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# Spectral and Thermal dimension in Quantum Gravity scenarios with UV/IR mixing


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**Natural units adopted (unless otherwise stated)**

$c = 1$

$\hbar = 1$

$k_B = 1$

**Acronyms**

| | |
|---|---|
| QG | Quantum Gravity |
| IR | Infrared |
| UV | Ultraviolet |
| QM | Quantum Mechanics |
| GR | General Relativity |
| QFT | Quantum Field Theory |
| SM | Standard Model |
| NCST | Non Commutative SpaceTime |
| DSR | Doubly Special Relativity |
| MDR | Modified Dispersion Relation |
| G&R | I. S. Gradshteyn and I. M. Ryzhik $8^{\text{th}}$ edition (2014), [2] |

**Computation shortcuts and conventions**

| | |
|---|---|
| $D$ | Topological dimension of spacetime, $D = 1 + d$ |
| $\vec{v}$ | Spatial part of a $D$-vector |
| $p$ | Norm of the spatial part of $D$-momentum, $\lVert \vec{p} \rVert$ |
| $S^{(d)}(1)$ | Surface of the $d$-dimensional sphere with unit radius |

$S_c^{(d)}(1,1)$ Surface of the $d$-dimensional cylinder with unit radius and height

$|a|$ Modulus of $a \in \mathbb{C}$, $\sqrt{a\bar{a}}$. If $a \in \mathbb{R}$ it is thought as $a \in \mathbb{C}$ with $\Im(a) = 0$

$\Re(a)$ Real part of $a \in \mathbb{C}$

$\Im(a)$ Imaginary part of $a \in \mathbb{C}$

$i$ Imaginary unit

$\odot$ Used to indicate multiplication when the equation has to be split on multiple lines

$a \propto b$ Used to indicate that $a$ is equal to $b$ up to numerical factors

$a \approx b$ Used to indicate that $a$ is of the same order of magnitude of $b$

$f(z) \simeq g(z)$ Used to indicate that $g(z)$ is the asymptotic expansion of $f(z)$ when $z \to 0$

$f(z) \sim g(z)$ Used to indicate that $g(z)$ is the asymptotic expansion of $f(z)$ when $z \to \infty$

**Special functions**

$D_\nu(z)$ Parabolic cylinder function of order $\nu$ and argument $z$

$\Gamma(z)$ Euler Gamma function of argument $z$

$K_\nu(z)$ Modified Bessel function of the second kind of order $\nu$ and argument $z$

$\psi_0(z)$ Polygamma function of argument $z$, $\psi_0(z) = \frac{\Gamma'(z)}{\Gamma(z)}$

$\zeta(z)$ Riemann Zeta function of argument $z$

$U(a,b,z)$ Tricomi Confluent Hypergeometric function of argument $z$

${}_pF_q\,(a_1,\dots,a_p;b_1,\dots,b_q;z)$ Generalized Hypergeometric function of argument $z$

$Li_n(z)$ Polylogarithm function of order $n$ and argument $z$.

$B_k$ Bernoulli numbers of second kind, $B_1 = \frac{1}{2}$

**Numerical sets**

$\mathbb{N}$ Natural numbers

$\mathbb{N}_0$ $\mathbb{N} \cup \{0\}$

$\mathbb{R}$ Real numbers

$\mathbb{C}$ Complex numbers

*Don't panic.*

*~ Douglas Adams, The Hitchhiker's Guide to the Galaxy*

# *Introduction*

The aim of the quantum gravity research program is to find a description of the dynamics of spacetime which is consistent with the laws of quantum mechanics. There are several different approaches to this problem [3]. They differ not only with respect to the mathematical frameworks involved but also on the logic and philosophical aspects of their foundations. It seems very difficult to find common ground between two or more of these approaches in this scenario, and it is not even known if one of these approaches is the "correct" one. In this regard, the few properties or predictions that arise from different approaches are very precious because they give us the opportunity to study properties that are more likely to be "true" characteristics of the quantum gravity realm, independently of the particular mathematical structure adopted to describe the theory.

One of these properties is that, at the fundamental level, spacetime is characterized by a "quantum nature", although this can imply very different things depending on the approach. With this in mind, it is no surprise that the (General Relativistic) Riemannian picture of the geometry of spacetime needs to be replaced by something else, able to manage its quantum nature. This implies, for example, that we cannot correctly define the dimension of spacetime as it is usually defined in Riemannian geometry with the topological or the Hausdorff dimension, so we need something to replace this notion.

The major efforts done in the literature to replace the notion of Hausdorff dimension focus on the spectral dimension as a suitable candidate; this geometric quantity is defined by means of a fictitious diffusion process on the spacetime and its definition is strictly linked to another property shared by some quantum gravity models: the deformed (or modified) dispersion law, which means that the spacetime is characterized by a dispersion law that is different from (and tends to, in the classical limit) the Lorentzian one. The reason why such a deformation occurs depends on the details of the various models.

Although the spectral dimension has been widely studied, some authors [26] have pointed out that this notion might be not physical because of its dependence on the off-shell modes and because it relies on the Euclideanization of spacetime and, consequently, of the dispersion law. For this reason the physical notion of thermal dimension has been introduced by the same authors. The thermal dimension and the spectral dimension agree in most cases and they differ only when the unphysical content of the latter plays a significant role. The thermal dimension is defined noting that there are some thermodynamical relations that are sensitive to the dimensionality of spacetime,

such as the Stefan-Boltzmann law. The thermal dimension is then extracted from these relations that are deduced by calculating the partition function for a photon gas, the on-shellness for the photons being enforced via the deformed dispersion law (not its Euclideanized version).
Both the spectral dimension and the thermal dimension depend on the scale of the probes used to investigate the properties of spacetime. This gives rise to the concept of "dimensional flow" which simply means this: the effective dimension of spacetime "felt" by the probes depends on their energy.

When studying Quantum Gravity, the scale which is considered to be the characteristic onset scale for the phenomena is the Planck scale. This scale is a really small length scale (or a really high energy scale), namely $10^{-35}$ $m$ ($10^{19}$ Gev), therefore the effects of the Quantum Gravity phenomena are expected to show up in this extreme ultraviolet regime. However, contrary to what could be expected, there are some cases where the effects of quantum gravitational phenomena are significant not only in the ultraviolet (i.e., high energy) regime, but also in the infrared (i.e., low energy) regime. This characteristic is called "UV/IR mixing". For our purposes the main effect of this property is that it gives rise to modified dispersion relations (MDRs) which are non-trivial both in the ultraviolet (UV) and in the infrared (IR) regimes.

The purpose of this thesis work is to study the IR behavior of the Spectral dimension and the Thermal dimension and to study the interplay between the UV/IR mixing and the dimensional flow: even though these two notions have been around for a while, there are no studies that link them in the quantum gravity literature.

The structure of the thesis is the following: chapter one is devoted to an (informal) introduction to the quantum gravity research program; in chapter two the notions of spectral dimension and thermal dimension are introduced; in chapter three and four the computation of spectral dimension and thermal dimension in different UV/IR mixing scenarios is presented; in chapter five two novel IR effects are considered, namely compactification and horizons; finally in the first appendix there is a proof of a mathematical method adopted in chapter three and the last two appendices give some background on Rindler space and noncommutative spacetime. The chapters from three to five are the bulk of this thesis in which the original material is shown.

*Learn from yesterday, live for today, hope for tomorrow. The important thing is not stop questioning.*

*~ Albert Einstein*

*First Chapter*

# *Why Quantum Gravity?*

The aim of this chapter is to (informally) answer the question: why should we care about Quantum Gravity at all?
The term "Quantum Gravity" (QG) collects under the same umbrella all the different attempts to find a consistent theory that describes the gravitational field with the laws of Quantum Mechanics (QM); given that in General Relativity (GR) the gravitational field is identified with spacetime dynamics, QG is the attempt to find a theory that describes the spacetime dynamics at really small scales and gives the usual GR dynamics at macroscopic scales.

Physicists succeeded in describing the dynamics at microscopic scales for all the (known) interactions and the matter: elementary particles (such as electrons and quarks) and Electromagnetic, Weak and Strong interactions are described together with the Higgs field in the Standard Model of particle physics (SM) and this model works experimentally really well. The framework in which this model works is Quantum Field Theory (QFT): it describes the dynamics of the fields, which in turn describe the interactions and the particles, with the laws of QM. What is the problem in doing the same with gravity? Can't we just repeat the trick?

The answer would seem: not at all. The problem in doing this with gravity is that, while QFT is based on the assumption that the fields live on a static background which may be curved as well giving rise to QFT on curved spacetime, in GR this assumption makes no sense: we want to find the theory that describes the dynamics of spacetime, that is the dynamics of the background itself which is consequently neither static nor, properly speaking, a "background". This is clearly not the only problem, there are several "technical" or mathematical problems, such as non-renormalizability, which I will not present here. There are also several somewhat "philosophical" issues when we stand in the common ground between the two theories, with "one foot in QM and the other in GR"; some of these will be briefly described below.

## 1.1 The Planck scale

Before going on, it should be analyzed what "microscopic" means in QG: we should talk about the Plank scale. The Plank scale is commonly accepted to be the characteristic scale of QG, that is the energy (or the distance) such that quantum gravitational effects become not negligible. There are several ways by which one could argue that

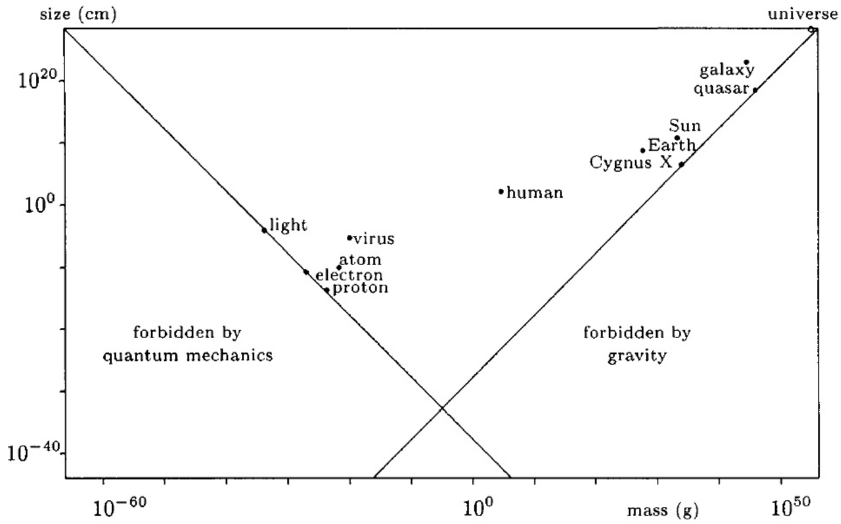



Figure 1.1: The emergence of Planck scale from the intersection between QM and GR forbidden regions (left-wedge and right-wedge respectively). Image taken from [9].

the Plank scale emerges as the scale of QG; the following is my favorite one.
Imagining to place all the known objects in a plane, plotting linear dimension vs mass, as shown in fig. 1.1, there are two lines that fix the boundaries of "what we know": the first line is the one defined by the Compton wavelength, the second line is the one fixed by the Schwarzschild radius. The Compton wavelength is the "QM boundary": there can be no object localized more sharply than its Compton wavelength, so we have $l \geq \frac{h}{Mc}$, where $h$ is the Planck constant, $M$ is the rest mass of the object and $c$ is the speed of light in vacuum. The Schwarzschild radius is the "GR boundary": there can be no object with a mass density greater than the one realized by an object with linear size given by this radius, that is a Schwarzschild black hole, so we have $l \geq \frac{2GM}{c^2}$ where $G$ is the Newton constant. Combining these two relations one gets [1] $\frac{\hbar}{lc} \lesssim \frac{lc^2}{G}$ that is $l \gtrsim l_P = \sqrt{\frac{\hbar G}{c^3}}$. Once the Planck length is obtained, all the other scales can be build by combining in the right way the fundamental constants $\hbar$, $G$, $c$ and the Boltzmann constant $k_B$: $M_P = \sqrt{\frac{\hbar c}{G}}$, $E_P = M_P c^2 = \sqrt{\frac{\hbar c^5}{G}}$, $t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}}$, $\Theta_P = \frac{E_P}{k_B} = \sqrt{\frac{\hbar c^5}{G k_B^2}}$ are respectively the Planck mass, energy, time and temperature. Putting all these scales

[1] I don't report the numerical factors because the order of magnitude is the only thing that matters.

together with their numerical values we have:

$$
\begin{aligned}
&l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}\,\text{m} \quad , \quad t_P = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44}\,\text{s} \\
&M_P = \sqrt{\frac{\hbar c}{G}} \approx 10^{19}\,\frac{\text{GeV}}{c^2} \quad , \quad E_P = \sqrt{\frac{\hbar c^5}{G}} \approx 10^{19}\,\text{GeV} \\
&\Theta_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} \approx 10^{32}\,\text{K}
\end{aligned}
\tag{1.1}
$$

Setting the constants $\hbar$, $c$ and $k_B$ to 1, these scales can be related with a simple relation

$$l_P = t_P = E_P^{-1} = M_P^{-1} = \Theta_P^{-1} \tag{1.2}$$

The important thing to notice here is the magnitude of these scales: they are either ridiculously small (Planck length and time) or ridiculously big (Planck energy, mass and temperature). This means that the quantum gravitational effects manifest themselves at scales that are hitherto undreamed of and that probably will never be attainable for humanity.

Given that the quest for QG seems to be very difficult from a technical/mathematical point of view and that the characteristic scales are way out of our league, why should we care about Quantum Gravity at all?

## 1.2 The Pragmatic point of view

If there was not a way to test the predictions of the approaches to QG with experiments we could forget about doing science in this field, indeed the only way to "solve" the problem would be to (arbitrarily) decide that the most "beautiful" theory from a mathematical/logical point of view is the correct one; we could equally well toss a coin. This was pretty much the situation in the 90's of the last century, but then people started to find a way to test the predictions of some of the approaches to QG [44], [45], [46], [47]. To understand how this is possible, we should first make a distinction between "steep onset" and "smooth onset" effects.

Steep onset effects are the ones that manifest themselves only at a given scale; for example if one wants to ionize an hydrogen atom, they should throw on it a photon with an energy $E \geq 13.6$ eV. If only photons with energies lesser than this value are available, it doesn't matter how many are thrown at the atom, it will not be ionized: the energy does not accumulate at every collision, we cannot "amplify" the effect. The only way to study such effects is to run experiments at the characteristic scale, so if this was the case for QG there would be no hope to achieve anything.

Smooth onset effects are those that are always there and that can be detected if the measures have the right sensitivity. They can also be amplified. The most paradigmatic example of this type is the Brownian motion. In 1827 the botanic Robert Brown observed a strange motion performed by pollen particles immersed in a sample of water. He had no clue about molecules and even if he had he could not see them because he had not a microscope sensitive enough to let him observe them. Yet he was (unknowingly) observing the effects of the collisions between pollen particles and the water molecules, as explained by Einstein almost 80 years later. How was it possible? The ratio between the linear dimension of a single molecule of water and a pollen particle is roughly $10^{-5}$ so the latter is a giant compared to the former, it should not feel the effects of those collisions; indeed it does not feel the single collision but rather the cumulative result of an Avogadro number ($\approx 10^{23}$) of collisions: the amplifier here is the number of molecules.

If the right amplifier can be found there is the possibility to test the predictions of those approaches to QG that are characterized by smooth onset effects. For example, in theories with modified dispersion relations $f(E^2, p^2) = 0$, corrections to the speed of photons can be derived such that, up to the leading order correction, it takes the form $c(E) = 1 - \eta \frac{E^n}{E_P^n}$ where $n$ is an integer (typically $n = 1$ or $n = 2$). The correction is of course negligible for the energies that can be reasonably achieved directly but, if photons with different energies emitted simultaneously from a source travel for enough time, differences in arrival times can be measured and these can be linked to the energy dependent speed of light. This can be done with astronomical sources such as gamma ray bursts [47]. The idea is to fight the smallness of the effects with the largeness of the Universe, to use the Universe itself as a laboratory.

Having dealt with the issue of experimental measures, I can now address the following: what kind of questions can we answer with QG?

### 1.2.1 Information paradox

The information paradox is one of the mysteries that arise when trying to put together QM and GR. It is known that all the matter and radiation that falls into a black hole is lost forever, there is no measure that can be made to get information from the region of spacetime beyond the event horizon. This would not be a problem per se. The problems come when the fact that black holes emit thermal radiation is taken into account; this result was obtained by Stephen Hawking [48] and it is based on a "semi-classical" approximation, that is studying a QFT on the (background) classical geometry of a black hole. The black hole mass decreases as an effect of the emission and eventually the black hole itself will evaporate, the only residual being the thermal radiation so that there will be no clue about what the black hole was made of, because a thermal radiation contains only one information: the temperature. A bit of formalism is needed to understand why this is a problem. In QM the evolution of a mixed state is given by the von Neumann equation

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho]$$

The solution to this equation is

$$\rho(t) = e^{-i\frac{Ht}{\hbar}}\rho(0)e^{i\frac{Ht}{\hbar}} \tag{1.3}$$

So if $\rho(0)$ is a pure state $\rho^2(0) = \rho(0)$, $\rho(t)$ is a pure state too. Defining the von Neumann entropy as

$$S = -k_b \mathrm{Tr}\left[\,\rho \log \rho\,\right]$$

it is clear that $S \geq 0$, with $S = 0$ only for pure states. The entropy so defined is preserved by the evolution eq. (1.3) because it is a unitary evolution and unitary transformations preserve von Neumann entropy; given that the entropy is a measure of the information contained in a system (the more the entropy the lesser the information) we have that information is preserved in QM.

Returning to the paradox, pure states can cross the event horizon and fall inside the black hole and the latter will give back a thermal radiation which is described by

$$\rho = \frac{e^{-\frac{H}{k_B T}}}{\mathrm{Tr}\left[e^{-\frac{H}{k_B T}}\right]}$$

which is not a pure state. The overall result is a loss of information that is not acceptable in QM for the analyses of the Von Neumann entropy.

Information is also trivially preserved in classical physics which includes GR. Indeed the only situation when information could be lost is when black holes are involved, but from a logical point of view there is no need for the information to be accessible, it is just needed that it is "stored" somewhere.

Another way of looking at the problem, which does not involve information at all and so could be less confusing, is noting that time evolution is reversible both in QM and in GR [2] while the process of black hole evaporation is not: we end up with thermal radiation which does not tell us anything besides the mass of the black hole, so a lot of different black holes (i.e., initial states) could give us the same radiation (i.e., final state). This is a mathematical inconsistency of the two theories when combined in a "semi-classical" way.

QG could solve the paradox by introducing a better description of the black hole physics, in particular of the final phases of evaporation, or it could give an explanation about why information is not preserved, maybe introducing a new notion, tailored for a new description of physics.

[2]Time reversibility is not to be confused with time reversal symmetry. Physical laws are reversible if there is a unique initial state for every final state while they are symmetric under time reversal if they are the same when time goes backward. It could be argued that the measurement process in QM violates time reversibility because different wavefunctions can give the same result after the measurement, but this is not the case since the time evolution, described for example by the Schrödinger equation, is time reversible; the result of a measurement is better understood as an initial state for the subsequent evolution not as a final state for the evolution prior to it.

To conclude it should be said that, although Hawking's calculation are very robust, the Hawking radiation phenomena has not been observed yet; it has been observed only in analog systems such as acoustic black holes [49], [50]. The problem for the observation of this phenomenon for actual black holes is that the estimated evaporation time is about $10^{67}$ years for a black hole with a mass equal to the mass of the Sun and about $10^{100}$ years for supermassive black holes.

### 1.2.2 Cosmology and the singularities problem

Other questions that could be answered with QG are the ones concerning the Early Universe. The modern cosmological models predict an expanding Universe with a singularity at the beginning of time: the Big Bang. The existence of this singularity is commonly accepted and is also rigorously demonstrated to be a necessity in the GR framework under some circumstances [51]. The problem is that we do not actually know what happens when distances and time intervals lesser than the Planckian ones are involved. The validity of GR in these regimes is highly questionable and the same argument can be used whenever a singularity emerges in GR: these are the regimes where the quantum mechanical and the gravitational effects become comparable. It is probable that, in order to investigate the Early Universe regime, as well as to get a complete description of black hole physics and all the situations when a singularity shows up in GR, a complete theory of QG is needed.

The hope to find such traces of quantum gravitational phenomena in the Early Universe physics relies on the observations of the remnants of that phase: the cosmic microwave background (CMB) and the gravitational wave background (GWB). The former is background electromagnetic radiation already widely studied, it is the furthest in time peek we can achieve. The observations of the latter are in their infancy; physicists have already detected gravitational waves: these are made up by ripples in the fabric of spacetime itself, originating from mergers of black holes and neutron stars or supernova explosions or any other phenomena involving these extreme energies. It is known that such phenomena have been happening for all of the cosmic history and probably in the early instants of life of the Universe insanely energetic events took place, producing gravitational waves. The Universe should be hence filled everywhere and all time with these ripples; this is the GWB. The human made apparatus are able to detect just a small part of the GWB: the wavelengths that are detectable with LIGO, for example, range from 10 Km to $10^5$ Km; it is not sensible to the gravitational waves with higher (or smaller) wavelength. To observe the entire GWB we need an apparatus sensible to ripples with single oscillations that span the solar system or the distance to the nearest star. In order to do that a galaxy spanning gravitational wave observatory is needed, returning to the idea of the Universe as a laboratory. This observatory is provided by a (galaxy spanning) collection of bizarre stars called pulsar timing arrays. This is exactly what has been done recently [52], [53], [68].

The hope for a theory of QG is that it could address questions concerning the birth of the Universe, fill in the gaps we have in our cosmological models and give us a better understanding of the most extreme phenomena we know, the singularities that arise in GR. The technology needed to study these situations is already here or it is reachable within reasonable time.

## 1.3 The Philosophical point of view

Science is done with facts and this is why I chose to put the pragmatic reasons to motivate the QG quest first. But I think that we should not forget to address foundational issues as well, which overlap in some extent with philosophy. QG probably offers the most fertile ground to reshape our understanding of reality.

