(\Gamma) = -2I + dL = 6 - 2N + (d-6)L \tag{4.5}$$

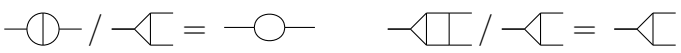
where there hold topological identities like $I - V = L - 1$ and $3V = 2I - N$, with notation $I = |\Gamma^{(1)}_{int}|$, $N = |\Gamma^{(1)}_{ext}|$, $V = |\Gamma^{(0)}|$.

4.2.2 Contraction

Amongst the operations that can be performed on a graph, it will be useful to reverse the insertion of a subgraph.

Definition 4.2.4. The contraction of a subgraph γ in a graph Γ is an operation which shrinks all internal edges of γ to a single point, leaving untouched the external structure. The result is the *cograph* Γ/γ.

Examples of contractions are:



(4.6)

At this point consider the graph in the main example (2.75).
Retaining the same notation, we know all the subgraphs γ_i with $i = 1, \dots, 5$ are divergent in $d = 6$, where indeed by (4.5) one finds $D(\gamma_i) = 0$.
Next we make use of the contraction to give some other definition for the forests of a graph and give some example on (2.75).

Definition 4.2.5. A 1PI graph that is void of any (superficially) divergent proper 1PI subgraph is said to be a *primitive* graph.

Definition 4.2.6. A forest $\mathcal{F}$ of a graph Γ is a collection of nonoverlapping divergent subgraphs $\gamma \subseteq \Gamma$, that is for any two subgraphs $\gamma, \gamma' \in \mathcal{F}$ one of the following conditions holds:

$$\gamma \subset \gamma' \qquad \gamma' \subset \gamma \qquad \gamma \cap \gamma' = \emptyset$$

[4]That is the number of cycles in the graph. On a pure graph-theoretical level this is its first Betti number. See the appendix B.

Definition 4.2.7. A forest $\mathcal{F}$ which does not contain the full graph Γ and the empty set $\emptyset$ are called *normal* forests. A normal forest $\mathcal{F}$ of a graph Γ is *maximal* if the cograph $\Gamma/\mathcal{F} := \Gamma/\cup_{\gamma\in\mathcal{F}}\gamma$ is primitive.

Definition 4.2.8. A maximal forest $\mathcal{F}$ of a graph Γ is *complete* if any $\gamma \in \mathcal{F}$ is either primitive or possesses a proper subgraph $\gamma' \in \mathcal{F}$ such that γ/γ' is primitive.

By these definitions we see:

- The primitive subgraphs of Γ are γ_1 and γ_2
- The maximal forests of Γ are $\{\gamma_3\}, \{\gamma_1, \gamma_3\}, \{\gamma_4\}, \{\gamma_2, \gamma_4\}, \{\gamma_5\} \equiv \{\gamma_1, \gamma_2\}$
- The complete forests of Γ are $\{\gamma_1, \gamma_3\}, \{\gamma_2, \gamma_4\}, \{\gamma_1, \gamma_2\}$

Contraction is also useful in implementing renormalization on graphs. In fact the subdivergences subtraction in the BPHZ procedure can be re-expressed in a rather convenient way:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma\subsetneq\Gamma} C(\gamma)U(\Gamma/\gamma) \tag{4.7}$$

where again $C(\gamma)$ is defined inductively as in (2.66).
Considering for example the graphs (2.56), in that notation we have:

$$\begin{aligned}\bar{R}(\Gamma_2) &= U(\Gamma_2) - T(\Sigma_1)U(\Gamma_2/\Sigma_1)\\ \bar{R}(\Gamma_4) &= U(\Gamma_4) - T(\gamma_3)U(\Gamma_4/\gamma_3) - T(\gamma_4)U(\Gamma_4/\gamma_4)\end{aligned} \tag{4.8}$$

4.3 Hopf Algebraic Renormalization

4.3.1 Hopf Algebra of Feynman Graphs

At this point we can put the puzzle pieces together.
First we need to endow the set of Feynman graphs with a Hopf algebra structure. Following [CM] we begin by describing the "discrete part" consisting only of divergent (sub)graphs and ignoring the external structure, which we shall discuss afterwards.

Definition 4.3.1. The algebra of Feynman graphs $\mathcal{H}$ is the (bi)graded free commutative algebra over $\mathbb{C}$ generated by 1PI graphs.

We know the vector space $\mathcal{H}$ of all Feynman graphs has a basis labelled by graphs Γ which are disjoint unions of 1PI graphs γ_j:

$$\Gamma = \bigcup_{j=1}^{n} \gamma_j \tag{4.9}$$

where the case $\Gamma = \emptyset$ is allowed.
One defines the multiplication in $\mathcal{H}$ on the generators as disjoint union:

$$m(\Gamma_1 \otimes \Gamma_2) \equiv \Gamma_1 \cup \Gamma_2 \equiv \Gamma_1 \cdot \Gamma_2 \tag{4.10}$$

It is manifestly bilinear, associative and moreover commutative. The empty set $\Gamma = \emptyset$ is the unit of the algebra, therefore we shall gladly set the notation $1 \equiv \emptyset$. A natural grading is induced on $\mathcal{H}$ by the loop number $\ell(\Gamma)$. One also has two other gradings by the (internal) line number $i(\Gamma) = |\Gamma^{(1)}_{int}|$ and the vertex number $v(\Gamma) = |\Gamma^{(0)}| - 1$. They satisfy:

$$deg(\Gamma_1 \cdots \Gamma_r) = \sum_i L(\Gamma_i) \qquad deg(1) = 0 \qquad \ell = i - v \tag{4.11}$$

hence the algebra $\mathcal{H}$ is in general bigraded. In general, the loop number ℓ renders the algebra graded connected, i.e. $\mathcal{H}_0 = \mathbb{C}1$, but the graded components $\mathcal{H}_n$ are not finite-dimensional. With the vertex number v one has the opposite. Endowed with the line number i the algebra is both.

