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Swampland: Quantum Gravity meets Particle Physics

Relatori:
Prof. Francesco Sannino
Dr. Franco Pezzella

Correlatore:
Prof. Wolfgang Mueck

Candidato:
Alessandra D'Alise
Matr. N94 000611

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Dedicated to my family. I fully, deeply love you.

Dedicata alla mia famiglia. Pienamente, profondamente vi amo.

Abstract

String Theory provides a huge number of consistent low-energy theories each of them can be considered as a point in the theory Landscape. The Swampland Program aims at characterising this Landscape. Through a number of string-inspired conjectures, the so-called Swampland conjectures. These have the role to reconcile quantum gravity arguments, phenomenological experience and String Theory arguments in order to constrain the Landscape. The latter is achieved by requiring that the low-energy effective field theories can be embedded into a quantum gravity theory. Among the Swampland conjectures, the thesis work focuses on the so-called No-Global Symmetry and the Weak Gravity conjectures. Regarding the first conjecture, the methodology used ranges from traditional String Theory arguments to modern AdS/CFT correspondence as well as black hole physics. For the second conjecture analysed in the thesis, beside arguing for it by using arguments born within previous methodology, we also considered the link between quantum gravity and conformal field theories, that was very recently put forward. The thesis work lays the foundations for a generalization of the Swampland conjectures beyond String Theory.

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Natural units adopted (unless otherwise stated)

$c = 1 = \hbar$

Acronyms

ST	String Theory
IR	Infrared
UV	Ultraviolet
CFT	Conformal field theory
QG	Quantum gravity
QFT	Quantum Field Theory
SP	Swampland Program
SM	Standard Model
EFT	Effective field theory
BH	Black hole
AdS	Anti-de Sitter spacetime
ASG	Asymptotically safe gravity
DT	Dynamical triangulations
CDT	Causal dynamical triangulations
EDT	Euclidean dynamical triangulations
QCD	Quantum Chromodynamics
FP	Fixed point
WF	Wilson-Fisher
WGC	Weak Gravity conjecture

NGSC	No-Global symmetry conjecture
KK	Kaluza-Klein
CEB	Covariant Entropy Bound
RN	Reissner-Nordstrom
LHS	Left-hand side
RHS	Right-hand side
WC	Weakly Coupled
SC	Strongly Coupled
ABJ	Adler- Bell- Jackiw

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Introduction

It ain't what you don't know that gets you in trouble
It's what you know for sure that just ain't so
— Mark Twain

Quantization of gravity surely plays a role among the most mysterious, ambitious, intriguing and interesting theoretical projects. One can definitely state that many efforts seem to have led to a dead end. To be more precise, nowadays there are several theories that are candidates for the description of quantum gravity, but none of them can be completely considered satisfying. Hence, this looks like a first fire signal of lacking completeness in comprehension of theory structure. Therefore, it is important to try to find and to fill up the essential keys necessary to shed light on this drawback. Moreover, it is still unclear which approach is the correct and consistent one. What is sure is that quantum gravity physics has kicked up a huge fuss during the last half century, driving the attention of theoretical physicists. However, the physicists community is not well balanced among the different proposed theories. One of them has received much more attention in recent years, namely the well-known String Theory. Despite of its occasional Pindaric leap or too abstract mathematical framework, this theory provides a unique description for the known fundamental physics as well as a new deeper landscape for quantization of gravity. Someone has observed that the beauty of the theory resides in its simplicity, hidden in the only one-dimensional parameter necessary for defining its action, the string length which in turn defines the sting tension.

While addressing the problem of quantum gravity, String Theory came up with new insights and new predictions, most of them not having evidence yet in our four-dimensional everyday world. Nevertheless, if on the one hand String Theory leaves open the door to speculations regarding its lack of descriptions in terms of less degrees of freedom, on the other hand it is extremely versatile and represents a lighthouse for all the other quantum gravity theories. Among the numerous new insights proposed by it, the theory naturally furnishes tools to an effective description in a four-dimensional spacetime. It is now consolidated and accepted the prediction of new dimensions as well as the mathematical elegant prescription of the extra-dimensions reduction. It is indeed due to the latter that String Theory is able to account for a low-energy description of the four-dimensional spacetime, presumably our world.

So, what are those drawbacks that actually weaken the theory itself? The theory lacks of determination in this range of energies because it predicts a huge number of vacuum states in which we could live. As a result, what is the right vacuum? Is this another fire signal of a satisfying predictive nature of the model at low energies? Actually, this is not as discouraging as it seems.

Objectives of the research

In order to reduce the number of effective description at lower energies, instead of pursuing the study and actual research at the energy scale at which the theory itself is defined, it could be useful to use a bottom-up approach. Therefore, the gaze is on the range of energy at which we actually live and properties of effective field theories become the new field of study. Hence, it could be interesting to clean up the vast low-energy description by investigating the properties that such effective field theories must have in order to be correctly ultraviolet-completed in a consistent quantum gravity theory. This in turn could provide a useful tool for research on quantum gravity: while at high energies self-consistency marks the structure of a given physical theory, the possible empirical constraints on effective field theories could help to "discriminate" among the possible descriptions at low energies. Hence, self-consistency plays the role at high energies where the empirical constraints become less accessible. In this sense, it could be interesting questioning if one can select those empirical constraints which already encode those crucial, essential properties for a given consistent effective field theory to be completed into quantum gravity in the ultraviolet. It is at this point that efforts have been made by the Swampland Program in the last years.

State of the art & Metodology

The Swampland Program has its roots in String Theory, but as already observed, the latter may lead to a large set of effective low-energy theories. Actually, already at this level, it could be interesting to study properties of such theories because they provide an example of consistent effective field theories completed into quantum gravity in the ultraviolet. In literature, these theories are said to live in the so-called *Landscape*. However, there is also a huge number of effective field theories that do not share those properties exhibited by the previous ones, for this reason they are said to live in the so-called *Swampland*. The properties of the Landscape theories represent the backbone for the Swampland conjectures, namely those empirical constraints imposed on an effective field theory in order to let it live in the Landscape. Actually, the Swampland could be rephrased by using a general notion of quantum gravity instead of specifically String Theory. Therefore, the Swampland can be defined in this more general sense.
Specifically, our research drives its attention on other quantum gravity theories that nowadays are considered valid candidates beyond String Theory.
Before going head first into a deep analysis of the conjectures, we first focused on the most consolidate ones. Furthermore, we are speculating by keeping feet firmly on the ground because potential consequences emerging from any action posed by such conjectures can deeply tear the place apart in the physical, consolidate landscape.
Exactly for this reason, the research project is in some sense one of a long-term work: the project itself is ambitious and requires several debates as well as careful analysis of all the aspects. Additionally, it aims to review all the necessary for a further analysis and to set the tone for future collaborations with local-experts of the quantum gravity theories under

current examination.
The heart of this research is a profound understanding of some of the Swampland conjectures by studying various reviews, considered the most complete in literature [BCV17], [Pal19], [Bee+21].

In particular, attention was focused on the so-called No-Global symmetry and the Weak Gravity conjectures whose supportive arguments range from String Theory literature to black hole physics. In fact, the latter provides another useful approach adopted to study Swampland criteria based on using established quantum gravity arguments. Among the Swampland conjectures, the Weak Gravity one is perhaps the richest in terms of actual speculations as well as already established tools. It was originally formulated in [AH+07] where it was conjectured that gravity is the weakest among the other interactions. By now, there are several statements applicable in different contexts. For this reason, it can be easily addressed to be the bête noire for the way it pervades different cases with different dynamics. In fact, a formulation of the Weak Gravity conjecture incorporates conditions for a black hole to decay and therefore, this can lead to an immediate test. However, the mere decay of a black hole is not sufficient for the Weak Gravity conjecture to hold. In fact, there is a world of motivations hidden in the previous statement and at this level it is sufficient to know that efforts have to be step up for a complete and a deeper understanding.
Perhaps, the conjecture that can actually and frighteningly undermine the physical landscape is the No-Global symmetry one [BS11]. Little is currently known about it, and it is actually expressed in a short and concise statement: exact global symmetries are not allowed in a theory with finite number of states when coupled to gravity. It is quite impressive that it does not put any constraints on models it applies to. Known supportive arguments for it either rely on String Theory or black hole physics too, but due to the wideness and lack of explicit field of application of the conjecture, we actually considered reasonable and extremely interesting to speculate on particle physics models. In fact, as it is stated, effective theories of particle physics can actually lead to relevant constraints or additional support to the Swampland Program.

This in turn gives us a wide range of inspection: asymptotically safe quantum gravity (ASG) and dynamical triangulations (DT) could provide a faithful and flourishing starting point. Additionally, they have a lot in common: both rely on the existence of an ultraviolet fixed point for gravity, providing renormalizability for the latter.

Purposes & Results

Hence, these two quantum gravity theories are playing an important role in this research. In fact, both black hole physics and fundamental physics can be equally bring into play. In particular, the first one seems to have proven a useful and well-handled topic by ASG, shown by the presence of a number of proposed papers on the argument [KS14]. In particular, we are currently focusing on the analysis of Reissner-Nordstrom black holes [GK16] which can probably reach out the Weak Gravity conjecture. However, due to subtleties of the latter, nothing can be strictly affirmed yet. In fact, it is not completely clear the position of the conjecture on the eventual presence of remnants and speculative efforts on it are in progress at present.

Switching sides, the non-perturbative lattice formulation of gravity provided by dynamical triangulations was considered the most appealing. In particular, initial efforts were made on the causal formulation which goes under the name of causal dynamical triangulations (CDT) by the fact that it has proved new compelling results through the last ten years [Lol20]. However, we then realised that vast literature on gravity-matter coupling pervades recent results in Euclidean dynamical triangulation (EDT). Additionally, maybe the latter belongs to the same universality class as CDT so all the previous work has poured grit into our own mill, in some sense. Based on the work of [CLUY18], interesting new insights can actually be used for a further analysis on the No-Global symmetry conjecture. The paper currently under examination proposes the presence of fermions on the triangulated spacetime and the coupling between matter and gravity is studied in the so-called "quenched approximation". The latter allows geometry to fluctuate without a back-reaction from the matter on it. However, this paper really represents a full speculative true challenge for the research project: the model predicts an exact $U(1)$ global symmetry.

Needless to say, we can only throw the ball for future deeply investigation on the topic through the Swampland Program-eye that we are aiming to highlight in our research. In fact, if this model turns out to be a precise formulation and firmly predicts a global exact symmetry, all of us should be prepared for what comes next.

OUTLINES

Future collaborations with experts of the quantum gravity theories under examination is surely in the to-do list of our research. Additionally, the aim of our program could be extended to other well-known quantum gravity theories such as loop quantum gravity. Moreover, beyond reaching more concreteness in the analysis currently in progress, natural extensions of the ongoing projects are several. First of all, one can try to understand whether or not the landscapes of these theories are disjointed. Hence, recollecting the Swampland ideas in a coherent framework could suggest possible new Physics. In fact for instance, if the No-Global symmetry conjecture could actually serve as a powerful constraint for the landscapes of several quantum gravity theories, then its arguments would reach more and more support. As a result, we will be quite sure, in the future, that when an effective theory is coupled to gravity its global symmetries are broken or gauged. Therefore, in this framework, the Standard Model should not provide global symmetries when gravity is taken into account. This in turn will lead us to further develop physics beyond Standard Model.
Furthermore, it could be interesting investigating and explaining the Swampland conjectures as consequences of the nature of fields in the framework of quantum gravity, due to the focus of the research on effective quantum field theories. Additionally, this could serve as a way in understanding the geometry of the Swampland, posing a modern emphasis on theory space. More precisely, Swampland geometry is based on the program of systematically curving out the effective field theory space, according to the basic ideas in [Cec21].

ABOUT THIS THESIS WORK

Therefore, this thesis work is based on the above described research project which is still ongoing and, obviously, it cannot cover all the topics that are currently boiling in the pot. I would like to apologize for this. My colleagues, Manuel Del Piano and Pellegrino Piantadosi, and I will cover part of the arguments under examinations. This thesis is thus structured as follows. There are three parts, the first furnishes the basic tools of String Theory and introduces the Swampland Program in chapters 1 and 2, the second part represents the heart of the Swampland Program and illustrates and motivates two Swampland conjectures: the No-Global symmetry conjecture in chapter 3 and the Weak Gravity conjecture in chapter 4. Finally, the third part focuses on implication for models of particle physics, in particular on the dynamics of particles. There will be suggestions for a possible Global Symmetry conjecture that, at the moment, will set the tone for future work and research. Hence this thesis can be seen as a compendium for the necessary notions of the ongoing research project which I am currently working on.

Conventions

In this thesis, if not otherwise specified, the mostly-plus signature is considered for the metric. Moreover, when p-form gauge fields $A_{\mu_1\cdots\mu_p}$ are considered, the associated field strength $F_{p+1} = \mathrm{d}A_p$ is defined as

$$F_q^2 \triangleq \frac{1}{q!} F_{\mu_1\cdots\mu_q}\ F^{\mu_1\cdots\mu_q} \implies \star F_q^2 = F \wedge \star F \tag{1}$$

with $\star$ the Hodge star [Nak03], the linear map $\star: \quad \Omega^p(\mathcal{M}) \to \Omega^{m-p}(\mathcal{M})$, where $m = \mathrm{dim}\mathcal{M}$, whose action on a basis vector is defined as

$$\star\left(\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\right) = \frac{\sqrt{|g|}}{(m-p)!}\ \epsilon^{\mu_1\mu_2\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_m}\ \mathrm{d}x^{\nu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_m}\ . \tag{2}$$

Let us now introduce some useful info about what is coming next. First of all, a given gauge field $A_{\mu_1\cdots\mu_p}$ has dimension p and thus its integral over a surface Σ_p is dimensionless. Moreover, concerning the gauge coupling $e^2_{p;D}$, its dimension is $2(p+1)-D$. Furthermore [HRR16], considering the action of general relativity coupled to a dilaton field and a p-gauge field in D dimensions, there will be the following conventions relating the Newton constant with the Planck mass

$$\frac{1}{k_D^2} = \frac{1}{8\pi G_D} \triangleq M_P^{D-2} \tag{3}$$

so that

$$S = \frac{1}{k_D^2} \int\ \mathrm{d}^D x\ \sqrt{-g}\ \left(R_D - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2e^2_{p;D}}\ e^{-\alpha_{p;D}\phi}\ F^2_{p+1}\right)\ , \tag{4}$$

where the dimension of the Ricci scalar in a D-dimensional theory is 2 so that k_D^2 has $2-D$ dimensions.

Additionally, given the electric charge of a general $p-1$-brane

$$Q = \frac{1}{e^2_{p;d}} \int_{S^{d-p-1}} e^{-\alpha_{p;D}\phi}\ \star F^2_{p+1} \tag{5}$$

the magnetic dual charge $\tilde{Q}$ is that associated to the dual $D-p-3$-brane, defined as

$$\tilde{Q} = \frac{1}{2\pi} \int_{S^{p+1}} F \tag{6}$$

satisfying the Dirac quantization condition $Q\tilde{Q} \in \mathbb{Z}$. Thanks to electromagnetic duality, the magnetic gauge dual charge and field strength are defined as

$$g_{p;D}^2 = \frac{4\pi^2}{e_{p;D}^2} \tag{7}$$

$$G_{D-p-1} = \frac{2\pi^2}{e_{p;D}^2} e^{-\alpha_{p;D}\phi} \star F_{p+1} \ . \tag{8}$$

In fact [Gai+15], in general for a $U(1)$ gauge theory without matter there are two $U(1)$ one-form symmetries. The first is the electric symmetry whose current is $J_e = \frac{2}{e^2} \star F$ and is generated by

$$U^E_{g=e^{i\alpha}}(R) = e^{i\frac{2}{e^2}\int_R \star F} \tag{9}$$

where $\int_R \star F$ is the electric flux through R. This symmetry shifts the electric gauge field by a flat connection. The second symmetry is the magnetic one with current being $J_m = \frac{1}{2\pi} F$ and generated by

$$U^M_{g=e^{i\eta}}(R) = e^{i\frac{\eta}{2\pi}\int_R F} \ , \tag{10}$$

with $\int_R F$ the magnetic flux through R. This symmetry shifts the magnetic foton by a flat gauge field. In D dimension, the electric one-form symmetry is still a one-form symmetry while the magnetic one is a $D-3$-form symmetry. The charged objects under these two symmetries are Wilson loops and 't Hooft loops, respectively.
Also, concerning chapter 4, we define

$$\gamma_{p;D}(\alpha) \triangleq \left(\frac{\alpha}{2} + \frac{p(D-p-2)}{D-2}\right)^{-1} \ . \tag{11}$$

Part I

Quantum Gravity & the Swampland Program

Chapter 1

The Swampland Program

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Introduction

In your life you surely must have all taken a Band-Aid off at some point. Maybe the following will be out of the blue but let me spill the beans for you by the way: the road to a quantum theory of gravity is devious and in some sense unknown by the simply fact that we are living in a world embedded at a different range of energies from that of a putative quantum theory of gravity. Grin and bear it. Nevertheless, nowadays the physical community is somehow on good terms with the high energy physics and still puts efforts in make speculations tick well. However Nature works in a different way compared with our human minds. The infrared (IR) behaviour is a consequence of the ultraviolet (UV) one. This is exactly the main difficulty in framing a quantum theory of gravity consistently: unexpected *surprises* arising in the low energy physics should not be swept under the carpet.
The Swampland Program touches wood in this attempt, impersonating Aristotle in his thinking discussing through syllogisms: figuring out some characteristics of the big picture things and narrowing it down through constraints imposed on the low energy theories.

1. 1 The Landscape and the Swampland

The Swampland Program moves into the cradle of String Theory. The latter is considered perhaps the most esteemed among theories of quantum gravity but in its framework there are *more than a lot* effective field theories consistent with quantum gravity considerations and principles. At this point the question naturally arises: what is the *exact* effective field theory that describes low-energy physics behaviour of its completion to the high-energy theory? Or, stated differently without jumping the gun, *is it possible to select among different effective theories those having the right characteristics and the passport to be putative quantum gravity theories in the ultraviolet regime?*
First of all, let us pull the plug about these issues for a moment and analyse the main aspects of an effective field theory in order to focus on the nails on which to beat for the understanding

of the Swampland Program.

The attempts of an effective field theory (EFT) are related to the representation and simplification of the dynamics of a given theory in the low-energy limit. Clearly, the immediate consequence is the validity of this description: since the actual theory is at higher energies, the field theoretic framework depicted by an EFT is valid up to a cutoff. EFTs have great importance in desperate aspects of the Standard Model for example. Likely the easy way to approach to them and learn the main characteristics and topics is the example of the chiral symmetry description in the framework of the QCD. In layman's term, above the scale of validity of the EFT the theory breaks down but below the cutoff it is capable of providing a useful description of low-energy physics.
If you smell something fishy about the purpose on an EFT you perhaps have an eye for subtleties. In fact, the purpose of an EFT is synonymous of counting the effective degrees of freedom at low energies. This in turn often implies a theoretical framework which is non-renormalizable.

Let us go back to the problem of quantum gravity (QG). For what previously said then someone can in principle glimpse the effective degrees of freedom of a given quantum gravity theory and hence write down a good and appropriate EFT. This is something that in theory seems reasonable. On the other hand if a friend of him/her has decided to walk the road in reverse, it is likely that it may appear to him/her that a quantum EFT can be completed to a consistent theory of quantum gravity at the drop of a hat. If this was possible, physicists would have already found the famous pot of gold at the end of the magic *quantization of gravity* rainbow, the Nobel Prize is waiting for him/her, to cite my supervisor.
The most important thing to appreciate here is the self-consistency of a theory, whether it is depicted at high energies or described at low ones. Thinking about String Theory, there is a huge number of EFT descriptions. Moreover, String Theory is not *the* theory of quantum gravity so in principle one should admit the other descriptions at low energies coming from other known (or who knows, future) theories. If you want, you could imagine living in a valley, call it *the Valley of EFTs of quantum gravity*. Here, you could also imagine that the valley is populated by different categories of animals: in theory every animal *could* live, eat and drink there. Like every fairy-tale, there are *good* and *bad* animals.

Nevertheless, like any self-respecting valley, there is the need of a diligent shepherd who cares and looks for the safety of the *good* animals, he aims to protect them and guides them in a paddock, leaving out the bad ones. Now, the paddock is metaphorically speaking the *Landscape* of the Swampland Program where the *good* EFTs live while all the remaining area is the so-called *Swampland*. Its perimeter can be regarded as constraints acting on them in order to critically safeguard the EFTs being in the running of a completion in the UV. Therefore the shepherd is, in this context, the Swampland Program itself that decides and makes a selection giving the green light to those *good* EFTs through his careful eyes: the Swampland conjectures.
Additionally, if you think about how a meticulous shepherd organises and keeps the right animals you will get on my same page concluding that he lets the animals in the paddock by carefully observing alike behaviours, alike shape, alike characteristics shared by the *good* ones. Analogously, the Swampland Program puts efforts in gather similar characteristics among EFTs candidate to be a good quantum theory of gravity.

The Swampland Program (SP) first (and until our efforts in the research) arena is String Theory, so it tries to select among different vacua solutions, which means different low-energy descriptions, those which can enter in the paddock from those which cannot by studying them as experimental data. Here there is the need to emphasize that the Swampland theories can come from other theories of quantum gravity.
Most importantly, the SP takes in consideration properties of the low-energy EFT itself not caring about their UV origin. Also, what should be clear at this level, sometimes just quantum gravity arguments suffices to distinguish between the *good* from the *bad* EFTs. For this reason the SP leans on phenomenology and naturally can lead to new exiting results in Cosmology. Without blinking an eye, it is easy to understand that the SP is ambitious and just only for this it is incredibly risky and subject to traps or controversies. Undeniably, it represents a new challenge and saying that it is still *unclear* is fairly to admit.
This uncertainty finds its roots in the fact that the Swampland criteria are not proven from a microscopic point of view, thus the SP results in a *binder* of conjectures. Some of these are believed to be well established while others rely on the "common sense" or if you want, on the sixth sense of a theoretical physicist. Regarding the latter, there is a long list of clichés stating that he/she does not even have the time to eat an apple, as a result, it seems reasonable to me if you are wondering how a physicist can actually study or investigate peculiarities of a given quantum theory of gravity. However, he/she is just as intelligent as he/she seems, in fact there is no need to worry: phenomenology comes to the rescue.
Some of the conjectures are just subjective in some sense and consequently black hole (BH) physics can be used.