### 1.3.1 Quantum foundations

One of the obstacles in the quest for QG might be that, while GR is a theory based on physical postulates, QM is a mathematical framework based on axioms. This is a major difference in my opinion because it means that we cannot compare the two theories on a foundational level.

Physicists stopped to ask foundational questions about QM, dismissing the problem by saying "shut up and calculate". This was a successful and rewarding behavior in the second half of the last century because it led to tremendous developments in particle physics, solid state physics, atomic physics and nuclear physics. Maybe it's time now to address the elephant in the room, to switch from "shut up and calculate" to "shut up and contemplate": we might be at a crossroad in fundamental physics that demands us to return to the investigation techniques of the first half of the last century, when addressing the foundational problems of physics led to the revolutions of QM and GR.

In the last 20 years there was an increasing interest in the quantum foundations research, the major efforts being in trying to reformulate QM from first (physical) principles [8], [54] and maybe disentangle it from the swampland of interpretations. There is also a recent interest in finding theories beyond QM in which the problem of locality of the wavefunction collapse is addressed, leading to the so called "superdeterministic theories" [56], [57],[58], [69].

Maybe it is no wonder that this research program developed alongside the quantum technologies one: in order to unlock the full potential of QM from a technological point of view, we might need first to truly understand it.

Maybe along the road to QG we could find a new understanding of quantum theory.

### 1.3.2 Spacetime and locality

Another issue is the one concerning the nature of spacetime. Part of the QG community thinks that QG demands for a reshaping of the notions of space and time. When probed at distances of the order of the Planck length or time intervals of the order of

Planck time, there is no intuition about the nature of spacetime. It might be something that differs dramatically from our current notion thought as a coarse-grained version of this fundamental entity, just like the dynamics of planets is not even comparable to the dynamics of the electrons and protons they are made of. The notions of "where" and "when" might even not make sense at all at super-Planckian scales. Among other things this means, for instance, that the classical notion of spacetime dimensionality should be replaced by some other effective notion, tailored to tame the quantum nature of spacetime.

There are also some approaches that predicts that the features of spacetime depend on the energy of the probes used to probe it; this means that the spacetime reconstructed by different observers depends on the energy of the probes they use [61], [62], [63].

### Time

Time occupies a peculiar position in this context. Phenomena evolve with time in physics, they are described by differential equations that relate a state at one instant of time to the same state at another instant; it may be a quantum state or a classical state, the difference being in the equations and the framework involved. There is no problem with time neither in QM nor in GR, the problems arise when trying to describe the dynamics of spacetime in the QM framework. In quantizing the structure of spacetime, being it the metric or spacetime itself since the two concepts get intertwined in GR, the notion of a quantum state representing spacetime at a given instant and its evolution make little sense because there is no classical time that can be used to evolve the state nor real "instants" [70].

There are also some authors [59] that questioned the nature of time as fundamental entity, claiming that it can emerge (together with gravity) as a sort of coarse-gained notion from a timeless non-dynamical space.

### Locality

Returning to the idea that the structure of spacetime might depend on the energy of particles, the proposal of relative locality might be the most interesting one [61], [62]. In this proposal it is claimed that we never actually "see" spacetime but rather we "see" particles with a given energy at given times, we as observers are the equivalent of "calorimeters with clocks" (setting aside the time issue). Indeed when we measure a distance we are actually observing photons coming from the object we are measuring the distance of. So the correct arena to describe (classical) physics is not spacetime but phase space, spacetime being a projection of this arena made by observers.

The main ingredient of this proposal is the idea that momentum space might be curved; this is an old idea, dating back to Born [60]: a theory of quantum gravity should accommodate a curved momentum space because in QM the descriptions on momentum and position space are dual to each other and in GR we have a curved spacetime.

Momentum preserving particle interactions are then studied taking into account such a curved structure for momentum space, interactions being understood as worldlines intersections. This leads quite straightforwardly to the prediction that worldlines intersect at one point only for an observer local to the interaction itself, while distant observers see the interaction as non local, with the coordinates of the particles being spread over a certain region. This is not of course a physical non locality because for every momentum preserving interaction there exists a local observer that sees the interaction to be local. This "just" means that different observers construct different spacetimes, with the coordinates of the particles being mixed with momentum space and becoming energy dependent.

Maybe along the road to QG we will find a new understanding of the notions of space, time, locality and maybe even causality itself.

## 1.4 Summary

This introduction has no claim to be a thorough discussion neither of all the problems with the QG research program nor of all the possible answers that might be addressed with QG. To be concise I only chose the ones that I personally think are the most interesting ones.

To sum up, why should we care about QG at all? The possible answer is that QG research program aims at:

- Deepening our understanding of the Early Universe and cosmological models, as well as the extreme phenomena of spacetime such as black holes.
- Reshaping our notions of time, space, locality and causality, possibly leading us to a new revolution in physics and philosophy, comparable to the ones produced by QM and GR about a century ago.

These two claims alone should be enough to address the answer which opened this introduction and to lure us to see how deep the rabbit hole goes.

*To arrive at abstraction, it is always necessary to begin with a concrete reality … You must always start with something. Afterward you can remove all traces of reality.*

*~ Pablo Picasso*

*Second Chapter*

# Spectral and Thermal dimensions

This chapter starts with an argument intended to convince the reader that the familiar notion of Hausdorff dimension cannot handle the problem of quantum spacetime dimensionality. I then introduce the concept of spectral dimension, showing why this definition makes sense when the classical limit is considered, developing the formalism and giving some examples in known cases. Lastly, I will introduce the notion of thermal dimension and the physical reasons that motivate the introduction of this notion.

## 2.1 Setting the Hausdorff dimension aside

In everyday life the notion of "number of dimensions" represents the minimum number of measurements an observer needs to do in order to locate an object. This everyday notion can be formalized and generalized with the notion of Hausdorff dimension.

Given a metric space $X$, a set $S \subseteq X$, a real positive number $d \in [0,\infty[$ and a collection of sets $\{E_i \subset X \,/\, i \in I\}$, where $I$ is a family of indexes, the Hausdorff $d$-dimensional outer measure of $S$ is defined as [1]

$$\mathscr{H}^d(S) := \lim_{r\to 0^+} inf \left\{ \sum_{i\in I} \left(diam\, E_i\right)^d \,/\, S \subseteq \bigcup_{i\in I} E_i \;\wedge\; 0 < diam\, E_i < r \,,\, \forall i \in I \right\}$$

Given this definition, the Hausdorff dimension of $S$ is defined by

$$\boxed{d_H(S) := inf \left\{ d \geq 0 \,/\, \mathscr{H}^d(S) = 0 \right\}} \tag{2.1}$$

This definition gives rise to the usual notion of dimension for smooth manifolds, so that the Hausdorff dimension of spacetime is 4 (or, more generally, $D = d+1$) in the General Relativistic picture, which simply means that a "volume" or a region in a $d$-dimensional space scales as $r^d$, where $r$ is the linear size of the region.

[1] The diameter of subset $A$ of a metric space $X$ is given by $diam\, A = sup\,\{g(x,y) \,/\, x, y \in A\}$ where $g$ is the metric defined on the metric space $X$. Usually the sets of the collection are balls $E_i = B(x_i, r_i)$ in which case the diameter is replaced with the radius $r_i$.

What happens when we turn to the Quantum Gravity realm? The quantum nature of the spacetime forbids us to give a reasonable definition of Hausdorff dimension, because it relies on some notions that are ill-defined in a spacetime with a non-classical behavior, such as sharp points or distances that can be infinitesimally small. I give two examples of this argument:

- In a discrete approach to Quantum Gravity, like the Causal Sets framework, the continuous spacetime is replaced with a discrete set of points, called "events". Being the set of events discrete there cannot be infinitesimally small distances between them.
- In the the Non-Commutative spacetime (NCST) approach to Quantum Gravity, the coordinates of spacetime are replaced with (non-commuting) operators on a Hilbert space; this gives rise to uncertainty principles between the coordinates, so the notion of a sharp point in spacetime becomes ill-defined, since all the coordinates cannot be defined with arbitrary precision. Moreover, the allowed distances between the (fuzzy) points in this spacetime are given by the spectrum of an operator and this spectrum is typically discrete and has a non-zero minimum eigenvalue. An example of this property can be found in Appendix B.

These features are obviously at odds with the definition in eq. (2.1). So, if we want to address the problem of the quantum spacetime dimensionality, we need a notion of dimension different from the Hausdorff one: we need to set the Hausdorff dimension aside.

## 2.2 Spectral dimension

### 2.2.1 Definition and premises

The spectral dimension is a geometrical observable that is defined by means of a fictitious diffusion process on the spacetime. Generally speaking this notion describes how "things" spread over (diffusion) time. For example an ink drop in a tank of water spreads differently according to how many closest neighbors each molecule of water has; if the tank is a common 3-dimensional region a cloud of ink grows as $\tau^{\frac{3}{2}}$, where $\tau$ is the diffusion time, while if the tank is a Sierpiński gasket the cloud grows as $\tau^{0.6826}$, which means a spectral dimension of 3 and 1.3652 respectively. For spacetime the idea is similar: we imagine to drop a probe into one building block of spacetime; from there the probe walks randomly and the number of other building blocks that are touched after a given period of (diffusion) time [2] depends on the the effective dimension of spacetime. After this simple introduction, it is now possible to give a formal definition of spectral dimension.

[2] Not to be confused with the time variable on spacetime which is part of the geometry that is being probed.

If $s$ is the fictitious diffusion time of the process, the spectral dimension can be defined as

$$\boxed{d_s(s) := -2\,\frac{d\log P(s)}{d\log s} = -2s\,\frac{P'(s)}{P(s)}} \tag{2.2}$$

where $P(s)$ is the return probability of the diffusion process. The parameter $s$ sets the energy scale of the probe used in the diffusion: high values of $s$ mean that the probe is characterized by low energies (IR regime) while low values of $s$ mean that the probe is characterized by high energies (UV regime); or, stated equivalently, the more the energy of the probe, the lesser the (fictitious) time it takes to return to the initial position.

Before going on with the spectral dimension, we should make some preliminary remarks in order to define the framework adopted in this thesis work. It is known [10], [11], [13], [18], [21] that the spectral dimension is sensible also to the curvature of spacetime; in this regard, three main regimes can be identified

- When the parameter $\sqrt{s}$ is comparable with the radius of curvature of spacetime $R$, the spectral dimension is sensible to the geometry of spacetime.
- When $l_P \ll \sqrt{s} \ll R$, the spacetime is effectively flat as seen by the probe and the energy of the latter is very high if compared to the Plack energy, so the spectral dimension should agree with the Hausdorff dimension. This is true unless the UV/IR mixing plays a significant role; this second regime is what is considered the IR limit in this thesis work, so we will deal only with flat spacetimes.
- When $\sqrt{s} \approx l_P$, the quantum nature of spacetime should undoubtedly manifest itself, so we expect that $d_s \neq d_H$, unless the properties of spacetime becomes trivial in the UV regime, as could happen with UV/IR mixing phenomena; this will be the UV regime in this thesis work.

### 2.2.2 The connection to MDRs

In the Introduction I said that a key ingredient for the spectral dimension is the MDR of the spacetime, but, looking at the definition in eq. (2.2), the former seems to be clueless about the latter. To make the connection between the two explicit it is necessary to calculate the return probability from the diffusion equation. This diffusion equation will be defined by the Euclideanized version of the deformed Laplacian of the theory; we can focus preliminarily on the case of a deformed Laplacian of the form $\Omega(\partial^2_{x^0}, -\nabla^2)$ so that the diffusion equation on a spacetime of Hausdorff dimension $d_H = d+1$ is defined by

$$\begin{cases} \left[\dfrac{\partial}{\partial s} + \Omega^{(E)}\left(-\partial^2_{x^0}, -\nabla^2\right)\right]\rho(x,x';s) = 0 \\[2ex] \rho(x,x';0) = \delta^{(d+1)}(x-x') \end{cases} \tag{2.3}$$

where $\Omega^{(E)}(-\partial^2_{x^0}, -\nabla^2)$ is the Euclideanized Laplacian obtained by a Wick rotation $x^0 \mapsto ix^0$ and $\delta^{(d+1)}(x-x')$ is the $(d+1)-$dimensional $\delta$ function; eq. (2.3) means that $\rho(x,x';s)$

is the probability density that the probe starts its diffusion in $x'$ and arrives in $x$ after a diffusion time $s$. Fourier analysis can be used to solve the equation on a flat spacetime

$$\rho(x,x';s)=\int\frac{d^d p\,dE}{(2\pi)^{d+1}}\;e^{iE(t-t')+i\vec{p}\cdot(\vec{x}-\vec{x}')}\tilde{\rho}(E,\vec{p};s)$$

Inserting this into eq. (2.3)

$$\begin{cases}\left[\frac{\partial}{\partial s}+\Omega^{(E)}\left(E^2,|\vec{p}|^2\right)\right]\tilde{\rho}(E,\vec{p};s)=0\\ \\ \tilde{\rho}(E,\vec{p};0)=1\end{cases}\implies\tilde{\rho}(E,\vec{p};s)=e^{-s\Omega^{(E)}\left(E^2,|\vec{p}|^2\right)}$$

The return probability is obtained from the density by setting $x=x'$ (definition of "return")

$$P(s)=\rho(x,x,s)=\int\frac{d^d p\,dE}{(2\pi)^{d+1}}\;e^{-s\Omega^{(E)}\left(E^2,|\vec{p}|^2\right)}$$

In this last result, the direct link between the MDR and the return probability can be easily seen; this expression can be generalized to theories in which the MDR is given by $\Omega(E,\vec{p})$ so that the return probability is given by

$$\boxed{P(s)=\int\frac{d^d p\,dE}{(2\pi)^{d+1}}\;e^{-s\Omega^{(E)}(E,\,\vec{p})}}\tag{2.4}$$

With this equation, we can forget about the diffusion equation and take this as a definition of return probability of the diffusion process. This also resolves a subtlety I did not mention: how can a diffusion process be defined with eq. (2.3), where the initial condition is defined with a $\delta$ function, implying that there is a notion of sharp point, if we are dealing, for instance, with a noncommutative spacetime, where sharp points do not exist? The answer is that in defining eq. (2.3) we are considering a sort of semi-classical limit of the theory in which the spacetime is trivial and yet the dispersion relation is deformed; once we obtain the result in eq. (2.4), which has no memory of eq. (2.3), we can infer that this is a good notion of return probability for the diffusion process on the quantum spacetime. It is also possible to define a smeared $\delta$ function, which is a Gaussian distribution, as initial condition as done by some authors [13].

### 2.2.3 The classical limit

In order for the definition in eq. (2.4) to make sense, it would be nice if it agreed with the classical value $d+1$ when the dispersion relation is not deformed, i.e. when $E^2+|\vec{p}|^2=m^2$ [3]. It is an easy calculation to show this:

$$P(s)=\int\frac{d^d p\,dE}{(2\pi)^{d+1}}\;e^{-s(E^2+|\vec{p}|^2)}=\frac{S^{(d)}(1)}{(2\pi)^{d+1}}\int_0^\infty dr\int_0^\infty dE\,e^{-sE^2}e^{-sr^2}r^{d-1}$$

[3] The mass term is not considered in the definition of the return probability because the diffusion equation is defined by means of the deformed Laplacian, the mass being its eigenvalue.

The two integrals can be computed by means of [4]

$$\int_0^\infty x^m e^{-px^2}dx = \frac{1}{2}p^{-\frac{m+1}{2}}\Gamma\big(\tfrac{m+1}{2}\big) \;\;,\;\; p>0 \;,\;\; m\in\mathbb{N}_0$$

with $m=0$ for the integral in the variable $E$, $m=d-1$ for the integral in the variable $r$ and $p=s$ for both integrals. Calling $a$ the product of all the numerical constants $P(s)$ is given by

$$P(s) = as^{-\frac{d+1}{2}}$$

so that

$$d_s(s) = -2s\frac{P'(s)}{P(s)} = -2s\left(-\frac{d+1}{2}\right)\frac{as^{-\frac{d-1}{2}}}{as^{-\frac{d+1}{2}}} = d+1$$

We end up with a result that is equal to the classical value of the Hausdorff dimension for all values of $s$.

### 2.2.4 Two examples with MDR's

I now consider two examples with MDR's: in the first example the spectral dimension agrees with the value given by the thermal dimension, in the second example the two notions of effective dimension give different results. The thermal dimension is discussed in the following section.

- Consider the MDR $E^2 = p^2 + l^{2\gamma}p^{2(1+\gamma)}$ where $l$ is a length scale and $\gamma>0$. The return probability is given by

  $$P(s) = \int \frac{d^d p\, dE}{(2\pi)^{d+1}}\;\exp\big[-s(E^2+p^2+l^{2\gamma}p^{2+2\gamma})\big]$$

  The integral over $E$ is trivial and gives $\sqrt{\pi}s^{-\frac{1}{2}}$ while the integral over $\vec{p}$ gives

  $$S^{(d)}(1)\int_0^\infty dp\, p^{d-1}\exp\big[-s(p^2+l^{2\gamma}p^{2+2\gamma})\big]$$

  This integral cannot be evaluated exactly for every value of $\gamma$ but for the calculation of the spectral dimension in the UV and IR limit this is not needed: there are some asymptotic expansion that can be considered. For large (small) values of $s$, corresponding to the IR (UV) limit, the smallest (biggest) power of $p$ in the exponential will give the major contribution. This means that [5]

  $$P(s) \sim s^{-\frac{1}{2}}\int_0^\infty dp\, p^{d-1}e^{-sp^2} \sim s^{-\frac{d+1}{2}} \quad (s\to\infty)$$

[4] G&R, 3.461.2$b$.

[5] I neglect all the numerical factors in $P(s)$ because they have no role in the computation of the spectral dimension.

$$P(s) \sim s^{-\frac{1}{2}} \int_0^{\infty} dp\, p^{d-1} e^{-sl^{2\gamma} p^{2+2\gamma}} \sim s^{-\frac{1}{2}} s^{-\frac{d}{2(1+\gamma)}} \quad (s \to 0)$$

where the second integral can be computed, defining $t := sl^{2\gamma} p^{2(1+\gamma)}$, as follows

$$\int_0^{\infty} dp\, p^{d-1} e^{-sl^{2\gamma} p^{2+2\gamma}} = \frac{l^{\frac{d\gamma}{1+\gamma}}}{2(1+\gamma)} s^{-\frac{d}{2(1+\gamma)}} \int_0^{\infty} dt\, e^{-t}\, t^{\frac{d}{2(1+\gamma)}}$$

The remaining integral is a dimensionless numerical factor (a $\Gamma$ function), the only useful information is the factor $s^{-\frac{d}{2(1+\gamma)}}$; note that this procedure can be applied also to the IR limit integral by setting $\gamma = 0$. The spectral dimension is then given by

$$d_s(\infty) = d + 1 \quad , \quad d_s(0) = 1 + \frac{d}{1+\gamma} \tag{2.5}$$

A general proof of the method adopted here can be found in Appendix A.

- As second example, consider the MDR $E^2 = p^2 + l^{2\gamma}(p^2 - E^2)^{1+\gamma}$ where $l$ and $\gamma$ have the same role as in the previous example. In this case $P(s)$ is given by

$$P(s) \propto \int_0^{\infty} d\rho\, \rho^d \exp\left[-s(\rho^2 + l^{2\gamma}\rho^{2+2\gamma})\right]$$

where $\rho := E^2 + p^2$. The same method applied before can be applied to this integral, giving

$$P(s) \sim \int_0^{\infty} d\rho\, \rho^d e^{-s\rho^2} \sim s^{-\frac{d+1}{2}} \quad (s \to \infty)$$

$$P(s) \sim \int_0^{\infty} d\rho\, \rho^d e^{-sl^{2\gamma}\rho^{2+2\gamma}} \sim s^{-\frac{d+1}{2(1+\gamma)}} \quad (s \to 0)$$

so that the spectral dimension is given by

$$d_s(\infty) = d + 1 \quad , \quad d_s(0) = \frac{d+1}{1+\gamma} \tag{2.6}$$

In both the above examples the spectral dimension coincides with the Hausdorff dimension in the IR limit; this is obvious and happens every time there is only a UV deformation in the MDR, which can be neglected in the IR limit. The opposite will happen, that is trivial (non-trivial) spectral dimension will show up in the UV (IR) limit, when the dispersion relation will be deformed with a IR deformation.

## 2.3 Thermal dimension

### 2.3.1 Benefits and flaws of the Spectral dimension

Spectral dimension is a useful notion of effective dimensionality in some cases. For instance it can be shown that, by means of a change of integration variables which transforms the dispersion relation into the trivial one, the spectral dimension can be interpreted as the Hausdorff dimension of the momentum space. In several approaches to QG there is the prediction of dimensionality reduction to the value of 2 in the UV using the spectral dimension [14]; this has some important consequences in cosmological models [15], [16], [17] and suggests that there could be the hope to construct a quantum field theory for gravity which is renormalizable.

The spectral dimension suffers however of some flaws:

- First of all, it is defined by means of a Euclideanization of spacetime, which is unphysical to begin with. The characteristic, properties and predictions of the Euclideanized version of a theory can differ significantly from the counterparts of the actual theory.
- Secondly, the integration is carried out over the whole momentum space, meaning that there is an important contribution given by the off-shell modes, which are unphysical.

The two observations above imply that the spectral dimension cannot be a physical definition of spacetime dimensionality. If there is the possibility that the dimensional flow is an actual characteristic of the "true" quantum theory of gravity, we should seek a physical notion of spacetime dimensionality, maybe linked to some observable quantities.