Since we have a grading, the coalgebra structure to be putted on must ensure this is respected. This means that the constructed Hopf algebra has to satisfy the relations in (3.15). Ultimately one needs to define an appropriate coproduct, from which the rest follows.

Definition 4.3.2. Let $\mathcal{F}(\Gamma)$ denote the complete forests of a 1PI graph Γ. The coproduct is defined as:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\substack{F \in \mathcal{F}(\Gamma) \\ \gamma \in F}} \gamma \otimes \Gamma/\gamma \tag{4.12}$$

Observe that the coproduct reflects the comibnatorics of the BPHZ formalism.

Theorem 4.3. *(Connes-Kreimer)*
Equipped with the coproduct (4.12) and the counit and the antipode defined below, the algebra of Feynman graphs $\mathcal{H}$ is a (bigraded) Hopf algebra of the form (3.15).

Proof. First one needs to check that Δ is an algebra morphism in the sense of the definition (3.2), i.e. that:

$$M \circ (\Delta \otimes \Delta) = \Delta \circ m \qquad\qquad M \equiv m \otimes m \circ (id \otimes \tau \otimes id)$$

where the multiplication over the tensor algebra $M : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is expressed in terms of the multiplication (4.10) and the transposition map.
Applying both sides on any given general 1PI graphs Γ_1, Γ_2, the identity follows observing that the subgraphs $\gamma \subset \Gamma_1 \cdot \Gamma_2$ in the complete forests of the product graph $\mathcal{F} \in \mathcal{F}(\Gamma_1 \cdot \Gamma_2)$ can all be factorized in terms of the (complete) forests of the single graphs $\mathcal{F}_1 \in \mathcal{F}(\Gamma_1)$ and $\mathcal{F}_2 \in \mathcal{F}(\Gamma_2)$ simply as $\gamma = \gamma_1 \cdot \gamma_2$
Here each subgraph $\gamma_i \subsetneq \Gamma_i$ runs over the complete forests $\gamma_i \in \mathcal{F}_i \in \mathcal{F}(\Gamma_i)$.
However we must also take into account that $\Gamma_i \subsetneq \Gamma_1 \cdot \Gamma_2$, extending the factorization to the cases $\gamma_i = \Gamma_i$ and $\gamma_i = \emptyset$

Next one has to prove that Δ is coassociative. Since it is well defined as algebra morphism, we shall just check the identity on 1PI graphs Γ:

$$(\Delta \otimes id)\Delta(\Gamma) = (id \otimes \Delta)\Delta(\Gamma)$$

After expanding both sides, most terms cancel each other out and we are left with:

$$\sum_{\substack{\gamma \in \mathcal{F}(\Gamma) \\ \varsigma \in \mathcal{F}(\gamma)}} \varsigma \otimes \gamma / \varsigma \otimes \Gamma / \gamma = \sum_{\substack{\gamma \in \mathcal{F}(\Gamma) \\ \tau \in \mathcal{F}(\Gamma/\gamma)}} \gamma \otimes \tau \otimes (\Gamma/\gamma)/\tau \tag{4.13}$$

Now suppose Γ admits m complete forests $\mathcal{F}_j(\Gamma) = \{\gamma_{j_1}, \dots, \gamma_{j_n}\}$.
By definition any γ_{j_k} is a divergent 1PI subgraph of Γ which is either primitive or it possesses a subgraph $\gamma_{j_\ell} \subsetneq \gamma_{j_k}$ contained in the same forest and such that the cograph $\gamma_{j_k}/\gamma_{j_\ell}$ is primitive.
For any fixed γ_{j_k} its complete forests are subsets of some $\mathcal{F}_j(\Gamma)$, i.e. they are of the form $\{\gamma_{j_1}, \dots, \gamma_{j_h}\}$ with $h < m$. This allows us to rewrite the left hand side of (4.13) as:

$$\sum_{\substack{j=1 \\ \gamma_{j_\ell} \subsetneq \gamma_{j_k}}}^{m} \gamma_{j_\ell} \otimes \gamma_{j_k}/\gamma_{j_\ell} \otimes \Gamma/\gamma_{j_k}$$

Retaining the same notation, for the cograph Γ/γ_{j_k} we work out the $r \leq m$ complete forests $\mathcal{F}_i(\Gamma/\gamma_{j_k}) = \{\tau_{i_1}, \dots, \tau_{i_s}\}$, where we put $\tau_{i_t} \equiv (\Gamma/\gamma_{j_k})_{i_t}$ for simplicity. Each of these terms is a proper divergent 1PI subgraph of Γ/γ_{j_k} thus we generally assume that the cograph's complete forests can be obtained from

those of Γ by contracting all of their graphs with γ_{j_k}:

$$F_i(\Gamma/\gamma_{j_k}) = F_i(\Gamma)/\gamma_{j_k} \equiv \begin{cases} 0 & \text{if} \quad \gamma_{j_k} \notin F_i \\ \bigcup_\ell \chi_{i_\ell} & \text{if} \quad \gamma_{j_k} \in F_i \end{cases}$$