1. 1.a The focus of the research

String-experimental data, quantum gravity arguments and BH physics all lay the foundations for supportive motivations to the Swampland conjectures. At the end of the day, how does the SP approach the task in practice? First of all, said Λ_{EFT} the cutoff of a given EFT under study, the SP gets its hands dirty by coupling the latter with gravity. Hence, the theory is expected to change, which means that the cutoff itself should be different. Experience shows that a new theory *should* lead to new insights if it is well defined, so one would imagine that the new cutoff Λ_{new} is lower than Λ_{EFT} if it was the case. In fact, if it was higher, the previous EFT would not be so *affected* by the new one, in some sense. However, the gap between the two values should not be so marked otherwise the new EFT will be inconsistent. Moreover, every Swampland constraint should call off performing the limit $M_{\text{Planck}} \to \infty$, when gravity is decoupled.

One should not act too shocked about the previous line, it is common in physics a similar situation. As already noted in [Bee+21], it may happen that a consistent EFT becomes inconsistent when coupled to a gauge field, due to the appearance of anomalies that need to cancel. Depending on the dynamics of a given theory, certain gauge symmetries develop a topological anomaly, the so-called *Witten anomaly*, emerging when the classical and the quantum lagrangian seem to be well defined but the *overall* theory itself no. This is exactly the case of a flavour global symmetry theory in the fundamental representation of $SU(2)$ with an odd number of fermions: when the symmetry is gauged by the coupling with a dynamical gauge field the theory becomes inconsistent.

It comes naturally then to accept that when a theory is coupled to gravity new anomalies may appear culminating in a non-consistent theory of gravity. What is less intuitive, and is the heart of the present thesis as well as the blooming interest for the SP, is that those gravitational anomalies do not suffice to conclude that the theory is consistent.
Here we are in the new market store of the Swampland conjectures.

1. 2 The String Theory Game

To be concrete, let us start to mention the mother-theory for this ambitious project: String Theory (ST). It can be considered an all-encompassing theory of the Universe: it contains all the main ingredients, it is a recipe that naturally leads to General Relativity, gauge theories as well as further theoretical answers of unknown proven existence such as supersymmetry or axions.
Needless to say, the beauty has always a prize, in the case of ST is the prediction of extra dimensions for the world we live in. Whether or not ST is *the* correct description of a quantum theory of gravity maybe we will never know, but it contains for sure all the characteristics one expects for the consolidate knowledge of the physics world. For this reason, the physics community is debating about the "String Universality" for which there is the likelihood that other quantum gravity theories all converge to ST in the UV. At the end of the day, how did one come up with the idea of ST?

By quantizing gravity with the tools of QFT the first issue is handing with a theory which is non-renormalizable. Among other suggestive proposals, the one on which ST is based on solves the problem beforehand by substituting zero-dimensional objects (points) with one-dimensional ones, strings. As a result, the short-range interactions that were the cause of divergent integrals are now removed by passing to a study of an extended object rather than a well *localised* point [Pol07a]. All of this is extremely interesting but where are particles? If before they were represented by points, where are they now in this new framework? Particles emerge from vibrational modes of the string, to be more precise, they correspond to particular and well defined oscillation modes of the string, i.e. quantum states of the string. The whole theory is based on the description of a *shoelace* embedded in a flat Minkowski space that, following the String-nomenclature, is called target space [Zwi06]. Well, but everyday experience has shown us that objects *span* trajectories during their dynamical motion. This naturally arises the next question: how does one describe the dynamics of particles? As long as they are represented by points their trajectories are worldlines in space. Now, the study converges to the dynamics of a string and all it takes is intuition to understand that from worldlines one moves to worldsheet, namely the string spans a two-dimensional object during its motion rather than a one-dimensional one. The mathematical description is thus split in two levels of space: the target space where the string lives and the worldsheet spanned by it during the motion.
This two-level description has interesting and important consequences, first and foremost concerns the symmetry theme; the latter has set the tone for the investigation regarding the oldest among the Swampland conjectures, the No-Global symmetry conjecture, as it will be later explained.

1. 2.a New horizons coming from String Theory

The picture provided by ST in the quantization of a one-dimensional object is highly persuasive. What renders this theory unique is the absence of adjustable parameters. For instance, the Standard Model (SM) theory of particle physics entails at least twenty parameters that need to be tuned to precise value with the help of experiments. Instead, ST has only one parameter that defines the string itself: the string length or equivalently the tension of the string.

Part of the physics community brings forward criticism of the spacetime-dependence distinctive of the theory due to the fact that gravity is, classically, the theory of the dynamics of the spacetime itself. Hence, one intuitively expects that a quantum theory of gravity shall share this property. On the other hand even if the target space is needed and set to have four dimensions, the ST naturally predicts the dimension of the spacetime itself, in other words the spacetime dimensions emerge from computations actually. And here ST risks life and limb because the result is 26 dimensions and the story is not even finished.
Thinking about a shoelace, one gets to grips with two possible configurations, an open string and a closed string and furthermore, the latter can be realized by matching the two endpoints of the former. For the open string, the two ends have to satisfy boundary conditions of the Neumann or of the Dirichlet type. The latter represents a spacetime hypersurface on which the string itself ends, as a result the only way to visualise the picture requires the presence of a physical object on which the string ends: the D-brane. Additionally, Nature is such that we distinguish between spin-integer particles, the bosons, and the spin-half particles, the fermions. ST arises from the so-called bosonic strings which can be either open or close strings and did not contemplate fermions. Alongside the development of the theory in the mid 1980s new symmetry-inspired ideas were coming down the pike by which a field theory should be such that bosonic fields are accompanied by fermionic counterparts: supersymmetry theories were on their path.
The mathematical description upon which they are based was further revealed to be encoded by ST. Nowadays there are at least five supersting theories, accounting for the presence of fermions in the framework. In this picture, the spacetime dimensions emerging from calculations is reduced to 10. The inclusion of supersymmetry opens the door to different theories, up to now they are five. For example, if one applies this feature to both right and left modes of a closed string he/she will end up with equal or opposite handedness for the movers. These in turn give rise to type IIA and type IIB theories respectively [BBS06]. Moreover, matching the 26-dimension formalism for the left-movers with the 10-dimensional one for the right-movers gives the so-called *heterotic* strings. At first glance, one may be surprised by the vastity of theories proposed by the framework of ST. Needless to say, ST has something up its sleeve: T-duality.
However, even though the feature of supersymmetry renders ST a consistent theory by accounting for fermions, it has not yet been discovered. This could be a double-edged sword since on the one hand it can be a signal that there is something wrong with current speculations; on the other, it may be a smoke signal that the scale at which supersymmetry breaking should occur (in order to give mass to the supersymmetry partners of particles) is above the experimentally already determined energy bounds.

Whether the presence of supersymmetry or not, the dimensions of the spacetime is higher than that characteristic of the world *we believe* to live in. Does ST come up with a solution

or a convincent explaination for these results? Things are easier said than done actually.
What is already established in Physics is the principle of conservation of energy and the fact that gravity becomes strongly coupled at the Planck scale, which in turn naturally leads to the awareness that with a good percentage we are not able to probe the *real* physics at that scale. As a consequence, we are allowed to sleep soundly in some sense, by just assuming that *for what we actually sense* conservation of energy still has to hold. Therefore, visualising in a *gedanken experiment* the presence of additional macroscopic dimensions we will run into contradiction with phenomena that imply symmetry arguments and consequently conservation of energy, i.e. the explosion of a supernova.
Six of one, half a dozen of the other, right? Additional microscopic dimensions are more reasonable.

1. 2.b Compactification of extra dimensions

Imagine to have a one-dimensional world populated by ants for which the latter are extremely irrelevant within the extension of the world they live in. They are thus allowed to move along only one dimension, name this as the x axis. If now, in the principle of an *gedanken experiment*, we imagine that for a point on the x axis there is a circle, the ants seem to live on a cylinder. Moreover, if this circle is extremely small compared with the extension of the x axis, from the point of view of the ants the world is *still* one-dimensional. The only way to be aware of this extra compactified dimension is encoded in a powerful experimental instrument capable of probing its characteristic length scale.

Ultimately, ST comes with a new concept of spacetime. Even if there are no direct experimental evidence of the tools provided by it and maybe we will never know in a near future, it seems that ST kills two birds with one stone: provides new dimensions which in turn lead to new insights in physics and it is also capable of restricting to a four-dimensional world with the reasonable and intelligent trick of compactification.
There is to mention how this idea emerged among the string-community. During the Roaring Twenties, Kaluza and Klein in the attempt of unifying electromagnetism and General Relativity, spread the compactification idea. Their original thought was exactly the example of the ants-world described above. The long dimension of the x axis is replaced by the four-dimentional spacetime while the other by a compact manifold of higher dimensions. On this path, in the mid 80s other type of compactifications were proposed, such as the so-called Calabi-Yau manifolds.

1. 2.c Two for one: Yang-Mills theory & Black Holes with D-branes

As already mentioned *en passant*, D-branes occupy a special role in ST. In particular, they emerge in type I and type II superstrings. Quantum field theory is based on perturbation theory, the probability of a given event, whether it is the creation or destruction of particles or even scattering processes, is computed by expanding the Dyson series in the T-product in terms of the coupling constant. Now, in ST, p-branes (with $p \geq 2$) become heavy when the latter goes to zero. This implies that for perturbation theory they cannot be advocated but when the coupling is large enough they become relevant. In fact, Yang-Mills theories are described in this context: their description is in terms of massless modes of open strings attached to D-branes.
The interest in this particular type of branes does not end here. Instead, there exist the

so-called *black p-branes* through which possible black hole description can be set. This in turn led to another discovery of ST: AdS/CFT duality. In fact, if one considers a "binder of p-branes" the spacetime geometry resembles that of a BH in the sense that a horizon appears. Now, in the vicinity of the latter the geometry can be approximated by a product of the maximally symmetric spacetime with negative curvature, namely the anti-de Sitter space, with a sphere.

Chapter 2

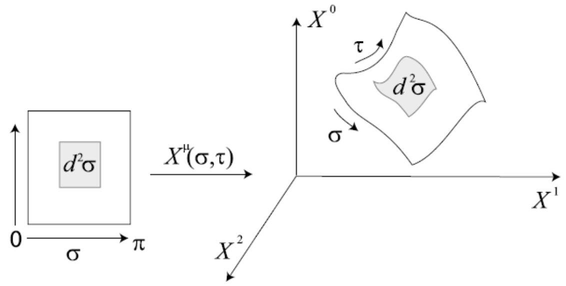
String Theory in a nutshell

Introduction

In this chapter the simplest mathematical description of a relativistic string is introduced: the bosonic string. We all have to buckle our seat belt because String Theory needs a little bit of imagination. Nevertheless, it can be visualized as a beautiful palette through which Standard Model, Quantum Gravity, Cosmology and so many others can be painted. The task is extremely vast and for what concerns in this thesis the main focus will be on the bosonic closed string, which will set the tone for the born of the well-believed Swampland conjectures. Therefore fundamental concepts are introduced, ranging from the visualization of the string to the compactification on a circle and the tower of states of Kaluza-Klein and winding modes.

2. 1 The bosonic string

In general, using the typical string-inspired nomenclature, a string can be visualised as a p-brane with $p = 1$. Hence, it comes naturally to describe a relativistic particle as a 0-brane. Needless to say, we consider a flat target space with D dimensions whose metric is denoted by $\eta_{\mu\nu}$. A p-brane sweeps out a $(p + 1)$-dimensional world-volume. In the case of a string the latter is two-dimensional and is called the *worldsheet*. The points on this worldsheet are parameterized by the set of two coordinates $\xi = (\tau, \sigma)$, where the former is time-like and the latter is space-like. To be more accurate, the parameter τ represents the worldsheet time while σ parameterizes the string itself at a given τ. As already said, the worldsheet is embedded in the target space and there is a two-level description of space.



The embedding of the world-sheet Σ
on flat Minkowski space is described by the following functions $X^\mu(\tau,\sigma)$: $\Sigma : (\tau,\sigma) \hookrightarrow X^\mu(\tau,\sigma) \in \mathbb{R}^{1,D-1}$.

The motion of a string in the spacetime can be described by using the principle of minimal action. This description is carried by an auxiliary metric on the worlsheet that will be denoted with $\gamma_{ab}(\tau,\sigma)$. Using this notation, the most general action for this aim is the so-called Brink- Di Vecchia- Howe- Deser- Zumino action usually referred to as the *Polyakov action* [Pol07a]:

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-\det\gamma_{ab}}\; \gamma^{ab}\partial_a X^\mu \partial_b X_\mu \tag{2.1}$$

where α' is a constant called the *Regge slope* and has units of length squared. Moreover, in flat Minkowski space, the functions X^μ can be visualised as a D-vector, as a result the (2.1) describes a two-dimensional theory for such a D-vector in the target space where the string propagates[1].
It is interesting to note that the auxiliary metric can be eliminated from (2.1) by using its equation of motion. Recalling the definition of the energy-momentum tensor [BBS06]

$$T_{ab} = -\frac{2}{T_P}\frac{1}{\sqrt{-\det\gamma_{ab}}}\frac{\delta S_P}{\delta\gamma^{ab}} \tag{2.5}$$

the latter implies that (2.5) has to vanish. Additionally, by the fact that $\delta\sqrt{-\det\gamma_{ab}} = -\frac{1}{2}\sqrt{-\det\gamma_{ab}}\;\gamma_{ab}\delta\gamma^{ab}$, the previous requirement leads to

$$\delta S_P = -\frac{T_P}{2}\int_\Sigma \mathrm{d}^2\sigma\;\; \delta\gamma^{ab}\left(\sqrt{-\gamma}\;\partial_a X^\mu\partial_b X^\nu - \frac{1}{2}\sqrt{-\gamma}\;\gamma_{ab}\gamma^{cd}\;\partial_c X^\mu \partial_d X^\nu\right)\eta_{\mu\nu} = 0 \tag{2.6}$$

$$\Longrightarrow\; T_{ab} = \partial_a X^\mu\partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu\partial_d X_\mu\;, \tag{2.7}$$

so, the worldsheet metric satisfies $\gamma_{ab} = 2\frac{\partial_a X^\mu\partial_b X_\mu}{\gamma^{cd}\partial_c X^\nu \partial_d X_\nu}$, namely that

$$\gamma_{ab} = \text{const}\cdot\;\partial_a X^\mu\partial_b X_\mu\;. \tag{2.8}$$

Two considerations are natural. Firstly, equation (2.8) manifests a symmetry encoded in the Polyakov action. In fact, said $h_{ab} = \partial_a X^\mu\partial_b X_\mu$, the previous relation leads to the following

[1]One can appreciate the presence of the unique parameter of the theory by generalising the (2.1) to the case of p-branes:

$$S_P = -T_P\int_{\mathcal{M}} \mathrm{d}\mu_P \tag{2.2}$$

with T_P the p-brane tension and $\mathrm{d}\mu_P$ the volume element which is written in terms of the metric

$$G_{ab} = \eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu \quad \text{where} \quad a,b = 0,\dots,p. \tag{2.3}$$

Therefore, since $\mathrm{d}\mu_p$ is the volume element it has units of $(\text{length})^{p+1}$, naturally leading to the dimension of the tension of the p-brane

$$[T_P] = (\text{length})^{-p-1} = \frac{\text{mass}}{(\text{length})^p} \xrightarrow{(2.1)} T_P \equiv \frac{1}{2\pi\alpha'}\;. \tag{2.4}$$

identity

$$\frac{h_{ab}}{\sqrt{-h}} = \frac{\frac{1}{2}\gamma_{ab}\gamma^{cd}\, h_{cd}}{\sqrt{-\det\left(\frac{1}{2}\gamma_{ab}\gamma^{cd}\, h_{cd}\right)}} = \frac{\gamma_{ab}}{\sqrt{-\gamma}}. \tag{2.9}$$

The (2.9) highlights that even though the induced metric h_{ab} is *fixed*, by the fact that the fields X^μ are given, the worldsheet metric is not. In fact, a field $\Omega(\tau, \sigma)$ could exist according to which $\gamma_{ab} \mapsto \Omega^2(\tau, \sigma)\, \gamma_{ab}$ and (2.9) is still satisfied. Stated differently, the Polyakov action is Weyl-invariant.
Secondly, the Euler-Lagrange equation for the fields X^μ is straightforward:

$$\partial_a \left(\sqrt{-\gamma}\, \gamma^{ab}\partial_b X^\mu\right) \ . \tag{2.10}$$

2. 1.a Symmetries of the Polyakov action

The Weyl symmetry is indeed a gauge symmetry for the worldsheet, the function Ω depends on the string coordinates indeed. As a result, the worldsheet metric is invariant under a *local* change of scale that preserves the angles between lines. This actually means that from the point of view of the Polyakov action, the two worldsheet metrics in figure 2.1 are equivalent. As a result, this in turn translates in a symmetry for the Polyakov action, a special case for

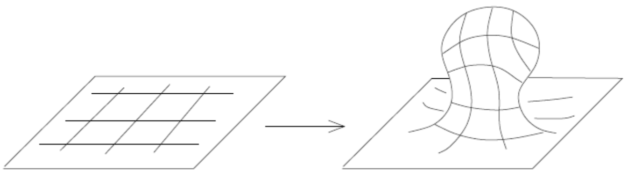


Figure 2.1. Illustration of a Weyl transformation.

two dimensions since the scaling of $\sqrt{-\gamma}$ is compensated by that of the inverse metric [Ton09]. Combining the symmetries of the target space and the worldsheet it is straightforward to state that the Polyakov action is indeed invariant under

1. D-dimensional Poincaré transformations on the target space

$$X'^\mu(\tau, \sigma) = \Lambda^\mu{}_\nu\, X^\nu(\tau, \sigma) + a^\mu \tag{2.11}$$
$$\gamma'_{ab}(\tau, \sigma) = \gamma_{ab}(\tau, \sigma) \ , \tag{2.12}$$

   it is easy to note that it is a *global* symmetry on the worldsheet by the fact that μ runs on internal indices. Moreover, the transformation (2.11) acts on the fields at fixed τ and σ.

2. Diffeomorphisms on the worldsheet

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma) \tag{2.13}$$
$$\gamma'_{ab}(\tau', \sigma') = \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b}\gamma_{cd}(\tau, \sigma) \tag{2.14}$$

3. Two-dimensional Weyl transformations on the worldsheet

$$X'^\mu(\tau, \sigma) = X^\mu(\tau, \sigma) \tag{2.15}$$
$$\gamma'_{ab}(\tau, \sigma) = e^{2\zeta(\tau,\sigma)}\gamma_{ab}(\tau, \sigma) \quad \text{with} \quad \zeta(\tau, \sigma) \quad \text{arbitrary.} \tag{2.16}$$

It is thus clear that from the point of view of the string there are two gauge symmetries and one global symmetry. The former implies a redundancy in the choice of coordinates so treatment simplify a lot by fixing an opportune gauge.

2. 1.b Gauge fixing and mode expansions

The worldsheet metric is defined by a 2×2 symmetric matrix so it has three independent parameters. To reduce redundancy we choose the so-called *conformal gauge* for which the metric is locally conformally flat, namely

$$\gamma_{ab} = e^{2\zeta(\tau,\sigma)}\eta_{ab} \ . \tag{2.17}$$

In this way [Ton09], two of the three parameters are fixed, the last independent one can be removed by choosing a null Weyl field, so that

$$\gamma_{ab} = \eta_{ab}. \tag{2.18}$$

Therefore, the metric on the worldsheet is just the Minkowski one. A primary consequence of this gauge choice regards the simplification of the equation of motion of the worldsheet metric. In fact, equation (2.7) in the conformal gauge becomes

$$T_{01} = \frac{\partial X^\mu}{\partial \tau}\frac{\partial X_\mu}{\partial \sigma} = 0 \tag{2.19}$$

$$T_{00} = T_{11} = \frac{1}{2}\left[\left(\frac{\partial X}{\partial \tau}\right)^2 + \left(\frac{\partial X}{\partial \sigma}\right)^2\right] = 0 \tag{2.20}$$

which together with the equation of motion for the fields [Pol07a]-[BBS06]

$$\partial_a \partial^a X^\mu = 0 \tag{2.21}$$

form a system of equations of motion for the string. The latter thus behaves as a free wave (2.21) subjected to constraints (2.19) and (2.20) which are called *Virasoro constraints*.