### 2.3.2 Thermodynamical preliminaries

The thermal dimension can be defined by noting that, in classical spacetime, there are some thermodynamical quantities which scale with the number of dimensions. Consider a gas of photons in a "cubic" box with volume $V = L^d$ in thermodynamical equilibrium at temperature $T$. The partition function is given by

$$Z = \sum_{\{n_{\vec{k},\epsilon}\}} e^{-\beta E(\vec{k},\epsilon)}$$

where $\{n_{\vec{k},\epsilon}\}$ symbolizes all the configurations of the system, $\epsilon$ is the polarization of the photons and $\beta := T^{-1}$. Given that $E(\vec{k},\epsilon) = \omega(\vec{k},\epsilon) n_{\vec{k},\epsilon}$, where $n_{\vec{k},\epsilon}$ is the number of photons with momentum $\vec{k}$ and polarization $\epsilon$, the partition function become [6]

$$Z = \prod_{\vec{k},\epsilon} \sum_{n_{\vec{k},\epsilon}=0}^{\infty} e^{-\beta\omega(\vec{k},\epsilon) n_{\vec{k},\epsilon}} = \prod_{\vec{k}} \left[ \frac{1}{1 - e^{-\beta\omega(\vec{k})}} \right]^{d-1}$$

so the logarithm of the partition function is

$$\log Z = -(d-1) \sum_{\vec{k}} \log\left(1 - e^{-\beta\omega(\vec{k})}\right)$$

[6] In $D = d + 1$ spacetime dimensions photons have $D - 2 = d - 1$ polarization states.

The momenta of the photons are quantized because they are in a finite box

$$k_i = \frac{\pi}{L} n_i \quad , \quad n_i \in \mathbb{N}_0 \quad , \quad i = 1, 2, ..., d$$

and the spacing of the modes is constant $\pi^d V^{-1}$. If the box is sufficiently big so that the mean energy is much bigger than the spacing between the modes, a continuum approximation can be applied. In order to do so, the infinitesimal density of states is needed; it is given by the ratio between the volume of the momentum space, $dV_{\vec{k}} = \frac{S^{(d)}(1)}{2^d} k^{d-1} dk$, and the spacing between modes

$$dn = V \frac{S^{(d)}(1)}{(2\pi)^d} k^{d-1} dk$$

In the continuum limit, the logarithm of the partition function is then given by

$$\begin{aligned} \log Z &= -(d-1)V \frac{S^{(d)}(1)}{(2\pi)^d} \int_0^\infty dp \int_0^\infty dE\, p^{d-1} \log\left(1 - e^{-\beta E}\right) \delta(E - p) = \\ &= -2\pi(d-1)V \int \frac{d^d p\, dE}{(2\pi)^{d+1}} 2E\theta(E)\delta(E^2 - p^2) \log\left(1 - e^{-\beta E}\right) \end{aligned} \quad (2.7)$$

where in the second line the integral is carried out on the whole momentum space and the $\theta$ and the $\delta$ functions constrain the variables on the physical submanifold of the momentum space (on-shell modes with positive energy). This is the standard manifest covariant expression for the logarithm of the partition function of a photon gas.

### 2.3.3 Thermal dimension: definition

The partition function in (2.7) can be used to compute all the thermodynamical variables of the system. The integral in the first line of (2.7) can be easily done

$$\begin{aligned} \log Z &= -(d-1)V \frac{S^{(d)}(1)}{(2\pi)^d} \int_0^\infty dE\, E^{d-1} \log\left(1 - e^{-\beta E}\right) \\ &= (d-1)(d-1)!\zeta(d+1) \frac{S^{(d)}(1)}{(2\pi)^d} V \beta^{-d} \propto \\ &\propto V \beta^{-d} \end{aligned} \quad (2.8)$$

From $\log Z$ the energy density and the pressure are defined as

$$\rho := -\frac{1}{V} \partial_\beta \log Z \quad , \quad P := \frac{1}{\beta} \partial_V \log Z \quad (2.9)$$

From these definition it is easy to see that

$$\rho \propto \beta^{-(d+1)} = T^{d+1} \quad , \quad w := \frac{P}{\rho} \propto \frac{1}{d} \quad (2.10)$$

The two parameters in eq. (2.10) scale with the number of dimensions of (classical) spacetime. There are therefore two physical observables that can be used to define an effective physical notion of spacetime dimensionality, called Thermal dimension of spacetime $d_T$ [26], [27], which is given by

$$\boxed{\rho \propto \beta^{-d_T} \text{ (UV or IR limit)}} \, , \quad \boxed{d_T(\beta) = \frac{1}{w(\beta)} + 1} \tag{2.11}$$

where the first definition, which is a generalization of the Stefan-Boltzmann law, can be used only in the limit of high/low temperature, while the second, which is linked to the measurable parameter $w$, can be used to see the flow of the dimensionality with the temperature. This is because, with general MDRs, the dependence from $\beta$ of $\rho$ and $P$ is not trivial, so the first relation in eq. (2.11) will hold only asymptotically and the parameter $w$ will not be a constant anymore. The only difference in (2.7) when a MDR is considered is the substitution of $E^2 - p^2$ with the MDR $\Omega(E, \vec{p})$ inside the $\delta$ function

$$\boxed{logZ = -2\pi(d-1)V \int \frac{d^d p \, dE}{(2\pi)^{d+1}} \, 2E\theta(E)\delta\Big(\Omega(E,\vec{p})\Big)\log\Big(1 - e^{-\beta E}\Big)} \tag{2.12}$$

From this definition it is easy to see that this is a pure physical notion: it is derived from physical quantities and everything depends only on the physical on-shell modes. It can be said that the thermal dimension is defined by the following statement: the thermodynamics of a photon gas behaves as if spacetime had a number of dimensions given by the thermal dimension. This new notion of effective dimensionality of spacetime agrees of course with the Hausdorff dimension in the classical case, as can be seen from (2.8) and it agrees also with the spectral dimension in most cases; the two notions give different results only when the unphysical characteristics of the latter play a major role as can be seen from the examples below.

### 2.3.4 Two examples with MDR's

The two examples are the same used in the case of the spectral dimension in order to compare the two notions.

- With $E^2 = p^2 + l^{2\gamma} p^{2(1+\gamma)}$ the logarithm of the partition function is given by

  $$\log Z \propto V \int_{-\infty}^{\infty} dE \int_0^{\infty} dp \, p^{d-1} \, E\theta(E) \, \delta\Big(E^2 - p^2 - l^{2\gamma} p^{2(1+\gamma)}\Big) \log\Big(1 - e^{-\beta E}\Big)$$

  defining $u := \beta p$ and $t := \beta E$

  $$\log Z \propto V\beta^{-(d+2)} \int_{-\infty}^{\infty} dt \int_0^{\infty} du \, u^{d-1} \, t\theta(t) \, \delta\Big(t^2\beta^{-2} - u^2\beta^{-2}(1 + l^{2\gamma}\beta^{-2\gamma}u^{2\gamma})\Big)\log\big(1 - e^{-t}\big)$$

  In the low temperature limit, which is the high $\beta$ limit, since $\gamma > 0$ the argument of the $\delta$ function can be approximated with $(t^2 - u^2)\beta^{-2}$ which gives the standard result $d_T(\infty) = d+1$ in the IR limit. In the high temperature/low $\beta$/UV limit, the dominant part of the argument of the $\delta$ function is $\beta^{-2}t^2 - l^{2\gamma}u^{2+2\gamma}\beta^{-2-2\gamma}$ so $\log Z$ takes the form

$$\log Z \simeq V\beta^{-(d+2)}\int_{-\infty}^{\infty} dt \int_0^{\infty} du\, u^{d-1}\, t\theta(t)\, \frac{\delta(u-\bar{u})}{\bar{u}^{2\gamma+1}}\beta^{2+2\gamma}\log\left(1-e^{-t}\right)$$

where $\bar{u} = t^{\frac{1}{1+\gamma}}\, l^{-\frac{\gamma}{1+\gamma}}\, \beta^{\frac{\gamma}{1+\gamma}}$ therefore

$$\log Z \simeq V\beta^{-(d+2)}\beta^{2+2\gamma}\beta^{-\frac{\gamma(2\gamma+1)}{1+\gamma}}\,\beta^{\frac{(d-1)\gamma}{1+\gamma}}\int_0^{\infty} dt\, t^{\frac{d-1}{1+\gamma}}\, t\, t^{-\frac{2\gamma+1}{1+\gamma}}\log\left(1-e^{-t}\right)$$

$$\log Z \simeq V\beta^{-\frac{d}{1+\gamma}}$$

From this partition function it is easy to see that $d_T(0) = 1 + \frac{d}{1+\gamma}$ which agrees with eq. (2.5).

- With the MDR $E^2 = p^2 + l^{2\gamma}(p^2 - E^2)^{1+\gamma}$ lesser effort is needed to see that the two notions of dimensionality do not give the same result. The logarithm of the partition function is given by

$$\log Z \propto V\int_{-\infty}^{\infty} dE \int_0^{\infty} dp\, p^{d-1}\, E\theta(E)\,\delta\left(E^2 - p^2 - l^{2\gamma}(p^2-E^2)^{1+\gamma}\right)\log\left(1-e^{-\beta E}\right)$$

The solution of the $\delta$ function gives

$$(E^2-p^2)^2[1 - l^{2\gamma}(E^2-p^2)^{\gamma}] = 0 \;\Rightarrow\; E = p \,\wedge\, E = \sqrt{p^2 + \frac{1}{l^2}}$$

At low temperature/low energies, only the first solution is accessible to the system, while at high energies, when also the second solution becomes accessible, there is no difference between $E = p$ and $E = \sqrt{p^2 + \frac{1}{l^2}}$ so we should expect that the thermal dimension is 4 in both limits. Defining $u := \beta p$ and $E := \beta E$ as in the previous example $\log Z$ becomes

$$\log Z \propto V\beta^{-d}\left[\int_0^{\infty} dt\, t^{d-1}\log\left(1-e^{-t}\right) + \int_{\beta l^{-1}}^{\infty} dt\, t\,(t^2 - \beta l^{-1})^{\frac{d}{2}-1}\log\left(1-e^{-t}\right)\right]$$

From this it is easy to see that in both UV and IR limits the behavior is the classical one: in the UV limit $\beta \to 0$ so the two integrals become the same integral and the leading order in $\beta$ is $\beta^{-d}$; in the IR limit $\beta \to \infty$ so the second integral becomes negligible compared to the first one. The thermal dimension is not trivial, however, there is indeed a running with the temperature in intermediate regimes.

The difference between the thermal dimension and the spectral dimension in this last example can be explained as follows. The thermal dimension depends only upon the on-shell modes and we have seen that in the UV and IR limits this modes coincide with the classical ones; the spectral dimension relies on a Euclideanization of the dispersion relation so we have $E^2 + p^2 + l^{2\gamma}(E^2+p^2)^{1+\gamma}$ in the MDR, and it is easy to understand where the problem is: in the UV limit we can neglect $E^2 + p^2$ with respect to $(E^2 + p^2)^{1+\gamma}$ while for the on-shell modes we cannot do the same for $E^2 - p^2$ and $(E^2 - p^2)^{1+\gamma}$.

This is one of the cases where the unphysical content of the spectral dimension plays a major role in the computations, showing that it is necessary to rely, in particular in these cases, on a different, physical notion of effective dimensionality for the spacetime, the thermal dimension being the best candidate so far presented in the literature.

*As long as we're living and breathing, there's more we can do. We just have to be strong enough.*

*~ James Holden, The Expanse*

*Third Chapter*

# *Spectral dimension with UV/IR mixing*

In this chapter the computations done for the spectral dimension in the scenarios with UV/IR mixing are shown. Several MDRs are considered, some of them previously introduced in the literature. Some MDRs, which are not present in the literature but are similar to the previous ones, are also considered in order to understand more deeply the mechanisms behind the dimensional flow in the scenarios with UV/IR mixing. The considered MDRs are:

- $E^2 = p^2 + \alpha p$ [1]
  This is the most interesting MDR considered for the obtained results. This MDR is widely considered in the literature [34], [36], [38] and emerges in two models: it arises in a quantum spacetime model inspired by the semiclassical limit of Loop Quantum Gravity [37] and in a model of NCST [42], $\theta$ light-like noncommutativity, that is
  $$[x_\mu, x_\nu] = i\theta_{\mu\nu} \quad , \quad \theta_{\mu\nu}\theta^{\mu\nu} = 0$$ [2]

- $E^2 = p^2 + \alpha p_1$
  This MDR can be derived in the same model of $\theta$ noncommutativity considered for the previous one [39].

- $E^2 = p^2 + \frac{4\alpha_1}{\theta^2\rho^2} + \alpha_2 m^2 \log\left(-\frac{1}{4}\theta^2\rho^2 m^2\right) \ , \ \rho^2 := \sum_{i=1}^{d-1} p_i^2$ [3]
  This MDR can be derived [40] in a model with $\theta$ noncommutativity where the matrix $\theta_{\mu\nu}$ is given by
  $$\theta_{0\nu} = 0 \quad , \quad \theta_{ij} = \frac{1}{2}\epsilon_{ijk}\delta^{kd}\theta$$

- $E^2 = p^2 + \alpha E$
  This is the first MDR we considered that has no justification in the literature and we considered it to for two reasons: the first one is that we wanted to point out that energy and momentum play the same role for the spectral dimension; more importantly is that this expression might be used to define a DSR with an IR deformation, which is something yet not studied in the literature. If this is realized

[1] $\alpha$ and all the deformation parameters have dimension of energy unless otherwise stated.
[2] The entries of the matrix $\theta$ have units of (lenght)$^2$
[3] $\alpha_1$ and $\alpha_2$ are dimensionless.

the computation of the spectral dimension could be repeated with a deformed integration measure.

- $E^2 = p^2 + \alpha\sqrt{p^2 - E^2}$
  As the previous one, this MDR is not justified in the literature; we considered it to show that a IR deformed MDR with deformation given by $f(p^2 - E^2)$ give the same differences between spectral and thermal dimension which arises with analogous UV deformations [26].

The computation of the spectral dimension for these scenarios is presented in the following sections.

## 3.1 Spectral dimension with $E^2 = p^2 + \alpha p$ MDR

Considering the MDR

$$E^2 = p^2 + \alpha p \tag{3.1}$$

the return probability eq. (2.4) is given by

$$P(s) = \frac{S^{(d)}(1)}{(2\pi)^{d+1}} \int_0^{+\infty} dp\, p^{d-1} e^{-sp^2 - s\alpha p} \int_{-\infty}^{+\infty} dE\, e^{-sE^2}$$

The integral in the variable $E$ is Gaussian and gives $\sqrt{\pi} s^{-\frac{1}{2}}$. The integral in the variable $p$ can be computed by means of [4]

$$\int_0^{+\infty} dx\, x^{\nu-1} e^{-\beta x^2 - \gamma x} = (2\beta)^{-\frac{1}{2}} \Gamma(\nu)\, exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right), \; Re(\beta) > 0\,, \; Re(\nu) > 0$$

Consequently, imposing $\beta = s\,,\ \gamma = s\alpha\,,\ \nu = d$

$$P(s) = \frac{S^{(d)}(1)}{(2\pi)^{d+1}} \sqrt{\pi}\,\Gamma(d)\, 2^{-\frac{d}{2}} s^{-\frac{d+1}{2}} e^{\frac{\alpha^2 s}{8}} D_{-d}\left(\sqrt{s}\frac{\alpha}{\sqrt{2}}\right)$$

Neglecting the numerical constant

$$P'(s) \propto \left[\frac{d+1}{2} - s\frac{\alpha^2}{8} - \frac{\alpha\sqrt{s}}{2\sqrt{2}} \frac{D'_{-d}\left(\sqrt{s}\frac{\alpha}{\sqrt{2}}\right)}{D_{-d}\left(\sqrt{s}\frac{\alpha}{\sqrt{2}}\right)}\right] s^{-\frac{d+1}{2}-1} e^{\frac{\alpha^2 s}{8}} D_{-d}\left(\sqrt{s}\frac{\alpha}{\sqrt{2}}\right)$$

From this, using eq. (2.2) and renaming $z := \frac{\alpha}{\sqrt{2}}\sqrt{s}$ the variable,

$$d_s(z) = d + 1 - z\left[\frac{z}{2} + \frac{D'_{-d}(z)}{D_{-d}(z)}\right]$$

[4] G&R , 3.462.1

The derivative of the Parabolic cylinder functions can be computed by means of $D'_p(z) = \frac{1}{2} z d_p(z) - D_{p+1}(z)$ [5] so the spectral dimension becomes

$$\boxed{d_s(z) = d + 1 - z\left[\frac{z}{2} + \frac{\frac{z}{2} D_{-d}(z) - D_{-d+1}(z)}{D_{-d}(z)}\right]} \tag{3.2}$$

For the UV limit it is sufficient to observe that [6]

$$D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^\infty e^{-xz-\frac{x^2}{2}} x^{-p-1}\, dx \ , \ Re(p) < 0$$

so that

$$D_p(0) = \frac{1}{\Gamma(-p)} \int_0^\infty e^{-\frac{x^2}{2}} x^{-p-1}\, dx < \infty$$

therefore $d_s(0) = d + 1$ since in eq. (3.2), setting $z = 0$, there would be a product of a finite number given by the square brackets and a factor which is equal to 0.
For the IR limit it is necessary to study the asymptotic behavior of the Parabolic cylinder functions; for this asymptotic behavior it is necessary to pay attention to the sign of the parameter $\alpha$. In particular [7]

$$D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left[1 - \frac{p(p-1)}{2z^2} + o\left(z^{-4}\right)\right] \ , \ |z| \gg 1 \ , \ |z| \gg |p| \ , \ |arg\, z| < \frac{3\pi}{4} \tag{3.3}$$

$$D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left[1 - \frac{p(p-1)}{2z^2} + o\left(z^{-4}\right)\right] - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{\frac{z^2}{4}} z^{-p-1} \left[1 + \frac{(p+1)(p+2)}{2z^2} + o\left(z^{-4}\right)\right]$$
$$|z| \gg 1 \ , \ |z| \gg |p| \ , \ \frac{\pi}{4} < arg\, z < \frac{5\pi}{4} \tag{3.4}$$

The first (second) expansion can be used when $\alpha > 0$ ($\alpha < 0$). In the first case

$$\begin{aligned} d_s(z) &\sim d + 1 - z\left[\frac{z}{2} + \frac{\frac{z}{2} z^{-d}\left(1 - \frac{d(d+1)}{2z^2}\right) - z^{-d+1}\left(1 - \frac{d(d-1)}{2z^2}\right)}{z^{-d}\left(1 - \frac{d(d+1)}{2z^2}\right)}\right] = \\ &= d + 1 - z^2\left[\frac{1}{2} + \frac{\frac{1}{2}\left(1 - \frac{d(d+1)}{2z^2}\right) - \left(1 - \frac{d(d-1)}{2z^2}\right)}{\left(1 - \frac{d(d+1)}{2z^2}\right)}\right] = d + 1 - \frac{z^2}{2}\left[1 - \frac{\left(1 - \frac{d(d-3)}{2z^2}\right)}{\left(1 - \frac{d(d+1)}{2z^2}\right)}\right] \sim \\ &\sim d + 1 - z^2\left[\frac{1}{2} + \frac{1}{2}\left(-1 + \frac{d(d-3)}{2z^2}\right)\left(1 + \frac{d(d+1)}{2z^2}\right)\right] \sim d + 1 - z^2\left[\frac{1}{2} - \frac{1}{2} - \frac{d}{z^2}\right] \end{aligned}$$

Therefore

$$d_s(\infty) = 2d + 1 \tag{3.5}$$

[5] G&R , 9.247.3
[6] G&R , 9.241.2
[7] G&R , 9.246.1 and G&R , 9.246.2

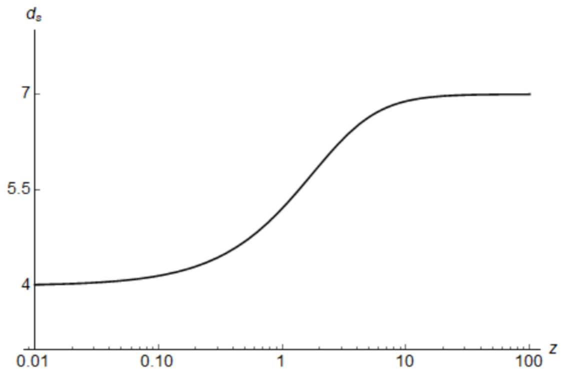
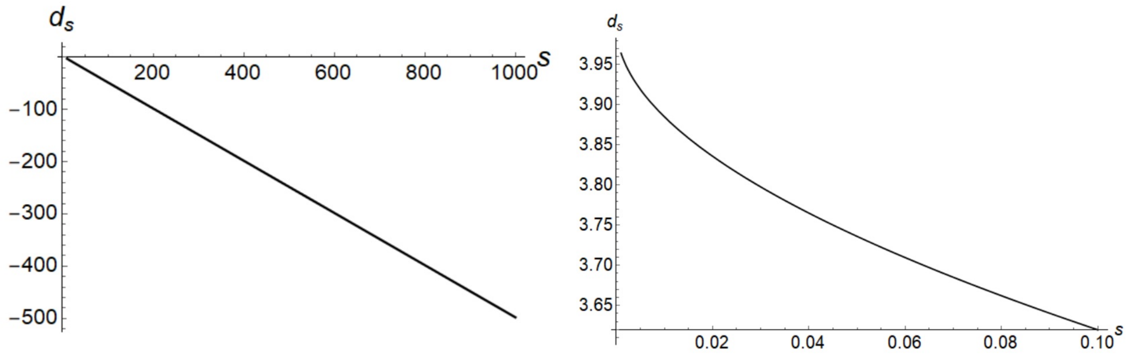



Figure 3.1: Spectral dimension eq. (3.2) with $d = 3$.




(a) IR behavior.

(b) UV behavior.

Figure 3.2: UV and IR behaviors of the spectral dimension eq. (3.2) with $\alpha = -1$ and $d = 3$.

This result can be also seen in fig. 3.1 where the full running of the spectral dimension eq. (3.2) with $d = 3$ is shown.

When $\alpha < 0$ the computations are analogous; the second square bracket in eq. (3.4) gives additional contributions to the spectral dimension, making it divergent at $-\infty$ as can be seen in fig. 3.2.