Observe that each γ_{j_k} is defined as element of $F_j(\Gamma)$, so that we need to consider only the associated forest:

$$F_j(\Gamma/\gamma_{j_k}) = \bigcup_\ell \chi_{j\ell} \qquad\qquad \chi_{j\ell} = \begin{cases} \gamma_{j_\ell}/\gamma_{j_k} & \text{if} \quad \gamma_{j_k} \subsetneq \gamma_{j_\ell} \\ \gamma_{j_\ell} & \text{otherwise} \end{cases}$$

The upshot is the right hand side of (4.13) becomes:

$$\sum_{\substack{j=1 \\ \gamma_{j_k} \subsetneq \gamma_{j_\ell}}}^{m} \gamma_{j_k} \otimes \chi_{j\ell} \otimes (\Gamma/\gamma_{j_k})/\chi_{j\ell}$$

Notice that the involved summation forces the form of $\chi_{j\ell}$ and moreover one has the equivalence $(\Gamma/\gamma_{j_k})/(\gamma_{j_\ell}/\gamma_{j_k}) = \Gamma/\gamma_{j_\ell}$. This implies the (4.13) can be finally expressed as:

$$\sum_{\substack{j=1 \\ \gamma_{j_\ell} \subsetneq \gamma_{j_k}}}^{m} \gamma_{j_\ell} \otimes \gamma_{j_k}/\gamma_{j_\ell} \otimes \Gamma/\gamma_{j_k} = \sum_{\substack{j=1 \\ \gamma_{j_k} \subsetneq \gamma_{j_\ell}}}^{m} \gamma_{j_k} \otimes \gamma_{j_\ell}/\gamma_{j_k} \otimes \Gamma/\gamma_{j_\ell}$$

which are indeed identical.
This coproduct clearly has the right grading property. Hence defining the counit by $\varepsilon(\emptyset) = 1$ and $\varepsilon(\Gamma) = 0$ the antipode is completely determined from (3.7). For instance the first equation therein gives on Γ:

$$m \circ \left[S(\Gamma) \otimes 1 + S(1) \otimes \Gamma + \sum_{\substack{F \in \mathcal{F}(\Gamma) \\ \gamma \in F}} S(\gamma) \otimes \Gamma/\gamma \right] = 0$$

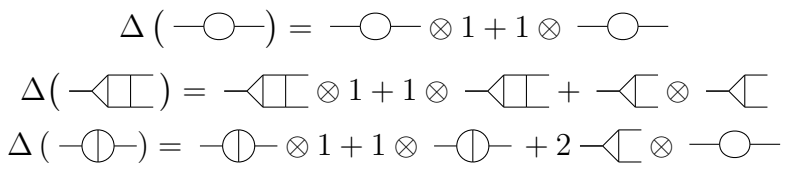
which is resolved by induction using the formula[5]:

$$S(\emptyset) = \emptyset \qquad\qquad S(\Gamma) = -\Gamma - \sum_{\substack{F \in \mathcal{F}(\Gamma) \\ \gamma \in F}} S(\gamma) \cdot \Gamma/\gamma \tag{4.14}$$

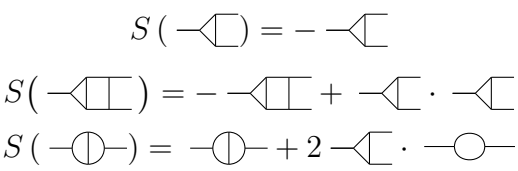
□

[5]Recall the notation $1 \equiv \emptyset$.

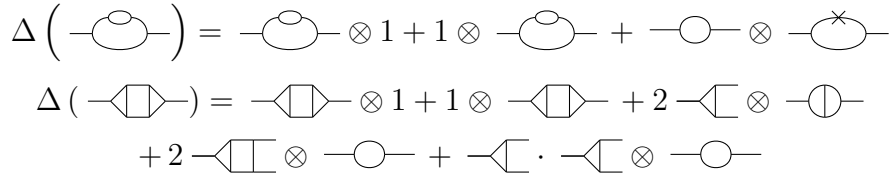
Let's give some explicit computations in ϕ^3 theory. The coproduct gives:



For the antipode we have:



Some more elaborated example for the coproduct:



We left behind the external structure of a general graph $\Gamma(p_1, \ldots, p_N)$. This can be seen as a distribution $\sigma \in \mathcal{C}_c^{-\infty}(E_\Gamma)$, that is the dual of the space of smooth functions $\mathcal{C}^\infty(E_\Gamma)$ where $E_\Gamma := \{(p_i)_{i=1,\ldots,N} : \sum p_i = 0\}$ is the set of possible external momenta subject to the conservation law. It generalizes as follow:

$$E := \bigcup_{\Gamma \in 1PI} E_\Gamma \qquad \mathcal{C}_c^{-\infty}(E) = \bigoplus_{\Gamma \in 1PI} \mathcal{C}_c^{-\infty}(E_\Gamma) \tag{4.15}$$

Thus the "full" Hopf algebra of Feynman graphs is the symmetric algebra $Sym(\mathcal{C}_c^{-\infty}(E))$ and is constructed as a direct extension of $\mathcal{H}$.

4.3.2 Renormalization of Characters

We now return to the discrete part of the Hopf algebra of Feynman graphs $\mathcal{H}$ and consider the group of characters $G_\mathcal{H}(\mathcal{K}) = Hom_{\mathcal{A}_\mathbb{C}}(\mathcal{H}, \mathcal{K})$. The intention is to interpret Feynman rules as characters from our Hopf algebra to Laurent series and accordingly recover the BPHZ procedure as a Birkhoff factorization in this group.