In the case of our interest of a closed string, periodic conditions must be imposed, namely σ is periodic $\sigma \equiv \sigma + 2\pi$. Consequently, this in turn implies that

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \ . \tag{2.22}$$

In order to simplify computations, we will introduce the so-called *lightcone coordinates*:

$$\xi^\pm = \tau \pm \sigma \implies \mathrm{d}s^2 = -\mathrm{d}\xi^+\,\mathrm{d}\xi^- \tag{2.23}$$

so that (2.21) becomes

$$\partial_+\partial_- X^\mu = 0 \tag{2.24}$$

while (2.19) reads as

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0 \ . \tag{2.25}$$

Intuitively, a closed string can be visualised as two open strings attached at their ends. As a result, we can decompose a general solution of the previous with two arbitrary functions

$$X^\mu(\tau,\sigma) = X^\mu_L(\xi^+) + X^\mu_R(\xi^-) \tag{2.26}$$

called *left* and *right movers*. They do not singularly satisfy the closed string constraint but their combination does. Said x^μ and p^μ the position and momentum of the center of mass of the string, a general periodic solution is

$$X_L^\mu(\xi^+) = \frac{x^\mu + \alpha' p^\mu \xi^+}{2} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in\xi^+} \tag{2.27}$$

$$X_R^\mu(\xi^-) = \frac{x^\mu + \alpha' p^\mu \xi^-}{2} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\xi^-} \tag{2.28}$$

for which $\alpha_n^\mu = \alpha_{-n}^{\mu\,*}$ and $\tilde{\alpha}_n^\mu = \tilde{\alpha}_{-n}^{\mu\,*}$ have to hold in virtue of the reality of X^μ. Additionally, the metric's constraints read as

$$\partial_- X_R^\mu = \frac{\alpha' p^\mu}{2} + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\xi^-} \xrightarrow{\alpha_0^\mu \triangleq \sqrt{\frac{\alpha'}{2}} p^\mu} = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\xi^-} \tag{2.29}$$

$$\implies \left(\partial_- X\right)^2 = \sqrt{\frac{\alpha'}{2}} \sum_{n,m} \alpha_n \cdot \alpha_m e^{-i(n+m)\xi^-} \xrightarrow{n+m=l} \triangleq \alpha' \sum_l L_l \; e^{-il\xi^-} \tag{2.30}$$

$$\left(\partial_+ X\right)^2 \triangleq \alpha' \sum_l \tilde{L}_l \; e^{-il\xi^+} \tag{2.31}$$

where the sum of oscillator modes have been introduced, or rather the Fourier modes of the constraints

$$L_l = \frac{1}{2} \sum_n \alpha_{l-n} \cdot \alpha_n \quad \text{and} \quad \tilde{L}_l = \frac{1}{2} \sum_n \tilde{\alpha}_{l-n} \cdot \tilde{\alpha}_n \; . \tag{2.32}$$

Performing a transformation in the lightcone coordinates actually leads to a transformation of the metric $\eta_{ab} \to \Omega^2(\tau,\sigma)\eta_{ab}$ obtained by combining a conformal transformation with a Weyl reperameterization. In order to fix this remaining reparameterization invariance we introduce the *spacetime lightcone coordinates*

$$X^\pm = \frac{X^0 \pm X^{D-1}}{\sqrt{2}} \implies \mathrm{d}s^2 = -2\mathrm{d}X^+\mathrm{d}X^- + \sum_i^{D-2} \mathrm{d}X^i\mathrm{d}X^i \tag{2.33}$$

where an explicit splitting of the time and space direction comes as a consequence. Using the same ansatz as before, the following choice of coordinates[2]

$$X_L^+ = \frac{x^+ + \alpha' p^+ \xi^+}{2} \quad \text{and} \quad X_R^- = \frac{x^+ + \alpha' p^+ \xi^-}{2} \implies X^+ = x^+ + \alpha' p^+ \tau \tag{2.34}$$

is called the *lightcone gauge*, where the X^+ component of the string lightcone coordinates $(X^+, X^-, X^1, \dots, X^{D-2})$ corresponds to the time coordinate as seen in a frame in which the string is travelling at infinite momentum. Moreover, this choice has the advantage that, being X^+ independent of σ, every point of the string is at *the same time*. Finally, the classical general solution is described in terms of $2D-4$ *transverse modes* α_n^i and $\tilde{\alpha}_n^i$ together with

[2]The residual invariance is due to the fact that a new worldsheet set of coordinates can be an arbitrary function of the previous one: $\xi^+ \mapsto \tilde{\xi}^+(\xi^+)$ and $\xi^- \mapsto \tilde{\xi}^-(\xi^-)$ resulting the following system of transformations for τ and σ: $\begin{cases} \tilde{\tau} &= \frac{\tilde{\xi}^+(\tau+\sigma)+\tilde{\xi}^-(\tau-\sigma)}{2} \\ \tilde{\sigma} &= \frac{\tilde{\xi}^+(\tau+\sigma)-\tilde{\xi}^-(\tau-\sigma)}{2} \end{cases}$. Once $\tilde{\tau}$ is fixed, $\tilde{\sigma}$ is determined. Furthermore, $\tilde{\tau}$ can be arbitrary, just as the free wave solution field. As a result, it can be chosen to be equal to X^+. Then the residual symmetry can be fixed setting the oscillation modes α_n^+ and $\tilde{\alpha}_n^+$ to zero, leading to the gauge (2.34).

the position and momentum of the center of mass of the string x^i, p^i, p^+ and x^-. After some algebra in this ansatz one ends up with the relations

$$\partial_+ X^i = \frac{\partial X^i}{\partial \tau} + \frac{\partial X^i}{\partial \sigma} \overset{\tilde{\alpha}_0^i = \sqrt{\frac{\alpha^i}{2}} p_L^i}{=} \sqrt{2\alpha'} \sum_n \tilde{\alpha}_n^i \, e^{-in(\tau+\sigma)} \tag{2.35}$$

$$\partial_- X^i = \frac{\partial X^i}{\partial \tau} - \frac{\partial X^i}{\partial \sigma} \overset{\alpha_0^i = \sqrt{\frac{\alpha^i}{2}} p_R^i}{=} \sqrt{2\alpha'} \sum_n \alpha_n^i \, e^{-in(\tau-\sigma)} \tag{2.36}$$

which, together with $\partial_\tau X^+ = \alpha' p^+$ coming from (2.34), lead to a new expression of the constraints in (2.25), that now read as

$$2\alpha' p^+ \left(\frac{\partial X^-}{\partial \tau} \pm \frac{\partial X^-}{\partial \sigma} \right) = \left(\frac{\partial X^i}{\partial \tau} \pm \frac{\partial X^i}{\partial \sigma} \right)^2 \iff \partial_\pm X^- = \frac{(\partial_\pm X^i)^2}{\alpha' p^+} \,. \tag{2.37}$$

2. 1.c Quantization of the string

In this gauge one can perform the quantization of the classical string action taking in consideration the physical degrees of freedom only. The result is the description of a theory in a Fock space. First of all, it is important to note that (2.32) is proportional to the square of the spacetime momentum p^μ. However, the momentum squared has a precise meaning in Minkowski space: it is the square of the rest mass of a particle. Moreover, constraints (2.25) together with the lightcone gauge (2.23) lead to

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \xrightarrow{(2.23)} \begin{cases} \partial_+ X_L^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \\ \partial_- X_R^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i \end{cases} \tag{2.38}$$

meaning that the function $X^-(\xi^+, \xi^-)$ is completely determined. Furthermore, rewriting the usual mode expansion for $X_{L/R}^-$ and using the previous equations, one can end up with the following expression for the oscillator modes α_n^-

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \, \alpha_m^i \,. \tag{2.39}$$

Now, recalling that $\alpha_0^- = \sqrt{\alpha'/2} p^-$ and $\tilde{\alpha}_0^- = \sqrt{\alpha'/2} p^-$:

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{\alpha' p^i p^i}{2} + \sum_{n \neq 0} \alpha_n^i \alpha_{-n}^i \right) \tag{2.40}$$

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{\alpha' p^i p^i}{2} + \sum_{n \neq 0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i \right) \tag{2.41}$$

we can write $p_\mu p^\mu = -M^2$

$$M^2 = 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \,. \tag{2.42}$$

Considerations are now in order. The oscillators α^i and $\tilde{\alpha}^i$ are called *transverse oscillators*, therefore, the general classical solution is described in terms of these transverse oscillators together with a number of zero modes defining the center of mass and momentum of the string, namely x^i, p^i, p^+ and x^+. Additionally, condition (2.23) imposes that a shift in x^+ can

be absorbed by a shift of τ, while p^- is fixed through (2.40) and (2.41).
It has come the time to quantize the theory by imposing commutation relations:

$$[x^i, p^j] = i\delta^{ij} \qquad , \qquad [x^-, p^+] = -i \tag{2.43}$$

$$\left[\alpha_n^i, \alpha_m^j\right] = \left[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\right] = n\ \delta^{ij}\delta_{n+m,0}\ . \tag{2.44}$$

In order to construct the Hilbert space, let $|0;p\rangle$ denote the vacuum state, therefore the Fock space is built by acting on the latter with the creation operators α_n^i and $\tilde{\alpha}_n^i$. However, having moved to quantization, attention must be payed on normal ordering of the creation and destruction operators. Hence, (2.42) reads as

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\underbrace{\alpha_{-n}^i\alpha_n^i}_{\triangleq N} - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\underbrace{\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i}_{\triangleq \tilde{N}} - a\right) \tag{2.45}$$

where N and $\tilde{N}$ are the number operators of the harmonic oscillator. Therefore [Ton09]

$$M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}\left(\tilde{N}-a\right) \tag{2.46}$$

and it can be demonstrated that $a = \frac{D-2}{24}$. Equation (2.46) can be used to study the string spectrum. The case of our interest concerns the first exited states, namely the $(D-2)^2$ particles arising from $\tilde{\alpha}_{-1}^i\alpha_{-1}^j\,|0;p\rangle$. These states have to fit into some representation of the full Lorentz $SO(1, D-1)$ group. Therefore, Wigner's classification of the Poincaré group has to be taken into consideration. If one supposes that these particles are massive, then one can sit at the rest frame and choose $p^\mu = (m, 0, \dots, 0)$ where particles are invariant under the little group $SO(D-1)$. However, there is no way to account for $(D-2)^2$ particles in any representation of $SO(D-1)$. Therefore, these particles must be massless.
In this case, one can choose $p^\mu = (p, 0, \dots, p)$ and the little group is now $SO(D-2)$. Hence, the only way to preserve the Lorentz group $SO(1, D-1)$ is when

$$D = 26 \tag{2.47}$$

which is the so-called *critical dimension*.
In quantum field theory, massless particles give rise to long range forces so let us focus more on these $(D-2)^2$ particles. The little group is $SO(24)$ so they transform under the $24 \otimes 24$ representation of $SO(24)$. Therefore, these can be decomposed into three irreducible representations:

$$\text{traceless symmetric} \quad \oplus \quad \text{anti-symmetric} \quad \oplus \quad \text{singlet}. \tag{2.48}$$

Hence, to each of these modes a massless field in spacetime can be associated, so the string oscillation is indeed identified with a quantum of these fields, which are

$$G_{\mu\nu}(X), \qquad B_{\mu\nu}(X), \qquad \Phi(X) \tag{2.49}$$

respectively. First of all, $B_{\mu\nu}$ is the so-called *Kalb-Ramond* field while Φ is the *dilaton*. The massless, symmetric, traceless field $G_{\mu\nu}$ represents a spin-2 particle and, following arguments of Feynmann and Weinberg that any theory of interacting massless spin-2 particles must be equivalent to General Relativity, we interpret it as the *graviton*.

2. 2 Non-linear sigma-models: low-energy effective action

The next logical step is to generalise the propagation of the string on a curved spacetime. This improvement can be achieved by the introduction of a target space-metric dependent on the embedding functions. The natural generalization of the (2.1) describing a string moving in curved spacetime is

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{-\gamma}\, \gamma^{ab}\partial_a X^\mu \partial_b X^\nu\ \Upsilon_{\mu\nu}(X) \tag{2.50}$$

which is a map from the worldsheet of the string into the curved target spacetime with metric $\Upsilon_{\mu\nu}(X)$. The field theory constructed upon (2.50) is such that the kinetic term is field-dependent. Such a theory is known as a *non-linear σ-model theory* [Ton09]. However, there is a subtle particularity in (2.50): the information about gravity is encoded in the graviton appearing as a state of the string. As a consequence, the background metric $\Upsilon_{\mu\nu}(X)$ should be constructed from these states. In other words, we should expect gravitons hidden in $\Upsilon_{\mu\nu}(X)$. Indeed, if one considers

$$\Upsilon_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X) \tag{2.51}$$

in the path integral formulation, the partition function is thus

$$\begin{aligned} Z = \int \mathcal{D}X\ \mathcal{D}g \exp\{-S_P - V\} &\simeq \int \mathcal{D}X\ \mathcal{D}g \exp\{-S_P\}\left(1 - V + \frac{1}{2}V^2 + \ldots\right) \\ \text{where} \quad V &= \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \sqrt{-\gamma}\, \gamma^{ab}\partial_a X^\mu \partial_b X^\nu\ \chi_{\mu\nu}(X) \end{aligned} \tag{2.52}$$

which can be demonstrated that in the conformal field theory-analysis is the vertex operator for the graviton state of the string, (see Appendix A). Therefore, the fluctuation $\chi_{\mu\nu}(X)$ can be written in terms of a plane wave (with momentum p) corresponding to a graviton whose polarization in given by a traceless tensor $s_{\mu\nu}$:

$$\chi_{\mu\nu} = s_{\mu\nu}\ e^{ip\cdot X}\ . \tag{2.53}$$

Hence, the curved spacetime is a coherent background of gravitons. Furthermore, in the conformal gauge, (2.50) is rewritten as

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \partial_a X^\mu \partial^a X^\nu\ \Upsilon_{\mu\nu}(X)\ , \tag{2.54}$$

meaning that the action is no longer quadratic in the embedding functions but constitutes an interacting two-dimensional field theory. We want to better understand the interaction term. First of all, let x_0^μ be a classical string solution, we introduce a massless, dimensionless, dynamical field fluctuation $W^\mu(\sigma)$, such that

$$X^\mu(\sigma) = x_0^\mu + \sqrt{\alpha'}\ W^\mu(\sigma)\ . \tag{2.55}$$

Then we expand the integrand of the (2.54)

$$\begin{aligned} \Upsilon_{\mu\nu}(X)\partial_a W^\mu \partial^a W^\nu =& \alpha'\Big[\Upsilon_{\mu\nu}(x_0) + \sqrt{\alpha'}\Upsilon_{\mu\nu,\ \omega}(x_0)\ W^\omega + \\ &+ \frac{\alpha'}{2}\Upsilon_{\mu\nu,\ \omega\rho}\ W^\omega\ W^\rho + \ldots\Big]\partial_a W^\mu \partial^a W^\nu \end{aligned} \tag{2.56}$$

from which it is clear that the coupling constants involve derivatives of the metric at x_0. We are interested in the weakly coupling regime in order to use low-energy effective field theory description. If r_c denotes the characteristic curvature radius of the target space, then $\partial_X \Upsilon \sim r_c^{-1}$, therefore we can define the *effective dimensionless coupling* as

$$\frac{\sqrt{\alpha'}}{r_c} \ , \tag{2.57}$$

in terms of which, if (2.57)$\lll 1$, it is defined the perturbation theory of the non-linear σ-model, which is a renormalizable theory.
Nevertheless till now we are at a classical level, what about the quantum level? We want to preserve conformal invariance and we already know that we have to conveniently regulate divergence by introducing an UV cutoff. For this aim, the focus moves on the study of the β-functions: the quantum theory is conformally invariant if

$$\beta_{\mu\nu} \sim \mu \frac{\partial \Upsilon_{\mu\nu}(X;\mu)}{\partial \mu} = 0 \ . \tag{2.58}$$

For what comes next, we will use the Riemann normal coordinates[3], in terms of which the geodesic equation leads to

$$0 = \Gamma^{\mu}_{\alpha\beta} \tag{2.59}$$
$$0 = \partial_\nu {\Gamma^{\mu}}_{\alpha\beta} + \partial_\alpha {\Gamma^{\mu}}_{\beta\nu} + \partial_\beta {\Gamma^{\mu}}_{\nu\alpha} \tag{2.60}$$

hence, by Taylor expanding the metric around x_0

$$\Upsilon_{\mu\nu} \approx \Upsilon_{\mu\nu} + \partial_\rho \partial_\lambda \ \Upsilon_{\mu\nu} \frac{X^\lambda X^\rho}{2} \xrightarrow{(2.59)(2.60)} \partial_\rho \partial_\lambda \ \Upsilon_{\mu\nu} = -\frac{1}{3}\left(R_{\mu\lambda\nu\rho} + R_{\mu\rho\nu\lambda}\right) \tag{2.61}$$

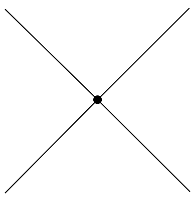
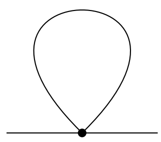
for which, evaluating at (2.55)

$$\Upsilon_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\lambda\nu\rho}(x_0) \ W^\lambda \ W^\rho \ . \tag{2.62}$$

Therefore, (2.50) becomes

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left(\partial W^\mu \cdot \partial W^\nu \ \eta_{\mu\nu}\right) - \frac{\alpha'}{3} R_{\mu\lambda\nu\rho} \ W^\lambda W^\rho \ \partial W^\mu \cdot \partial W^\nu \ , \tag{2.63}$$

where, in terms of Feynman diagrams, the quartic interaction and the divergence comes from

 $\sim R_{\mu\lambda\nu\rho}$ and  $\Longleftrightarrow$ one-loop diagram .

(2.64)

[3]They are used to define the coordinates of nearby points through geodesics. For example, given a point O and a nearby point P, defining with a^μ the components at O of the unit tangent vector of the geodesic joining O to P, the coordinates in P are $x^\mu = s \ a^\mu$, with s the geodesic arc length.

Using dimensional regularization with $D = 2+\epsilon$, it can be demonstrated that the counterterm of $\propto R_{\mu\lambda\nu\rho}\, W^\lambda W^\rho\, \partial W^\mu \cdot \partial W^\nu$ can be expressed through a renormalization of Υ, requiring that

$$\Upsilon_{\mu\nu} \to \Upsilon_{\mu\nu} + \frac{\alpha'}{\epsilon}\, R_{\mu\nu} \tag{2.65}$$

then the (2.58) leads to the condition of conformal invariance

$$\beta_{\mu\nu} = \alpha'\, R_{\mu\nu} = 0 \tag{2.66}$$

thus the background spacetime, the target space where the string propagates, has to obey the vacuum Einstein equations. An important consequence follows when discussing the Weyl invariance. In performing the latter

$$\gamma_{ab} \mapsto e^{2\zeta}\, \gamma_{ab} \implies S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma\ \partial X^\mu \cdot \partial X^\nu \left[\Upsilon_{\mu\nu}(X) + \alpha'\ \zeta\, R_{\mu\nu}\right] \tag{2.67}$$

a breakdown of Weyl invariance appears in the last term.
This is not the end of the story yet. Recall that the closed string has other massless states associated to the Kalb-Ramond field, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton Φ. So, besides graviton, in (2.50) one has to account for $B_{\mu\nu}$ and Φ. How strings couple to them? First of all, there is the need to highlight that $B_{\mu\nu}$ is the analogous to the gauge potential A_μ in electromagnetism, as a result, the coupling with the Kalb-Ramond field is explicated by saying that the string is *charged* under $B_{\mu\nu}$. In fact, a charged particle couples to A_μ through the following terms

$$\int \mathrm{d}\tau\ A_\mu(X)\ \dot{X}^\mu \tag{2.68}$$

which actually means the pullback of the one-form $A_\mu\ \mathrm{d}X^\mu$ in spacetime onto the worldline of the particle. In order to translate this to the two-dimensional worldsheet one has to define the two-form in spacetime, the antysimmetric tensor field $B_{\mu\nu}$, whose pullback onto the worldsheet gives

$$\int \mathrm{d}^2\sigma\ \ B_{\mu\nu}(X)\ \partial_a X^\mu \partial_b X^\nu\ \epsilon^{ab}\ . \tag{2.69}$$

As well as (2.68) is gauge invariant under $A_\mu \to A_\mu - \partial_\mu \lambda$, the same happens for (2.69) under $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu$. Additionally, starting from the gauge field one defines the two-form (the strength field tensor), in the same way we define the three-form

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{2.70}$$

called the *torsion*. The direct consequence of (2.69) is that the total action describing the motion of the string is thus

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma\ \left[\Upsilon_{\mu\nu}(X)\ \partial_a X^\mu \partial_b X^\nu\ \gamma^{ab} + iB_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\ \epsilon^{ab} + \alpha' \Phi(X) R^{(2)}\right] \tag{2.71}$$

with Φ the dilaton and $R^{(2)}$ the Ricci scalar on the worldsheet. Anyway, we previously stressed that the term proportional to the latter breaks the Weyl invariance. How can we *impose* the Weyl invariance? Note that the breakdown comes with the Regge slope. Going to the one-loop level there could emerge contribution coming from $\Upsilon_{\mu\nu}$ and $B_{\mu\nu}$ proportional to the latter that may resolve the problem. Indeed, there exists a subtle trick and makes use of the one-loop β-functions:

$$\beta_{\mu\nu}(\Upsilon) = \alpha' \ R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} \ H_{\mu\lambda\rho} \ H_\nu{}^{\lambda\rho} \tag{2.72}$$

$$\beta(B) = -\frac{\alpha'}{2} \ \nabla^\lambda \ H_{\lambda\mu\nu} + \alpha' \ \nabla^\lambda \ \Phi \ H_{\lambda\mu\nu} \tag{2.73}$$

$$\beta(\Phi) = -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \ \nabla_\mu \Phi \nabla^\mu \Phi - \frac{\alpha'}{24} \ H_{\mu\nu\lambda} \ H^{\mu\nu\lambda} \tag{2.74}$$

and imposes the Weyl invariance on the worldsheet theory through $\beta(\Upsilon) = \beta(B) = \beta(\Phi) = 0$. Hence, one can rewrite (2.71) as a D-dimensional spacetime action for the equation of motion for the (2.72), (2.73) and (2.74), namely [Ton09]-[Pal19]

$$S = 2\pi M_S^{D-2} \int \mathrm{d}^D X \ \sqrt{-\Upsilon} \ e^{-2\Phi} \left[R - \frac{1}{12} H_{\mu\nu\rho} \ H^{\mu\nu\rho} + 4\partial_\mu \Phi \partial^\mu \Phi \right] \tag{2.75}$$

which is the low-energy effective action for the bosonic string written for a general dimension D and M_S is the mass of the string.