To summarize, the value of the spectral dimension in the IR limit, which depends on the sign of $\alpha$ but not on the particular value of the parameter, is given by

$$\boxed{d_s(\infty) = \begin{cases} 2d+1 \ , \ \alpha > 0 \\ d+1 \ , \ \alpha = 0 \\ -\infty \ , \ \alpha < 0 \end{cases}} \tag{3.6}$$

The last result means that the probabilistic interpretation of $P(s)$ is lost since its derivative needs to be positive to have a negative value for the spectral dimension in eq. (2.2), which means that $P(s)$ is monotonically increasing and exceeding the value 1.

Physically the MDR in eq. (3.1) might be useful only with $\alpha > 0$ because if $\alpha < 0$ in the IR there is the emergence of imaginary energies, unless it is not postulated that there is a lower bound on the momentum, $p \geq \alpha$. This is why there is the loss of probabilistic interpretation for $P(s)$ with $\alpha < 0$: the integration in the definition of $P(s)$ eq. (2.4) is carried out on the whole momentum space, so this physical lower bound on $p$ is ignored, resulting in the negative divergence in eq. (3.6).

## 3.2 Spectral dimension with $E^2 = p^2 + \alpha p_1$ MDR

Considering the MDR [8]

$$E^2 = p^2 + \alpha p_1 \tag{3.7}$$

the return probability eq. (2.4) is given by

$$P(s) = \frac{S_c^{(d)}(1,1)}{(2\pi)^{d+1}} \int_{-\infty}^{+\infty} dE\, e^{-sE^2} \int_{-\infty}^{+\infty} dp_1\, e^{-sp_1^2 - s\alpha p_1} \int_0^{+\infty} d\rho\, \rho^{d-2}\, e^{-s\rho^2} \;,\; \rho^2 := \sum_{n=2}^{d} p_n^2$$

The firt two integrals are Gaussian while the last one can be computed using [9]

$$\int_0^{+\infty} x^m\, e^{-wx^2}\, dx = \frac{1}{2}\, w^{-\frac{m+1}{2}}\, \Gamma\left(\tfrac{m+1}{2}\right)$$

Setting $m = d-2$ and $w = s$ and neglecting the numerical factor as done before

$$P(s) \propto s^{-\frac{d+1}{2}}\, e^{\frac{\alpha^2 s}{4}}$$

thus

$$P'(s) \propto s^{-\frac{d+1}{2}-1}\, e^{\frac{\alpha^2 s}{4}} \left(\frac{d+1}{2} - \frac{s\alpha^2}{4}\right)$$

so the spectral dimension is given by

$$d_s(s) = d + 1 - s\,\frac{\alpha^2}{2} \tag{3.8}$$

From eq. (3.8) it is easy to get $d_S(0) = d+1$ e $d_s(\infty) = -\infty$.

As happened with the previous MDR, $P(s)$ can be greater than 1 because of the factor $e^{\frac{\alpha^2 s}{4}}$ and the reasons are the same: $p_1$ can be both positive and negative and there is no constraint to its value in the integral defining $P(s)$, so this also means that imaginary energies are considered in the IR limit. There are two possible ways out; the first one is considering $\alpha$ as an energy scale, supposing the necessity to consider a Wick rotation also for this parameter $\alpha \mapsto -i\alpha$ [25] so that the previous relations become

$$P(s) \propto s^{-\frac{d+1}{2}}\, e^{-\frac{\alpha^2 s}{4}} \;\Rightarrow\; d_s(s) = d + 1 + s\,\frac{\alpha^2}{2}$$

therefore $d_s(\infty) = +\infty$ e $0 \leq P(s) \leq 1 \;\; \forall \;\; s$.
The second possibility is to consider a slightly different, "physical" MDR as done below.

[8] Obviously it does not matter what direction is chosen.
[9] G&R , 3.461.2b

### 3.2.1 $|p_1|$ deformation

Consider the MDR

$$E^2 = p^2 + \alpha |p_1| \ , \ \alpha > 0 \tag{3.9}$$

so the return probability eq. (2.4) is given by

$$P(s) = 2\frac{S_c^{(d)}(1,1)}{(2\pi)^{d+1}} \int_{-\infty}^{+\infty} dE\, e^{-sE^2} \int_0^{+\infty} dp_1\, e^{-sp_1^2 - s\alpha p_1} \int_0^{+\infty} d\rho\, \rho^{d-2}\, e^{-s\rho^2}$$

The first and last integral are trivial, so they will contribute as $d$ in total to the spectral dimension; the second integral could be computed with the same formula used for the first MDR [10] or with the (complementary) error function [11]

$$\text{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}$$

The same techniques of asymptotic expansion could be adopted in this case. However, there is a general technique, whose proof can be found in Appendix A, by which it is possible to compute the spectral dimension in the UV and IR limits simply by looking at the biggest and the smallest power of $p$ in the MDR. Adopting the same notations used in the proof in the apppendix, the second integral will contribute to the spectral dimension as $\frac{1}{\gamma_M+1}$ in the UV limit and as $\frac{1}{\gamma_m+1}$ in the IR limit, where $\gamma_M = 0$ and $\gamma_m = -\frac{1}{2}$. Therefore

$$d_s(0) = d+1 \ , \ d_s(\infty) = d+2 \tag{3.10}$$

The IR limit in this scenario is useful to observe that every component of $\vec{p}$ gives a double counting in the dimension when the deformation is with a modulus: with the MDR considered in the first section, the result is $1+2d$ because in the deformation there are all the $d$ components of $\vec{p}$; here the result is $d+2$ because there is just one component of $\vec{p}$ in the deformation. From this we get the result that "each dimension counts twice" in the IR limit when a linear deformation is considered. Given the symmetry between energy and momentum in the definition eq. (2.4) this statement can be also extended to the temporal dimension, as can be seen with the following MDR.

## 3.3 Spectral dimension with $E^2 = p^2 + \alpha E$ MDR

Considering the MDR

$$E^2 = p^2 + \alpha E \tag{3.11}$$

the return probability eq. (2.4) is given by

$$P(s) = \frac{S^{(d)}(1)}{(2\pi)^{d+1}} \int_0^{+\infty} dp\, p^{d-1} e^{-sp^2} \int_{-\infty}^{+\infty} dE\, e^{-sE^2 - is\alpha E}$$

[10] G&R , 3.462.1

[11] The methods are actually equivalent since $D_{-1}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}}\, \text{erfc}(\frac{z}{\sqrt{2}})$

The second integral is Gaussian

$$\int_{-\infty}^{+\infty} e^{-bx^2+cx}\,dx = \sqrt{\frac{\pi}{b}}\, e^{\frac{c^2}{4b}}$$

while the first integral can be computed with the same integral used in the previous sections (G&R , 3.461.2b); setting $b = s$, $c = is\alpha$, $m = d-1$ and $p = s$

$$P(s) \propto s^{-\frac{d+1}{2}}\, e^{-s\frac{\alpha^2}{4}}$$

therefore

$$d_s(s) = d + 1 + s\frac{\alpha^2}{2} \Rightarrow d_s(0) = d + 1\,,\; d_s(\infty) = \infty \tag{3.12}$$

In this case the dimension is divergent but is positive: this happens because $E$ is the variable that is Wick rotated, so there is a factor $i$ in the computation of the Gaussian integral. It should be noted that if $\alpha$ is considered an energy parameter, so that $\alpha \mapsto i\alpha$ should be considered, there is again a negative divergence in the spectral dimension and a probability that can be greater than one:

$$P(s) \propto s^{-\frac{d+1}{2}}\, e^{s\frac{\alpha^2}{4}}$$

$$d_s(s) = d + 1 - s\frac{\alpha^2}{2} \Rightarrow d_s(0) = d + 1\,,\; d_s(\infty) = -\infty$$

So the opposite of what happened with the deformation linear in $p_1$ happens with this MDR. As done in the previous section, there is the possibility of avoiding these troubles by considering a "physical" deformation.

### 3.3.1 $|\alpha E|$ deformation

As done in the previous section consider the MDR

$$E^2 = p^2 + |\alpha E|\ \ ,\ \ \alpha > 0 \tag{3.13}$$

which is the same thing of saying that $E$ should be restricted to positive values. Considering a Wick rotation of both $\alpha$ and $E$ the return probability is given by [12]

$$P(s) = 2\frac{S^{(d)}(1)}{(2\pi)^{d+1}} \int_0^{+\infty} dp\, p^{d-1} e^{-sp^2} \int_0^{+\infty} dE\, e^{-sE^2 - s\alpha E}$$

As happened with the deformation in $p_1$ it is possible to do the computations of the second integral using the complementary error function but to study the IR and UV limits it is not necessary, thanks again to the machinery developed in Appendix A. The first integral will give a contribution of $d$ to the spectral dimension while the second

[12] It is also possible to consider a MDR of the form $E^2 = p^2 + \alpha|E|\ ,\ \alpha > 0$ and considering only a Wick rotation of $E$. The return probability and hence the spectral dimension are the same.

integral will contribute as $\frac{1}{\gamma_M+1}$ in the UV limit and $\frac{1}{\gamma_m+1}$ in the IR limit, where $\gamma_M = 0$ and $\gamma_m = -\frac{1}{2}$. Therefore

$$d_s(0) = d+1 \quad , \quad d_s(\infty) = d+2 \tag{3.14}$$

As already stated in the previous section, the temporal dimension gives a double counting too in the IR limit. It is easy to guess what will happen if a deformation with $\sqrt{E^2 - p^2}$ is considered: the IR limit should give $2+2d$ if this double counting is met when first order deformations are considered. This is shown in the following section.

## 3.4 Spectral dimension with $E^2 = p^2 + \alpha\sqrt{E^2 - p^2}$ MDR

Considering the MDR

$$E^2 = p^2 + \alpha\sqrt{E^2 - p^2} \tag{3.15}$$

the return probability eq. (2.4) is given by [13]

$$P(s) \propto \int_0^\infty d\rho\, \rho^d e^{-s(\rho^2+\alpha\rho)}$$

This integral can be easily computed with Mathematica®; the return probability is

$$P(s) \propto s^{-\frac{d}{2}}\, U\!\left(1+\frac{d}{2}, \frac{3}{2}, \frac{\alpha^2 s}{4}\right)$$

The derivative of this probability reads

$$P'(s) \propto \frac{1}{8} s^{-1-\frac{d}{2}} \left[ -4\, d\, U\!\left(1+\tfrac{d}{2}, \tfrac{3}{2}, \tfrac{\alpha^2 s}{4}\right) - \alpha^2 \left(1+\tfrac{d}{2}\right) U\!\left(2+\tfrac{d}{2}, \tfrac{5}{2}, \tfrac{\alpha^2 s}{4}\right)\right]$$

the spectral dimension is then given by

$$d_s(s) = d + s\frac{\alpha^2\,(2+d)\, U\!\left(2+\frac{d}{2}, \frac{5}{2}, \frac{\alpha^2 s}{4}\right)}{4\, U\!\left(1+\frac{d}{2}, \frac{3}{2}, \frac{\alpha^2 s}{4}\right)} \tag{3.16}$$

Therefore, computing the limits of this expression

$$d_s(\infty) = 2d+2 \quad , \quad d_s(0) = d+1 \tag{3.17}$$

This is in perfect agreement with what has been found for the MDRs of the previous sections; for every component of the $D$-momentum in the deformation there is a "+1" in the spectral dimension counting in the IR limit as can be seen from eq. (3.6), eq. (3.10), eq. (3.14) and eq. (3.17).

[13] In order to obtain an Euclidean dipsersion relation we consider a Wick rotation of both the energy $E \to -iE$ and $\alpha$, $\alpha \to -i\alpha$ [25].

## 3.5 Non-analytic MDRs

In this section we consider the other class of MDRs which is considered in phenomenological scenarios in quantum gravity; this MDRs are peculiar because they are characterized by a non analytic behavior for small momenta, and this is probably the reason why, in the end, we obtain a divergent spectral dimension.

Consider firstly the MDR

$$E^2 = p^2 + \frac{4\alpha_1}{\theta^2\rho^2} \ , \ \rho^2 := \sum_{n=1}^{d-1} p_n^2$$

the return probability eq. (2.4) is given by

$$P(s) = \frac{S_c^{(d)}(1,1)}{(2\pi)^{d+1}} \int_{-\infty}^{+\infty} dp_d \, e^{-sp_d^2} \int_{-\infty}^{+\infty} dE \, e^{-sE^2} \int_0^{+\infty} d\rho \, e^{-s\rho^2 - s\frac{4\alpha_1}{\theta^2\rho^2}} \rho^{d-2}$$

The first two integrals are Gaussian, the last one can be computed by means of

$$\int_0^{+\infty} x^{\nu-1} e^{\frac{\beta}{x} - \gamma x} dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\gamma}\right) \ , \ Re(\beta) > 0 \ , \ Re(\gamma) > 0$$ [14]

setting $\rho^2 = x$, $\beta = \frac{4\alpha_1 s}{\theta^2}$ and $\nu = \frac{d-1}{2}$ the return probability is

$$P(s) \propto s^{-1} K_{\frac{d-1}{2}}\left(2s\sqrt{\frac{4\alpha_1}{\theta^2}}\right)$$

therefore, calling $u := 2\sqrt{\frac{4\alpha_1}{\theta^2}}$ the spectral dimension reads

$$d_s(u) = 2\left[1 - u\frac{K'_{\frac{d-1}{2}}(u)}{K_{\frac{d-1}{2}}(u)}\right]$$

and using the relation

$$K'_\rho(u) = -K_{\rho-1}(u) - \frac{\rho}{u} K_\rho(u)$$

it becomes

$$d_s(u) = 2\left[1 + u\frac{K_{\frac{d-3}{2}}(u) + \frac{d-1}{2u} K_{\frac{d-1}{2}}(u)}{K_{\frac{d-1}{2}}(u)}\right] \tag{3.18}$$

The asymptotic expansions of the Bessel functions are given by

$$K_\rho(u) \simeq \frac{1}{2}\Gamma(\rho)\left(\frac{u}{2}\right)^{-\rho} \ , \ u \to 0$$

[14] G&R , 3.471.9

$$K_\rho(u) \sim \sqrt{\frac{\pi}{2u}}e^{-u}\left[1+\frac{4\rho^2-1}{8u}+o(u^{-2})\right]\ ,\ u\to\infty$$

therefore

$$d_s(u) = d+1+2u\frac{K_{\frac{d-3}{2}}(u)}{K_{\frac{d-1}{2}}(u)} \simeq d+1+2u\frac{\Gamma\left(\frac{d-3}{2}\right)2^{-\frac{d-3}{2}}u^{-\frac{d-3}{2}}}{\Gamma\left(\frac{d-1}{2}\right)2^{-\frac{d-1}{2}}u^{-\frac{d-1}{2}}} \Rightarrow d_s(0) = d+1$$

when $d \le 3$ the previous calculation is not well defined because of the divergent $\Gamma\left(\frac{d-3}{2}\right)$ in the numerator, so the limit has to be computed manually without the auxiliary asymptotic expansion of the numerator giving

$$d_s(u) \simeq d+1+2u\frac{K_{\frac{d-3}{2}}(u)}{\Gamma\left(\frac{d-1}{2}\right)2^{-\frac{d-1}{2}}u^{-\frac{d-1}{2}}} \to d+1 \text{ for } u\to 0$$

which holds for $d=2$ and $d=3$. For $d=1$ the denominator suffers the same problem, however using $K_{-\rho}(u) = K_\rho(u)$ the same behavior of the case $d=3$ is obtained.

In the $u\to\infty$ limit the spectral dimension reads

$$\begin{aligned} d_s(u) &\sim 2\left[1+u\frac{1+\frac{(d-3)^2-1}{8u}+\frac{d-1}{2u}\left(1+\frac{(d-1)^2-1}{8u}\right)}{1+\frac{(d-1)^2-1}{8u}}\right]\sim \\ &\sim 2+2u\left[1+\frac{(d-3)^2-1}{8u}\right]\left[1-\frac{(d-1)^2-1}{8u}\right]+(d-1)\left[1+\frac{(d-1)^2-1}{8u}\right]\left[1-\frac{(d-1)^2-1}{8u}\right]\sim \\ &\sim 3+2u \Rightarrow d_s(\infty)=\infty \end{aligned} \tag{3.19}$$

As already mentioned, the spectral dimension diverges with this MDR in the IR limit.

### 3.5.1 Logarithmic term

Another MDR with this type of deformation is

$$E^2 = p^2+\frac{4\alpha_1}{\theta^2\rho^2}+\alpha_2 m^2\log\left(\frac{1}{4}\theta^2\rho^2m^2\right) \tag{3.20}$$

the return probability eq. (2.4) is given by

$$\begin{aligned} P(s) &= \frac{S_c^{(d)}(1,1)}{(2\pi)^{d+1}}\int_{-\infty}^{+\infty}dp_d\, e^{-sp_d^2}\int_{-\infty}^{+\infty}dE\, e^{-sE^2}\int_0^{+\infty}d\rho\, e^{-s\rho^2-s\frac{4\alpha_1}{\theta^2\rho^2}-s\alpha_2m^2\log\left(\frac{1}{4}\theta^2\rho^2m^2\right)}\rho^{d-2} = \\ &= \frac{S_c^{(d)}(1,1)}{(2\pi)^{d+1}}\int_{-\infty}^{+\infty}dp_d\, e^{-sp_d^2}\int_{-\infty}^{+\infty}dE\, e^{-sE^2}\int_0^{+\infty}d\rho\, e^{-s\rho^2-s\frac{4\alpha_1}{\theta^2\rho^2}}\rho^{d-2}\left(\frac{\theta^2\rho^2m^2}{4}\right)^{-s\alpha_2m^2} \end{aligned}$$

The first two integrals are Gaussian therefore, calling $\frac{\theta^2\rho^2m^2}{4} := x^2$ the return probability is

$$P(s) \propto s^{-1}\int_0^{+\infty}x^{-2\alpha_2sm^2+d-2}\, e^{-s\left(\frac{4x^2}{m^2\theta^2}+\frac{\alpha_1m^2}{x^2}\right)}$$

which can be computed using the same integral of the previous section (G&R , 3.471.9)

$$P(s) \propto s^{-1}\left(\alpha_1\theta^2 m^4\right)^{-\frac{\alpha_2 s m^2}{2}} K_{-\alpha_2 s m^2+\frac{d}{2}}\left(2s\sqrt{\frac{4\alpha_1}{\theta^2}}\right)$$

Unfortunately, this expression is too messy to derive the spectral dimension because the variable $s$ is both in the argument and the order of the Bessel function $K$. Therefore we restricted ourselves to a MDR which has only the logarithmic deformation.

Using the MDR

$$E^2 = p^2 - \alpha\log(\theta^2\rho^2\beta) \tag{3.21}$$

which is the previous MDR with $\alpha_1 = 0$, $\alpha_2 = -\alpha$ and $\beta = \frac{m^2}{4}$, the return probability is

$$\begin{aligned} P(s) &= \frac{S_c^{(d)}(1,1)}{(2\pi)^{d+1}}\int_{-\infty}^{+\infty} dp_d\, e^{-sp_d^2}\int_{-\infty}^{+\infty} dE\, e^{-sE^2}\int_0^{+\infty} d\rho\, e^{-s\rho^2+s\alpha\log\left(\theta^2\rho^2\beta\right)}\rho^{d-2} \propto \\ &\propto s^{-1}\int_0^{+\infty} d\rho\, e^{-s\rho^2+s\alpha\log\left(\theta^2\rho^2\beta\right)}\rho^{d-2} \end{aligned}$$

The last integral can be computed with [15]

$$\int_0^{+\infty} dx\, x^m e^{-\nu x^n} = \frac{\Gamma\left(\frac{m+1}{n}\right)}{n\nu^{\frac{m+1}{n}}} \ , \ Re(n)>0 \ , \ Re(m)>0 \ , \ Re(\nu)>0$$

Setting $n = 2$, $m = d-2+2s\alpha$, $\nu = s$ and supposing $d \geq 2$ the return probability becomes

$$P(s) \propto s^{-\frac{d+1}{2}}\, e^{s\alpha\log\left(\frac{\theta^2\beta}{s}\right)}\Gamma\left(\frac{d-1}{2}+s\alpha\right)$$

The derivative reads

$$P'(s) \propto -s^{-\frac{d+1}{2}-1} e^{s\alpha\log\left(\frac{\theta^2\beta}{s}\right)}\Gamma\left(\frac{d-1}{2}+s\alpha\right)\left[\frac{d-1}{2}+s\alpha\left(1+\log\left(\frac{s}{\theta^2\beta}\right)\right)-s\alpha\,\psi_0\left(\frac{d-1}{2}+s\alpha\right)\right]$$

Therefore the spectral dimension is

$$d_s(s) = d+1+2s\alpha\left[1+\log\left(\frac{s}{\theta^2\beta}\right)-\psi_0\left(\frac{d-1}{2}+s\alpha\right)\right] \tag{3.22}$$

For the UV limit it is sufficient to observe that $s\to 0$, $s\log\left(\frac{s}{\theta^2\beta}\right)\to 0$ and $\psi_0\left(\frac{d-1}{2}\right)<\infty$, therefore $d_s(0) = d+1$. For the IR limit, instead, the asymptotic expansion of the Polygamma function is

$$\psi_0(z) \sim \log z - \sum_{k=1}^{\infty}\frac{B_k}{kz^k} \ , \ z\to\infty$$

[15] G&R , 3.326.2

hence

$$\begin{aligned} d_s(s) &\sim d+1+2s\alpha\left[1+\log s-\log(\theta^2\beta)-\log(\tfrac{d-1}{2}+s\alpha)+\tfrac{1}{d-1+2s\alpha}\right]\sim \\ &\sim d+1+2s\alpha\left[1+\log s-\log(\theta^2\beta)-\log(s\alpha)+\frac{1}{2s\alpha(\frac{d-1}{2s\alpha}+1)}\right]\sim \\ &\sim d+1+2s\alpha\left[1-\log(\theta^2\beta\alpha)-\frac{d-1}{2s\alpha}+\frac{1}{2s\alpha}\right]=3+s\alpha\log\big(\tfrac{e}{\theta^2\beta\alpha}\big) \end{aligned}$$

The spectral dimension in the IR limit is therefore given by

$$d_s(\infty)=\begin{cases}\infty, & \theta^2\alpha\beta\neq e\\ 3\ , & \theta^2\alpha\beta=e\ ,\ \forall\ d\geq 2\end{cases} \tag{3.23}$$

Thus the spectral dimension is always divergent in this case too except when the numerical coincidence in the second line of eq. (3.23) occurs.