We start by proving the general result:

Theorem 4.4. *(Connes-Kreimer)*
Let H be a positively graded connected commutative Hopf algebra. Then every Hopf character $\phi \in G_H(\mathcal{K}) = Hom_{\mathcal{A}_{\mathbb{C}}}(H, \mathcal{K})$ admits a unique Birkhoff factorization as in (3.28), that is it can be written in the form:

$$\phi = (\phi_- \circ S) * \phi_+$$

Proof. All we need to do is to recollect some results from the previous chapters. The recursive formula (3.21) defines Hopf characters in any commutative algebra A, since by the definitions of convolution product (3.8) and of Rota-Baxter operators (3.20) they are indeed elements in the group of characters $G_H(A)$.
Upon the identification $\phi_- \equiv S^{\phi}_{\mathcal{R}}$ we have from (3.24):

$$\phi_- * \phi = \phi_- + \phi_- * (\phi - \phi \circ \eta \circ \varepsilon)$$

Notice that $\phi \circ \eta \circ \varepsilon = e$ the neutral element. We multiply from the left by the group inverse of ϕ_- (cf. 3.22) to obtain:

$$\phi = (\phi_- \circ S) * \phi_- * \phi$$

Finally the identification $\phi_+ \equiv \phi_- * \phi$ gives exactly (3.28).
Therefore making an opportune choice for the operator $\mathcal{R}$ one can fulfill the conditions:

$$A = \mathcal{K} \qquad \phi_+ \in G_H(\mathcal{O}) \qquad \phi_- \in G_H(\mathcal{Q}) \qquad \varepsilon_- \circ \phi_- = \varepsilon$$

□

This result also provides an explicit recursive formula for the factorization. Hence by applying this result to the Hopf algebra of Feynman graphs, one can reconnect to renormalization by following the steps we outlined, thus reconstructing the BPHZ procedure.

Theorem 4.5. *(Connes-Kreimer)*
Let $\mathcal{H}$ be the discrete Hopf algebra of Feynman graphs. Then the expressions for the Birkhoff factorization of loops $\phi \in G_{\mathcal{H}}(\mathcal{K}) = Hom_{\mathcal{A}_{\mathbb{C}}}(\mathcal{H}, \mathcal{K})$ are given by the recursive formula (2.64) for the BPHZ procedure:

$$R(\Gamma) = \bar{R}(\Gamma) - T \circ \bar{R}(\Gamma)$$

Proof. The starting point is to observe that the T operation previously defined (2.61) is a Rota-Baxter operator in the sense of the definition (3.20). Indeed it is a endomorphism on $\mathcal{K}$ viewed as a $\mathbb{C}-$algebra which in virtue of (2.72) respects the defining property.
Provided such an operator we then consider a character $U \in G_H(\mathcal{K})$ and as in (3.21) we define the linear map $S_T^U : \mathcal{H} \to \mathcal{K}$ recursively by setting $S_T^U(\emptyset) := 1$ and:

$$S_T^U(\Gamma) = -T \circ \left[(S_T^U * (U \circ \mathcal{P}))(\Gamma) \right] = -T \circ \left[U(\Gamma) + \sum_{\substack{F \in \mathcal{F}(\Gamma) \\ \gamma \in F}} S_T^U(\Gamma) U(\Gamma/\gamma) \right] \tag{4.16}$$

where the projector is such that $\mathcal{P}(\emptyset) = 0$ and $\mathcal{P}(\Gamma) = \Gamma$, whilst the convolution product is as (3.8) in terms of the coproduct (4.12) of $\mathcal{H}$.
The get the conclusion one just has to make the identifications:

$$\bar{R}(\Gamma) \equiv (S_T^U * (U \circ \mathcal{P}))(\Gamma) \qquad\qquad R(\Gamma) \equiv (S_T^U * U)(\Gamma) \tag{4.17}$$

This returns exactly the expression (2.64) □

The theorem can be extended to involve the external structure of $\mathcal{H}$. More details can be found in [CM] .

4.3.3 Further Developments

This is far from the end of the story.
The Connes-Kreimer theory is a wide and currently active framework in the context of noncommutative geometry. Its principal aims address towards a depeer understanding of the mathematical structures underlying perturbative approach to quantum field theory and field theories in general. The arguments we treated in the simplest nontrivial case can be expanded and generalized on both the geometrical and the algebraic sides, in order to cover other major topics of phyisical interest. Above all the Hopf algebra of Feynman graphs can be constructed for gauge theories [K06], enlightening the fact that symmetries can be understood as an Hopf ideal [vS].

Particularly newsworthy is a consequence of the Milnor-Moore theorem, which states that connected graded cocommutative Hopf algebras such as the one we discussed is isomorphic to the dual of the universal enveloping algebra of the graded Lie algebra built up on its primitive elements. [MM]

Beyond its pure mathematical glamor, this allows to highligt the Lie algebra structure on the group of Hopf characters, where the graphs are equipped with the Lie bracket respect to a precisely defined insertion operation of subgraphs. Then another important result of the Connes-Kreimer theory shows that this can be related to the renormalization group flow of the theory [CK], [CM].

Ultimately it provides a far reaching and striking connection between the mathematical and physical perspective on perturbative quantum field theories and renormalization which helps to demystify some clumsy machinery usually performed.