2. 3 Kaluza-Klein reduction

We will consider a theory with gravity in $\mathcal{D} = D + d$ dimensions with the aim to reduce it to a d-dimensional torus [Arg98]. So consider a $\mathcal{D}$-dimensional Riemannian spacetime $\mathcal{M}^{\mathcal{D}} = \mathbb{R}^D \times T^d$. A point in this spacetime will be denoted by $X^M = (x^\mu, y^i)$, where the greek letters run on $\mu = (0, \dots, D-1)$ while the latin ones on $i = (1, \dots, d)$, so x^μ denotes a coordinate on $\mathbb{R}^D$ while y^i is a coordinate on T^d. We would like to write the Einstein-Hilbert action in D dimensions including the fields that will arise during the procedure of the reduction. The procedure is called *Kaluza-Klein reduction* and the main assumption is that all the functions appearing in the following metric tensor G_{MN} are independent from the coordinates y^i on the torus:

$$\mathrm{d}s^2 \triangleq G_{MN} \ \mathrm{d}X^M \ \mathrm{d}X^N = g_{\mu\nu} \ \mathrm{d}x^\mu \ \mathrm{d}x^\nu + h_{ij} \left(\mathrm{d}y^i + A^i_\mu \mathrm{d}x^\mu \right) \left(\mathrm{d}y^j + A^j_\nu \mathrm{d}x^\nu \right) \tag{2.76}$$

as a result the metric tensor can be decomposed as

$$G_{\mu\nu} = g_{\mu\nu} + h_{ij} A^i_\mu \ A^j_\nu \ , \qquad G_{\mu i} = h_{ij} A^j_\mu \ , \qquad G_{ij} = h_{ij} \tag{2.77}$$

The inverse metric G^{MN} clearly is such that $G^{MP} G_{PN} = \delta^M_N$, this implies that

$$G^{\mu\nu} = g^{\mu\nu}, \qquad G^{\mu i} = -A^{i\mu} = -g^{\mu\nu} A^i_\nu, \qquad G^{ij} = h^{ij} + A^{i\lambda} A^j_\lambda. \tag{2.78}$$

Denoting by $M_P^{\mathcal{D}}$ the Planck mass and by $R[G]$ the Ricci scalar in $\mathcal{D}$ dimensions, the action for pure gravity in $\mathcal{D}$ dimensions is

$$S = \frac{\left(M_P^{\mathcal{D}}\right)^{\mathcal{D}-2}}{2} \int \mathrm{d}^{\mathcal{D}} X \sqrt{-G} R[G] \tag{2.79}$$

where, using (2.77), the determinant of the metric is

$$\sqrt{-G} = \sqrt{-g} \sqrt{h}. \tag{2.80}$$

First of all, in order to reduce (2.79) in D dimensions, there is the need to explicit the D-dimensional Ricci tensor. So, remembering that the Christoffel symbols are

$$\Gamma^A{}_{BC} = \frac{G^{AL}}{2} \left[\partial_B G_{LC} + \partial_C G_{LB} - \partial_L G_{BC} \right] \tag{2.81}$$

the Ricci tensor is the contraction between the first and third indices of the Riemann tensor

$$R_{AB} = R^{C}{}_{ACB} = \partial_L \Gamma^L{}_{BA} - \partial_B \Gamma^L{}_{LA} + \Gamma^L{}_{LK}\Gamma^K{}_{BA} - \Gamma^L{}_{BK}\Gamma^K{}_{LA} \ , \tag{2.82}$$

so, using (2.77) to compute $G^{AB}R_{AB} = R$ and then recalling (2.78), one ends up with

$$R[G] = R[g] - \frac{3}{4}\partial_\mu h^{ij}\partial^\mu h_{ij} - \frac{1}{4}h^{ij}\partial_\mu h_{ij}\ h^{kl}\partial^\mu h_{kl} - h^{ij}\Box h_{ij} - \frac{1}{4}h_{ij}\ F^i_{\mu\nu}F^{j\ \mu\nu} \tag{2.83}$$

where:

1. $R[g]$ and $\Box$ are the D-dimensional curvature and Dalambertian respectively
2. $F^i_{\mu\nu}$ is the field strength tensor written in terms of the gauge field A^i_μ as $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$.

We will use a trick in order to simplify (2.83)

$$-h^{ij}\Box h_{ij} = -\frac{2}{\sqrt{h}}\Box\sqrt{h} + \partial_\mu h^{ij}\partial^\mu h_{ij} + \frac{1}{2}h^{ij}\partial_\mu h_{ij}\ h^{kl}\partial^\mu h_{kl} \tag{2.84}$$

so that substituting (2.84) in (2.83) and then in (2.79), neglecting boundary terms, the action is just [Arg98]

$$S = \int\ \mathrm{d}^D x\sqrt{-g}\sqrt{h}\left[R[g] + \frac{1}{4}\partial_\mu h^{ij}\partial^\mu h_{ij} + \frac{1}{4}h^{ij}\partial_\mu h_{ij}\ h^{kl}\partial^\mu h_{kl} - \frac{1}{4}h_{ij}\ F^i_{\mu\nu}F^{j\ \mu\nu}\right]\ . \tag{2.85}$$

However we want to get rid of $\sqrt{h}$. This can be achieved by performing a Weyl transformation of $g_{\mu\nu}$

$$g_{\mu\nu} = e^{2\Phi}\tilde{g}_{\mu\nu} \tag{2.86}$$

consequently, considering that

$$\sqrt{-G} = \sqrt{-g}\sqrt{h} \mapsto e^{D\Phi}\sqrt{-g}\sqrt{h}, \qquad \text{and} \qquad g^{\mu\nu} = e^{-2\Phi}\tilde{g}^{\mu\nu} \tag{2.87}$$

then

$$\begin{aligned} S = \int\ \mathrm{d}^D x\sqrt{-g}\sqrt{h}\ e^{(D-2)\Phi}\Big[R[\tilde{g}] + \tilde{g}^{\mu\nu}\frac{1}{4}\partial_\mu h^{ij}\partial_\nu h_{ij} + \tilde{g}^{\mu\nu}\frac{1}{4}h^{ij}\partial_\mu h_{ij}\ h^{kl}\partial_\nu h_{kl} + \\ - \tilde{g}^{\mu\rho}\tilde{g}^{\sigma\nu}\frac{1}{4}h_{ij}\ F^i_{\rho\sigma}F_{j\ \mu\nu}\Big]\ . \end{aligned} \tag{2.88}$$

Therefore, the value of Φ such that $\sqrt{h}$ disappears is then [Arg98]-[Pal19]

$$e^{(D-2)\Phi}\sqrt{h} = \exp\left\{(D-2)\Phi + \frac{1}{2}\log h\right\} \equiv 1 \qquad \Longrightarrow \qquad \Phi = \frac{-\log h}{2(D-2)} \tag{2.89}$$

so repeating the computation by using

$$g_{\mu\nu} \mapsto e^{-\frac{\log h}{D-2}}\tilde{g}_{\mu\nu} \tag{2.90}$$

we can again neglect boundary terms ad use the identity $\partial_\mu \log h = h^{ij}\partial_\mu h_{ij}$. The final result is

$$S = \int\ \mathrm{d}^D x\ \sqrt{-g}\left[R + \frac{1}{4}\partial_\mu h^{ij}\partial^\nu h_{ij} + \frac{1}{4(D-2)}h^{ij}\partial_\mu h_{ij}\ h^{kl}\partial^\mu h_{kl} - \frac{1}{4}\ h^{\frac{1}{D-2}}\ h_{ij}\ F^i_{\mu\nu}F^{j\ \mu\nu}\right]\ . \tag{2.91}$$

2. 3.a Compactification on a circle and Kaluza-Klein modes

Now we will specialize to the case of $d = 1$ thus we will compactify on a circle, whose metric is $\mathrm{d}s^2_{\mathcal{C}^1} = e^{2\sigma}\ \mathrm{d}y\ \mathrm{d}y$. So

$$h_{ii} = h = e^{2\sigma} \implies h^{ii} = e^{-2\sigma} \implies \partial_\mu h_{ii} = 2(\partial_\mu \sigma)e^{2\sigma} \quad \text{and} \quad \partial_\mu h^{ii} = -2(\partial_\mu\sigma)e^{2\sigma} \tag{2.92}$$

$$\begin{aligned}\frac{1}{4}\partial_\mu h^{ij}\partial^\nu h_{ij} + \frac{1}{4(D-2)}h^{ij}\partial_\mu h_{ij}\ h^{kl}\partial^\mu h_{kl} &= -(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{D-2}(\partial_\mu\sigma)(\partial^\mu\sigma) = \\ &= -\frac{D-1}{D-2}(\partial_\mu\sigma)(\partial^\mu\sigma)\end{aligned} \tag{2.93}$$

the result can be replaced in (2.91) obtaining

$$S = \int \mathrm{d}^D x\ \sqrt{-g}\left[R + -\frac{D-1}{D-2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{4}e^{2\frac{D-1}{D-2}\sigma}\ F_{\mu\nu}F^{\mu\nu}\right] . \tag{2.94}$$

Notation can be simplified further with the introduction of a scalar field $\phi(x)$ such that

$$\phi(x) \triangleq a\sigma \implies \frac{D-1}{D-2}\partial_\mu\left(\frac{\phi}{a}\right)\partial^\mu\left(\frac{\phi}{a}\right) \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \iff a = \sqrt{\frac{2(D-1)}{D-2}}\ , \tag{2.95}$$

therefore $\phi(x)$ will play the role of a dynamical field in the theory as can be appreciated in the final pure gravity reduced action

$$S = \int \mathrm{d}^D x\ \sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{a\phi}\ F_{\mu\nu}F^{\mu\nu}\right] . \tag{2.96}$$

Using $\phi(x)$, the metric of the Riemannian $\mathcal{D}$-dimensional manifold is

$$\begin{aligned}&\mathrm{d}s^2 = e^{-\frac{\log h}{D-2}}g_{\mu\nu}\ \mathrm{d}x^\mu\ \mathrm{d}x^\nu + h\ \left(\mathrm{d}y\ + A_\mu \mathrm{d}X^\mu\right)^2 \xrightarrow[(2.95)]{h=e^{2\frac{\phi}{a}}} \\ &\mathrm{d}s^2 = e^{-2\frac{1}{\sqrt{2(D-1)(D-2)}}\ \phi}g_{\mu\nu}\ \mathrm{d}x^\mu\ \mathrm{d}x^\nu + e^{2\sqrt{\frac{D-2}{2(D-1)}}\ \phi}\ \left(\mathrm{d}y\ + A_\mu \mathrm{d}X^\mu\right)^2\end{aligned} \tag{2.97}$$

that can be reduced by introducing

$$\alpha = -\frac{1}{\sqrt{2(D-1)(D-2)}} \quad \text{and} \quad \beta = \sqrt{\frac{D-2}{2(D-1)}}\ . \tag{2.98}$$

Furthermore, the dynamical field ϕ is also related to the radius of the circle $\mathcal{C}^1$ by definition of length of a curve [Pal19]:

$$e^{\beta\phi} = \int_0^1 \mathrm{d}y\ e^{\beta\phi} = \int_0^1 \mathrm{d}y\ \sqrt{h} \equiv 2\pi R\ . \tag{2.99}$$

If we want to introduce a $\mathcal{D}$-dimensional *massless* scalar field, $\Psi(x)$, it has to be periodic in the y coordinate due to the fact that the latter is periodic and defines the circle $\mathcal{C}^1$. This implies that Ψ has to be single-valued on the circle, namely that its momentum in that direction is expected to be quantized. We can decompose Ψ as a Block function underlining the periodic dependence on y[4]:

$$\Psi(x) = \sum_{n=-\infty}^{+\infty} \psi_n(x)\ ^{i\ 2\pi n y} \tag{2.100}$$

[4] This decomposition emphasizes the quantization of the momentum in the y direction, in fact $-i\frac{\partial\Psi}{\partial y} = 2\pi n\Psi$.

where $\psi_n(x)$ are the so-called *Kaluza-Klein (KK) modes.* For the sake of simplicity we now restrict to the case of $g_{\mu\nu} = \eta_{\mu\nu}$. By the fact that Ψ is a scalar field, it has to satisfy the Klein-Gordon equation

$$\partial_M \Psi \partial^M \Psi = \left(\underbrace{e^{-2\alpha\phi}}_{(2.98) \implies \left(\frac{1}{2\pi R}\right)^{\frac{2}{2-D}}} \partial^\mu \partial_\mu + \underbrace{e^{-2\beta\phi}}_{(2.99) \implies \frac{1}{(2\pi R)^2}} \nabla_y^2 \right) \Psi = 0 \tag{2.101}$$

resulting in

$$\left[\partial^\mu \partial_\mu - \left(\frac{1}{2\pi R}\right)^{2+\frac{2}{D-2}} (2\pi n)^2 \right] \psi_n = 0 \implies M^2_{KK,\ n} = \left(\frac{n}{R}\right)^2 \left(\frac{1}{2\pi R}\right)^{\frac{2}{D-2}} \tag{2.102}$$

which shows that the KK modes represents a *tower of states* [Pal19].

2. 3.b Compactification of a string on a circle

To be in line with the spirit of this chapter, we will consider the compactification of the bosonic closed string on a circle, in a 25-dimensional spacetime. The Riemannian manifold is thus $\mathcal{M}^{25} = \mathbb{R}^{1,24} \times \mathcal{C}^1$ and a point in this spacetime is represented by $X^M = (x^\mu, y)$, where according to the previous subsection the interpretation is straightforward. The y direction is periodic, namely

$$y \sim y + 2\pi R\ . \tag{2.103}$$

Recalling the discussion made at the beginning of the chapter, for a closed bosonic string propagating along the non-compact dimensions x^μ (2.21), (2.22) have to hold and the general periodic solution is

$$X^\mu = X^\mu_L + X^\mu_R \quad \text{with} \quad \begin{cases} X^\mu_L &= \frac{x^\mu + \alpha' p^\mu_L \xi^+}{2} + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\tilde{\alpha}^\mu_n}{n}\, e^{-in\xi^+} \\ X^\mu_R &= \frac{x^\mu + \alpha' p^\mu_R \xi^-}{2} + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\alpha^\mu_n}{n}\, e^{-in\xi^-} \end{cases} \quad \text{with} \quad p^\mu_L = p^\mu_R. \tag{2.104}$$

The innovation here is the possibility that the string can wind along the compactified direction as can be appreciated in figure 2.2. Hence, this condition is achieved by imposing periodicity

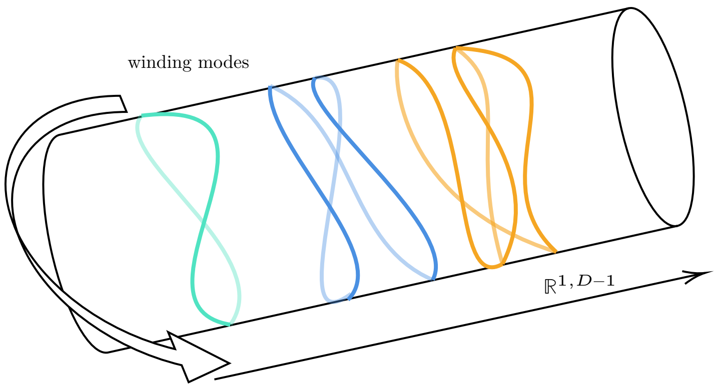


Figure 2.2. Visual illustration of winding strings.

on the worldsheet coordinate σ in the string coordinate $Y(\tau, \sigma)$ along $\mathcal{C}^1$:

$$Y(\tau, \sigma) \mapsto Y(\tau, \sigma + \pi) = Y(\tau, \sigma) + 2\pi R\, w \quad \text{with} \quad w \in \mathbb{Z}\ . \tag{2.105}$$

Therefore, along the y direction, the embedding string function can be written as

$$\begin{cases} Y_L &= \frac{y+\alpha' p_L^y \xi^+}{2} + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\tilde{\alpha}_n^y}{n} e^{-in\xi^+} \\ Y_R &= \frac{y+\alpha' p_R^y \xi^-}{2} + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\alpha_n^y}{n} e^{-in\xi^-} \end{cases} \xrightarrow[p^y = \frac{p_L^y + p_R^y}{2} \quad \text{and} \quad 2R\ w = \frac{\alpha'\ 2(p_L^y - p_R^y)}{2}]{\xi^\pm \triangleq \tau \pm \sigma} \tag{2.106}$$

$$Y = y + \alpha' p^y\ \tau + 2R\ w\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\tilde{\alpha}_n^y}{n} e^{-in(\tau+\sigma)} + i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\alpha_n^y}{n} e^{-in(\tau-\sigma)} \ . \tag{2.107}$$

Clearly, as already noted for the scalar field Ψ, the quantum-mechanical wave function of the center of mass of the string will be composed by an oscillating factor $\propto \exp\{ip^y\ y\}$, hence, increasing the periodic coordinate y, this means that p^y has to be quantized

$$p^y = \frac{k}{R} \quad \text{with} \quad k \in \mathbb{Z} \ . \tag{2.108}$$

Using the (2.106), one can relate the winding number w to the left and right momenta

$$p_L^y = p^y + \frac{R\ w}{\alpha'} \overset{(2.108)}{=} \frac{k}{R} + \frac{R\ w}{\alpha'} \tag{2.109}$$

$$p_R^y = p^y - \frac{R\ w}{\alpha'} \overset{(2.108)}{=} \frac{k}{R} - \frac{R\ w}{\alpha'} \tag{2.110}$$

and then use (2.109) and (2.110) to define the zero modes for the compact dimension, in the same way as done for the non-compact case

$$\alpha_0^y \triangleq \sqrt{\frac{\alpha'}{2}} \left(\frac{k}{R} + \frac{R\ w}{\alpha'}\right) \implies L_0 = \frac{1}{2}\left[\alpha_0 \cdot \alpha_0 + \sum_{n=1}^{+\infty} \alpha_{-n} \cdot \alpha_n\right] \tag{2.111}$$

$$\tilde{\alpha}_0^y \triangleq \sqrt{\frac{\alpha'}{2}} \left(\frac{k}{R} - \frac{R\ w}{\alpha'}\right) \implies \tilde{L}_0 = \frac{1}{2}\left[\tilde{\alpha}_0 \cdot \tilde{\alpha}_0 + \sum_{n=1}^{+\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n\right] \tag{2.112}$$

from which, using the fact that $M^2 = -\sum_{\mu=0}^{D-1} p_\mu\ p^\mu$, solving the equations $L_0 = 1 = \tilde{L}_0$ one gets

$$\frac{\alpha'}{4}\left(\frac{k}{R} - \frac{R\ w}{\alpha'}\right)^2 + N_R - \frac{\alpha'\ M^2}{4} = 1 \tag{2.113}$$

$$\frac{\alpha'}{4}\left(\frac{k}{R} + \frac{R\ w}{\alpha'}\right)^2 + N_L - \frac{\alpha'\ M^2}{4} = 1 \ . \tag{2.114}$$

Hence, taking in consideration the states for which $N_R = N_L = 1$, the final result is

$$M_{KK,\ w}^2 = \left(\frac{k}{R}\right)^2 + \left(\frac{R\ w}{\alpha'}\right)^2 \ . \tag{2.115}$$

However, this result is valid in the frame where we are not performing a Weyl transformation on the tensor metric, namely we are not considering the dilaton yet. In order to re-write (2.115) in this frame, one should multiply the latter by $\exp\{2\alpha\phi\}$ as it is done in equation (2.102). Nevertheless, in (2.115) in the winding term there is the contribution of $(\alpha')^{-2}$. The latter is related to the length of the string and thus to the mass of the string that can be written as

$$M_S = \frac{1}{2\pi\sqrt{\alpha'}} \ . \tag{2.116}$$

In some sense, *renormalization* of α' has to be taken in consideration, therefore the last term in (2.115) needs additional multiplicative terms. Recalling (2.57), its inverse can be seen as a dimensionless length of the theory, in this case written in terms of the compactification radius R:

$$l_{\text{dimensionless}} = \frac{R}{\sqrt{\alpha'}} \overset{(2.116)}{=} 2\pi R M_S \ . \tag{2.117}$$

Now, let us introduce the dilatonic scalar field in D dimensions, Φ^D. It is easy to find the relation between Φ^D and the dilaton in $\mathcal{D}$ dimensions:

$$\sqrt{-g}\ e^{-2\Phi^D} \equiv \sqrt{-G}\ e^{-2\Phi} = \sqrt{-\det G_{DD}\ l^2_{\text{dimensionless}}}\ e^{-2\Phi} = \sqrt{-g}\ \frac{R}{\sqrt{\alpha'}}\ e^{-2\Phi} \tag{2.118}$$

hence

$$\Phi^D \equiv \Phi - \frac{1}{2}\log\left(2\pi R M_S\right) \ . \tag{2.119}$$

Therefore, observing that Φ^D is independent from the compactified dimension, it must not depend on R. So, imposing this constraint, one ends up with a relation between the dilaton in $\mathcal{D}$ dimensions and the compactification radius

$$\partial_{l_{\text{dimensionless}}} \Phi^D \equiv 0 \implies \partial_{l_{\text{dimensionless}}} \Phi = \frac{1}{l_{\text{dimensionless}}} \iff e^{-2\Phi} \sim \frac{1}{2\pi R M_S} \ . \tag{2.120}$$

Hence, comparing (2.75) in $\mathcal{D}$ dimensions with (2.79) in D dimensions, it is straightforward that

$$\frac{\left(M_P^D\right)^{D-2}}{2} \equiv 2\pi M_S^{\mathcal{D}-2}\ e^{-2\Phi} \tag{2.121}$$

which, imposing $M_P^D = 1$ in the frame of our interest, reads as

$$M_S \sim (2\pi R)^{\frac{1}{D-2}} \ . \tag{2.122}$$

Multiplying (2.122) with the last term in (2.115) and denoting with α'_0 the renormalized value of α, one has

$$(2.115) \cdot \frac{1}{\alpha'^2} \xrightarrow{e^{-2\alpha\phi}(=(2.102))\cdot(2.122)} (2\pi R)^{\frac{2}{2-D}}\ (2\pi R)^{\frac{2}{D-2}}\ \frac{1}{\alpha'^2} \triangleq (2\pi R)^{\frac{2}{D-2}}\ \frac{1}{\alpha'^2_0} \tag{2.123}$$

and as a consequence, equation (2.115) becomes

$$M^2_{KK,\ w} = \left(\frac{1}{2\pi R}\right)^{\frac{2}{D-2}} \left(\frac{k}{R}\right)^2 + (2\pi R)^{\frac{2}{D-2}} \left(\frac{R\ w}{\alpha'_0}\right)^2 \ . \tag{2.124}$$

Moreover, one can wonder about the relation between the Planck mass in the non-compactified and compactified cases. Are there any consequences on the number of states in the KK and winding modes tower of states? Indeed, we can immediately compare the Planck mass in the spatial dimensions with that in the total non-compactified regime, without the necessity to introduce the D-dimensional dilaton, namely by just considering the metric as

$$ds^2 = g_{\mu\nu}\ dX^\mu dX^\nu + (2\pi R)^2\ \left(dX^D\right)^2 \tag{2.125}$$

so that

$$\left(M_P^D\right)^{D-2} = \left(M_P^{\mathcal{D}}\right)^{\mathcal{D}-2}\ 2\pi R \ . \tag{2.126}$$

Therefore, once we have compactified the D-th dimension we are actually living in a D-dimensional world. This means that, in our conventions, $M_P^D \equiv 1$, then as a result $M_P^{\mathcal{D}} \sim R^{\frac{1}{2-\mathcal{D}}}$.