*Compare yourself to who you were yesterday, not to who someone else is today.*

*~ Jordan B. Peterson, 12 Rules for Life: An Antidote to Chaos*

Fourth Chapter

# *Thermal dimension with UV/IR mixing*

In this chapter the computations done for the thermal dimension with UV/IR mixing MDRs are shown. The considered MDRs are the same used for the spectral dimension with the exception of the ones that have singularities in $p = 0$ which are rather pathological and give difficulties in interpretation while studying the on-shellness of the modes. In what follows the IR regime will be $T \to 0$ or $\beta \to \infty$ and the UV regime will be $T \to \infty$ or $\beta \to 0$. All the series expansions were done using the software Mathematica®. We will not focus on the UV results because they are trivial, that is the thermal dimension in the UV limit is always $d+1$ as it should be since we are considering IR deformations.

## 4.1 Thermal dimension with $E^2 = p^2 + \alpha p$ MDR

Considering the MDR

$$E^2 = p^2 + \alpha p \tag{4.1}$$

the logarithm of the partition function eq. (2.12) is

$$\log Z \propto V \int_{-\infty}^{+\infty} dE \int_0^{+\infty} dp\, p^{d-1}\, E \log\left(1 - e^{-\beta E}\right) \theta(E)\, \delta(E^2 - p^2 - \alpha p)$$

Expressing in dimensionless form the integral in $E$ by means of $\beta E := t$, $\log Z$ becomes

$$\log Z \propto V \int_{-\infty}^{+\infty} dt \int_0^{+\infty} dp\, p^{d-1}\, \beta^{-2} t \log\left(1 - e^{-t}\right) \theta(t)\, \delta(t^2 \beta^{-2} - p^2 - \alpha p)$$

The next step is solving the $\delta$ with respect to p

$$\delta(t^2 \beta^{-2} - p^2 - \alpha p) = \frac{\delta(p - \bar{p})}{\left|2\bar{p} + \alpha\right|}$$

where $\bar{p}$ is the only positive solution of $t^2 \beta^{-2} - p^2 - \alpha p = 0$ given by

$$\bar{p} = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta^2 t^2}}{2}$$

This solution shows the differences between the two cases $\alpha > 0$ and $\alpha < 0$ in the IR regime: when $\alpha > 0$, the lowest allowed momentum is $\bar{p} = 0$ while when $\alpha < 0$ the

lowest allowed momentum is $\bar{p} = -\alpha$; this is what was observed for the spectral dimension, whose problem for $\alpha < 0$ is that the integration is carried out over the whole momentum space, allowing $p = 0$ as lowest momentum and this results in a negative divergence of the spectral dimension. In this case, this lowest bound will be taken into account for the integration boundaries.
For both $\alpha > 0$ and $\alpha < 0$ the resolution of the $\delta$ function gives

$$\delta(t^2\beta^{-2} - p^2 - \alpha p) = \frac{\delta(p-\bar{p})}{\sqrt{\alpha^2 + 4\beta^{-2}t^2}}$$

so that $\log Z$ becomes

$$\log Z \propto V \int_0^{+\infty} dt \left(-\alpha + \sqrt{\alpha^2 + 4\beta^{-2}t^2}\right)^{d-1} \left(\alpha^2 + 4\beta^{-2}t^2\right)^{-\frac{1}{2}} \beta^{-2} t \log\left(1 - e^{-t}\right)$$

In the UV regime we consider an expansion around $\beta = 0$ and carry out the computations to leading order in $\beta$. With this expansion the integrand of $\log Z$ becomes

$$\left(-\alpha + \sqrt{\alpha^2 + 4\beta^{-2}t^2}\right)^{d-1} \left(\alpha^2 + 4\beta^{-2}t^2\right)^{-\frac{1}{2}} \beta^{-2} t \log\left(1 - e^{-t}\right) \simeq \beta^{-d} 2^{-2+d} t^{d-1} \log\left(1 - e^{-t}\right)$$

therefore

$$\log Z \simeq V \beta^{-d} \ , \ d > 1$$

which is the same behavior in $\beta$ and $V$ of the classical case, therefore the thermal dimension is $d+1$ in the UV limit, as it is expected since the MDR is IR deformed.

In the IR regime, the expansion is done in $\beta^{-1}$ around $\beta^{-1} = 0$, carrying out the integration to the leading order in $\beta^{-1}$. The expansion of the integrand of $\log Z$ depends on the sign of $\alpha$ in this case leading to

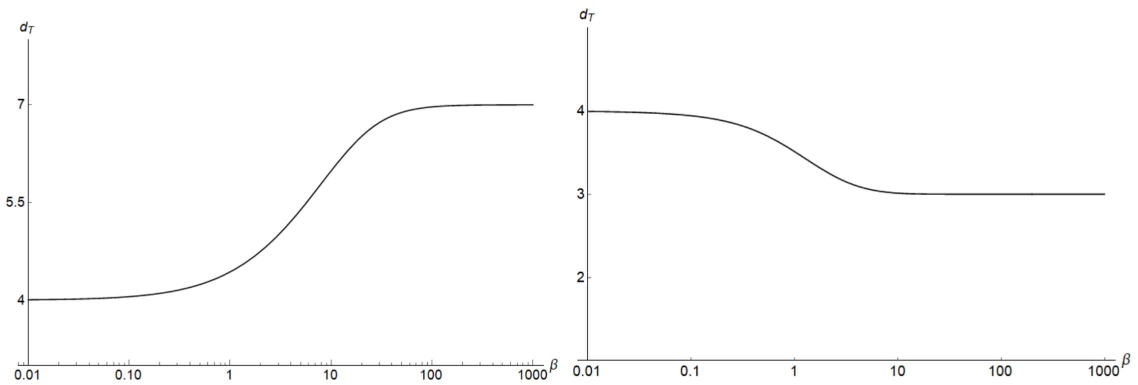
$$\log Z \sim \begin{cases} V \beta^{-2d} \int_0^\infty dt\, t^{2d-1} \log\left(1 - e^{-t}\right) \ , \ \alpha > 0 \\ V \beta^{-2} \int_0^\infty dt\, t \log\left(1 - e^{-t}\right) \ , \ \alpha < 0 \end{cases} \tag{4.2}$$

with $d > \frac{1}{2}$ in the first line so that the integral converges. The derivative with respect to $\beta$ is

$$-\partial_\beta \log Z \sim \begin{cases} V\, 2d\, \beta^{-(2d+1)} \int_0^\infty dt\, t^{2d-1} \log\left(1 - e^{-t}\right) \ , \ \alpha > 0 \\ V \beta^{-3} \int_0^\infty dt\, t \log\left(1 - e^{-t}\right) \ , \ \alpha < 0 \end{cases} \tag{4.3}$$

Since $-V^{-1}\partial_\beta \log Z$ is the energy density, we can see already here that the Stefan-Boltzmann law scales with dimensionality $2d+1$ for $\alpha > 0$ and 3 for $\alpha < 0$. Computing the derivatives of eq. (4.2) with respect to the volume to obtain the pressure defined in eq. (2.9) and computing the ratio $w$ in eq. (2.10) we get the equation of state in the IR limit

$$w = \begin{cases} \frac{1}{2d} \ , \ \alpha > 0 \, , \, d > \frac{1}{2} \\ \frac{1}{2} \ , \ \alpha < 0 \ , \ \forall\, d \end{cases} \tag{4.4}$$




(a) Thermal dimension for eq. (4.2) with $\alpha = 1$ and $d = 3$.

(b) Thermal dimension for eq. (4.2) with $\alpha = -1$ and $d = 3$.

Figure 4.1: Plots of the thermal dimension for eq. (4.2) for the two possible signs of $\alpha$. The deformation parameter has been set to unity therefore $\beta$ is dimensionless, or equivalently the thermal dimension is plotted as a function of $\beta|\alpha|$.

therefore the thermal dimension $d_T$ is

$$\boxed{d_T = \begin{cases} 2d+1 \ , \ \alpha > 0 \ , \ d > \frac{1}{2} \\ 3 \ , \ \alpha < 0 \ , \ \forall \, d \end{cases}} \tag{4.5}$$

from both the equation of state and the Stefan-Boltzmann law.

From eq. (4.5) and fig. 4.1 it can be seen that the thermal dimension agrees with the spectral dimension section 3.1 when $\alpha > 0$ and it gives a finite result when $\alpha < 0$ while the spectral dimension eq. (3.2) gives the unphysical result $-\infty$. As already discussed, this difference arises since the physical momentum is bounded in the IR and this information gets lost in the spectral dimension due to the integration over the whole momentum space.

## 4.2 Thermal dimension with $E^2 = p^2 + \alpha p_1$ MDR

With the MDR

$$E^2 = p^2 + \alpha p_1 \tag{4.6}$$

the logarithm of the partition function is

$$\log Z \propto V \int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} dp_1 \int_0^{+\infty} d\rho \, \rho^{d-2} \, E \log\left(1 - e^{-\beta E}\right) \theta(E) \, \delta(E^2 - \rho^2 - p_1^2 - \alpha p_1)$$

Expressing in dimensionless form all the integrals with $\beta E := t$, $\beta \rho := u$ e $\beta p_1 := v$ the logarithm of $Z$ becomes

$$\log Z = a V \beta^{-d} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dv \int_0^{+\infty} du \, u^{d-2} \, t \log\left(1 - e^{-t}\right) \theta(t) \, \delta(t^2 - u^2 - v^2 - \alpha \beta v)$$

Solving the $\delta$ with respect to $u$

$$\delta(t^2 - u^2 - v^2 - \alpha\beta v) = \frac{\delta(u-\bar{u})}{2\bar{u}}$$

where $\bar{u} = \sqrt{t^2 - v^2 - \alpha\beta v}$ must be real, hence

$$\frac{-\alpha\beta - \sqrt{\alpha^2\beta^2 + 4t^2}}{2} \leq v \leq \frac{-\alpha\beta + \sqrt{\alpha^2\beta^2 + 4t^2}}{2}$$

Consequently $\log Z$ becomes

$$\log Z \propto V\beta^{-d} \int_0^{+\infty} dt \int_{\frac{-\alpha\beta-\sqrt{\alpha^2\beta^2+4t^2}}{2}}^{\frac{-\alpha\beta+\sqrt{\alpha^2\beta^2+4t^2}}{2}} dv \left(t^2 - v^2 - \alpha\beta v\right)^{\frac{d-3}{2}} t \log\left(1 - e^{-t}\right) \tag{4.7}$$

The integral in $v$ can be solved using Mathematica®

$$\int_{\frac{-\alpha\beta-\sqrt{\alpha^2\beta^2+4t^2}}{2}}^{\frac{-\alpha\beta+\sqrt{\alpha^2\beta^2+4t^2}}{2}} dv \left(t^2 - v^2 - \alpha\beta v\right)^{\frac{d-3}{2}} = -2^{2-d}\pi^{\frac{3}{2}}(\alpha^2\beta^2 + 4t^2)^{-1+\frac{d}{2}} \frac{\sec(\frac{d\pi}{2})}{\Gamma(\frac{3}{2}-\frac{d}{2})\Gamma(\frac{d}{2})} \ , \ d > 1$$

Before throwing away the numerical constants it is important to observe that

$$\frac{\sec(\frac{d\pi}{2})}{\Gamma(\frac{3}{2}-\frac{d}{2})}$$

is finite for odd values of $d$ when $d \geq 3$ since both the numerator and the denominator have a pole for these values and the order of the pole is the same, so that the ratio in finite. Therefore $\log Z$ becomes

$$\log Z \propto V\beta^{-2} \int_0^{+\infty} dt\, t\left(\alpha^2 + \tfrac{4t^2}{\beta^2}\right)^{-1+\frac{d}{2}} \log\left(1 - e^{-t}\right) \tag{4.8}$$

The UV expansion around $\beta = 0$ of this integral gives

$$\log Z \simeq V\beta^{-d} \int_0^{\infty} dt\, t^{d-1}$$

so that the UV value of the thermal dimension is $d+1$ in this case too.

For the IR expansion, in this case, the sign of $\alpha$ does not play any role because $\alpha^2 + \frac{4t^2}{\beta^2} > 0$. Expanding in series around $\beta^{-1} = 0$ $\log Z$ becomes

$$\log Z \sim V\beta^{-2}$$

thus

$$\rho \sim 2\beta^{-3} \ , \ P \sim \beta^{-2} \ \Rightarrow \ w = \frac{1}{2} \tag{4.9}$$

consequently

$$d_T = 3 \;\; \forall\, d > 1 \tag{4.10}$$

The spectral dimension in this case eq. (3.8) diverges negatively, unless a Wick rotation of $\alpha$ is considered too; the thermal dimension, instead, gives a finite result and we get that thermodynamics behaves as if spacetime were 3-dimensional for every value of $d$. The failure of the spectral dimension in this case is again the integration over the whole momentum space; the thermal dimension restricts the integration to the physical part of the momentum space as can be seen in eq. (4.7).

### 4.2.1 $|p_1|$ deformation

Considering the MDR

$$E^2 = p^2 + \alpha |p_1| \tag{4.11}$$

$\log Z$ is given by

$$\log Z \propto V \int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} dp_1 \int_0^{+\infty} \rho^{d-2}\, E \log\left(1 - e^{-\beta E}\right) \theta(E)\, \delta(E^2 - \rho^2 - p_1^2 - \alpha |p_1|)$$

Expressing in dimensionless form in the same way as done before $\log Z$ becomes

$$\log Z \propto V \beta^{-d} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dv \int_0^{+\infty} du\, u^{d-2}\, t \log\left(1 - e^{-t}\right) \theta(t)\, \delta(t^2 - u^2 - v^2 - \alpha\beta |v|)$$

Next we solve the $\delta$ with respect to $u$

$$\delta(t^2 - u^2 - v^2 - \alpha\beta|v|) = \frac{\delta(u - \bar{u})}{2\bar{u}}$$

where $\bar{u} = \sqrt{t^2 - v^2 - \alpha\beta|v|}$ must be real therefore

$$\frac{\alpha\beta - \sqrt{\alpha^2\beta^2 + 4t^2}}{2} \le v \le \frac{-\alpha\beta + \sqrt{\alpha^2\beta^2 + 4t^2}}{2}$$

Since $v \le 0 \Rightarrow |v| = -v$ and $v \ge 0 \Rightarrow |v| = v$ the integral in $v$ becomes the sum of two integrals

$$\begin{aligned}
\log Z &= a V \beta^{-d} \int_{-\infty}^{+\infty} dt\, t \log\left(1 - e^{-t}\right) \left[ \int_{\frac{\alpha\beta - \sqrt{\alpha^2\beta^2 + 4t^2}}{2}}^{0} dv \left(t^2 - v^2 + \alpha\beta v\right)^{\frac{d-3}{2}} + \right. \\
&\left. + \int_0^{\frac{-\alpha\beta + \sqrt{\alpha^2\beta^2 + 4t^2}}{2}} dv \left(t^2 - v^2 - \alpha\beta v\right)^{\frac{d-3}{2}} \right]
\end{aligned}$$

The sum of the two integrals in $v$ can be done with Mathematica® and gives

$$\begin{aligned}
&\int_{\frac{\alpha\beta - \sqrt{\alpha^2\beta^2 + 4t^2}}{2}}^{0} dv \left(t^2 - v^2 + \alpha\beta v\right)^{\frac{d-3}{2}} + \int_0^{\frac{-\alpha\beta + \sqrt{\alpha^2\beta^2 + 4t^2}}{2}} dv \left(t^2 - v^2 - \alpha\beta v\right)^{\frac{d-3}{2}} = \\
&= \frac{2^{1+\frac{d+1}{2}}}{d-1} \beta^{-1} t^{-1+d} \left(\alpha^2 + \tfrac{4t^2}{\beta^2}\right)^{\frac{-3+d}{4}} \left(\alpha + \sqrt{\alpha^2 + \tfrac{4t^2}{\beta^2}}\right)^{\frac{1-d}{2}} {}_2F_1\left(\tfrac{d-3}{2}, \tfrac{-1+d}{2}, \tfrac{1+d}{2}, \tfrac{1}{2} - \frac{a}{2\sqrt{\alpha^2 + \frac{4t^2}{b^2}}}\right)
\end{aligned}$$

with $d > 1$. This result holds both when $\alpha > 0$ and when $\alpha < 0$.

In the UV ($\beta = 0$) limit the same result of the previous example is obtained, that is a trivial thermal dimension.

The IR series expansion around $\beta^{-1} = 0$ will depend upon the sign of $\alpha$; $\log Z$ and its $\beta$ derivative are given by

$$\log Z \sim \begin{cases} -V\beta^{-d-1}4\,\alpha^{-1}\Gamma(d)\zeta(2+d) \ , \ \alpha > 0 \\ \\ V\beta^{-2}4\,(-\alpha)^{-2+d}\pi\,\Gamma\left(\frac{d+1}{2}\right)\zeta(3)\frac{\sec\left(\frac{d\pi}{2}\right)}{\Gamma\left(\frac{3-d}{2}\right)\Gamma(d)} \ , \ \alpha < 0 \end{cases} \tag{4.12}$$

$$-\partial_\beta \log Z \sim \begin{cases} -V\beta^{-d-2}4\,\alpha^{-1}\Gamma(d+1)\zeta(2+d) \ , \ \alpha > 0 \\ \\ V\beta^{-3}8\,(-\alpha)^{-2+d}\pi\,\Gamma\left(\frac{d+1}{2}\right)\zeta(3)\frac{\sec\left(\frac{d\pi}{2}\right)}{\Gamma\left(\frac{3-d}{2}\right)\Gamma(d)} \ , \ \alpha < 0 \end{cases} \tag{4.13}$$

therefore the equation of state is

$$w = \begin{cases} \frac{1}{d+1} \ , \ \alpha > 0 \, , \, d > 1 \\ \\ \frac{1}{2} \ \forall\, d > 1 \ , \ \alpha < 0 \end{cases} \tag{4.14}$$

and the thermal dimension is

$$d_T = \begin{cases} d+2 \ , \ \alpha > 0 \, , \, d > 1 \\ \\ 3 \ \forall\, d > 1 \ , \ \alpha < 0 \end{cases} \tag{4.15}$$

In the case $\alpha > 0$ we get the same result obtained with the spectral dimension. The case $\alpha < 0$ was not considered for the spectral dimension because it gave the same problems of the MDR without the module; for the thermal dimension we get a sensible result as happened for eq. (4.6). The problem of the spectral dimension is once again the integration domain.

## 4.3 Thermal dimension with $E^2 = p^2 + \alpha E$ MDR

Considering the MDR

$$E^2 = p^2 + \alpha E \tag{4.16}$$

focusing firstly on the case $\alpha > 0$, $\log Z$ is given by

$$\log Z \propto V \int_{-\infty}^{+\infty} dE \int_0^{+\infty} dp\, p^{d-1}\, \delta\left(E^2 - p^2 - \alpha E\right)\theta(E)\, E\, \log\left(1 - e^{-\beta E}\right)$$

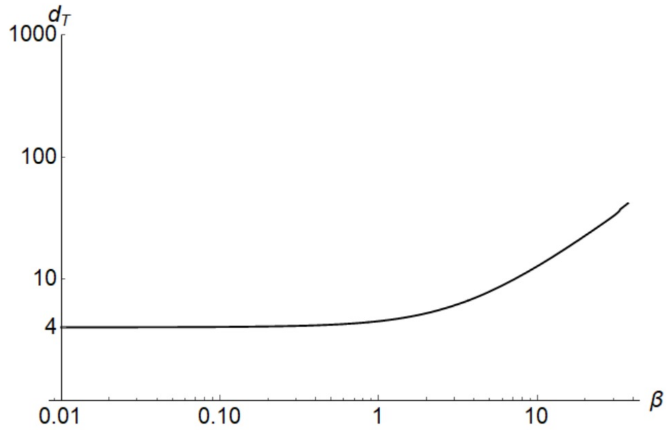



Figure 4.2: Thermal dimension for eq. (4.16) with $\alpha = 1$ and $d = 3$.

Solving the $\delta$ with respect to $p$

$$\delta(E^2 - p^2 - \alpha E) = \frac{1}{2|p|_{p=\sqrt{E^2-\alpha E}}}\delta(p - \sqrt{E^2 - \alpha E})$$

and considering that with $\alpha > 0$ energy must be greater than $\alpha$ so that the square root is real, $\log Z$ becomes

$$\log Z \propto V \int_{\alpha}^{+\infty} dE\, E\, (E^2 - \alpha E)^{\frac{d-2}{2}} \log(1 - e^{-\beta E})$$

Expressing in dimensionless form by setting $\beta E = t$ and setting $\alpha = 1$ by redefining energies and temperatures $\alpha^{-1}\beta \mapsto \beta$ , $\alpha E \mapsto E$ we get

$$\log Z \propto V \beta^{-\frac{d-2}{2}} \int_{\beta}^{+\infty} dt\, t\, (\beta^{-1} t^2 - t)^{\frac{d-2}{2}} \log(1 - e^{-t})$$

The $\beta$ derivative is composed of two pieces: the first one is the derivative of the lower bound of the integral which is zero since the integrand is zero in $t = \beta$; the other piece comes from the derivative of the integrand which is

$$-\partial_\beta \left[ t\, (\beta^{-1} t^2 - t)^{\frac{d-2}{2}} \log(1 - e^{-t}) \right] = -\frac{\beta^{-(d+1)} (t^2 - \beta t)^{\frac{d}{2}} [\beta(2+d) - 2d\, t]}{2t(\beta - t)^2}\, t \log(1 - e^{-t})$$

Therefore the thermal dimension computed via the equation of state eq. (2.10) is given by

$$d_T(\beta) = 1 + \beta^{-\frac{d+2}{2}} \frac{\int_{\beta}^{+\infty} dt\, (t^2 - \beta t)^{\frac{d}{2}} [\beta(2+d) - 2d\, t] [2t(\beta - t)^2]^{-1}\, t \log(1 - e^{-t})}{\int_{\beta}^{+\infty} dt\, t\, (\beta^{-1} t^2 - t)^{\frac{d-2}{2}} \log(1 - e^{-t})} \tag{4.17}$$

The plot of this result for $d = 3$ is shown in fig. 4.2.