Appendix A

Some Calculations

This appendix is devoted to some standard manipulation in quantum field theory. Further details can be found in the references, such as [C], [PS], [Wei].

A.1 One-loop Self-energy

Here we want to reformulate the integral (2.6) introducing the lattice regulator $\varkappa = \frac{a^2\mu^2}{1+a^2\mu^2}$. The overall aim is to show the following:

$$\frac{ig^2}{2(2\pi)^d}\int \frac{1}{(k^2-\mu^2)^2}\,d^dk = \frac{g^2\mu^{d-4}}{2(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_1^{\varkappa}\left(\frac{1-y}{y}\right)^{\frac{d}{2}-1}dy \tag{A.1}$$

One starts with the Wick rotation[1] $k^0 = i\ell^0$

$$\frac{ig^2}{2(2\pi)^d}\int \frac{1}{(k^2-\mu^2)^2}\,d^dk = -\frac{g^2}{2(2\pi)^d}\int \frac{1}{(\ell^2+\mu^2)^2}\,d^d\ell$$

We rewrite this in terms of the unitary $d-$sphere:

$$-\frac{g^2}{2(2\pi)^d}\int d\Omega_d\int_0^{1/a}\frac{\ell^{d-1}}{(\ell^2+\mu^2)^2}\,d\ell = -\frac{g^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^{1/a}\frac{\ell^{d-1}}{(\ell^2+\mu^2)^2}\,d\ell$$

and then consider the square $d(\ell^2) = 2\ell d\ell$:

$$-\frac{g^2}{2(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^{1/a^2}\frac{(\ell^2)^{\frac{d}{2}-1}}{(\ell^2+\mu^2)^2}\,d(\ell^2)$$

At this point one makes a change of variables:

$$y = \frac{\mu^2}{\ell^2+\mu^2}, \qquad \ell^2 = \mu^2\left(\frac{1}{y}-1\right), \qquad d(\ell^2) = -\frac{\mu^2}{y^2}dy$$

[1] This trick is somehow subtler than it may seems, but we won't linger on this, referring to the literature for any clarification.

After substituting, this simplifies as:

$$-\frac{g^2}{2(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_1^{\varkappa}\frac{\mu^{d-2}\left(\frac{1}{y}-1\right)^{\frac{d}{2}-1}}{\left(\frac{\mu^2}{y}\right)^2}\left(-\frac{\mu^2}{y^2}\right)\,dy=\frac{g^2\mu^{d-4}}{2(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_1^{\varkappa}\left(\frac{1-y}{y}\right)^{\frac{d}{2}-1}\,dy$$

which indeed gives the $(A.1)$.

A.1.1 Some Results

Varying the dimension d one may seek for patterns...

$d=1$

$$\Sigma_1=-\frac{g^2\mu^{-3}}{4\pi}\left(\frac{a\mu}{1+a^2\mu^2}+\tan^{-1}\left(\frac{1}{a\mu}\right)\right)\to-\frac{g^2\mu^{-3}}{8}$$

$d=2$

$$\Sigma_1=-\frac{g^2\mu^{-2}}{8\pi}\frac{1}{(1+a^2\mu^2)}\to-\frac{g^2\mu^{-2}}{8\pi}$$

$d=3$

$$\Sigma_1=\frac{g^2\mu^{-1}}{8\pi^2}\left(\frac{a\mu}{1+a^2\mu^2}-\tan^{-1}\left(\frac{1}{a\mu}\right)\right)\to-\frac{g^2\mu^{-1}}{16\pi}$$

$d=4$

$$\Sigma_1\approx-\frac{g^2}{16\pi^2}\ln\frac{1}{a}+finite\qquad\text{as } a\to 0$$

$d=5$

$$\Sigma_1\approx-\frac{g^2}{24\pi^3}\frac{1}{a}+finite\qquad\text{as } a\to 0$$

$d=6$

$$\Sigma_1\approx-\frac{g^2}{256\pi^3}\frac{1}{a^2}+\frac{g^2\mu^2}{64\pi^3}\ln\frac{1}{a}+finite\qquad\text{as } a\to 0$$

$d=7$

$$\Sigma_1\approx-\frac{g^2}{720\pi^4}\frac{1}{a^2}+\frac{g^2\mu^2}{120\pi^4}\frac{1}{a}+finite\qquad\text{as } a\to 0$$

$d=8$

$$\Sigma_1\approx-\frac{g^2}{6144\pi^4}\frac{1}{a^4}+\frac{g^2\mu^2}{1536\pi^4}\frac{1}{a^2}-\frac{g^2\mu^4}{512\pi^4}\ln\frac{1}{a}+finite\qquad\text{as } a\to 0$$

$d=9$

$$\Sigma_1\approx-\frac{g^2}{16800\pi^5}\frac{1}{a^5}+\frac{g^2\mu^2}{5040\pi^5}\frac{1}{a^3}-\frac{g^2\mu^4}{1120\pi^5}\frac{1}{a}+finite\qquad\text{as } a\to 0$$

$d=10$

$$\Sigma_1\approx-\frac{g^2}{2^{14}9\pi^5}\frac{1}{a^6}+\frac{g^2\mu^2}{2^{14}3\pi^5}\frac{1}{a^4}-\frac{g^2\mu^4}{2^{14}\pi^5}\frac{1}{a^2}+\frac{g^2\mu^6}{2^{11}3\pi^5}\ln\frac{1}{a}+finite\qquad\text{as } a\to 0$$