Finally, how many KK modes we can have in the effective theory before reaching the UV scale, $M_P^{\mathcal{D}}$? Let N_s be the number of these states, then

$$N_s \sim \frac{M_P^{\mathcal{D}}}{M_{KK}} \xrightarrow{(2.126)} R^{\frac{D-2}{\mathcal{D}-2}} \implies R = N_s^{\frac{\mathcal{D}-2}{D-2}} \xrightarrow[(2.126)]{\text{substituting in}} M_P^{\mathcal{D}} \sim \frac{1}{N_s^{\frac{1}{D-2}}} \ . \tag{2.127}$$

Immediate considerations are now in order. Firstly, (2.126) highlights how the true scale at which gravity becomes strongly coupled is lower than one would have been expected. Moreover, this leads us to define a reference scale Λ_s at which gravity is strongly coupled

$$\Lambda_s \triangleq \frac{M_P^D}{N_s^{\frac{1}{D-2}}} \tag{2.128}$$

for a D-dimensional theory. Precisely, (2.128) is the heart of the so-called *Species conjecture*.

Part II

The Swampland Conjectures

Chapter 3

No-Global Symmetry Conjecture

Contents: **3. 1 The conjectures: loose arguments. 3. 2 Global symmetry in Quantum Field Theory. 3. 3 Gauge symmetry.** 3. 3.a Lattice gauge theory. **3. 4 Proof of the conjectures in Holography.** 3. 4.a The No-Global symmetry conjecture – 3. 4.b The Completeness conjecture.

Introduction

All the Swampland conjectures are currently under deep speculations. Before diving head in the conjectures, it is important to bear in mind that some of them can be argued in a more convincing way while others are still on a speculative path. Since the starting point is either ST or BH physics, in some way they are all connected. This chapter aims to provide a detailed and complete definition of global symmetries and also to highlight how quantum gravity actually limits symmetries in nature.

3. 1 The conjectures: loose arguments

It is well established nowadays [Pol07b] that in models of quantum gravity there are no global symmetries and that all continuous gauge symmetries are compact. Regarding the former, a straightforward demonstration comes from String Theory. In fact, if we have a global symmetry on the worldsheet, there is a conserved global worldsheet charge that, following the discussion in Appendix A, is

$$Q = \frac{1}{2\pi i} \oint \left[J_z \mathrm{d}z - J_{\bar{z}} \mathrm{d}\bar{z} \right] \ . \tag{3.1}$$

However, the previous charge has also to be a symmetry of the physical spectrum and moreover has to be conformally invariant. As a result, when (3.1) is applied to physical states of the string spectrum, by virtue of the *state-operator correspondence*, this operation leads to an expression of the integrand of the (3.1) in terms, for example, of the following conformal fields

$$J_z \bar{\partial} X^\mu e^{ik\cdot X} \quad \text{and} \quad \partial X^\mu J_{\bar{z}} e^{ik\cdot X} \ . \tag{3.2}$$

Therefore, as exposed in definition A. 3.4, (3.2) create massless vectors that couple with the left and right-moving part of the charge Q. Hence, due to the description of string interactions, *it comes naturally* that a global symmetry on the worldsheet is gauged in the spacetime. In

conclusion, String Theory shows that there are no continuous global symmetries in the target space where they are gauged even coming from a global symmetry on the worldsheet.
In this section the statements of the conjectures will be presented along with loose arguments coming from black hole physics. Attention will be focused on two such conjectures:

Conjecture 1.1: No-Global symmetry conjecture

An effective field theory with finite number of states cannot provide exact global symmetries when coupled to gravity.

Conjecture 1.2: Completeness conjecture

If a quantum gravity theory provides a gauge theory with compact gauge group G at low energies, there must exist physical states that transform in all finite dimensional irreducible representations of G.

Despite of the simplicity of their statements, these conjectures are non-trivial: there exist examples of effective field theory of matter coupled to gravity that actually exhibit global symmetries. In order to appreciate their reasonable validity, let us consider a semi-classical argument of BH physics. Let us assume the existence of a continuous global symmetry with symmetry group being G. Now, if r labels the irreducible representation of G according to which light particles with mass m transform, in theory we could arrange a number of these particles to form a multi-particle state. As a result [BS11], the latter will carry a non-trivial representation of G, $\mathrm{R} \subset \mathrm{r} \otimes \mathrm{r} \otimes \ldots \mathrm{r}$. In particular, this multi-particle state can be represented by a black hole which in turn can evaporate according to Hawking discussion [Haw76]. The work of Hawking underlines how the evaporation process is actually thermal: it only depends on geometry and on the temperature. Therefore, the complexity of the black hole in terms of the representation R carried by it will not decrease during this process.
Roughly speaking, this means that during the evaporation process an observer can achieve information only on the mass, the angular momentum and the electric charge of the BH. In literature this result is encoded in the so-called *No-hair theorem*. Hence, the global information of the BH remains hidden in the BH in some sense. Moreover, there is the possibility according to which the BH evaporates till reaching the Planckian size, becoming a *remnant* (the process is illustrated in figure 3.1). At this point two are the possibilities, whether or not the remnant has a mass m:

1. $m \neq 0$: the remnant is stable due to the fact that it carries a large representation of the global group G which is actually heavier than the remnant itself.
2. $m = 0$: the only way to emit global information through particles with finite energy is via bremsstrahlung processes. Nevertheless, neither the remnant nor the decay products can carry global charge by the fact that the only gauge field involved in the process is the gravitational one and gravitons are neutral under global charge.

Someone could be interested in understanding the role of these remnants within the global symmetry scenario. In the subsequent chapter the so-called *remnant problem* will be introduced as well as a conjecture concerning entropy: the covariant entropy bound (CEB). For what

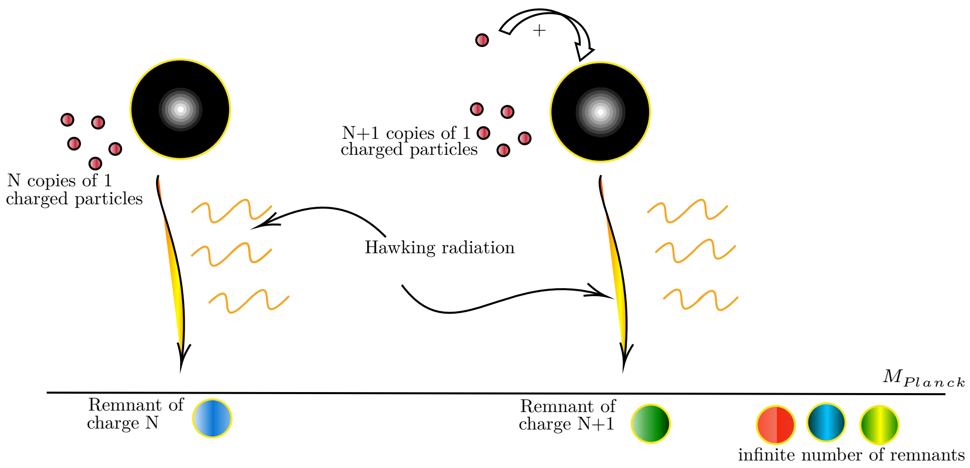


Figure 3.1. Illustration of Hawking radiation for a charged black hole and creation of charged remnants.

concerns now, it suffices to know that dimensionality constraints can be put on the global symmetry group G by using the CEB. In fact, let Σ represent the sum of the dimensionalities of the irreducible unitary representations of G. Each of these irreducible representations could be a possible state for the remnant and a straightforward proportionality between Σ and the partition function of the system can be written down as follows [BS11]

$$\Sigma \propto Z \propto e^{-\beta F} < e^{S} \implies \Sigma < e^{S} \xrightarrow[S< \pi X^2]{\text{CEB}} \Sigma < e^{\pi X^2} \tag{3.3}$$

where F represents the free energy, S the entropy and X the length-size of the remnant. Equation (3.3) is thus a soft bound on the maximal size of a finite global symmetry group G. Note the term *finite* in the statement of conjecture 1.1. A clear explanation is given in [GnH21], where the difference between abelian and non-abelian symmetry groups is emphasized. For example, if the symmetry group is $SU(2)$, which is non-abelian, all the states of the theory are classified in irreducible representations of the group each of which is labelled with j, hence, its dimension is $2j + 1$. Therefore, a multi-particle state created by combining an appropriate number of particles charged under non-trivial representations of $SU(2)$ would have an arbitrary large j. Due to the fact that there is no preferred charge carried by the particles emitted through Hawking radiation, the black hole will loose the mass but not its charge. Hence, the finite behaviour is just encoded in the finite dimensions of the irreducible representation of the group.

On the contrary, if the symmetry group is abelian, for example $U(1)$, the global charge Q of the black hole could be assembled in an infinite way. Therefore, in principle, for every black hole having charge Q there would be a remnant. As a result, in order to preserve the physics and to not invalidate the theory, *it is necessary* to conjecture that at the final stage of the Hawking radiation there would be a *finite* number of remnants, labelled with different values of q. In conclusion, the term *finite* in the statement of conjecture 1.1 accounts for the possibility of abelian groups as symmetry groups.

Additionally, a loose argument based on this discussion can also be used in order to justify conjecture 1.2. In fact, given an arbitrary compact gauge group, any finite-dimensional faithful representation of the group is such that every finite-dimensional reppresentation appears in the tensor power of the latter and its conjugate.

However, these BH arguments do not tell why conjectures 1.1 and 1.2 are correct.

As a result, in order to justify them, the subsequent sections aim to organize the knowledge about quantum gravity theories through group language. In particular, the Noether theorem itself will be discussed. Additionally, in the definition of global symmetry *approximate* global symmetries and spontaneously broken ones will be included. In fact, in quantum field theory both of them are linked to a conservation of information: the former appear in a low-energy effective action until new terms appear in the lagrangian which can lead to a possible *explicit* break of the symmetry while the latter are simply transformed in (pseudo-)Goldstone bosons. A famous example can help to clarify the above identification. In the Standard Model it happens that the difference between the barion and lepton number is indeed a global symmetry, as a result it is actually interesting investigating whether there exist some lower bounds on the coefficients of the operators entering the lagrangian which violate it.

3. 2 Global symmetry in Quantum Field Theory

In quantum mechanics the definition of global symmetry includes a Hilbert space $\mathcal{H}$, the hamiltonian of the theory H and clearly the symmetry group G. Therefore, the action of G on $\mathcal{H}$ is mediated through a faithful homomorphism U from G to a set of unitary operators on $\mathcal{H}$ for which the operator $U(g)$ commutes with the hamiltonian H, with g an element of the group G. Moreover, $U(g)$ has to satisfy a composition law explaining how the symmetry is actually projected on the Hilbert space:

$$U(g) * U(g') = e^{i\ \alpha(g,g')}\ U(g * g')\ . \tag{3.4}$$

Nevertheless, this definition needs the presence of a hamiltonian formulation of a given theory. Therefore it could be helpful to be more generic and give a definition based on a group-theoretic approach.

Definition 3. 2.1

Consider a d-dimensional Lorentz-invariant quantum field theory. The latter has a global symmetry under group G if the followings are true [HO21]:

(i) *For every time slice Σ_t of a given flat spacetime manifold $\mathbb{R}^d$ there exists a homomorphism $U(g, \Sigma_t)$ from the group G to the set of unitary operators on $\mathcal{H}$. Continuity of $U(g, \Sigma_t)$ is not required in order to include symmetry that could be spontaneously broken[1].*

(ii) *Said $\mathcal{A}[R]$ the algebra of operators with domain $D[R]$ on a subregion R of Σ_t, then for any $g \in G$ and $R \subset \Sigma_t$*

$$U^\dagger(g, \Sigma_t)\mathcal{A}[R]U(g, \Sigma_t) = \mathcal{A}[R] \tag{3.5}$$

namely that the group G provides a linear action on the set of local operators at each point through $U(g, \Sigma_t)$.

(iii) *$U(g, \Sigma_t)$ are topological operators in the sense that for any $g \in G$ their action remains unchanged by arbitrary deformation of Σ, namely*

$$U^\dagger(g, \Sigma_t)T_{\mu\nu}(x)U(g, \Sigma_t) = T_{\mu\nu}(x) \tag{3.6}$$

with $T_{\mu\nu}(x)$ being the stress tensor of the theory.

(iv) Said $\mathcal{O}$ a local operator of the theory, then for any element $g \in G$

$$U^{\dagger}(g,\Sigma_t)\mathcal{O}(x)U(g,\Sigma_t) \neq \mathcal{O}(x) \ , \tag{3.7}$$

so a global symmetry maps a local operator into a local operator, not necessarily implying the existence of gauge invariant charged local operators.

Definition 3. 2.1 also includes global symmetries with 't Hooft anomaly, a famous example is related to the chiral phase rotation of a massless Dirac fermion in $1+3$ dimensions:

$$\psi \mapsto \psi = e^{i\gamma^5\theta}\psi \quad \text{with action} \quad S = i\int \mathrm{d}^4x \ \bar{\psi}\ \not{\partial}\ \psi \ . \tag{3.8}$$

However, 3. 2.1 is valid with a flat manifold. How can it be generalised to whatever manifold? So, consider a more general spatial geometry Σ, then [HO21]

Definition 3. 2.2

A quantum field theory is said to preserve a global symmetry on Σ if after quantization there still exists a homomorphism $U(g,\Sigma)$ from G to the set of unitary operators whose action by conjugation preserves the local algebras as in (3.5). Moreover, the energy-momentum tensor is still preserved.

Having established these details, the next step regards a formulation of the Noether theorem in a non-lagrangian form. The Noether theorem states that any quantum field theory having a continuous global symmetry has a conserved current whose integral infinitesimally generates that symmetry [HO21], in particular the mathematical expression that a conserved current has to satisfy is $\nabla_\mu J^\mu_a = 0$. Now, observe that if G is a compact connected Lie group, an element $g \in G$ can be expressed through the exponential map in terms of the generators of the algebra. Therefore, one can consider the following expression for $U(g,\Sigma)$

$$U(e^{i\epsilon^a T_a}, R) = e^{i\epsilon^a \int_R \sqrt{\gamma}\ \mathrm{d}^{d-1}x\ n_\mu J^\mu_a} = e^{i\epsilon^a \int_R \star J_a} \tag{3.9}$$

which in turn can help us in defining the Noether theorem in a non-lagrangian formalism. In fact, equation (3.9) satisfies the following definition:

Definition 3. 2.3

A quantum field theory satisfying definition 3. 2.2 is said to be splittable if $\forall R \subset \Sigma$ and $\forall g \in G$ $\exists$ a unitary operator $U(g,R)$ such that

$$U^{\dagger}(g,R)\mathcal{O}U(g,R) = \begin{cases} U^{\dagger}(g,R)\mathcal{O}U(g,R) & \forall \mathcal{O} \in \mathcal{A}[R] \\ \mathcal{O} & \forall \mathcal{O} \in \mathcal{A}[Int(\Sigma - R)] \end{cases} \tag{3.10}$$

if R_i are a finite disjoint set of open sub-regions of Σ whose boundaries do not intersect, then

$$\Pi_i U(g,R_i) = U(g,\cup_i R_i) \ . \tag{3.11}$$

[1]An example can be the scalar theory invariant under $\mathbb{Z}_2$ symmetry which is not continuous, whose action is $S = \frac{1}{2}\int \ \mathrm{d}^3x\,(\partial^\mu\phi\partial_\mu\phi + m^2\phi^2 + \frac{\lambda}{6}\phi^4)$.

Hence, taking in consideration the definition 3. 2.3 and the expression (3.9), if a compact, connected Lie group provides a global symmetry with Noether current preserved on Σ and the symmetry is always splittable on the latter then if all global symmetries were splittable we would achieve an abstract definition of the Noether's theorem.
Using (3.9), one can properly focus the attention on definition (3.7). In fact a simple example [Bee+21] can be provided in order to better appreciate what the last sentence really means. Imagine having a scalar field ϕ and that the global symmetry under examination is the invariance under translations, namely that

$$\phi \to \phi + c \,, \tag{3.12}$$

whose associated Noether current is $J = \mathrm{d}\Phi$. Therefore the homomorphism from the group G to the set of unitary operators can be expressed as

$$U(g, \Sigma) = e^{i\alpha \int \, \star J} \,. \tag{3.13}$$

Now, if Q denotes the associated charge to the current J, a local operator can be considered to be of the form

$$\mathcal{O}(x) = e^{iQ\phi} \,. \tag{3.14}$$

According to conjecture 1.1 this symmetry is not allowed in quantum gravity, hence, or it is *explicitly* broken or it is gauged. Following the latter approach, if $A = A_\mu \mathrm{d}x^\mu$ labels the gauge field, the term which takes into consideration the gauging of the current J is

$$\int A \wedge \star J \qquad \Longleftrightarrow \qquad L \propto (\mathrm{d}\phi - A)^2 \tag{3.15}$$

which is still invariant under $\phi \to \phi + c$ and $A \to A - \mathrm{d}c$. However, the local operator (3.14) it self is no longer gauge invariant. This condition in fact does not agree with definition (3.7).
The previous example throws the ball for another one that highlights the connection between splittability and existence of global symmetries. Take in consideration a pure gauge theory whose gauge group is $\mathbb{R} \times \mathbb{R}$ with action

$$S = \frac{1}{4} \int \, \mathrm{d}^d x \, \sqrt{-g} \, F_{a\ \mu\nu} F_b^{\ \mu\nu} \ \delta^{ab} = \frac{1}{2} \int \, F_a \wedge \star F_b \ \delta^{ab} \tag{3.16}$$

whose global symmetry is provided by the $U(1)$ group which rotates the two gauge fields into each other. The Noether current is thus

$$\star \, J = \epsilon^{ab} \, A_a \wedge \star F_b \tag{3.17}$$

and if one performs a gauge transformation

$$A_a \to A_a + \mathrm{d}\lambda_a \implies \star J \to \star J + \epsilon^{ab} \, \mathrm{d}\lambda_a \wedge \star F_b \xrightarrow{\mathrm{d}\star F_a = 0} \star J + \mathrm{d}\left(\epsilon^{ab} \, \lambda_a \star F_b\right) \,. \tag{3.18}$$

The Noether current is no gauge invariant! What is the reason? Equation (3.18) in terms of local charge is expressed as

$$Q(R) = \int_R \star J \xrightarrow{(3.18)} Q(R) \to Q(R) + \epsilon^{ab} \int_{\partial R} \lambda_a \star F_b \tag{3.19}$$

therefore the gauge non-invariance is encoded in an operator supported only on ∂R. As a result, one can modify the charge operator in order to cancel out the latter contribution. This

is a subtlety: what should be modified, equation (3.17) or the last term in equation (3.18)? Following the former procedure, in principle we should improve the current by adding to the expression a local gauge non-invariant total derivative in order to cancel out the non-invariance of $\star J$ under gauge transformations. However, such a transformation should involve a term with λ_a without any derivatives and it cannot be written down. Is there any deep reason for which one cannot improve the current itself?
Actually, the symmetry under examination is not splittable. In particular, remember that associated with gauge fields there are the so-called Wilson lines which are line operators. Roughly speaking, there exists an exact one-form symmetry under which these lines are charged. Therefore, these Wilson lines are addressed to be *unbreakable* and are responsible for the gauge non-invariance of the current. As emphasized in equation (3.18), the gauge non-invariance is hidden in the border of R, and a way to address the problem is in re-routing Wilson lines around the boundary. This re-arrangement is only possible if ∂R is connected. Furthermore, this represents a nice geometrical interpretation to what can actually be done in order to provide the gauge invariance. In fact, let x_0 be a reference point on the connected ∂R and consider a whatever curve γ connecting x_0 to another point x on ∂R and define the operator

$$I_a(x) \triangleq \int_{\gamma_{x,x_0}} A_a \tag{3.20}$$

whose gauge transformation is $I_a \to I_a + \lambda_a(x) - \lambda_a(x_0)$. Hence, defining

$$C[\partial R] \triangleq \epsilon^{ab} \int_{\partial R} I_a \star F_b \to C[\partial R] + \epsilon^{ab} \int_{\partial R} \lambda_a \star F_b \tag{3.21}$$

equation (3.19) can be improved subtracting (3.21) and

$$\tilde{Q}(R) \triangleq Q(R) - C[\partial R] \tag{3.22}$$

is therefore gauge invariant and $U(\alpha, R) \triangleq e^{i\alpha\tilde{Q}(R)}$ is a set of local symmetry generators which split the symmetry.