From fig. 4.2 we see that the thermal dimension in this case is constantly four until $\beta \simeq 1$ which means, remembering the redefinition of temperature, $\beta \simeq \alpha^{-1}$; for this value of $\beta$ the thermal dimension begins to increase and for large values of $\beta$ it is divergent. We can compare this result with the two cases considered for the spectral dimension. If we compare it to eq. (3.12) we get that both spectral and thermal dimension are divergent in the IR; if we compare it to eq. (3.14) we get a completely different behavior, because in eq. (3.14) the IR value of the spectral dimension is $d+2$. It could then be argued that the deformation linear in $E$ produces a divergent notion of dimensionality for the spacetime, and this could be used, for instance, to bound the value of $\alpha$: given that in everyday life, which is the IR regime of quantum gravity for sure, we see four dimensions, it can be said that $\alpha \geq k_B T_{max}$ where $T_{max}$ is the maximum temperature experimentally accessible.

### 4.3.1 $\alpha < 0$

When $\alpha < 0$ in eq. (4.16) we can follow the same steps as before but the square root $\sqrt{E^2 - \alpha E}$ will always be well-defined, so there is no constraint on the values that $E$ can take, therefore

$$\log Z \propto, V\beta^{-\frac{d-2}{2}} \int_0^{+\infty} dt\, t\, (\beta^{-1} t^2 + t)^{\frac{d-2}{2}} \log(1 - e^{-t})$$

where $-\alpha = 1$ with the same redefinition of temperatures and energies done before $-\alpha^{-1}\beta \mapsto \beta$ and $-\alpha E \mapsto E$. Expanding in series the integrand around $\beta^{-1} = 0$ (IR limit) we get

$$\log Z \sim V\beta^{-\frac{d}{2}-1} \int_0^{+\infty} dt\, t^{\frac{d}{2}} \log(1 - e^{-t}) \propto V\beta^{-\frac{d}{2}-1}$$

The $\beta$ derivative is

$$-\partial_\beta \log Z \propto V\left(1 + \frac{d}{2}\right)\beta^{-\frac{d}{2}-2}$$

therefore the thermal dimension is

$$d_T = 2 + \frac{d}{2} \tag{4.18}$$

that is different from eq. (3.12) which gives a divergent result.

## 4.4 Thermal dimension with $E^2 = p^2 + \alpha\sqrt{E^2 - p^2}$ MDR

Considering the MDR

$$E^2 = p^2 + \alpha\sqrt{E^2 - p^2} \tag{4.19}$$

focusing firstly on $\alpha > 0$, $\log Z$ is given by

$$\log Z \propto V \int_{-\infty}^{+\infty} dE \int_0^{+\infty} dp\, p^{d-1}\, \delta\left(E^2 - p^2 - \alpha\sqrt{E^2 - p^2}\right)\theta(E)\, E \log\left(1 - e^{-\beta E}\right)$$

The $\delta$ can be rewritten as

$$\delta\left(E^2 - p^2 - \alpha\sqrt{E^2 - p^2}\right) = \frac{\delta(E-p)}{p\left|-2 + \frac{\alpha}{\sqrt{E^2-p^2}}\right|_{p=E}} + \frac{\delta\left(p - \sqrt{E^2-\alpha^2}\right)}{p\left|-2 + \frac{\alpha}{\sqrt{E^2-p^2}}\right|_{p=\sqrt{E^2-\alpha^2}}}$$

The derivative in the denominator of the first term diverges so we introduce a regulator

$$\delta\left(E^2 - p^2 - \alpha\sqrt{E^2 - p^2}\right) = \frac{\delta(E-p)}{E\left|-2 + \alpha\epsilon\right|} + \frac{\delta\left(p - \sqrt{E^2-\alpha^2}\right)}{\sqrt{E^2-\alpha^2}}$$

where $\epsilon$ has dimension of inverse energy and $\alpha\epsilon \to \infty$; in the second term $E \geq \alpha$ must hold. Therefore $\log Z$ becomes

$$\log Z \propto V\left[|-2+\alpha\epsilon|^{-1}\int_0^{+\infty} dE\, E^{d-1}\log\left(1 - e^{-\beta E}\right) + \int_\alpha^{+\infty} dE\,(E^2-\alpha^2)^{\frac{d}{2}-1}\,E\log\left(1-e^{-\beta E}\right)\right]$$

The first integral gives

$$\int_0^{+\infty} dE\, E^{d-1}\log\left(1-e^{-\beta E}\right) = -\beta^{-d}\,(d-1)!\,\zeta(d+1)$$

in the second integral we chose $\alpha = 1$, making $\epsilon$ dimensionless and redefining the temperature and energy scales: $\beta\alpha \to \beta$ and $\alpha^{-1}E \to E$; expanding the logarithm in the integrand we get the series

$$\int_1^{+\infty} dE\,(E^2-1)^{\frac{d}{2}-1}\,E\log\left(1-e^{-\beta E}\right) = -\frac{2^{\frac{d-1}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{d}{2}\right)\beta^{\frac{1-d}{2}}\sum_{n=1}^{\infty} n^{-\frac{d+1}{2}}K_{\frac{d+1}{2}}(n\beta)$$

therefore $\log Z$ becomes

$$V^{-1}\log Z \propto |-2+\epsilon|^{-1}\beta^{-d}(d-1)!\,\zeta(d+1) + \frac{2^{\frac{d-1}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{d}{2}\right)\beta^{\frac{1-d}{2}}\sum_{n=1}^{\infty} n^{-\frac{d+1}{2}}K_{\frac{d+1}{2}}(n\beta) \tag{4.20}$$

and its $\beta$ derivative is

$$-\partial_\beta \log Z \propto \beta^{-d-1}|-2+\epsilon|^{-1}\,d!\,\zeta(d+1) +$$
$$+\frac{2^{\frac{d-3}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{d}{2}\right)\beta^{\frac{1-d}{2}}\sum_{n=1}^{\infty}\left[(d-1)n^{-\frac{d+1}{2}}K_{\frac{d+1}{2}}(n\beta) + \beta n^{-\frac{d-1}{2}}\left(K_{\frac{d-1}{2}}(n\beta) + K_{\frac{d+3}{2}}(n\beta)\right)\right] \tag{4.21}$$

With eq. (4.20) and eq. (4.21) the parameter $w$ in the equation of state can be computed with eq. (2.10) and the thermal dimension is $d_T = 1 + \frac{1}{w}$; moreover, the limit $\epsilon \to \infty$ has to be considered, therefore the thermal dimension is given by fig. 4.3.

From fig. 4.3 the thermal dimension is 4 for $d = 3$ in the UV regime and it diverges in the IR regime. Actually it is constantly equal to 4 until $\beta \simeq 1$ as happened for the linear in $E$ MDR. It is important to remember that we have redefined the scales so this means that the dimension is constantly 4 until $\beta \simeq \alpha^{-1}$ that is $T \simeq k_B^{-1}\alpha$.

This is a completely different behavior with respect to the spectral dimension eq. (3.17) for which we obtained $2d+2$ in the IR; as happened for the previous MDR, this result could be used to bound the parameter $\alpha$ in the same way.

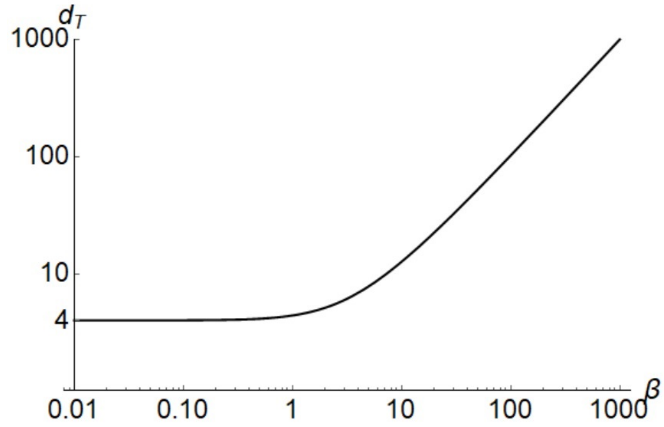



Figure 4.3: Thermal dimension for eq. (4.19) with $\alpha = 1$ and $d = 3$.

### 4.4.1 $\alpha < 0$

When $\alpha < 0$ we can perform the same computations done before but this time the $\delta$ has only one solution indeed

$$E^2 = p^2 + \alpha\sqrt{E^2 - p^2} \;\Rightarrow\; \sqrt{E^2 - p^2}\left(\sqrt{E^2 - p^2} - \alpha\right) = 0$$

therefore only the solution $E = p$ is available and this means that in eq. (4.20) we have only the first term; setting $-\alpha = 1$, $\log Z$ is thus given by

$$\log Z \propto V\,|-2+\epsilon|^{-1}\,\beta^{-d}\,(d-1)!\,\zeta(d+1)$$

therefore the thermal dimension is trivially

$$d_T = d + 1 \tag{4.22}$$

*Nothing happened.*

*~ Roronoa Zoro, One Piece*

*Fifth Chapter*

# Compactified dimensions and Horizons

In the previous two chapters we presented the results obtained in UV/IR mixing scenarios for spectral and thermal dimension. The most interesting result is that with the MDR $E^2 = p^2 + \alpha p \ , \ \alpha > 0$ the two notions of effective dimension give the same result. What would be interesting to study now is the interplay between this UV/IR mixing phenomena with curvature effect: the difference between UV running and IR running of effective dimensions is that the latter allows one to study the concomitant effects of curvature, which is naturally an infrared scale. In this way it would be possible to set bounds on the values of the parameters which appear in the UV/IR mixing MDRs.

The study of the two notions of effective dimensionality in curved spacetimes is however rather problematic as explained in the next section. Nonetheless, what we have found in the attempt of studying curved scenarios is perhaps more interesting: we found that the spectral dimension is influenced in the IR regime also by the presence of "boundaries" (horizons) on the spacetime manifold and both spectral and thermal dimension are affected in the IR regime by compactified dimensions.

## 5.1 Curved spacetime is tricky

The spectral dimension relies on the Euclideanized version of the Lorentzian spacetime manifold and this is a problem in GR since it is always possible to define new coordinates on the spacetime and then the notion of Wick rotation is challenged by the fact that time coordinate is not a physical quantity and there is not a unique time that can be Wick-rotated [20]. Even setting this question aside, there is the problem of solving the heat equation on a curved Riemannian manifold: this problem is studied in the literature mostly by means of series expansions [11], [22], [23], [24] whose coefficients are written in terms of geometrical invariant quantities; the problem with these solutions is that they can be used only for smooth manifolds while for quantum spacetime scenarios the method adopted to calculate the heat trace depends strongly on the QG model one is considering [11], [12], [13]. This is expected because once the curvature is introduced there is no way that the details of the theory can be disentangled from the computation of the spectral dimension since we are taking into account the details of the spacetime structure.

Another problem is the fact that in flat spacetime momentum and energy are the conserved charges associated to translation symmetry and the dispersion relation comes from wave equations solved with monochromatic solutions (plane waves) in order to build general solutions with Fourier transform. All of this works just because there is translation symmetry and the equations are linear so that can be solved with Fourier analysis. On a general curved background there is neither translation symmetry nor the equations are linear, since the derivative must be replaced with the covariant derivative. Only when spacetime is maximally symmetric one can compute the Casimir element of the symmetry algebra and interpret it as a mass-shell condition, but even restricting to these scenarios (basically, de Sitter spacetime and anti-de Sitter spacetime) there is the problem of interpreting the deformation of the dispersion relation because, given that there is no plane wave solution like eq. (2.4) to the diffusion equation, deformations should be introduced at the level of the diffusion equation and this would add more problems to the solution of this equation making it unsolvable for all practical purposes.

The problems with the thermal dimension are lesser mathematical and more physical. The problem that should be addressed is: what is the thermodynamics of a photon gas on a curved spacetime? Once again we are challenged by different problems. To compute a partition function there is the necessity of a mode counting to get the density of states and to do this counting one has to compute the volume in momentum space and the definitions of momentum and energy suffer the same problems as before. There are anyway some studies of thermodynamics on curved spacetime [28] which focus on some particular cases. However, in these cases the Stefan-Boltzmann law in eq. (2.10) does not hold anymore, so the very definition of thermal dimension might need a revision, since the only way to compute it would be with the equation of state, so the statement "the thermodynamics of a photon gas behaves as if spacetime had a number of dimensions given by the thermal dimension" is questionable.

It is clear from these considerations that curved spacetime is a slippery ground on which little intuition is guaranteed; this is rather unfortunate since curvature is an IR scale. Having this in mind, in this chapter we focus on two different IR scenarios. Firstly we consider compactified dimensions, finding that the compactification scale is an IR scale with effects similar to the ones introduced by other compact, but curved, scenarios such as the sphere. Then we consider the "boundary" scenario, that is a flat spacetime with boundaries (or horizons), focusing on the Rindler spacetime.

## 5.2 Compactified dimensions

The simplest non-trivial example that can be considered is a flat spacetime with compactified dimensions. This scenario is particularly interesting for string theory in which compactified dimensions show up. The compactified dimensions are considered as regular dimensions with periodic conditions, that is

$$x_i \cong x_i + 2\pi R_i \quad , \quad i = 1, \ldots, D-p \tag{5.1}$$

In this case, momentum on the compactified dimensions is quantized. Indeed the laplacian is undeformed since spacetime is flat so the wave equation is solved by plane waves but we have to enforce the periodicity in eq. (5.1)

$$e^{ik_i x_i} = e^{ik_i(x_i+2\pi R_i)} \quad \Rightarrow \quad k_i = \frac{n_i}{R_i} \ , \ n_i \in \mathbb{Z}$$

so the return probability is given by

$$P(s) = 2^{D-p}\, S^{(p)}(1) \prod_{i=1}^{D-p} \sum_{n_i=0}^{\infty} \int_0^{+\infty} dk\, k^{p-1}\, e^{-s\left(\frac{n_i^2}{R_i^2}+k^2\right)}$$

The integral in the variable $k$ is

$$\int_0^{+\infty} dk\, k^{p-1}\, e^{-sk^2} = \frac{1}{2} s^{-\frac{p}{2}} \Gamma\left(\frac{p}{2}\right)$$

therefore

$$P(s) = 2^{D-p-1}\, S^{(p)}(1)\Gamma\left(\frac{p}{2}\right) s^{-\frac{p}{2}} \prod_{i=1}^{D-p} \sum_{n_i=0}^{\infty} e^{-s\frac{n_i^2}{R_i^2}} \propto s^{-\frac{p}{2}} \prod_{i=1}^{D-p} \sum_{n_i=0}^{\infty} e^{-s\frac{n_i^2}{R_i^2}} \tag{5.2}$$

With this result it is straightforward to compute the spectral dimension. The derivative with respect to $s$ is

$$P'(s) \propto -s^{-\frac{p}{2}} \left[ -\frac{p}{2} s \prod_{i=1}^{D-p} \sum_{n_i=0}^{\infty} e^{-s\frac{n_i^2}{R_i^2}} + \sum_{j=1}^{D-p} \left( \sum_{n_j=0}^{\infty} \frac{n_j^2}{R_j^2} e^{-s\frac{n_j^2}{R_j^2}} \prod_{i\neq j} \sum_{n_i=0}^{\infty} e^{-s\frac{n_i^2}{R_i^2}} \right) \right]$$

With this equation and (5.2), defining the dimensionless quantities

$$\frac{s}{\bar{R}^2} := \sigma \ , \ \frac{R_i}{\bar{R}} := \ell_i \ , \ \bar{R} := \sum_{i=1}^{D-p} R_i \tag{5.3}$$

the spectral dimension is given by

$$\boxed{d_s(\sigma) = p + 2\sigma \sum_{i=1}^{D-p} \frac{1}{\ell_i^2} \left( \frac{\sum_{n=0}^{\infty} n^2 e^{-\frac{\sigma}{\ell_i^2} n^2}}{\sum_{n=0}^{\infty} e^{-\frac{\sigma}{\ell_i^2} n^2}} \right)} \tag{5.4}$$

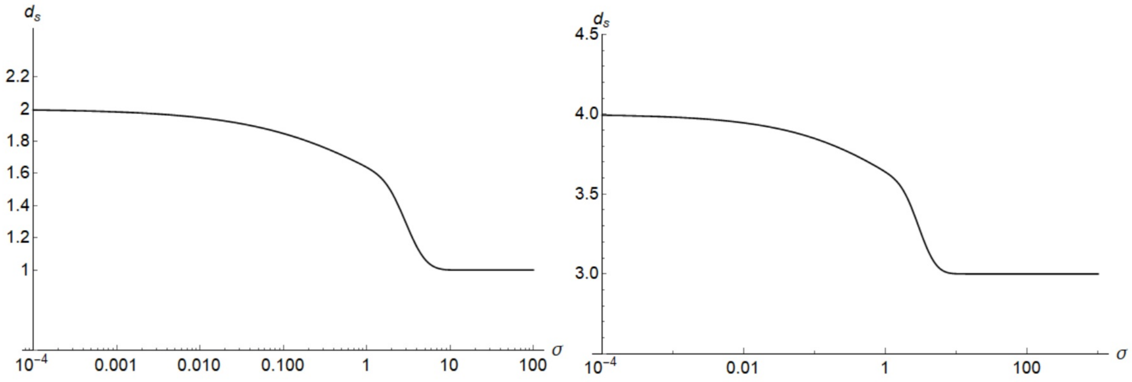
It is important to note that

$$\ell_i = 1 - \frac{\sum_{j\neq i} R_j}{\sum_{i=1}^{D-p} R_i} \quad \Rightarrow \quad 0 < \ell_i \leq 1$$

and

$$\sum_{i=1}^{D-p} \ell_i = 1$$

In what follows we refer to $1+d$ spacetimes with one spatial compactified dimension as "$(1,d)$-cylinders".




(a) Spectral dimension of the (1, 1)-cylinder.

(b) Spectral dimension of the (1, 3)-cylinder.

Figure 5.1: Spectral dimension of the (1, 1)-cylinder and the (1, 3)-cylinder. The series were summed up to $n = 10000$.

### 5.2.1 (1, 1)-cylinder

The simplest example is a (1, 1)-cylinder. This is not an interesting example per se but it is a preliminary example useful to understand what is going on with compactified dimensions.

We can use eq. (5.4) with $D = 2$, $p = 1$ and $\ell = 1$ to obtain

$$d_s(\sigma) = 1 + 2\sigma \frac{\sum_{n=0}^{\infty} n^2 e^{-\sigma n^2}}{\sum_{n=0}^{\infty} e^{-\sigma n^2}}$$

The series cannot be summed analytically, so numerical evaluation is needed. The plot of this spectral dimension is shown in fig. 5.1a.

It is easy to see from fig. 5.1a that in the UV limit the spectral dimension is 2 while in the IR limit it is 1; this result can be interpreted by saying that in the IR limit the compactified dimension disappears and this happens because, if $\sqrt{s}$ is interpreted as a "wavelength" of the probe, the IR limit means $\sqrt{s} >> R$ so a single oscillation of the particle spans the entire dimension, therefore the latter becomes point-like. In particular the transition between $d_s = 2$ and $d_s = 1$ occurs around $\sigma = 1$ that is when $\sqrt{s}$ is of the order of the radius of the radius of the compact dimension.

### 5.2.2 (1, 3)-cylinder

We study now the (1, 3)-cylinder. For this example also the thermal dimension and the spectral dimension with the MDR in eq. (3.1) will be studied.

We can use eq. (5.4) once again with $D = 4$, $p = 3$ and $\ell = 1$

$$d_s(\sigma) = 3 + 2\sigma \frac{\sum_{n=0}^{\infty} n^2 e^{-\sigma n^2}}{\sum_{n=0}^{\infty} e^{-\sigma n^2}} \tag{5.5}$$

The full running of eq. (5.5) is the same as before, as can be seen from fig. 5.1b.