A.2 Renormalized Integral

Here we want to manipulate (2.3) following the insight provided by (2.16). Then imposing (2.15) one can recover (2.19). From (2.17), (2.18):

$$\frac{\partial \Sigma_{1R}}{\partial p^2} = \frac{p^\mu}{2p^2}\frac{\partial \Sigma_1}{\partial p^\mu} = -\frac{ig^2}{2(2\pi)^d p^2}\int \frac{p\cdot(p+k)}{(k^2-m^2)((p+k)^2-m^2)^2}d^dk \tag{A.2}$$

First we introduce the Feynman parameters:

$$\frac{1}{(k^2-m^2)((p+k)^2-m^2)^2} = \iint_0^1 \frac{2y\,\delta(x+y-1)}{(xk^2-xm^2+y(p+k)^2-ym^2)^3}dx\,dy$$

The delta functions sets $y = 1-x$.

$$-\frac{ig^2}{(2\pi)^d p^2}\int_0^1\int \frac{(1-x)(p^2+p\cdot k)}{((p+k)^2-2x(p\cdot k)-xp^2-m^2)^3}dx\,d^dk$$

One makes the change of variables:

$$q = k+(1-x)p, \qquad p\cdot k = p\cdot q-(1-x)p^2, \qquad d^dq = d^dk$$

then we substitute and simplify:

$$-\frac{ig^2}{(2\pi)^d p^2}\int_0^1\int \frac{(1-x)(xp^2+p\cdot q)}{(q^2+x(1-x)p^2-m^2)^3}dx\,d^dq$$

Antisymmetrizing, the term $\propto q^\mu$ gives no contribution. Hence:

$$-\frac{ig^2}{(2\pi)^d}\int_0^1 x(1-x)\int \frac{1}{(q^2+x(1-x)p^2-m^2)^3}d^dq\,dx$$

Wick rotating $q^0 = i\ell^0$, we pose $\Theta = -x(1-x)p^2+m^2$:

$$-\frac{g^2}{(2\pi)^d}\int_0^1 x(1-x)\int \frac{1}{(\ell^2+\Theta)^3}d^d\ell\,dx$$

Going to the unitary $d-$sphere:

$$-\frac{g^2}{(2\pi)^d}\int_0^1 x(1-x)\int d\Omega_d\int_0^\infty \frac{\ell^{d-1}}{(\ell^2+\Theta)^3}d\ell\,dx = -\frac{2g^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^1 x(1-x)\int_0^\infty \frac{\ell^{d-1}}{(\ell^2+\Theta)^3}d\ell\,dx$$

We then shift to the square $d(\ell^2) = 2\ell d\ell$:

$$-\frac{g^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^1 x(1-x)\int_0^\infty \frac{(\ell^2)^{\frac{d}{2}-1}}{(\ell^2+\Theta)^3}d(\ell^2)\,dx$$

and make another change of variables:

$$z = \frac{\Theta}{\ell^2+\Theta}, \qquad \ell^2 = \Theta\left(\frac{1}{z}-1\right), \qquad d(\ell^2) = -\frac{\Theta}{z^2}dz$$

Substituting, simplifying:

$$-\frac{g^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^1 x(1-x)\int_1^0 \frac{\Theta^{\frac{d}{2}-1}\left(\frac{1}{z}-1\right)^{\frac{d}{2}-1}}{(\frac{\Theta}{z})^3}\left(-\frac{\Theta}{z^2}\right)dz\,dx =$$

$$= -\frac{g^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^1 x(1-x)\Theta^{\frac{d}{2}-3}\int_0^1 (1-z)^{\frac{d}{2}-1}z^{2-\frac{d}{2}}\,dz\,dx$$

We recognize the Euler's Beta function and make use of its properties:

$$\int_0^1 (1-z)^{\frac{d}{2}-1}z^{2-\frac{d}{2}}\,dz = \frac{\Gamma(3-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(3)}$$

Therefore, making Θ explicit, we get:

$$-\frac{g^2}{2(4\pi)^{\frac{d}{2}}}\Gamma(3-\frac{d}{2})\int_0^1 \frac{x(1-x)}{(-x(1-x)p^2+m^2)^{3-\frac{d}{2}}}dx$$

but this can be written as:

$$-\frac{g^2}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(3-\frac{d}{2})}{4-d}\frac{\partial}{\partial p^2}\int_0^1 \left(-x(1-x)p^2+m^2\right)^{\frac{d}{2}-2}dx \qquad d\neq 4$$

$$\frac{g^2}{32\pi^2}\frac{\partial}{\partial p^2}\int_0^1 \ln(-x(1-x)p^2+m^2)\,dx \qquad d=4$$

Thus we get the equivalence with $(A.2)$ up to constants of integration.