Having established what it is meant by a global symmetry in quantum field theory, we now turn the gaze on gauge symmetry: conjecture 1.1 will be proven by using the previous definitions and the notion of *long-range gauge symmetry.*

3. 3 Gauge symmetry

Gauge symmetry is omnipresent in physics and sometimes it is usually referred to be a merely redundancy of degrees of freedom. Additionally, for a given QFT there exist equivalent formulations in terms of gauge groups. For this reason, the question "what is the gauge group of a given QFT?" does not provide a unique answer, unlike the case of a global symmetry where definition 3. 2.1 is unambiguous. Nevertheless, there exist physical phenomena associated with gauge symmetry, for example the existence of gauge bosons or loop operators. In [HO21] these are said to be characteristic of a so-called *free-charge phase*, a notion first introduced in the context of lattice gauge theory because of some advantages provided by the latter[2]. The notion of long-range gauge symmetry will be formalised in the next subsection after introducing some basics of lattice gauge theory.

[2] Lattice gauge theory may be defined for any compact Lie group G, whether continuous or discrete.

3. 3.a Lattice gauge theory

First of all, by lattice it is meant a regular set of points in $\mathbb{R}^n$ and in lattice gauge symmetry the notion also includes a graph connecting all these points. In this graph, the vertices are said *sites* while the edges connecting them are the *links* [HO21]. These can be identified by a pair $(\vec{x}, \vec{\delta})$, where $\vec{x}$ labels the starting point of the link while $\vec{\delta}$ is the displacement vector to its endpoint. Hamiltonian lattice gauge theory considers each edge as a gauge field and each site comes with a gauge transformation where any matter field is living. In particular the Hilbert space is defined as

$$\mathcal{H} = \bigotimes_{e\in E} \mathcal{H}_e \bigotimes_{\vec{x}\in X} \mathcal{H}_{\vec{x}} \tag{3.23}$$

where $\mathcal{H}_{\vec{x}}$ is the Hilbert space of the matter fields at $\vec{x}$ while $\mathcal{H}_e$ resembles the Hilbert space $\mathcal{H}_G$ of a quantum-mechanical particle moving on the group manifold G. Labelling with $|g\rangle$ the set of states of $\mathcal{H}_G$, three family of operators can be introduced

$$W_{\alpha,\ ij}\,|g\rangle = D_{\alpha,\ ij}(g)\,|g\rangle \tag{3.24}$$

$$L_h\,|g\rangle = |hg\rangle \tag{3.25}$$

$$R_h\,|g\rangle = |gh\rangle \tag{3.26}$$

where a list of notation is now in order[3]:

- α denotes an irreducible representation of the group G whose representation matrices are $D_{\alpha,\ ij}(g)$.
- $W_{\alpha,\ ij}$ is the Wilson link in representation α and the indices i, j run over the dimensionality of the latter.
- L_h and R_h are the left and right multiplication operators which are the analogous to the momentum operators in single-particle quantum mechanics.

Therefore, if $V_g(\vec{x})$ labels an additional unitary operator implementing the gauge transformation on any charged matter fields at site $\vec{x}$, the action of a gauge transformation by a group element g at site $\vec{x}$ on Hilbert space (3.23) is implemented via

$$U_g(\vec{x}) \triangleq \Pi_{\vec{\delta}}\ R_g^\dagger(\vec{x}, \vec{\delta}) V_g(\vec{x})\ . \tag{3.28}$$

Under the action of (3.28) there are classes of gauge-invariant operators (see figure 3.2):

1. Wilson loops: defined as $W_\alpha(C) \triangleq \mathrm{Tr}\{W_\alpha(l_N) \dots W_\alpha(l_1)\}$ where C is a closed curve consisting of the links $l_N \dots l_1$.
2. Wilson lines: defined by $W_\alpha(C) \triangleq W_\alpha(l_N) \dots W_\alpha(l_1)$.
3. Wilson lines ending on charges: only one end pierces boundary while the other ends on a matter operator charged under representation α or $\alpha*$.

[3]The hermiticity properties of these operators are

$$\begin{cases} W^\dagger_{\alpha,\ ij} & = W_{\alpha*,\ ji} \\ L^\dagger_h & = L_{h^{-1}} \\ R^\dagger_h & = R_{h^{-1}} \end{cases} \xrightarrow{\text{whose algebra is}} \begin{cases} L_h L_{h'} & = L_{hh'} \\ R_h R_{h'} & = R_{h'h} \\ L_h R_{h'} & = R_{h'} L_h \\ R^\dagger_{h\,\alpha} R_h & = W_\alpha D_\alpha(h) \\ L^\dagger_{h\,\alpha} L_h & = D_\alpha(h) W_\alpha \end{cases} \xrightarrow[\text{under}]{\text{invariant}} \begin{cases} L_h & \leftrightarrow R_{h^{-1}} \\ W_\alpha & \leftrightarrow W^\dagger_\alpha \end{cases} . \tag{3.27}$$

4. Localised asymptotic symmetries: defined as $U(g,R) \triangleq \Pi_{l\in R} L_g(l)$.

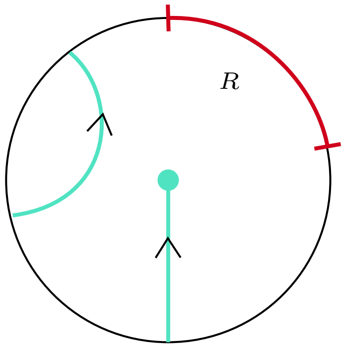


Figure 3.2. Illustration of gauge invariant operators explained in the text.

With all these preliminaries, we can introduce the definition of long-range gauge symmetry. Consider a QFT defined on an infinite-volume spatial manifold Σ having asymptotic boundary $\partial\Sigma$ such that in any state the energy density vanishes approaching the latter, then

Definition 3. 3.1

The theory is said to have a long-range gauge symmetry with symmetry compact group G if

(i) For each spatial closed curve C in the interior of Σ there exist Wilson loops and Wilson lines.

(ii) For every sub-region R of $\partial\Sigma$ and every $g \in G$ $\exists$ $U(g,R)$ unitary operator defined on the Hilbert space, which commutes with all operators supported only in the interior of Σ. Said W_α any Wilson line starting at point $x \in \partial\Sigma$ and ending at $y \in \partial\Sigma$ then [HO19]

$$U^\dagger(g,R) W_\alpha U(g,R) = \begin{cases} D_\alpha(g) W_\alpha D_\alpha(g^{-1}) & x,y \in R \\ W_\alpha D_\alpha(g^{-1}) & x \in R, y \notin R \\ D_\alpha(g) W_\alpha & x \notin R, y \in R \\ W_\alpha & x,y \notin R \end{cases} \tag{3.29}$$

for arbitrary R the $U(g,R)$ are called the localised asymptotic symmetry operators while if R is connected are said asymptotic symmetry operators.

(iii) Despite global symmetry definition 3. 2.1, the ground state is invariant under $U(g,\partial\Sigma)$ and there are finite-energy charged states which transform under $U(g,\partial\Sigma)$.

An immediate consideration follows. While global symmetries have well-defined *local* consequences because local operators transform non-trivially (3.7) and the stress tensor is invariant (3.6), it never happens that global symmetries are not present on the simplest background $\mathbb{R}^d$ while are present on others. By contrast, long-range gauge symmetries are properties of the phase of the theory. In the sense of definition 3. 3.1, the operators introduced in 3 are such that their endpoints are charged operators in the representation α: they actually stand for the so-called *dynamical charges in representation α*.

We should now move to quantum gravity. In this scenario, definition 3. 2.1, which is valid in quantum field theory, has to be modified taking in consideration the fact that there is diffeomorphism invariance in General Relativity, it is actually a long-range spacetime gauge symmetry. Therefore the next task concerns the definition of global symmetries in gravitational theories, effective field theories coupled to gravity both perturbatively and non-perturbatively.

3. 4 Proof of the conjectures in Holography

The previous two sections will be considered in the context of holography. Precisely, conjecture 1.1 will be demonstrated by using the principle of AdS/CFT correspondence. The basic claim of the latter is that any theory of quantum gravity in asymptotically-AdS_{d+1} spacetime, namely a theory which approaches the AdS_{d+1} metric

$$\mathrm{d}s^2 = -\left(1+r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r}{\left(1+r^2\right)} + r^2\,\mathrm{d}\Omega^2_{d-1} \tag{3.30}$$

as $r \to \infty$, is non-perturbatively equivalent to a conformal field theory living on a boundary cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$ at $r = \infty$ (see Appendix B). Moreover, for any bulk field ϕ there is a CFT primary operator $\mathcal{O}$ having scaling dimension Δ such that [Har18]

$$\mathcal{O}(t,\Omega) = \lim_{r\to\infty} r^{\Delta}\,\phi(r,t,\Omega)\ . \tag{3.31}$$

An important result is the following

Theorem 3. 4.1

A global symmetry with group symmetry G of a holographic asympotically-AdS quantum gravity theory is also a global symmetry with the same group symmetry G of the dual CFT.

Moreover, the set of diffeomorphisms preserving (3.31) is formed by those transformations belonging to the so-called *asymptotic conformal symmetry*. Hence, the group $SO(d,2)$ is the spacetime version of a long-range gauge symmetry. As a consequence, we now need to introduce the gravitational analogue of the Wilson lines which extend from the boundary to an interior point that carry gauge charge. These operators in a bulk EFT coupled perturbatively to gravity are constructed as *gravitationally-dressed* versions of ordinary local operators. In the case of a bulk EFT coupled non-perturbatively to gravity one introduces the so-called *quasilocal bulk operators* [HO19]. Therefore, the analogue in quantum gravity of definition 3. 2.1 has as basic idea the definition of a global symmetry with symmetry group G in terms of a homomorphism from G to a set of unitary operators on Hilbert space. These operators faithfully act by conjugation on the set of gravitationally-dressed local operators. Moreover, the symmetry operators commute with the *boundary* stress tensor, so they commute with the asymptotic conformal symmetry [4].

3. 4.a The No-Global symmetry conjecture

In this context, a straightforward definition of global symmetry is the following one:

Definition 3. 4.1

A quantum gravity theory in asymptotically-AdS space has a global symmetry with symmetry group G if [HO19]

(i) if $\partial\Sigma$ denotes any boundary time-slice of the full-invariant diffeomorphisms Hilbert space, then there exists a homomorphism $U(g,\partial\Sigma)$ from G to the set of unitary operators defined in that Hilbert space.

[4] Additionally, in order to take also in consideration condition (3.5), global symmetries act locally on the so-called *gravitationally-dressed surface operators*.

(ii) *$U(g,\partial\Sigma)$ acts locally: if ϕ is a gravitational-dressed operator then both it and $U^\dagger(g,\partial\Sigma)\phi U(g,\partial\Sigma)$ are dressed by the same gravitational Wilson line.*

(iii) *Said $R \subset \partial\Sigma$, $U(g,\partial\Sigma)$ acting by conjugation on $\mathcal{A}[R]$ gives another element of the algebra $\mathcal{A}[R]$.*

(iv) *$U(g,\partial\Sigma)$ commutes with the boundary stress tensor.*

(v) *$U(g,\partial\Sigma)$ acts faithfully on the set of those gravitationally-dressed operators in the bulk, the so-called gauge singlets [HO19], which transform non-trivially under $U(g,\partial\Sigma)$.*

(vi) *For every invariant subgroup $H \subset G$ there exist two gauge singlets transforming under the same representation of the asymptotic conformal symmetry but different representations of H.*

Now, all the definitions introduced are going to be discussed in the structure of AdS/CFT. It will be shown that any global symmetry in the bulk would lead to a contradiction on the boundary and the arguments used are based on the so-called *entanglement wedge reconstruction* [HO19]. The latter is a property of the AdS/CFT correspondence and simply states that there exists a sub-duality between any spatial sub-region R of the boundary CFT and a sub-region in the bulk, the entanglement wedge of R. The duality is shown in figure 3.3.

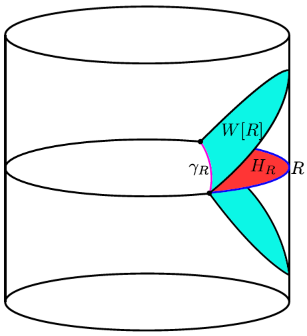


Figure 3.3. Sub-duality in AdS/CFT correspondence with the visual illustration of the entanglement wedge.

Therefore, any bulk operator with support in the entanglement wedge of R, $W[R]$, can be represented by an operator with support only in R in the CFT. This actually means that if we are observers sitting at the boundary we will have access to complete information on what is going on in the bulk only in $W[R]$.
It is condition (vi) which is inconsistent with the entanglement wedge reconstruction, providing a proof for conjecture 1.1. In fact, consider the symmetry operators $U(g,\partial\Sigma)$. By virtue of theorem 3. 4.1, there exists a boundary symmetry operator in the CFT. Moreover, in QFT, definition of global symmetry 3. 2.1 and splittability 3.11 one has

$$U(g,\partial\Sigma) = U(g,R_1)\dots U(g,R_n)\; U_{\text{edge}} \tag{3.32}$$

where U_{edge} just fixes arbitrary choices in defining $U(g,R_i)$. By entanglement wedge reconstruction, in the bulk $U(g,R_i)$ just implements the global symmetry only on those operators having support in $W[R_i]$. Following condition (vi), in the bulk there are two gravitationally-dressed operators charged under G and transforming under the same representation of the asymptotic conformal symmetry (vi). However, equation (3.32) highlights that the charge is expressed in terms of CFT operators whose spatial supports is such that their entanglement wedges can lead access only to those operators living there. Now, according to (v), the two dressed local operators can be arranged such that their only support in the entanglement wedge is their gravitational Wilson lines. Therefore, the charge has support in those regions whose entanglement wedges have only access to the gravitational Wilson lines of the two operators. As a result, by the fact that the two operators transform under the same asymptotic conformal symmetry, they clearly have identical gravitational Wilson lines. Nevertheless, there is no way for condition (vi) to be satisfied: there is no way for the two dressed local operators to transform according to two different representations of the global symmetry group, there is no global information in the entanglement wedge.

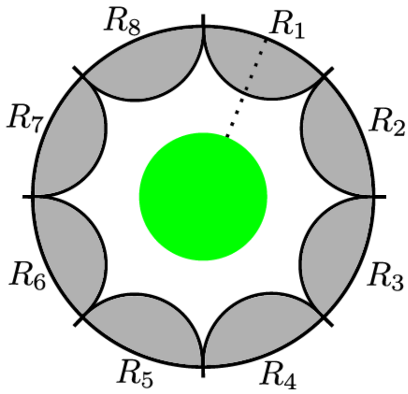


Figure 3.4. Illustration of a bulk timeslice: the dressed local operator does not live in the grey regions, namely the entanglement wedges of regions R_i. The only way to make a correspondence between bulk and boundary is through a gravitational Wilson line.

After all this pingponting from global symmetry definitions to long-range gauge symmetry one, the entanglement wedge reconstruction simply implies that we can choose the regions R_i small enough that the associated surfaces γ_R do not reach the center of the bulk where an operator charged under the symmetry lives, so that there is no chance for $U(g, R_i)$ to commute with it, see figure 3.4. At the end of the day, the final result is that there can be no localized operators charged under the global symmetry, which is clearly a contradiction for the initial hypothesis of the existence of a global symmetry in the bulk.
On the contrary, this contradiction is avoided for the case of a long-range gauge symmetry in the bulk by the fact that any operator charged needs to be attached to the asymptotic boundary by a Wilson line which in turn intersects in the $W[R_i]$ region and so $U(g, R_i)$ is able to detect it.
Having established that a global symmetry on the boundary cannot be dual to a global symmetry on the bulk, a question naturally arises: a global symmetry on the boundary to what is dual in the bulk?

3. 4.b The Completeness conjecture

In this subsection [Pal19]-[HO19] will be argued that a splittable global symmetry on the boundary CFT is indeed dual to a *long-range gauge symmetry* in the bulk and this will be sufficient to prove conjecture 1.2. For what already said, this seems reasonable by the fact that on operator which creates a charged object in the bulk must have a Wilson line attaching it to the boundary which, in turn, intersects the entanglement wedge of at least one of the

boundary regions R_i. Moreover, by the fact that the asymptotic symmetry operators $U(g,R)$ are supported only at the boundary, one can assume that the bulk long-range gauge symmetry $U(g,\partial\Sigma)$ acts only on the algebra $\mathcal{A}[R]$, for any boundary region R.
In order to better settle the duality between a global symmetry on the boundary and a long-range gauge symmetry on the bulk, one should ensure that definitions 3. 2.1 and 3. 3.1 are indeed both satisfied and related. The first guess is that a bulk long-range gauge symmetry implies a splittable global symmetry on the boundary with the same symmetry group and, additionally, the $U(g,R)$ of the bulk is exactly the $U(g,R)$ of the boundary global symmetry. Therefore, for the previous assumption, condition 3.5 of definition 3. 2.1 is satisfied. Furthermore, condition 3. 2.1.3.6 is naturally satisfied due to AdS/CFT correspondence: the boundary stress tensor is the limit of the bulk metric (see Appendix B) which in turn is invariant under any (internal) long-range gauge symmetry transformations. The last step is the non-trivial one: condition 3. 3.1.(iii). In fact, this condition is necessary by the fact that a CFT operator transforming non-trivially under the global symmetry would be dual to a state of finite energy. The latter, in turn, is charged under the long-range gauge symmetry. So it seems that charged states transforming under all irreducible representations of the symmetry group are allowed. However, this does not mean that they actually exist. It is exactly this the argument of conjecture 1.2.
Hence, the next step is proving that in the boundary CFT there are states in the Hilbert space living on a spatial $\mathbb{S}^{d-1}$ that transform according to all finite-dimensional irreducible representations of the symmetry group, which in turn is dual to the long-range symmetry.
For what comes next, it is necessary the introduction of some useful notions and theorems concerning compact Lie groups (definition 3. 3.1 was given supposing the symmetry group being compact). So,

Theorem 3. 4.2

If G is a compact Lie group and U is a finite dimensional representation of G $\implies$ U is unitary.

Theorem 3. 4.3

If G is a compact Lie group and U is a unitary representation of G $\implies$ U is the direct sum of a set of finite-dimensional irreducible representations.

Moreover, in order to prove the Completeness conjecture, it is necessary including the faithfulness of the representation. Therefore, the following theorems are necessary:

Theorem 3. 4.4

If G is a compact Lie group and U is a faithful unitary representation of G $\implies$ $\exists$ a finite invariant subspace of the Hilbert space on which U acts faithfully, so
If G is a compact Lie group and U is a faithful unitary representation of G $\implies$ U has a finite-dimensional faithful sub-representation.

Theorem 3. 4.5

If G is a compact Lie group and U is a finite-dimensional faithful unitary representation of G and U^* is its conjugate representation $\implies$ $\forall$ finite-dimensional irreducible representation α of G $\exists\, n,m>0$ / α appears in the direct sum decomposition of the tensor product $U^{\otimes n}\otimes U^{*\otimes m}$.

Therefore, the idea is to apply these theorems to the action of a representation of G on the set of local operators. In fact, condition 3.7 of definition 3. 2.1 together with theorem 3. 4.4

leads to the conclusion that a finite-dimensional subset of local operators transforming under a faithful representation of G exists. Hence, by virtue of theorem 3. 4.5, if one acts with these operators and their conjugates on the vacuum then one can prepare states transforming in any irreducible representation of G. As a result, if one is able to demonstrate that the long-range gauge symmetry acts faithfully on the Hilbert space of the CFT on the boundary, conjecture 1.2 can be proven via AdS/CFT principle and state-operator correspondence. So, let us consider the case of a theory where a long-range gauge symmetry is such that Wilson line operators extend from one connected component of the spatial boundary to the opposite spatial boundary. For example, to have a visualization, imagine that this Wilson line starts on the left of the manifold Σ_L and ends on the right Σ_R. The action on the right component of the spatial boundary Σ_R is given by condition 3. 3.1.(ii)

$$U^\dagger(g, \Sigma_R) W_\alpha U(g, \Sigma_R) = D_\alpha(g) W_\alpha \tag{3.33}$$

valid for any particular irreducible representation α. Recalling the algebra of the Wilson line operator, then

$$U^\dagger(g, \Sigma_R) W_\alpha U(g, \Sigma_R) W_\alpha^\dagger = D_\alpha(g) \ . \tag{3.34}$$

Now, one has to prove that $U(g, \Sigma_R)$ acts non-trivially for every $g \in G$ in order to prove that it acts faithfully. However this is immediate: $U(g, \Sigma_R)$ is necessarily non-trivial, otherwise the Wilson line operators on the left side of (3.34) cancel each other and as a result $D_\alpha(g)$ would be the identity matrix, invalidating condition 3.7 of definition 3. 2.1.

All this chapter relies on the important hypothesis of the compactness of the Lie symmetry group G. Therefore this chapter ends with another important conjecture connected with conjectures 1.1 and 1.2, namely

Conjecture 4.1: The Compactness conjecture

If a quantum gravity theory at low energies includes a gauge theory with gauge group G, then the latter must be compact.

Chapter 4

Weak Gravity Conjecture

Introduction

The Weak Gravity Conjecture (WGC) is perhaps the most debated but also well believed among the conjectures of the SP. This because it *should be* the haunt for all the quantum gravity theories: it is a matter of fact that at low energies the gravitational interaction is the weakest among the others. Nevertheless, there is a world behind the motivations and support for this conjecture which finds its roots both in ST and BH physics, it is not just the previous reduction to low levels. However, it makes the idea. If on the one hand the WGC, like all other conjectures, arises from ST, it is clearly not satisfying staying in its shadows and it is for this reason that efforts are made for BHs' arguments. Needless to say, if on the one hand this is indisputable, on the other it does not put emphasis on the fact that the WGC has gathered evidence through the years by itself. In this chapter a *zooming out-point of view* of the origin of the conjecture is presented.