### 5.2.3 (1,3)-cylinder with $E^2 = p^2 + \alpha p$ MDR

The previous example is very useful to study what happens if we consider the interplay between the scale of IR deformation of the MDR

$$E^2 = p^2 + \alpha p$$

and the compactification scale of the cylinder. We need to compute the return probability which is given by

$$P(s) \propto \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dE \int_0^{\infty} dk\, k\, e^{-sE^2 - s\left(k^2 + \frac{n^2}{R^2} + \alpha\sqrt{k^2 + \frac{n^2}{R^2}}\right)}$$

The integral in the variable $E$ is Gaussian, therefore it gives a factor $s^{-\frac{1}{2}}$; the integral over $k$ can be rewritten as

$$\int_0^{\infty} dt\, e^{-s\left(t + \frac{n^2}{R^2} + \alpha\sqrt{t + \frac{n^2}{R^2}}\right)}$$

therefore, calling $R := H^{-1}$

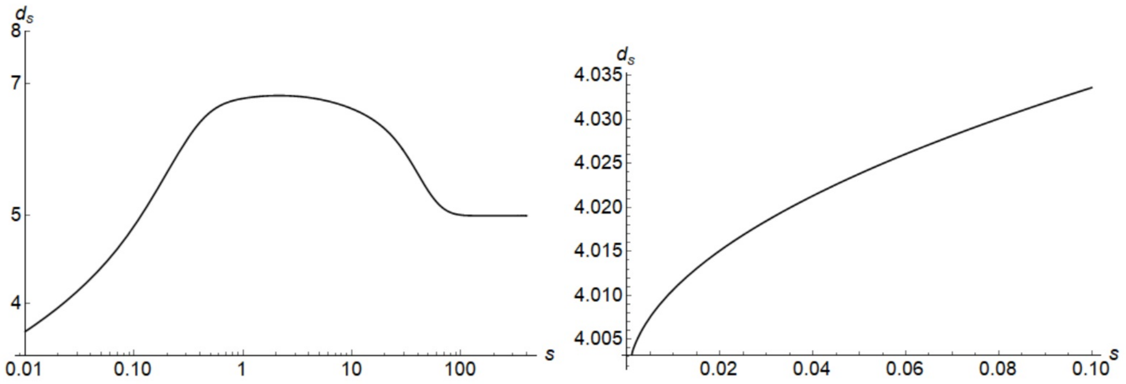
$$\boxed{P(s) \propto -s^{-\frac{3}{2}} e^{\frac{\alpha^2}{4}s} \sum_{n=0}^{\infty} \left\{ -2e^{-s\left(nH + \frac{\alpha}{2}\right)^2} + \alpha\sqrt{\pi}\sqrt{s}\,\mathrm{Erfc}\left[\sqrt{s}\left(nH + \frac{\alpha}{2}\right)\right]\right\}} \tag{5.6}$$

From this the derivative with respect to $s$ can be computed and the spectral dimension is

$$\begin{aligned} d_s(s) &= \frac{2s}{\sum_{n=0}^{\infty}\left\{-2e^{-s\left(nH+\frac{\alpha}{2}\right)^2} + \alpha\sqrt{\pi}\sqrt{s}\,\mathrm{Erfc}\left[\sqrt{s}\left(nH+\frac{\alpha}{2}\right)\right]\right\}} \odot \\ &\odot \sum_{n=0}^{\infty}\left\{\alpha e^{-(\frac{\alpha}{2}+Hn)^2 s}(\frac{\alpha}{2}+Hn) - 2e^{-(\frac{\alpha}{2}+Hn)^2 s}(\frac{\alpha}{2}+Hn)^2 + \right. \\ &- \frac{\alpha\sqrt{\pi}\mathrm{Erfc}\left[\sqrt{s}\left(nH+\frac{\alpha}{2}\right)\right]}{2\sqrt{s}} + \\ &+ 3\frac{-2e^{-(\frac{\alpha}{2}+Hn)^2 s} + \alpha\sqrt{\pi}\sqrt{s}\mathrm{Erfc}\left[\sqrt{s}\left(nH+\frac{\alpha}{2}\right)\right]}{2s} + \\ &+ \left.\alpha^2\frac{2e^{-(\frac{\alpha}{2}+Hn)^2 s} - \alpha\sqrt{\pi}\sqrt{s}\mathrm{Erfc}\left[\sqrt{s}\left(nH+\frac{\alpha}{2}\right)\right]}{4}\right\} \end{aligned} \tag{5.7}$$

Two possible scenarios can be considered: $\alpha > H$ and $\alpha < H$. These two cases are shown in fig. 5.2 and fig. 5.3.

What we get from these results is that there is an interplay between the two IR scales but when $\sqrt{s}$ reaches the IR scale for the curvature, the compactified dimension disappears as in the undeformed case.




(a) Complete plot with $n = 100$, $H = 0.01$ and $\alpha = 10$.

(b) UV magnification computed with $n = 10000$, $H = 0.01$ and $\alpha = 0.1$.

Figure 5.2: IR and UV behavior of the spectral dimension for the (1,3)-cylinder with MDR and $H < \alpha$.

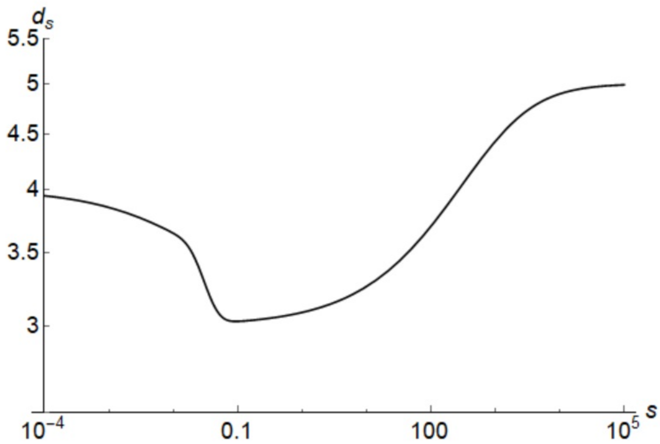



Figure 5.3: Spectral dimension for the (1,3)-cylinder with MDR and $H > \alpha$. $n = 100$, $\alpha = 0.1$ and $H = 10$ were considered.

When $H < \alpha$, which means $R > \alpha^{-1}$[1], we see from fig. 5.2a that there is a UV regime for both $\alpha$ and $H$ where the spectral dimension is four; then the spectral dimension goes up to seven, which is the value we found in section 3.1 without the compactification, in an intermediate regime which is "UV" for the compactification scale and "IR" for the deformation scale; finally there is the IR regime for both scales where the spectral dimension goes down to 5, which means that the compactified dimension has disappeared but the other dimensions are anyway "doubled" by the deformation. In fig. 5.2b we did a focus on the UV zone because the problem in truncating the series up to some value of $n$ is that resolution is lost for small values of $\sqrt{s}\left(nH + \frac{\alpha}{2}\right)$, so we did another plot with more terms in the series but with a smaller interval for $s$ to see that

[1] In this case we would have a large compactified dimension.

in the far UV regime the spectral dimension reaches the value of four.

When $H < \alpha$, which means $R < \alpha^{-1}$ [2], we see from fig. 5.3 that a behavior similar to the previous one is exhibited: there is a far UV regime where the dimension is four, an intermediate regime where the dimension is three and a far IR regime where it reaches the value of five as happened before.

In both cases the compactified dimension disappears in the far IR regime and so it is not even doubled by the deformation. What changes is the intermediate regime: it depends upon which IR scale is turned on first by the probe.

### 5.2.4 Thermal dimension for the $(1,3)$-cylinder

The last case we considered for the compactified dimension scenario is the computation of the thermal dimension for the $(1,3)$-cylinder, to see if the intuition that "compactified dimensions disappear in the IR regime" is preserved by the physical notion of dimensionality.

The logarithm of partition function for the photon gas on the cylinder is

$$\log Z \propto V \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dE \int_{0}^{+\infty} dk\, kE\theta(E)\delta\big(E^2 - k^2 - \tfrac{n^2}{R^2}\big) \log\Big(1 - e^{-\beta E}\Big) \propto$$

$$\propto V \sum_{n=0}^{\infty} \int_{\frac{n}{R}}^{+\infty} dE\, E \log\Big(1 - e^{-\beta E}\Big)$$

where the $\delta$ function has been used for integration over $k$ and the integration over $E$ has lower bound $\frac{n}{R}$ since the $\delta$ function gives $k = \sqrt{E^2 - \frac{n^2}{R^2}}$. We can change variables by $E \to RE$ and $\beta \to \beta R^{-1}$ to get rid of $R$ so that

$$\log Z \propto V \sum_{n=0}^{\infty} \int_{n}^{+\infty} dE\, E \log\Big(1 - e^{-\beta E}\Big)$$

therefore

$$\boxed{\log Z \propto V \beta^{-2} \sum_{n=0}^{\infty} \Big[\beta\, n\, Li_2\big(e^{-\beta n}\big) + Li_3\big(e^{-\beta n}\big)\Big]} \tag{5.8}$$

From this the $\beta$ derivative can be computed and the equation of state is

$$w = \frac{\beta^{-2} \sum_{n=0}^{\infty} \Big[\beta\, n\, Li_2\big(e^{-\beta n}\big) + Li_3\big(e^{-\beta n}\big)\Big]}{\sum_{n=0}^{\infty} \Big\{2\beta^{-2}\Big[\beta\, n\, Li_2\big(e^{-\beta n}\big) + Li_3\big(e^{-\beta n}\big)\Big] - n^2 \log\big(1 - e^{-\beta n}\big)\Big\}}$$

[2] In this case we would have a small compactified dimension.

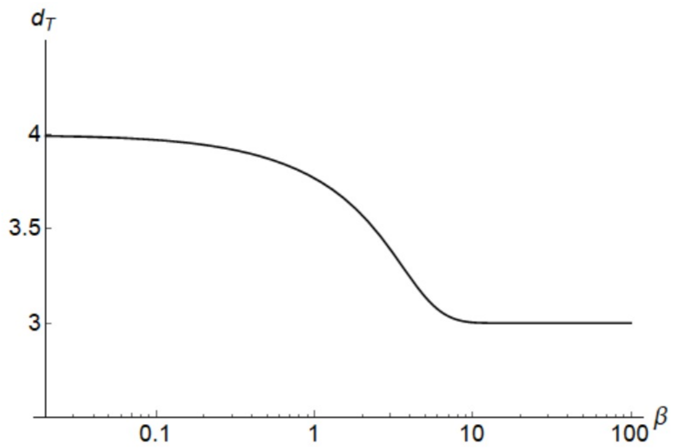



Figure 5.4: Thermal dimension for the (1,3)-cylinder. The series where truncated to $n = 750$.

therefore the thermal dimension is

$$d_T(\beta) = 1 + \frac{\sum_{n=0}^{\infty}\left\{2\beta^{-2}\left[\beta\, n\, Li_2\left(e^{-\beta n}\right) + Li_3\left(e^{-\beta n}\right)\right] - n^2 \log\left(1 - e^{-\beta n}\right)\right\}}{\beta^{-2}\sum_{n=0}^{\infty}\left[\beta\, n\, Li_2\left(e^{-\beta n}\right) + Li_3\left(e^{-\beta n}\right)\right]} \tag{5.9}$$

whose plot is shown in fig. 5.4.

We see from fig. 5.4 that the UV and IR limits for the thermal dimension are the same of the spectral dimension eq. (5.5): we get a dimension of four (three) in the UV (IR) regime; that is, the compactified dimension disappears in the IR regime.

## 5.3 Spectral dimension of Rindler space

The last scenario we considered is the one with an accelerated (or Rindler) observer. Spacetime is flat, there is no compactification nor deformation of dispersion relation and yet there is a nontrivial spectral dimension; this is because Rindler observers are characterized by a causal horizon which means that the manifold has a boundary. Remarkably enough, the presence of boundaries is a $IR$ effect just like curvature and compactification. A short review of the necessary concepts for what follows can be found in Appendix C .

In Rindler coordinates the metric is

$$ds^2 = e^{2a\xi}\left(-d\eta^2 + d\tau^2\right) + dy^2 + dz^2$$

We now focus on the nontrivial part of this spacetime: the $y$ and $z$ variables are "spectators", they give nothing different from the Minkowski case; therefore we do the calculation in $\eta - \xi$ coordinates only and add a +2 to the dimension at the end. Considering the change of variables

$$T := a^{-1}e^{a\xi}\sinh(a\eta) \quad , \quad X := a^{-1}e^{a\xi}\cosh(a\eta) \tag{5.10}$$

we get a spacetime with a boundary

$$ds^2 = -dT^2 + dX^2 \quad , \quad X > |T|$$

and considering a Wick rotation $T$, $T \to i\,T$ this manifold becomes an Euclidean half-plane. Therefore the diffusion equation is the Minkowskian one but the eigenvectors of the Laplacian must satisfy the correct boundary condition [3].

$$\begin{cases} \left(\partial_T^2 + \partial_X^2\right) f_\lambda = -\lambda^2 f_\lambda \\ \\ f_\lambda(0, X) = 0 \end{cases} \tag{5.11}$$

We consider as a solution

$$f_\lambda(T, X) \equiv f_{\omega,p}(T, X) = e^{i\omega T}\sin(pX) \tag{5.12}$$

where $\lambda^2 = \omega^2 + p^2$. Consequently

$$\left[\partial_s - \left(\partial_T^2 + \partial_X^2\right)\right]\rho(\vec{X}, \vec{X}'; s) = 0 \tag{5.13}$$

is solved by

$$\rho(\vec{X}, \vec{X}'; s) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\omega\, \tilde{\rho}(\omega, p; s)\, e^{i\omega(T-T')}\sin(pX)\sin(pX')$$

Inserting this in eq. (5.13) we get as in the Minkowskian case

$$\tilde{\rho}(\omega, p; s) = A\, e^{-s(\omega^2+p^2)} \tag{5.14}$$

Setting $\rho(\vec{X}, \vec{X}'; 0) = \delta^{(2)}(\vec{X} - \vec{X}')$ we can find the constant $A$

$$\frac{A}{2}\int_{-\infty}^{\infty} dp\left[e^{ip(X-X')} - e^{ip(X+X')}\right] = \delta(X - X')$$

The second term in the integral gives $\delta(X + X')$ which is identically 0 since the $X$ coordinates are positive. Thus we get $A = 2$ and

$$\rho(\vec{X}, \vec{X}'; s) = 2\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\omega\, e^{-s(\omega^2+p^2)}\, e^{i\omega(T-T')}\sin(pX)\sin(pX')$$

Therefore the return probability is

$$P(s; X) = 2\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\omega\, e^{-s(\omega^2+p^2)}\left[\sin(pX)\right]^2 = \frac{\pi}{s}\, e^{-\frac{X^2}{s}}\left(1 - e^{\frac{X^2}{s}}\right) \tag{5.15}$$

[3] The boundary is $-T^2 + X^2 > 0$ , $X > 0$; however, after the Wick rotation, this boundary becomes $T^2 + X^2 > 0$ , $X > 0$.

With this probability it is straightforward to get

$$d_s(s,X) = 2 + \frac{2X^2}{s\left(-1+e^{\frac{X^2}{s}}\right)}$$

hence, reinserting the trivial dimensions

$$\boxed{d_s(s,X) = 4 + \frac{2X^2}{s\left(-1+e^{\frac{X^2}{s}}\right)}} \tag{5.16}$$

Defining $X^2 s^{-1} := z$ we get

$$\lim_{z\to 0} \frac{2z}{\left(-1+e^z\right)} = 2 \ , \ (IR) \tag{5.17}$$

$$\lim_{z\to \infty} \frac{2z}{\left(-1+e^z\right)} = 0 \ , \ (UV) \tag{5.18}$$

hence

$$d_s(UV) = 4 \ \ , \ \ d_s(IR) = 6 \tag{5.19}$$

We can interpret this result by observing that

$$z = \frac{X^2}{s} = \frac{a^{-2}e^{2a\xi}\left[\cosh(a\eta)\right]^2}{s} = \frac{\lambda_u^2}{s}\left[(2\pi)^{-2}\cosh(a\eta)\right]^2 \tag{5.20}$$

where $\lambda_u := (2\pi)^2\alpha^{-1} = (2\pi)^2 a^{-1}e^{a\xi}$ is the Unruh wavelength. With this identification there are two ways of interpreting this result. Using the $X$ variable the UV (IR) limit is $s << X^2$ ($s >> X^2$) which can be interpreted as a walker far from (near) the boundary or horizon. In the IR limit the walker probes the structure of the horizon which increases the spectral dimension. Using the Unruh wavelength, the UV (IR) limit is $\sqrt{s} << \lambda_u$ ($\sqrt{s} >> \lambda_u$); therefore in the IR limit the probe can "resolve" the Unruh radiation, while in the UV it has no clue about the latter and everything looks like Minkowski as far as the probe is concerned.

### 5.3.1 Hints for the thermal dimension of Rindler space

We also started to investigate the problem of finding the thermal dimension of Rindler space. There are some works which focus on the thermodynamics in the presence of horizons [31], [32] and also specifically for Rindler space [29]. We tried to do something similar to [29] but in a slightly different way: we studied a gas of photons in a cubic box as seen from a Rindler observer; the results obtained are still not clear enough to be shown with a good interpretation, in particular we found that

- The thermodynamical quantities, such as pressure and energy density, depend upon the size of the box.
- The energy density is proportional to $T^4$, suggesting a thermal dimension of 4.
- The equation of state for the photon gas suggests a thermal dimension which depends upon the size of the box and the Unruh wavelength; in particular a box which is large compared to the Unruh wavelength (IR regime) gives a thermal dimension of 5.5 while a box which is small compared to the Unruh wavelength (UV) gives a thermal dimension of 4.

The apparent contradictory result given by the energy density and the equation of state is what concerns us the most and it deserves a deeper scrutiny.

*You can go the distance, you can run the mile*
*You can walk straight through hell with a smile.*

*~ The Script, Hall of Fame*

# *Conclusions and outlook*

UV/IR effects in quantum gravity might produce a modification of the dispersion relation governed by a IR scale. We studied the behavior of the spectral and the thermal dimensions for such models, finding that the effective number of dimensions is modified, more frequently it is increased, in the IR regime. The most interesting MDRs are the ones which contain "linear" terms in momenta or energy, while the other class of MDRs, namely the ones with logarithms or $p^{-2}$ terms, do not give sensible results; this should be expected since these MDRs are divergent in the IR regime, therefore they are pathological from the start.

We found that in the IR regime every linear term in momenta (or energy) gives rise, for the spectral dimension, to a doubling of the corresponding dimension and this happens also, to some extent, for the thermal dimension; in particular the most interesting result in this case is that both spectral and thermal dimension agree on the value $2d+1$ for the effective dimension of a spacetime with a MDR with a deformation linear in the modulus of the spatial part of momentum.

We then shifted our attention to curved spacetime scenarios, in order to study the interplay between deformation and curvature scales. We understood very soon that the tools used in flat spacetime are not very well tailored for curved cases. The problems for the spectral dimension are the Wick rotation, which is not uniquely defined, and the resolution of the eigenvalue equations for deformed Laplacians in curved spacetime, which in any case is a problem that descends from the interpretation of momentum components as derivative operator and this is another problem per se. For thermal dimension the problems mainly concern the fact that the thermodynamics of a photon gas on curved spacetime is needed in order to define the required quantities; this is a rather non trivial task and preliminary results on Anti-de Sitter spacetime [28] suggest also that the Stefan-Boltzmann law is no longer valid on curved spacetime.

Despite these difficulties we were able to study some simple cases where the aforementioned problems are more manageable. While studying these scenarios we found novel IR effects. Compactified dimensions lead to the same results that were obtained for the sphere and the torus [11], [19] even when a flat manifold is considered, as we found for the $(1,d)$-cylinders; what happens for both the spectral and the thermal dimension is that the compactified dimensions disappear in the IR regime. We found also that the spectral dimension is affected by the presence of horizons on the spacetime, in particular we studied the Rindler space, which is also a good test ground for the near-horizon region of Schwarschild spacetime [30].

The outlooks of this research program are mainly aimed to tame the curved scenarios and to find a formulation of DSR in the IR regime.
First of all, finding an interplay between curvature effects and deformation, taming the aforementioned problems, for at least one of the two notions of effective dimensionality would give us the opportunity to set some boundaries on the deformation parameters.
Secondly, the project of finding a IR Doubly Special Relativity theory is an independent line of research but it would make it possible to repeat all the analyses done in this work with a deformed measure on momentum space, which could modify the predictions for both notions of effective dimensionality and it could also give sensible results for the "pathological" MDRs.

*Part of the journey is the end.*

*~ Tony Stark, Avengers Endgame*

*Appendix A*

# *A general technique for spectral dimension limits*

It is known [10] that if MDRs such as

$$E^2 = c_w p^{2w} + c_{w+1} p^{2w+2} + \ldots + c_{z-1} p^{2z-2} + c_z p^{2z}$$

are considered, with $1 \leq w < z$ , $w, z \in \mathbb{N}$ it is possible to find asymptotic expansions that allow to find the sectral dimension in the UV and IR limits without even computing the return probability $P(s)$ but simply looking at the MDR. In particular, calling $f(p^2)$ the polinomial expression in p in the MDR,

$$\exp(-sf(p^2)) \sim \exp(-sc_w p^{2w})[1 - sc_{w+1} p^{2w+2} + \ldots] \tag{A.1}$$

$$\exp(-sf(p^2)) \simeq \exp(-sc_z p^{2z})[1 - sc_{z-1} p^{2z-2} + \ldots] \tag{A.2}$$

when $s \to \infty$ and $s \to 0$ respectively. This means that in the IR (UV) limit the smallest (biggest) power of $p$ is the dominant one. In the series in the square brackets there are terms of the form $s^n p^{2m}$ with $m \geq n(w+1)$ for eq. (A.1) and $m \leq n(z-1)$ for eq. (A.2).

Since the MDR is trivial in the energy, it is convenient to write the spectral dimension as

$$d_s(s) = 1 - 2s \frac{d \log Z(s)}{ds}$$

where the factor 1 comes from the $E^2$ term in the MDR and

$$Z(s) := \int_0^{+\infty} dp\, p^{d-1} \exp(-sf(p^2))$$

To compute $Z(s)$ it is possible to exploit the expansions eq. (A.1) and eq. (A.2):

$$Z(s) \simeq \frac{C}{s^{\frac{d}{2z}}} \left[ \sum_{n=0}^{N} a_n s^{\frac{n}{z}} + O\left(s^{\frac{N+1}{z}}\right) \right] \ , \ s \to 0$$

$$Z(s) \sim \frac{C}{s^{\frac{d}{2w}}} \left[ \sum_{n=0}^{N} b_n s^{-\frac{n}{w}} + O\left(s^{-\frac{N+1}{w}}\right) \right] \ , \ s \to \infty$$

from these expansions it is straightforward to get

$$d_s(0) = 1 + \frac{d}{z} \quad , \quad d_s(\infty) = 1 + \frac{d}{w} \tag{A.3}$$

This method can actually be used also with general deformations of the dispersion relation $f(p^2)$ with some hypothesis on the function $f$: there cannot be poles in zero and the pole in $\infty$ has to be of finite order and the coefficient of the pole in $\infty$ must have a real part greater than zero. If $w$ and $z$ are the infinitesimal order in $p^2 = 0$ and the order of infinity in $p^2 = \infty$ respectively, the two relations in eq. (A.3) hold. For instance, for the MDR eq. (3.1) with $\alpha > 0$ the results $d_s(0) = d + 1$ and $d_s(\infty) = 2d + 1$ can be obtained by observing that in this case $z = 1$ and $w = \frac{1}{2}$.
A proof of the validity of this method for a polynomial MDR can be found in the section below.