A.3 DimReg Self-Energy

We give a prototipical example of dimensional regularization working with the one-loop self-energy (2.3). Some motivation are given before (2.34) together with some assumption. We also assume the validity of the "generalized Gaussian integral" with non-integer d:

$$\int e^{k^2}\, d^d k = i \int e^{-\omega^2 - k^2}\, d^{d-1} k d\omega = i\pi^{\frac{d}{2}} \tag{A.3}$$

then we introduce Schwinger representation for each propagator:

$$\frac{1}{(m^2-k^2)}\frac{1}{(m^2-(p+k)^2)} = \int_0^\infty \int_0^\infty e^{-a(m^2-k^2)} e^{-b(m^2-(p+k)^2)}\, da\, db$$

Observe that because of the Wick rotation we treat k^2 as negative. Then we exchange the order of integration to obtain:

$$\Sigma_1 = \frac{ig^2}{2(2\pi)^d} \int_0^\infty \int_0^\infty \int e^{-(a+b)m^2 + bp^2 + 2bp\cdot k + (a+b)k^2}\, d^d k\, da\, db$$

We shift k^μ by an amount $-\dfrac{p^\mu b}{a+b}$ and change variables to $z = a+b$, $x = \frac{a}{z}$:

$$\Sigma_1 = \frac{ig^2}{2(2\pi)^d} \int_0^1 \int_0^\infty z \int e^{-z(m^2 - p^2 x(1-x)) + zk^2}\, d^d k\, dz\, dx$$

After scaling k by a factor $z^{\frac{1}{2}}$ we find that:

$$\Sigma_1 = \frac{ig^2}{2(2\pi)^d} \int_0^1 \int_0^\infty z^{1-\frac{d}{2}} e^{-z(m^2 - p^2 x(1-x))} \int e^{k^2}\, d^d k\, dz\, dx$$

The integral in $d^d k$ is of the form $(A.3)$:

$$\Sigma_1 = -\frac{g^2}{2(4\pi)^{\frac{d}{2}}} \int_0^1 \int_0^\infty z^{1-\frac{d}{2}} e^{-z(m^2 - p^2 x(1-x))}\, dz\, dx$$

The integral in dz can be rearranged as a Gamma function with $z\xi = t$:

$$\xi^{\frac{d}{2}-2} \int_0^\infty t^{1-\frac{d}{2}} e^{-t}\, dt, \qquad \xi = (m^2 - p^2 x(1-x))$$

Finally:

$$\Sigma_1 = -\frac{g^2}{2(4\pi)^{\frac{d}{2}}} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 (m^2 - p^2 x(1-x))^{\frac{d}{2}-2}\, dx$$

A.4 DimReg Vertex

We follow the same procedure of $A.3$ for the one-loop vertex (2.50):

$$\mathcal{V}_1 = i\frac{(-ig)^3}{2(2\pi)^d}\int \frac{i}{(k^2-m^2)}\frac{i}{((p+k)^2-m^2)}\frac{i}{((q+k)^2-m^2)}\,d^dk$$

Taking account of Wick rotation, we rewrite:

$$\mathcal{V}_1 = -\frac{ig^3}{2(2\pi)^d}\int \frac{1}{(m^2-k^2)}\frac{1}{(m^2-(p+k)^2)}\frac{1}{(m^2-(q+k)^2)}\,d^dk$$

In Schwinger representation:

$$\mathcal{V}_1 = -\frac{ig^3}{2(2\pi)^d}\int_0^\infty\int_0^\infty\int_0^\infty\int e^{-(a+b+c)m^2+bp^2+2bp\cdot k+cq^2+2cq\cdot k+(a+b+c)k^2}\,d^dk\,da\,db\,dc$$

At this point we shift $k^\mu \to k^\mu - \dfrac{p^\mu b}{a+b+c} - \dfrac{q^\mu c}{a+b+c}$:

$$\mathcal{V}_1 = -\frac{ig^3}{2(2\pi)^d}\int_0^\infty\int_0^\infty\int_0^\infty\int e^{-(a+b+c)m^2+bp^2+cq^2+(a+b+c)k^2-\frac{(bp+cq)^2}{(a+b+c)^2}}\,d^dk\,da\,db\,dc$$

and make a change of variables $z = a+b+c+$, $x = \frac{a+c}{z}$, $y = \frac{a+b}{z}$:

$$\mathcal{V}_1 = -\frac{ig^3}{2(2\pi)^d}\int_0^1\int_0^1\int_0^\infty z^2\int e^{-z\eta+zk^2}\,d^dk\,dz\,dx\,dy$$

where: $\eta = m^2-(1-x)p^2-(1-y)q^2+\big((1-x)p+(1-y)q\big)^2$
As before, we scale k of a factor $z^{\frac{1}{2}}$ making the d^dk-integral of the form $(A.3)$:

$$\mathcal{V}_1 = \frac{g^3}{2(4\pi)^{\frac{d}{2}}}\int_0^1\int_0^1\int_0^\infty z^{2-\frac{d}{2}}e^{-z\eta}\,dz\,dx\,dy$$

Again we put $z\eta = t$ and recognize a Gamma function:

$$\int_0^\infty z^{2-\frac{d}{2}}e^{-z\eta}\,dz = \eta^{\frac{d}{2}-3}\int_0^1 t^{2-\frac{d}{2}}e^{-t}\,dt = \eta^{\frac{d}{2}-3}\Gamma\left(3-\frac{d}{2}\right)$$

Substituting:

$$\mathcal{V}_1 = \frac{g^3}{2(4\pi)^{\frac{d}{2}}}\Gamma\left(3-\frac{d}{2}\right)\int_0^1\int_0^1\eta^{\frac{d}{2}-3}\,dx\,dy$$

Since we are interested in the limit $d \to 6$, we rewrite as:

$$\mathcal{V}_1 = \frac{g^3}{128\pi^3}(4\pi\mu^3)^{3-\frac{d}{2}}\Gamma\left(3-\frac{d}{2}\right)\int_0^1\int_0^1 \eta^{\frac{d}{2}-3}\,dx\,dy$$

thus asymptotically:

$$\mathcal{V}_1 \approx \frac{2}{6-d} - \gamma_E + \ln 4\pi\mu^3 - \int_0^1\int_0^1 \ln\eta\,dx\,dy$$

and the counterterm is given by:

$$\delta g = \mu^{3-\frac{d}{2}}(pole) = \mu^{3-d/2}\left(\frac{g^3}{64\pi^3(d-6)}\right) + O(g^4)$$

Appendix B

On Graph Theory

In this appendix we give a series of formal definitions for standard graph-theoretical notions we used through this work, without any claim to completeness. For details we refer to some Graph theory textbook, for instance [Die].