4. 1 Loose arguments and first encounter with the WGC

First of all, the discussion emerges from the consideration of a $U(1)$ gauge field theory coupled to gravity in a four-dimensional world.
To be precise, the conjecture naturally emerges from String Theory arguments. In fact, recalling equation (2.96) or equivalently (2.97), the reduced action in D dimensions is

$$\int \mathrm{d}^{\mathcal{D}}X \ \sqrt{-G}\ R^{\mathcal{D}} \equiv \int \mathrm{d}^{D}X \ \sqrt{-g}\ \left[R^{D} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\ \underbrace{e^{-2(D-1)\alpha\phi}}_{\equiv g_{(A)}^{-2}}\ F_{\mu\nu}F^{\mu\nu}\right] \tag{4.1}$$

where the associated metric is

$$\mathrm{d}s^2 = e^{2\alpha\phi}\ g_{\mu\nu}\mathrm{d}X^\mu\ \mathrm{d}X^\nu + e^{2\beta\phi}\left(\mathrm{d}X^D + A_\mu\ \mathrm{d}X^\mu\right)^2 \tag{4.2}$$

with $D+1$-metric tensor being

$$G_{MN} = \begin{pmatrix} e^{2\alpha\phi}\ g_{\mu\nu} + e^{2\beta\phi}\ A_\mu A_\nu & e^{2\beta\phi}\ A_\mu \\ e^{2\beta\phi}\ A_\mu & e^{2\beta\phi} \end{pmatrix} = e^{2\beta\phi} \begin{pmatrix} e^{2(\alpha-\beta)\phi}\ g_{\mu\nu} + e^{2\beta\phi}\ A_\mu A_\nu & A_\mu \\ A_\mu & 1 \end{pmatrix} . \tag{4.3}$$

Therefore, the lower-dimensional theory has a propagating $U(1)$ gauge field whose gauge coupling, written in terms of the compactification radius R, is

$$g_{(A)} = e^{(D-1)\alpha\phi} \xrightarrow{(2.98)+(2.101):e^{\alpha\phi}=\left(\frac{1}{2\pi R}\right)^{\frac{1}{D-2}}} \frac{1}{2\pi R}\left(\frac{1}{2\pi R}\right)^{\frac{1}{D-2}} . \tag{4.4}$$

What is the gauge symmetry related to this propagating $U(1)$ gauge field? If $\lambda(X^\nu)$ is the local gauge parameter, the gauge symmetry is actually a circle isometry:

$$A_\mu \to A_\mu - \partial_\mu \lambda(X^\nu) \quad \text{with} \quad X^D \to X^D + \lambda(X^\nu) \tag{4.5}$$

hence, the charge of the Kaluza-Klein modes is periodic and quantized

$$q_n^{(A)} = 2\pi n\ . \tag{4.6}$$

Comparing (2.124) with (4.6), the mass of the Kaluza-Klein modes tower can be written in terms of the charge and the gauge coupling related to the gauge field A_μ

$$g_{(A)}\ q_n^{(A)} = M_{n,0}\ . \tag{4.7}$$

This is not the end of the game yet. Additionally, also the anti-symmetric Kalb-Ramond field gives rise to a gauge symmetry by a generalization of the Kaluza-Klein mechanism. In fact, following the discussion in chapter 2, let us separate B_{MN} in $B_{\mu\nu}$ and $V_\mu \triangleq B_{\mu D}$. So, considering (2.75), in particular the term proportional to the Kalb-Ramond field, one has a first contribution coming from the square of the minus of the determinant of the metric when compactifying the theory on a circle. In particular, starting from

$$\propto \int \mathrm{d}^{\mathcal{D}}X\ \sqrt{-G}\ H_{MNL}\ H^{MNL} \xrightarrow[\text{proportional to}]{\text{we want something}} \int \mathrm{d}^D X\ \sqrt{-g}\ ?\ F_{\mu\nu\ (V)}\ F^{\mu\nu}_{(V)} \tag{4.8}$$

computing $\sqrt{-G}$ with (4.3), one has

$$\begin{aligned} G &= e^{2\beta\phi\cdot(D+1)}\left[e^{2(\alpha-\beta)\phi\cdot D}\ g + gA^{2D} - gA^{2D}\right] \\ &= g\exp\{2\beta\phi D + 2\beta\phi + 2\alpha\phi D - 2\beta\phi D\} \\ &\xrightarrow{(2.98):\beta=-(D-2)\alpha} g\exp\{-2D\alpha\phi + 4\alpha\phi + 2\alpha\phi D\} \\ &= g\ e^{4\alpha\phi}\ . \end{aligned} \tag{4.9}$$

Moreover, a second contribution comes from the reduced Kalb-Ramond field on the compactified dimension, namely

$$H_{MNL}\ H^{MNL} = H_{\mu\nu D}\ H_{\rho\sigma D}\ G^{\mu\rho}\ G^{\nu\sigma}\ G^{DD} \quad \text{with} \quad H_{\mu\nu D} = \partial_{[\mu}\ B_{\nu D]} = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{4.10}$$

so that

$$G_{\mu\nu} = e^{2\alpha\phi}\ g_{\mu\nu}\left(1 + A_\mu A^\mu\ e^{2\alpha(1-D)\phi}\right) \xrightarrow{\text{leading to}} \left(G^{\mu\nu}\right)^2 \propto \left(g^{\mu\nu}\right)^2\ e^{-4\alpha\phi}\ . \tag{4.11}$$

Hence, the ? term in (4.8) comes from the reduction of $\sqrt{-G}H_{MNL}\ H^{MNL}$ and is

$$\sqrt{-G}\ \left(G^{\mu\nu}\right)^2\ G^{DD} \propto e^{2\alpha\phi}\ e^{-4\alpha\phi}\ e^{-2\beta\phi} = e^{-2(\alpha+\beta)\phi}\ . \tag{4.12}$$

Therefore, the reduced Kalb-Ramond field is actually a new gauge field whose coupling is thus

$$g_{(V)} = e^{(\alpha+\beta)\phi} = 2\pi R \left(\frac{1}{2\pi R}\right)^{\frac{1}{D-2}} \tag{4.13}$$

and the next natural step is finding out what are the states charged under V_μ. Nevertheless, recalling (2.69), one has

$$\int \mathrm{d}\tau\ \mathrm{d}\sigma\ B_{\mu\nu}\ \partial_a X^\mu \partial_b X^\nu\ \epsilon^{ab} \overset{\nu=D}{\propto} \int \mathrm{d}\tau\ \mathrm{d}\sigma\ V_\mu \partial_\tau X^\mu \partial_\sigma X^D\ , \tag{4.14}$$

and as a particular choice of τ and σ one can have $\tau = X^0$ and $\sigma = \frac{2\pi}{w}\ X^D$, where the latter is equivalent in imagining the string wrapping w times around the compactified dimension, as can be seen in figure 4.1.

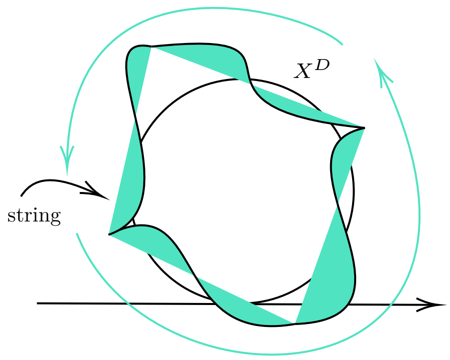


Figure 4.1. Illustration of a string wrapping around the compactified dimension w times.

Hence, (4.14) leads to the following term in the effective Polyakov action

$$S_P \xrightarrow{(2.75)-(4.14)} = -\frac{T}{2} \int \mathrm{d}\tau\ \mathrm{d}\sigma \left[2i\ V_\mu \partial_\tau X^\mu \partial_\sigma \left(\frac{w\sigma}{2\pi}\right)\right] = -i\frac{w}{2\pi\alpha'} \int \mathrm{d}\tau (\partial_\tau X^\mu)\ V_\mu \tag{4.15}$$

which is exactly of the same form as the (2.68). What is missing here is the correct dimensionality of the charge. In fact, the term in front the integral in the (4.15) has the dimension of the inverse of a length squared and, in order to be properly defined, has to be renormalised with a multiplication with a squared mass. Furthermore, (2.122) gives a measure of the string mass, as a result the charge of a particle coupled to the gauge field V_μ is

$$q_w^{(V)} = \frac{w}{2\pi\alpha'}\,(2\pi R)^{\frac{2}{D-2}}\ . \tag{4.16}$$

It is thus clear that, comparing with (2.124), one has

$$g_{(V)}\ q_w^{(V)} = M_{0,w}\ . \tag{4.17}$$

Therefore, String Theory naturally provides two gauge fields in D dimensions, A_μ coming from the reduction of the metric tensor and V_μ coming from the reduction of the Kalb-Ramond field instead. Under these two gauge fields there are charged states, the Kaluza-Klein modes for the former and the winding modes for the latter. For both of them, their mass is written as the product between the respective gauge coupling and gauge charge, $m = gq$. Additionally, from (2.124), one does not have just one Kaluza-Klein mode or just one winding mode: there is a tower of states instead, each of which depending on the periodicity number n and the winding number w. For this reason, at some given energy there will be some charged states, lowering or increasing the energy there will be less or more states respectively. As a consequence, one can also make a rough estimate of an energy cutoff Λ of the theory when considering the n-th Kaluza-Klein charged state or the w-th winding mode charged state, equivalently, given by

$$\Lambda \sim gM_P \tag{4.18}$$

in four dimensions.

Now, with this premise, everyone would agree that the effective theory that will arise is valid up to a cutoff below the Planck mass, by a landslide. This because gravity becomes strongly coupled at that scale, clearly. Are there any constraints on the value of the gauge coupling g? In principle nothing could prevent us from choosing a lower value for g.

What was argued for the first time in [AH+07] is that in doing this, a new *lower* EFT cutoff will appear. A theoretical physicist interested in this phenomenon could never be able to detect it.

Let us now, finally, introduce the general statement of the Weak Gravity conjecture.

CONJECTURE 1.1: WEAK GRAVITY

Consider a four-dimensional spacetime, a theory coupled to gravity with a $U(1)$ gauge symmetry with gauge coupling g described by the action

$$S = \int_{\mathcal{M}} \mathrm{d}^4X \, \sqrt{-G} \left[\frac{M_P^2}{2} R - \frac{1}{4g^2} F^2 \right] \tag{4.19}$$

then [AH+07]

1. There exists a state in the theory with mass m and charge q such that

$$m \leq \sqrt{2}\, gq\, M_P \qquad \text{ELECTRIC WGC} \tag{4.20}$$

2. The cutoff of the EFT is lowered and has to satisfy an upper bound defined by the gauge coupling

$$\Lambda \lesssim gM_P \qquad \text{MAGNETIC WGC.} \tag{4.21}$$

First of all, the (4.20) does not say anything about the spin of the state. Moreover, it makes clear the origin of the name of the conjecture [Urb18]: the Coulomb-like repulsion which is proportional to g^2q^2 overcomes the gravitational attraction, proportional to m^2/M_P^2, hence, *gravity is the weakest force*. In addition, there are currently speculations on extension of the 1.1 to a whole tower of heavy charged states [HRR17]-[And+18]. This stronger version of the WGC is connected to the Distance Conjecture[1] and the Completeness one. From a heuristic point of view, what are the basis of this conjecture 1.1?

[1] A very detailed study of the Distance conjecture is made by my colleague P. Piantadosi in his master thesis.

First of all, as stated by the No-Global symmetry conjecture 1.1, global symmetries are not allowed in QG. If g is the coupling constant of a gauge theory, one can in principle send it to zero in order to recover the global symmetry. However, and here lies the link between the two conjectures, this is prohibited by the spirit of the 1.1: if $g \to 0$ then also $\Lambda \to 0$, invalidating the EFT.
In some sense, (4.20) is named according to the spirit of the equation itself, the existence of a charged *particle* in the theory. So the question naturally arises: (4.21) sets constraint on the EFT cutoff, but why is it called "magnetic" WGC? The name refers to supportive arguments coming from the existence of a magnetic charge. The starting point is the conjecture presented in [BS11] according to which "all continuous gauge groups are compact". An immediate consequence is that in constructing a $U(1)$ gauge theory, one can write down magnetically charged BH solutions too. Clearly, the magnetic charge has to be coherent within Dirac quantization, namely $Q^M = \frac{2\pi n}{g}$, where $n \in \mathbb{Z}$. Moreover, if there exist such solutions, the next speculative step concerns magnetic monopole solutions. What are the leading arguments for such a prediction? The work [FSS15] lays qualitative backgrounds in this direction. It is argued that a magnetically charged BH has zero temperature and an entropy inversely proportional to the gauge coupling $S \sim 1/g^2$, so the result is a system corresponding to a large number of degenerate quantum states. Furthermore, the latter is quite unreasonable due to the fact that it is unusual that magnetic charge can only appear in these entropic configurations. In some sense it would be more natural if they originate in a low-energy non-gravitational scenario as *fundamental* states.
Moreover, heuristic supportive arguments for 1.1 come from BH physics. In fact, the WGC offers a solution to the so-called *remnants problem* underlined by Susskind in [Sus95]. In fact, all the difficulties arose by the problem are overcome if there exist macroscopic BHs that are able to completely evaporate all their charge. This results in gravitational unstable objects. As a consequence, following [Kra19], one could simply expect that if a BH is able to decay all by itself then there are no grounds for such remnants to appear. Also, this argument builds a bridge between the No-Global symmetry and Weak Gravity conjectures. However, if such BHs do not exist, the focus moves on the forces at stake. The main argument is indeed the need for the existence of a *particle* upon which the gravitational force is the weakest. Nevertheless, in order to settle in the motivation's arguments, it is important to plunging further in BH phenomenology.

4. 2 Arguments from BH physics

First of all, the following motivations do not have to be considered as stated arguments, they are rather phenomenological "landmarks" for testing IR physics.
A BH solution for (4.19) is the Reissner-Nordstrom (RN) one[2]:

$$f(r) = 1 - \frac{2M}{r} + \frac{2g^2Q^2}{r^2} \tag{4.22}$$

where M and Q have the straightforward meaning of mass and quantized charge of the BH. RN's BHs are characterised by two horizons due to the quadratic behaviour of the (4.22), located at $r_\pm = M \pm \sqrt{M^2 - Q^2}$. Moreover, additional characteristics are the area of the event horizon being $A_{RN} = 4\pi r_+^2$, the Bekenstein-Hawking entropy $S_{RN} = \pi r_+^2$ and then the

[2]The details of computations can be found in literature, for example in [Car19], and it is well recovered in the master thesis work of my colleague M. Del Piano.

BH temperature

$$T_{RN} = \frac{r_+ - r_-}{4\pi r_+^2}. \tag{4.23}$$

Indeed, their values are well suited in the first thermodynamics law

$$\mathrm{d}M = T\,\mathrm{d}S + \xi\,\mathrm{d}Q, \quad \text{where} \quad T = \left(\frac{\partial M}{\partial S}\right)_Q \quad \text{and} \quad \xi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{Q}{r_+} \tag{4.24}$$

therefore ξ is the electric potential and clearly plays the role of a chemical potential. In particular, the latter can be expressed in terms of the gauge field $A_\mu \mathrm{d}x^\mu = A(r)\mathrm{d}t$

$$A(r) = Q\left(-\frac{1}{r} + \frac{1}{r_+}\right) \tag{4.25}$$

when $r \to \infty$.
Those BHs for which the outer horizon shrinks to the inner are called *extremal black holes*. As a consequence, the latter naturally satisfies the 1.1

$$M \geq \sqrt{2}\, gQ\, M_P. \tag{4.26}$$

However, what does this actually mean? In a $U(1)$ gauge theory we can imagine [AH+07] that the coupling is extremely low, for example $g \simeq 10^{-100}$. As a result, the charge of the BH can range from 0 up to $\simeq 1/g$ in order to still satisfy the 1.1. If the mass of the BH is $\simeq 10M_P$, there will be a number of BH states that satisfies the bound (4.26). Precisely, If we label with Λ the upper mass allowed by (4.26) then the number of possible states N_{BH} is

$$N_{BH} \propto \frac{\Lambda}{g\, M_P}. \tag{4.27}$$

This precisely marks the link between the No-Global symmetry conjecture and the WGC: N_{BH} can indeed be considered as the number of remnants that can be possibly formed from a BH with mass M. In fact, if $g \to 0$, from (4.27) the number of remnants goes to infinity and as a result, an external observer cannot "quantify" the information that lies inside the BH. Additionally, if $g \to 0$ then $Q \to \infty$ and still today it is not clear how to measure the BH charge [BJS06] or how to handle with the magnetic WGC [Sar17]. An important question naturally follows: are remnants to be expected? What does the WGC state about remnants?

4. 2.a A proposal for the remnants problem

The *remnants problem* arises because any thermodynamic system would find extremely favorable to turn all the energy into zero temperature remnants, even the Sun for example. So, how to deal with the remnants? The first idea came as a proposal to prove that a sufficient large number of remnants would invalidate the Covariant Entropy Bound (CEB), an entropy bound conjectured in 1999 valid in all spacetime admitted by Einstein's equation. The statement of the conjecture can be easily qualitatively expressed by saying that the entropy S of a system restricted to a sphere of radius R, area $A(R)$, has to satisfy the bound

$$S \leq \frac{A(R)}{4}. \tag{4.28}$$

By definition, the entropy of a Schwarzschild BH precisely saturates the bound. Now, we can imagine to fill the sphere with a number N_s of species of particles[3]. The minute that it is

[3] In the original paper [Bou02] the sphere was filled by a gas composed by massless particles but here the argument is enlarged in order to emphasise the link between remnants and the WGC. Here we assume remnants to be small objects of about the Planck size.

done, another thermodynamic bound has to be taken in consideration: the Bekenstein bound [Bek73] according to which

$$S_{\text{matter}} \leq 2\pi E R \tag{4.29}$$

where E is the total energy of the matter system. Now, we want the N_s particles to be gravitational stable, thus we require (4.28)≤(4.29):

$$2\pi E R \leq \frac{A}{4} = \frac{4\pi R^2}{4} \implies 2E \leq R. \tag{4.30}$$

If we consider the particles weakly interacting with temperature T we can immediately relate the energy with the temperature as follows

$$E \sim N_s\ R^3\ T^4 \tag{4.31}$$

$$S \sim N_s\ R^3\ T^3\ , \tag{4.32}$$

then, applying (4.30) to (4.31), one has

$$2N_s\ R^2\ T^4 \leq 1 \implies T \sim \left(\frac{1}{2N_s\ R^2}\right)^{\frac{1}{4}} \xrightarrow{(4.32)} S \sim N_s^{\frac{1}{4}}\ R^{\frac{3}{2}}. \tag{4.33}$$

As a result, using (4.33) for S and putting it in the constraint (4.30), gravitational stability can be expressed in terms of

$$N_s \lesssim R^2 \quad \iff \quad N_s \lesssim A. \tag{4.34}$$

On the one hand, the (4.34) seems to resolve the species problem but on the other it does not assure that the number of species has to be exactly it.
Nevertheless, it can be shown [Bou02] that the time for a Hawking evaporation of a BH is proportional to $A^{\frac{3}{4}}$ and if N_s is large then it is modified to

$$t \propto \frac{A^{\frac{3}{2}}}{N_s} \xrightarrow{\text{if} N_s \gtrsim A} t \lesssim A^{\frac{1}{2}} \tag{4.35}$$

but the time needed to form a BH with area A is at least $A^{\frac{1}{2}}$.
Note also that, in support of the previous WGC argument there is the Species Conjecture, for which in $D = 4$ dimensions

$$\Lambda_s = \frac{M_P}{N_s^{\frac{1}{2}}} \xrightarrow{(4.34)} \Lambda \sim \frac{1}{R} \tag{4.36}$$

which is exactly the Hubble scale of the sphere. In conclusion, the WGC does not propose a rigid answer “remnants yes” or “remnants no” but it assures that in order to have a consistent EFT, the number of remnants has to be low and finite, obeying (4.34). Otherwise the cutoff of the theory would be too lowered compromising both the EFT’s validity and the CEB, whose validity is at a thermodynamical level.

4. 2.b Discarge of a Black Hole

In general, a BH immersed in vacuum [Gib75] may possess a charge and may lose it, under particular circumstances, in the form of spontaneous production of particles of opposite charges [Sch51]. The chemical potential introduced in (4.24), favours the production of oppositely charged particles, thus we can distinguish between two mechanisms of pair production. One due to the thermal Hawking radiation and the other by a Schwinger production due to the presence of ξ. One prevails on the other whether the BH is hot or is cold. However, in both cases the BH is able to discarge if there exists a charged particle in the theory that it can emit. This statement follows immediately by conservation of energy and charge, in fact, if we name with m_i and q_i the masses of the set of particles the BH can decay into, then

$$M \geq \sum_i m_i \quad \text{and} \quad Q = \sum_i q_i \tag{4.37}$$

must hold and hence

$$\frac{M}{Q} \geq \frac{\sum_i m_i}{Q} = \frac{1}{Q} \sum_i \frac{m_i}{q_i}\, q_i \geq \left(\frac{m_i}{q_i}\right)\Big|_{\text{min}} \frac{\sum_i q_i}{Q} \implies \frac{M}{Q} \geq \left(\frac{m_i}{q_i}\right)\Big|_{\text{min}} . \tag{4.38}$$

Equivalently, another formulation of the (4.20) is

Conjecture 2.1: Minimal Electric Weak Gravity

In a theory defined by the action (4.32) there exists an electrically charged state with mass m and charge q satisfying

$$\mathcal{O}(1) \sim \frac{Q}{M} \leq \frac{q}{m} \tag{4.39}$$

where Q and M are the charge and the mass of an extremal BH.