## A.1 Proof for a polynomial MDR

Consider a MDR

$$\boxed{E^2 = p^2\left[1 + \sum_{\gamma\in\Gamma} \lambda_\gamma p^{2\gamma}\right]} \tag{A.4}$$

where

$$\Gamma := \left\{\gamma \in \left]-1, 0\right[ \,\cup\, \left]0, +\infty\right[ \;:\; \lambda_\gamma \neq 0\right\}$$

is a countable subset of $\left]-1, +\infty\right[ \setminus \{0\}$, $card(\Gamma) \leq \aleph_0$. In other terms, $\Gamma$ is the set of the indexes such that a power of $p^2$ appears in the sum and it is postulated to be countable for the (discrete) sum to be meaningful.
Let

$$\gamma_m := \min\left[0, \min_{\gamma\in\Gamma}\{\gamma\}\right] \quad , \quad \gamma_M := \max\left[0, \max_{\gamma\in\Gamma}\{\gamma\}\right]$$

where it is assumed that $\lambda_{\gamma_m} > 0$ and $\lambda_{\gamma_M} > 0$ so that the dispersion relation is physically meaningful both in the IR and in the UV regime; imaginary energies are avoided in both regimes with this assumption. In the MDR eq. (A.4), in the limit $s \to \infty$ the modes which give a major contribution are those with $p \to 0$ (IR) and the opposite will happen in the limit $s \to 0$ (UV). The return probability $P(s)$ is given by

$$P(s) = \frac{S^{(d)}(1)}{(2\pi)^{d+1}} \int_{-\infty}^{+\infty} dE\, e^{-sE^2} \int_0^{+\infty} dp\, p^{d-1}\, e^{-sp^2\left[1+\sum_{\gamma\in\Gamma}\lambda_\gamma p^{2\gamma}\right]}$$

Calling $a$ the numerical constant, in which all subsequent numerical constants will be absorbed,

$$P(s) = a\, s^{-\frac{1}{2}} \int_0^{+\infty} dp\, p^{d-1}\, e^{-sp^2\left[1+\sum_{\gamma\in\Gamma}\lambda_\gamma p^{2\gamma}\right]}$$

with these considerations, we can study the two limits focusing on the integral in $p$:

- In the UV limit $\gamma_M + 1$ is the order of the pole in $p^2 = +\infty$ therefore

$$e^{-sp^2\left[1+\sum_{\gamma\in\Gamma}\lambda_\gamma p^{2\gamma}\right]} \sim e^{-s\lambda_M p^{2(\gamma_M+1)}}\left[1+\frac{\lambda_{prec(\gamma_M)}}{\lambda_{\gamma_M}}p^{2(prec(\gamma_M)-\gamma_M)}+\ldots\right]$$

where $prec(\gamma_M)$ indicates the element which precedes $\gamma_M$ in $\Gamma$ and by definition $\lambda_0 := 1$. Consequently the integral in $p$ reads

$$\int_0^{+\infty} dp\, e^{-s\lambda_M p^{2(\gamma_M+1)}}\, p^{d-1} + \int_0^{+\infty} dp\, e^{-s\lambda_M p^{2(\gamma_M+1)}}\, \frac{\lambda_{prec(\gamma_M)}}{\lambda_{\gamma_M}} p^{d-1+2(prec(\gamma_M)-\gamma_M)} + \ldots$$

calling $s\lambda_M p^{2(\gamma_M+1)} := t$

$$\alpha\int_0^{+\infty} dt\, e^{-t}\, t^{\frac{d}{2(\gamma_M+1)}-1}\, s^{-\frac{d}{2(\gamma_M+1)}} +$$

$$+\beta\int_0^{+\infty} dt\, e^{-t}\, t^{\frac{d+2(prec(\gamma_M)-\gamma_M)}{2(\gamma_M+1)}-1}\, s^{-\frac{d+2(prec(\gamma_M)-\gamma_M)}{2(\gamma_M+1)}} + \ldots$$

where $\alpha$ and $\beta$ are numerical constants in which all subsequent numerical constants are absorbed. The two integrals give two Gamma functions $\Gamma(\nu)$, in particular $\Gamma\left(\frac{d}{2(\gamma_M+1)}\right)$ and $\Gamma\left(\frac{d+2(prec(\gamma_M)-\gamma_M)}{2(\gamma_M+1)}\right)$ respectively. The return probability becomes

$$P(s) \sim a\, s^{-\frac{1}{2}}\, s^{-\frac{d}{2(\gamma_M+1)}}\left(\alpha + \beta\, s^{-\frac{2(prec(\gamma_M)-\gamma_M)}{2(\gamma_M+1)}} + \ldots\right)$$

therefore

$$\boxed{d_s(0) = 1 + \frac{d}{\gamma_M+1}} \tag{A.5}$$

- In the IR limit $p^{2(\gamma_m+1)}$ is the zero of smallest order of the pole in $p^2 = 0$ therefore

$$e^{-sp^2\left[1+\sum_{\gamma\in\Gamma}\lambda_\gamma p^{2\gamma}\right]} \simeq e^{-s\lambda_m p^{2(\gamma_m+1)}}\left[1+\frac{\lambda_{succ(\gamma_m)}}{\lambda_{\gamma_m}}p^{2(succ(\gamma_m)-\gamma_m)}+\ldots\right]$$

where $succ(\gamma_m)$ indicates the element which comes after $\gamma_m$ in $\Gamma$ and by definition $\lambda_0 := 1$. Consequently the integral in $p$ reads

$$\int_0^{+\infty} dp\, e^{-s\lambda_m p^{2(\gamma_m+1)}}\, p^{d-1} + \int_0^{+\infty} dp\, e^{-s\lambda_m p^{2(\gamma_m+1)}}\, \frac{\lambda_{succ(\gamma_m)}}{\lambda_{\gamma_m}} p^{d-1+2(succ(\gamma_m)-\gamma_m)} + \ldots$$

calling $s\lambda_m p^{2(\gamma_m+1)} := t$

$$\alpha\int_0^{+\infty} dt\, e^{-t}\, t^{\frac{d}{2(\gamma_m+1)}-1}\, s^{-\frac{d}{2(\gamma_m+1)}} +$$

$$+\beta\int_0^{+\infty} dt\, e^{-t}\, t^{\frac{d+2(succ(\gamma_m)-\gamma_m)}{2(\gamma_m+1)}-1}\, s^{-\frac{d+2(succ(\gamma_m)-\gamma_m)}{2(\gamma_m+1)}} + \ldots$$

where $\alpha$ and $\beta$ are numerical constants in which all subsequent numerical constants are absorbed. The two integrals give two Gamma functions $\Gamma(\nu)$, in particular $\Gamma\left(\frac{d}{2(\gamma_m+1)}\right)$ and $\Gamma\left(\frac{d+2(succ(\gamma_m)-\gamma_m)}{2(\gamma_m+1)}\right)$ respectively. The return probability becomes

$$P(s) \simeq a\, s^{-\frac{1}{2}}\, s^{-\frac{d}{2(\gamma_m+1)}}\left(\alpha+\beta\, s^{-\frac{2(succ(\gamma_m)-\gamma_m)}{2(\gamma_m+1)}}+\ldots\right)$$

therefore

$$\boxed{d_s(\infty) = 1+\frac{d}{\gamma_m+1}} \tag{A.6}$$

With the results eq. (A.5) and eq. (A.6) all the results obtained for the MDRs eq. (3.1), eq. (3.7) can be recovered. In particular for $E^2 = p^2+\alpha p$ , $\alpha > 0$ setting $\Gamma = \{-\frac{1}{2}\}$ and $\lambda_{-\frac{1}{2}} = \alpha$, we read $\gamma_m = -\frac{1}{2}$ and $\gamma_M = 0$ which inserted in eq. (A.6) and eq. (A.2) respectively give $2d+1$ in the IR and $d+1$ in the UV. The methods adopted here can be extended also to MDRs which are not trivial in the energy, the only requirement is that the functions which appear in the integrals respect the hypothesis as happens, for istance, for the MDRs eq. (3.11) and eq. (3.15).

*Appendix B*

# *Noncommutative Spacetime*

## B.1 The problem of localizability

To understand which are the premises of the NCST approach, we can consider as an example the problem of localizability of spacetime events. The Einstenian setup to measure the position of something in space is shown in fig. B.1.

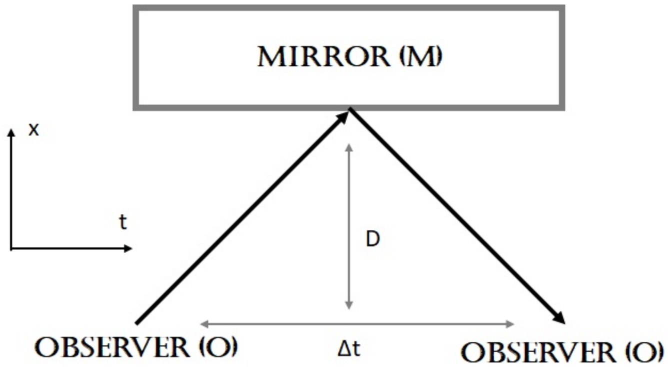



Figure B.1: The Einsteinian setup for the measurement of the position.

The observer ($O$) sends a light beam towards the object of which the position has to be measured then this beam is reflected, as example by a mirror ($M$), and then it is observed again by $O$ after a time $\Delta t$; the distance $D$ of the observer from the mirror is then $D = \frac{c\Delta t}{2}$. This is what happens classically, the position can be measured with arbitrary precision. What happens if the probe used to measure the position is a quantum particle? In that case we have an uncertainty on the initial position, $\delta x_0$, and we know that $m\delta x_0 \delta v_0 \gtrsim \hbar$, where $m$ is the mass of the probe, hence

$$\delta D = \delta x_0 + \delta v_0 \Delta t \gtrsim \delta x_0 + \frac{\hbar}{m\delta x_0}\Delta t \gtrsim \sqrt{\frac{\hbar \Delta t}{m}}$$

where the last inequality comes from minimizing $\delta D$, which happens when $\delta x_0 = \sqrt{\frac{\hbar\Delta t}{m}}$. We see therefore that there is a limit on the localizability of an event in space and this limit would disappear only when $m \to \infty$ in which case the probe is not a "probe" anymore, but it is more kind of a black hole.

## B.2 $\kappa$-Minkowski and $\theta$-NCST

NCST approach to QG arises (also) from this "hint" on the non localizability of spacetime events. The idea is that it is possible to implement directly the non localizability on the spacetime coordinates if these are taken to be non trivial objects which do not commute; as happens in QM with phase space, where $[\hat{q}, \hat{p}] \neq 0$ implies a limit on the localizability in phase space, in NCST $[\hat{x}^\mu, \hat{x}^\nu] \neq 0$ implies that there is a limit on the localizability in spacetime.

The most general form of this noncommutative behavior would be

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Gamma^{\mu\nu}(\hat{x}) \tag{B.1}$$

where $\Gamma^{\mu\nu}$ are general functions of the noncommutative coordinates. However, there are some constraints that could be imposed. First of all, all the terms on the right-hand side of eq. (B.1) must have dimension of $length^2$; secondly, it is postulated that the quantum gravitational effects, in this case the noncommutativity of spacetime, manifest themselves at the Planck scale, therefore we could impose that $[\hat{x}^\mu, \hat{x}^\nu] \to 0$ when $\ell_P \to 0$. Consequently the possible scenarios for NCST are collected under

$$[\hat{x}^\mu, \hat{x}^\nu] = i\ell_P^2\theta^{\mu\nu} + i\ell_P\gamma_\rho^{\mu\nu}\hat{x}^\rho \tag{B.2}$$

Usually, two classes of noncommutativity are studied, the canonical noncommutative spacetime [1]

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \tag{B.3}$$

and $\kappa$-Minkowski spacetime

$$[\hat{x}^0, \hat{x}^i] = i\ell_P\hat{x}^i \quad , \quad [\hat{x}^i, \hat{x}^j] = 0 \tag{B.4}$$

## B.3 Quantization of distances in Moyal plane

An example of geometrical properties induced by the noncommutativity can be seen in the Moyal plane [41] which is a $1+2$ NCST with

$$[\hat{x}^0, \hat{x}^i] = 0 \quad , \quad [\hat{x}^1, \hat{x}^2] = i\theta \tag{B.5}$$

These coordinates, of course, cannot be interpreted as coordinates of points but they should be rather interpreted as operators on a Hilbert space. This Hilbert space is an auxiliary space in which there are no particles, and it is called "pregeometric". The expected values of the coordinates operators on the states of this Hilbert space give fuzzy coordinates of points in spacetime.

The Moyal plane is quite simple to study because the Hilbert space induced by eq. (B.5) is the same Hilbert space of a single one dimensional particle in ordinary QM,

[1] $\ell_P^2$ is absorbed in $\theta^{\mu\nu}$ which is then dimensionful.

indeed we could identify $\hat{x}^1$ with $\hat{q}$, $\hat{x}^2$ with $\hat{p}$ and $\theta$ with $\hbar$. If we want to define a notion of distance on this spacetime we need at least two points, thus we need a Hilbert space which is the tensor product of two copies of the Hilbert space of single point [2]. Calling $A$ and $B$ the points

$$[\hat{x}^1_A, \hat{x}^2_A] = i\theta = [\hat{x}^1_B, \hat{x}^2_B]$$

The action of these operators on the states of the Hilbert space in the "$x^1$" representation is

$$\hat{x}^1_{A,B}\psi(x^1_A, x^1_B) = x^1_{A,B}\psi(x^1_A, x^1_B) \quad , \quad \hat{x}^2_{A,B}\psi(x^1_A, x^1_B) = -i\theta\frac{\partial}{\partial x^1_{A,B}}\psi(x^1_A, x^1_B)$$

With these tools, we can analyze what happens to the distances on this spacetime. An intuitive definition of distance operator is

$$\widehat{d^2} = (\hat{x}^1_A - \hat{x}^1_B)^2 + (\hat{x}^2_A - \hat{x}^2_B)^2$$

Defining $\hat{Q} := \hat{x}^1_A - \hat{x}^1_B$, $\hat{P} := \hat{x}^2_A - \hat{x}^2_B$, $M := \frac{1}{2}$ and $\omega := 2$ we have

$$[\hat{Q}, \hat{P}] = 2i\theta$$

and

$$\widehat{d^2} = \frac{\hat{P}^2}{2M} + \frac{1}{2}M\omega^2\hat{Q}^2 \tag{B.6}$$

which is the hamiltonian of a harmonic oscillator; therefore we know already that the spectrum is discrete and that it is given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad , \quad n \in \mathbb{N}_0$$

if $[\hat{Q}, \hat{P}] = i\hbar$; therefore, identifying $\hbar$ with $2\theta$ and using $\omega = 2$ we have for the distance operator eq. (B.6) that the spectrum is given by

$$\boxed{d^2_n = 4\theta\left(n + \frac{1}{2}\right) \quad , \quad n \in \mathbb{N}_0} \tag{B.7}$$

From this spectrum we see that the distances are quantized and that there is a minimum distance of $\sqrt{2\theta}$; this is expected since the quantization condition eq. (B.5) implies that it is not possible to define with arbitrary precision both coordinates, therefore it is not possible to say that two "points" have distance zero because this statement would require the knowledge of both coordinates for both "points".

[2] This is the same thing that happens in QM: when two particles are considered, the states of the system live in tensor product of two copies of the one particle Hilbert space.

*Appendix C*

# *Rindler space*

## C.1 Kinematics and Rindler coordinates

Consider an accelerated point particle in Minkowski. First of all, we have to address the problem of which acceleration we are dealing with. Indeed there are different notions of acceleration that can be considered. First of all, there is the 4-acceleration; calling $\tau$ the proper time of the point particle, the 4-acceleration is

$$\alpha^{\mu} = \frac{d^2 x^{\mu}}{d\tau^2}$$

which is the simplest geometrical object that can be defined because it is a 4-vector whose norm is Lorentz invariant. Given a reference frame $S$ of coordinates $(t, \vec{x})$, the 3-acceleration is defined by

$$\vec{a} = \frac{d^2 \vec{x}}{dt^2}$$

Remembering that $dt = \gamma d\tau$ the link between these two notions can be derived

$$a^{\mu} = \gamma_u^2 \Big( \gamma_u^2 \vec{u} \cdot \vec{a}, \gamma_u^2 (\vec{a} \cdot \vec{u}) \vec{u} + \vec{a} \Big)$$

where $\vec{u}$ is the velocity of the point particle in the reference frame $S$ with respect to which the 4-acceleration is computed and $\gamma_u^2 := (1 - \vec{u} \cdot \vec{u})^{-1}$. There is a third notion of acceleration; at each given instant of time we can consider a reference frame $S'$ which is instantaneously inertial with a velocity equal to $\vec{u}$: the 3-acceleration measured in this inertial frame with an accelerometer is by definition the proper acceleration of the point particle. By hypothesis the coordinate time measured in $S'$ is equal to the proper time of the point particle, that is the proper time depends only upon the (instantaneous) velocity and it is independent of the acceleration [1]. Therefore we have that in the reference frame $S'$ the four acceleration is

$$a^{\mu} = (0, \vec{\alpha})$$

where $\vec{\alpha}$ is the proper acceleration (it is the 3-acceleration measured in $S'$). Given that the norm of a 4-vector is Lorentz invariant we have that

$$a^{\mu} a_{\mu} = \vec{\alpha} \cdot \vec{\alpha} := \alpha^2 \tag{C.1}$$

[1] This is called "clock hypothesis".

holds in every reference frame.

Consider now two observers, A is at rest while B is accelerating with proper acceleration $\alpha$ along the $x$ axis of A. The equation of motion for B in the A frame are

$$a^0 = \frac{du^0}{d\tau} \quad , \quad a^1 = \frac{du^1}{d\tau} \quad , \quad a^2 = a^3 = 0 \tag{C.2}$$

Remembering that $u^\mu u_\mu = -1$ we have $u^\mu a_\mu = 0$, hence $u^1 = \frac{a^0 u^0}{a^1}$. From section C.1 we have also $-(a^0)^2 + (a^1)^2 = \alpha^2$. Putting these equations all together

$$\begin{cases} -(a^0)^2 + (a^1)^2 = \alpha^2 \\ (u^0)^2 - (u^1)^2 = 1 \\ u^1 = \frac{a^0 u^0}{a^1} \end{cases}$$

therefore $a^1 = \alpha u^0$ and $a^0 = \alpha u^1$; using these relations in eq. (C.2) we get

$$\frac{d^2 u^0}{d\tau^2} = \alpha^2 u^0 \quad , \quad \frac{d^2 u^1}{d\tau^2} = \alpha^2 u^1$$

which together with $(u^0)^2 - (u^1)^2 = 1$ give us

$$u^0 = \cosh(\alpha\tau) \quad , \quad u^1 = \sinh(\alpha\tau)$$

that is

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau) \quad , \quad x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau) \tag{C.3}$$

From eq. (C.3) it is understood that the trajectories are hyperbolas given by $-t^2 + x^2 = \frac{1}{\alpha^2}$. From this we see that when $\alpha \to \infty$ the trajectories become the light-like paths $x = \pm t$; thus everything that lies outside the wedge of the Minkowski plane between these two lines (called Rindler wedge or Rindler space) is inaccessible for the Rindler observer, therefore these two lines define the horizons of the Rindler space.

It is possible to define a set of coordinates for Rindler observers, preserving the hyperbolic nature of eq. (C.3). Defining

$$t = \frac{1}{a}e^{a\xi}\sinh(a\eta) \quad , \quad x = \frac{1}{a}e^{a\xi}\cosh(a\eta) \tag{C.4}$$

we have $\alpha = ae^{-a\xi}$ and $a\eta = \alpha\tau$. The coordinates in eq. (C.4) are called "Rindler coordinates": an observer with constant value of $\xi$ is at rest in this reference frame and has a proper acceleration $\alpha = ae^{-a\xi}$ with respect to the Minkowski reference frame. We see that the horizon $\alpha \to \infty$ becomes $\xi \to -\infty$. In these coordinates from

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

it is easy to get

$$\boxed{ds^2 = e^{2a\xi}\left(-d\eta^2 + d\xi^2\right) + dy^2 + dz^2} \tag{C.5}$$

## C.2 Unruh effect

Considering the vacuum state of a quantum field for a Minkowskian observer, the Unruh effect is the prediction that a Rindler observer will observe a thermal bath out of this vacuum. More precisely, if the solutions to mass-less Klein-Gordon equation are considered in the Minkowski reference frame and in the Rindler reference frame

$$\phi(t,x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2|k|}} \left( a_k e^{-i\omega_k t + ikx} + a_k^\dagger e^{i\omega_k t - ikx} \right)$$

$$\phi(\eta,\xi) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\sqrt{2|p|}} \left( \alpha_p e^{-i\omega_p \eta + ip\xi} + \alpha_p^\dagger e^{i\omega_p \eta - ip\xi} \right)$$

it can be shown that the average value in the Minkowski vacuum, $a_k |0\rangle^{(M)} = 0$, of the Rindler occupation number operator, $n_p^{(R)} := \alpha_p^\dagger \alpha_p$, is

$$\langle 0| n_p^{(R)} |0\rangle \propto \left( e^{\frac{\hbar\omega_p}{k_B T_u}} - 1 \right)^{-1} \tag{C.6}$$

where the dimensional units were restored and

$$\boxed{T_u = \frac{\hbar\alpha}{2\pi k_B c}} \tag{C.7}$$

is the Unruh temperature. In natural units $T_u = \frac{\alpha}{2\pi}$; moreover we can turn this temperature into a wavelength, that is the wavelength of the thermal radiation seen by the Rindler observer with temperature eq. (C.7)

$$k_B T_u = \hbar\omega_u = 2\pi\hbar \frac{c}{\lambda_u} \;\Rightarrow\; \lambda_u = c^2 \frac{(2\pi)^2}{\alpha}$$

In natural units this becomes $\lambda_u = (2\pi)^2 \alpha^{-1}$. This is what is called Unruh wavelength.

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