B.1 Basic Terminology

Definition B.1.1. A graph is a pair (V, E), where V and E are finite sets, whose elements we call vertices and edges respectively.

Definition B.1.2. A undirected graph is a tuple (V, E, r) consisting of a graph (V, E) and a map $r : E \to \{\{x, y\} : x, y \in V\}$ assigning to each edge $e \in E$ its unordered pair of endpoints. An elements of E is called a loop if $r(e)$ is a single point.

Remark. We represent a graph pictorially by drawing a point for each vertex and an arc connecting points x and y for each $e \in E$ such that $r(e) = \{x, y\}$.

Definition B.1.3. A directed graph is a tuple (V, E, s, t) where (V, E) is a graph and $s, t : E \to V$ are two maps called source and target maps respectively.

Remark. Pictorially we draw an arrow from x to y for each edge $e \in E$ such that $s(e) = t(e)$.

Definition B.1.4. In a graph (risp. directe graph), a walk (risp. directed walk) is a sequence $v_0 e_0 v_1 e_1 \dots v_k e_k v_{k+1}$ of vertices $v_i \in V$ and edges $e_i \in E$ such that one has $r(e_i) = \{v_i, v_{i+1}\}$ for all $0 \leq i \leq k$ (risp. $s(e_i) = v_i$ and $t(e_i) = v_{i+i}$).

Definition B.1.5. A graph is said to be connected if for each pair of vertices there is a walk joining them.

Definition B.1.6. A cycle is a walk such that $v_0 = v_{k+1}$ and no other vertices or edges repeated. A loop is a cycle with one only edge.

Remark. Notice that this is what is generally called "loop" in QFT. The different terminology between physics and mathematics is listed at the end of the appendix.

Definition B.1.7. A tree is a connected undirected graph with no cycles, that is any two distinct vertex of the graph are connected exactly by one walk with no repeated edges or vertex.

Remark. Given a tree, it is useful to define the valency of a vertex as the number of edges emitted by that vertex. This allows us to distinguish between external vertex, with valency 1 and internal vertex, with valency ≥ 2.

Definition B.1.8. A rooted tree is a pair (t, r) where t is a tree and r is a specified vertex of t called root. A forest is a disjoint union of trees.

B.2 Topological Graphs

Let X be a topological space.

Definition B.2.1. A parametrized curve in X is a continuous map $\sigma : [0, 1] \to X$; its image $\sigma([0, 1])$ is called arc, where $\sigma(0)$ and $\sigma(1)$ are called endpoints.

Definition B.2.2. A topological graph in X is a pair (V, E) where V is a set of points and E is a set of arcs such that:

$i)$ every $e \in E$ has endpoints in V;

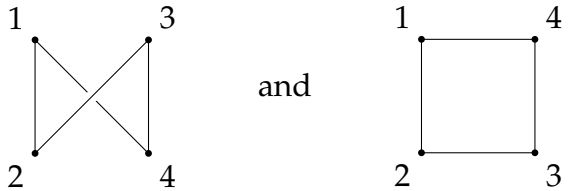
$ii)$ two arcs can only meet at endpoints;

$iii)$ no interior point of an arc belongs to V.

Definition B.2.3. An embedding of a graph (V, E, r) in X is the datum of a topological graph (V', E') in X and a pair of bijections $f : V \to V'$ and $g : E \to E'$ such that f sends $r(e)$ to the endpoints of $g(e)$ for all $e \in E$.

Definition B.2.4. A plane graph is a topological graph in $\mathbb{R}^2$. A planar graph is a graph admitting an embedding in $\mathbb{R}^2$.

Let's look at an example:



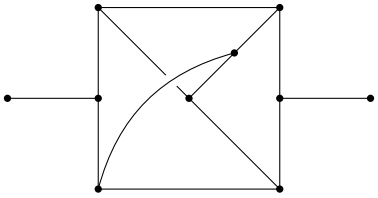
(B.1)

The two drawings represent the same graph, but one is a planar embedding and the other is not. This motivates the introduction of the following.

Definition B.2.5. Two graphs (V, E, r) and (V', E', r') are said to be isomorphic if there exist bijections $f : V \to V'$ and $g : E \to E'$ that preserve the endpoint relations of G and G', i.e. such that $f \circ r = r' \circ g$.

Remark. Graph isomorphism is an equivalence relation, whose equivalence classes are those of isomorphic graphs. Thus we can consider just one representat for each class.

Despite we do not deal with non-planar graphs, one should not forget they also exist. An examples of graph that can not be embedded in $\mathbb{R}^2$ is:



(B.2)

Definition B.2.6. In a plane graph, let F, E, V denote respectively the number of internal faces (regions bounded by internal arcs), internal edges and internal vertices. Then we have the Euler's formula:

$$F - E + V = 1 \tag{B.3}$$

In particular, in a plane graph the number F of internal faces does not depend on the embedding.

Finally, here we have a brief account of the two dictionaries discrepancies:

math	**phys**
loop	bubble
cycle	loop
internal faces	independent loops

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