Thus 2.1 is an equivalent formulation of the WGC: it simply gives a necessary condition for extremal BH to be able to decay into fundamental charged particles.
When one talks about the decay of a BH, he/she actually refers to the absence of a BH stable gravitational state. Hence, the study of a decay of a BH can be carried by focusing on the main consequences that such an absence could lead to. First of all, in our Einstein-Maxwell theory, in line with 2.1, consider the particle which has the largest charge to mass ratio among the others and call it $P1$. Imagine to have two copies of $P1$. We want to study the forces between them. By the fact that these particles are massive, for sure there is the attractive gravitational force acting. Moreover, they are charged. All in all, the two main forces are the gravitational force and the electromagnetic one, in formulas

$$F_{\text{gravity}} = \frac{m^2}{8\pi M_P^2\, r^2} \tag{4.40}$$

$$F_{\text{electro}} = \frac{g^2\, q^2}{4\pi r^2} , \tag{4.41}$$

that immediately bring an equivalent statement of the 1.1:

$$F_{\text{gravity}} \leq F_{\text{electro}} . \tag{4.42}$$

Equation (4.42) states that the attractive force is weaker that the repulsive one. What could happen if the WGC is not valid? Via *reductio ad absurdum*, if $F_{\text{gravity}} \geq F_{\text{electro}}$ the immediate consequence would be the formation of a bound state since the two particles prefer to be attracted rather than repealing each other. By charge conservation, the charge of the final state is $2q$ for sure. Clearly, by energy conservation the mass of the final state is less than $2m$. As a result, the bound state will have a charge to mass ratio greater than that of the particle $P1$. This is a contradiction for the initial hypothesis. In addition, this bound state cannot discarge by emitting particles by those conservation theorems meaning that it is gravitational stable. In summary:

CONJECTURE 2.2: WEAK GRAVITY PARTICLE

In a four-dimensional $U(1)$ gauge theory coupled to gravity there exists a particle with the greater charge to mass ratio among the others in the spectrum. This immediately leads to state that

$$F_{\text{gravity}} \leq F_{\text{electro}} \tag{4.43}$$

the gravitational (attractive) force is weaker than the electromagnetic (repulsive) one. As a consequence, the theory does not admit the formation of bound states.

An important observation has to be stressed now. The bound state in this particle treatment of the WGC resembles a remnant for the BH arguments. For this reason it is *still* unclear whether or not it is problematic the presence of bound states in the theory.

4. 3 WGC AND SCALAR FIELDS

Conjecture 2.2 has an elegant physical explanation when scalar fields are taken in consideration. In order to highlight the latter, the first thing to do is to generalize the formulation 2.1 when such fields are included. All the discussion is carried in the $\mathcal{N} = 2$ supergravity context, see Appendix F. So let us start with the lagrangian (F.24)

$$L = \frac{R}{2} + \frac{1}{4}\,\Im(\mathcal{N}_{IJ})\; F^I_{\mu\nu}\; F^{J\;\mu\nu} - \frac{1}{8}\Re(\mathcal{N}_{IJ})\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\; F^I_{\mu\nu}\; F^J_{\rho\sigma} - g_{i\bar{j}}\partial_\mu\tau^i\partial^\mu\bar{\tau}^{\bar{j}} \tag{4.44}$$

using the notation in Appendix F, in literature [Dal13] is shown that the ADM mass of a $\mathcal{N} = 2$ supersymmetric extremal BH is given by the central charge with fields evaluated at infinity, namely

$$M_{\text{ADM}} = |Z|_\infty \;. \tag{4.45}$$

Therefore, identity (F.28) has a straightforward generalization to this case, leading to the following

$$\mathcal{Q}^2 = M^2_{\text{ADM}} + g^{ij}D_iZ\bar{D}_j\bar{Z} \tag{4.46}$$

which can be equally be written as [Pal17]

$$\mathcal{Q}^2 = M^2_{\text{ADM}} + 4g^{ij}\partial_i M_{\text{ADM}}\bar{\partial}_j M_{\text{ADM}} \;. \tag{4.47}$$

We would like to rephrase the WGC for a BH to decay in terms of a massive particle. Hence, if m labels the mass of the particle, the generalization is straightforward. In fact, despite of the RN black hole condition (4.22) (expressing the extremal condition), in (4.45) and (4.47) it is implied the fact that these BHs are actually super-extremal. Moreover, the last term on the right-hand side (RHS) of (4.47) is positive definite and non-vanishing and as a result, the particle we are looking for should also be a BPS state, namely its mass is equal to the central charge. In conclusion, we are free to declare that the following should be satisfied for a BH to decay

$$\mathcal{Q}^2 \geq m^2 + 4g^{ij}\partial_i m\bar{\partial}_j m \;. \tag{4.48}$$

The inequality (4.48) implicitly states that between two WGC states, the $U(1)$ gauge force should be at least equal to the sum of their gravitational and scalar forces. Additionally, by the fact that the last term on the RHS of (4.48) is definite positive, the scalar force between

equal charged particles is attractive. Therefore, the inequality then states that these two states should be self-repulsive rather than self-attractive [Pal19]: the gauge field repulsion must overcome both gravitational and scalar field attractions [Pal17], see figure 4.2.
Interestingly, from a particle physics point of view, the last term on the RHS of (4.48) comes from a cubic-interaction in the lagrangian. If we denote by h the WGC particle, then this term arises from the following contribution

$$V = m_0^2|h|^2 + 2m_0\mu\ t \tag{4.49}$$

with t the real part of a scalar field z and μ is the non-relativistic coupling of the WGC state to t.

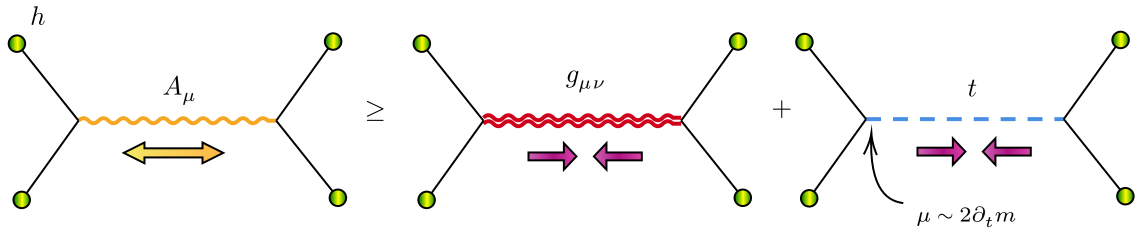


Figure 4.2. The $U(1)$ gauge repulsion between two WGC states h with mass m must be at least equal to the sum of the attractive gravitational and scalar forces.

Hence, condition (4.48) is generalised as

CONJECTURE 3.1: WEAK GRAVITY FOR SCALARS

A theory defined by the lagrangian (4.44) should admit a particle with mass $m(t)$ satisfying the bound

$$\mathcal{Q}^2 \geq m^2 + g^{ij}\mu_i\mu_j \qquad \Longleftrightarrow \qquad \mu_i = \partial_{t^i} m \tag{4.50}$$

where $\mathcal{Q}$ is defined as in (F.26).

4. 3.a A convex charge conjecture

The WGC as stated by 2.2 and 3.1 has an elegant and straightforward formulation when applied to conformal field theories (see also Appendix D). Precisely, it is possible to conjecture a precise property that local operators in unitary CFT with continuous global symmetries have to satisfy. The pioneering work [AP21] proposes such formulation. Firstly, the WGC can be formulated in terms of the so-called binding energy. In fact, according to 2.2, in the spectrum of the theory a self-repulsive particle must exist such that it cannot form a stable bound state with n copies of itself. A self-binding energy is thus defined as the difference of energies between the lowest two-particle state and twice the energy of the one-particle state (see figure 4.3). As shown in [AP21], the following generalizations for the WGC have been tested only for WC gravitational theories (see Appendix B) where explicit computations can be made with perturbation theory or semi-classical methods.

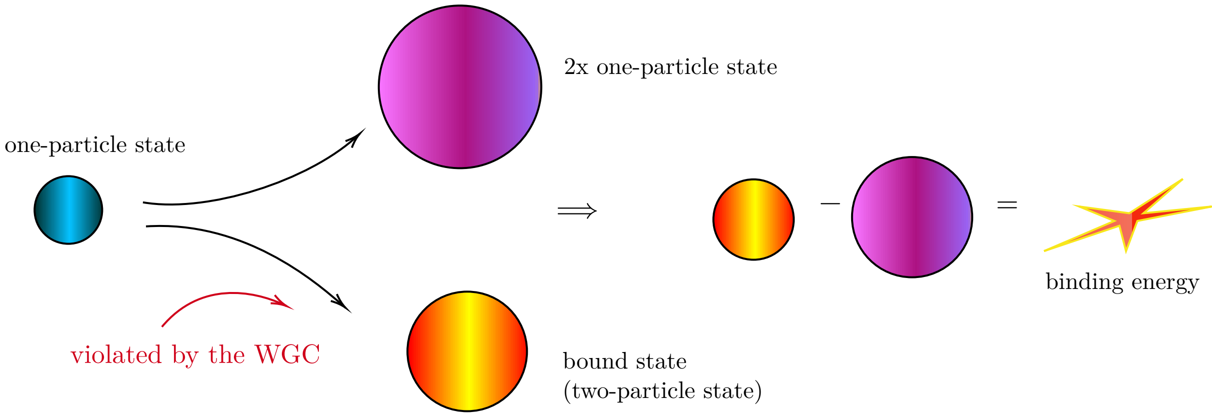


Figure 4.3. Illustration of the definition of binding energy coming from the creation of a bound state and twice the sum of the initial one-particle state.

Conjecture 3.2: Weak Gravity as Positive Binding

For a gravitational theory with a $U(1)$ gauge field, there should exist at least one particle in the spectrum, having charge of order one, that has a non-negative self-binding energy.

Now, due to the AdS/CFT correspondence (see Appendix B), the energy of a given state in a global AdS space is mapped to the conformal dimension of its dual local operator in the CFT. Hence, conjecture 3.2 can be formulated in terms of CFTs operators' properties. In particular, these operators are dual to the fields appearing in the action of a given theory:

Conjecture 3.3: Weak Gravity as Abelian Convex Charge

Let $\Delta(q)$ be the conformal dimension of the lowest dimension operator, with charge q, in any CFT having $U(1)$ global symmetry, then

$$\Delta(n_1 q_0 + n_2 q_0) \geq \Delta(n_1 q_0) + \Delta(n_2 q_0)\ . \tag{4.51}$$

Therefore, if $\mathcal{O} = \Phi$ labels a charged primary operator under a global symmetry, then the dimension of the first lowest dimension primary operator, Φ^n, must satisfy 3.3, namely

$$\Delta(\Phi^{n_1+n_2}) - [\Delta(\Phi^{n_1}) + \Delta(\Phi^{n_2})] \triangleq \gamma_{n_1,n_2} \overset{3.3}{\geq} 0\ . \tag{4.52}$$

A straightforward example, which serves as preliminary test for conjecture 3.3, can be made with theories in $d = d_0 - \epsilon$ dimensions in the presence of Wilson-Fisher fixed points. Following the discussion in Appendix B, let us consider the $U(1)$ global invariant lagrangian

$$L = \partial_\mu \bar{\Phi} \partial^\mu \Phi + \frac{\lambda_0}{4} (\bar{\Phi}\Phi)^2 \tag{4.53}$$

in $4-\epsilon$ dimensions, there is a WF fixed point occurring at λ_* given in (B.18). Computing the anomalous dimension γ_{Φ^n} by mapping $\mathbb{R}^d \mapsto \mathbb{R} \times \mathbb{S}^{d-1}$, the contribution to the 0-loop is given by [Bad+19]-[Ant+21b]:

$$\frac{\Delta_{-1}}{\lambda_* n} = \frac{1}{4}\left(\frac{3^{\frac{2}{3}} x^{\frac{1}{3}}}{3^{\frac{1}{3}} + x^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}(3^{\frac{1}{3}} + x^{\frac{2}{3}})}{x^{\frac{1}{3}}}\right) \quad \text{where} \quad x = \frac{9}{(4\pi)^2}\lambda_* n + \sqrt{-3 + 81\frac{\lambda_* n}{(4\pi)^2}} \tag{4.54}$$

which, summed with the contribution at 1-loop [Bad+19]-[Ant+21b] in accordance to (B.27), furnishes the scaling dimension Δ_{Φ^n}

$$\Delta_{\Phi^n} = n\left(\frac{d}{2}-1\right) + \frac{\epsilon}{10}n(n-1) - \frac{\epsilon^2}{50}n(n^2-4n) + \ldots \tag{4.55}$$

meaning that, using (B.19), the anomalous dimension is

$$\gamma_{\Phi^n} = \frac{\epsilon}{10}n(n-1) - \frac{\epsilon^2}{50}n(n^2-4n) + \ldots\ . \tag{4.56}$$

Therefore, computing (4.52), the final result reads as

$$\gamma_{n_1,n_2} = \frac{\epsilon}{5}n_1 n_2 - \frac{\epsilon^2}{50}n_1 n_2(3n_1+3n_2-8) + \mathcal{O}(\epsilon^2) \tag{4.57}$$

which satisfies (4.52) at leading order in ϵ.

4. 4 Multi-field WGC and naturalness problem

In quantum field theory renormalization is daily bread. So, viewing 1.1 and 2.1 through the eye of QFT there is the subtly issue regarding what type of masses are involved in the statement of the conjectures. Are they the physical ones? Are they the renormalized ones? In [AH+07] it is stressed that the appropriate scale to evaluate the charge to mass ratio is the physical mass of the particle. In this section naturalness principle will be taken in consideration. Hence, we will undergo speculations on particle-field theory and its connection to the WGC. First of all, imagine to have numerous $U(1)$ fields, each of which labelled with the i index. Thus, we will denote with m_i and q_i the mass and the charge relative to the i-th field. For what comes next, we define the dimensionless charge to mass ratios for each particle as

$$z_i = \frac{q_i\, M_P}{m_i}\ , \tag{4.58}$$

in terms of which 1.1 is stated through $z_i > 1$. Untill now, the WGC was introduced at a tree level, we will now investigate radiative contributions. Therefore, consider the classical ϕ^4 theory

$$L = (D_\mu\phi)(D^\mu\phi)^* - m^2\phi\phi^* - \frac{\lambda}{4}(\phi\phi^*)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.59}$$

where ϕ is a fundamental scalar. The renormalization algorithm assures the modification of its mass as

$$m^2 \to m^2 + \delta m^2 \quad \text{where} \quad \delta m^2 = \frac{\Lambda^2}{16\pi^2}(aq^2 + b\lambda)\ . \tag{4.60}$$

How large has to be m in comparison with δm? This question has answer in the context of the naturalness principle. The latter assures that those operators not protected by symmetry are unstable to quantum fluctuations induced at the cutoff of the EFT [CR14]. Therefore, in a natural theory the physical mass of the fundamental scalar cannot be smaller than the radiative corrections. For this reason, we now consider $m^2 \sim \delta m^2$. As a result, computing (4.58) and imposing the validity of the WGC, the results are

$$z = \frac{4\pi M_P}{\Lambda}\frac{1}{\sqrt{a + b\frac{\lambda}{q^2}}} \xrightarrow{z>1} \begin{cases} \Lambda & < \frac{4\pi M_P}{\sqrt{a}} \qquad \text{when} \quad q^2 \gg \lambda \\ \Lambda & < 4\pi M_P\sqrt{\frac{q^2}{b\lambda}} \qquad \text{when} \quad q^2 \ll \lambda \end{cases}\ . \tag{4.61}$$

In the case (4.61) b has to be well tuned if $q^2/\lambda \to 0$ in order to preserve the WGC. Therefore it seems machinery this tuning of coefficients and as a consequence scalar QED in this limit is inconsistent with WGC. According to the SP, it is thus inconsistent with a quantum theory of gravity. Therefore, how these issues can be solved? How can the WGC and naturalness be reconciled? One can eliminate the root hierarchy problem by assuming a new Physics below the Planck scale with the presence of new light states. There are two main possibilities:

1. these in turn can eliminate the divergences of ϕ by coupling to it regulating its *quadratic* divergences. Hence, in line with the WGC, they effectively low the cutoff of the EFT
2. they do not couple to ϕ. If one of these new states satisfies (4.20) then ϕ becomes irrelevant and as a consequence, WGC and naturalness are reconciled.

The WGC is stated for a $U(1)$ gauge theory until now. What happens if one has a product gauge symmetries? So consider a theory whose symmetry gauge group is $\Pi_{A=1}^{N}\, U(1)_A$. The theory is such that for each gauge group there is a number of charged particles. Hence, if i labels the i-th set of particles belonging to the A-th $U(1)$ gauge group, then $q_{i\ A}$ and m_i represent the charge and mass of the particles belonging to that set, respectively. It is a *second-quantization* notation. We will gather all of the sets having the same mass under a unique vector, namely we will consider a *charge-vector* whose components are the charges of the particles running over the sets, each of it transforms under $U(1)_A$. In this notation, each component of such a vector contains charged particles that can be transformed into each other under transformations of $SO(N)$. So:

$$\forall i\text{-th set } \in U(1)_A \text{with mass } m_i, \quad q_{i\ A} \mapsto \mathrm{q}_i \triangleq (g_1 q_1, g_2 q_2, \dots, g_N q_N), \quad \mathrm{z}_i \triangleq \frac{\mathrm{q}_i\, M_P}{m_i}\ . \tag{4.62}$$

It is naive thinking that for each $U(1)$ there exists a particle satisfying the (4.20) as well as requiring that there exists at least one set of species such that $z_i > 1$. We have to repeat the conditions for an allowed decay of a BH. If Q and M are the charge and mass of the BH and n_i denotes the number of total sets of particles into which the BH can decay, then conservation theorems imply

$$\mathrm{Q} = \sum_i n_i \mathrm{q}_i \quad \text{and} \quad M > \sum_i n_i m_i; \quad \sigma_i \triangleq \frac{n_i m_i}{M} \implies \mathrm{Z} = \sum_i \sigma_i \mathrm{z}_i\ . \tag{4.63}$$

The latter expression in equation (4.63) is a weighted average. To be precise, it is a sub-unitary weighted average since $\sum \sigma_i < 1$. From a geometrical point of view, in the charge-space the sub-unitary weighted average defines a convex hull. As a result, the decay is allowed only if the latter contains the unitary ball whose boundary represents extremal BHs. In order to get a visual representation of the argument above, let us consider the case of only two $U(1)$ gauge groups with only two sets of particles. Constraints for WGC validity impose the following

$$(\mathrm{z}_1^2 - 1)(\mathrm{z}_2^2 - 1) > (1 + |\mathrm{z}_1 \cdot \mathrm{z}_2|)^2\ , \tag{4.64}$$

which is more stringent than the case of a singular gauge group symmetry. In fact, if you consider the case where $\mathrm{z}_1 \perp \mathrm{z}_2$ and $|\mathrm{z}_1| = |\mathrm{z}_2| = z$, equation (4.64) imposes $z > \sqrt{2}$. In figure 4.4 the latter situation is depicted.

The left panel represents a theory consistent with the WGC while the right panel a theory inconsistent with the WGC. In the latter, the regions of the unit disc not within the convex hull represent stable black hole remnants. As a consequence, generalising to the case of N

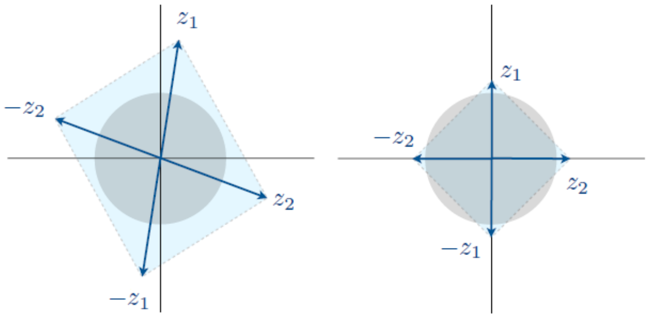


Figure 4.4. Representation in the charge-space of the charge to mass ratios for a $U(1) \times U(1)$ gauge theory. WGC is satisfied when the unit disc is contained in the convex hull defined by the values of z_1 and z_2.

$U(1)$ symmetry gauge groups, equation (4.64) imposes $z > \sqrt{N}$. Therefore, the more the number of the gauge groups the more stringent becomes the constraint. The generalization to multi-field WGC can be summarised as

CONJECTURE 4.1: MULTI-FIELD WEAK GRAVITY

A four-dimensional theory coupled to gravity and invariant under multiple $U(1)$s gauge symmetries must have a spectrum of particles with charge to mass ratio as defined in the (4.63), such that, in the charge-space the convex hull must include the unit ball.

Starting from 4.1, let us generalise the lagrangian for N fields in order to study the consistency between the WGC and the naturalness principle. So starting from

$$L = \sum_i \left[(D_\mu \phi_i)(D^\mu \phi_i)^* - m_i^2 \phi_{\phi i}^* - \frac{\lambda_i}{4} (\phi_i \phi_i^*)^2 \right] - \frac{1}{4} \sum_A F_{\mu\nu\ A} F^{\mu\nu\ A} \tag{4.65}$$

and repeating the same steps as before for the radiative corrections, the charge to mass ratio vector for the i-th field ϕ_i is

$$\mathrm{z}_i = \frac{4\pi M_P}{\Lambda} \frac{\mathrm{q}_i}{|\mathrm{q}_i|} \frac{1}{\sqrt{a_i + b_i \frac{\lambda_i}{\mathrm{q}_i^2}}} \tag{4.66}$$

so that the upper limit for the cutoff of the EFT is lowered by a factor of $\sqrt{N}$.

From what has been said until now, it is a matter of fact that there are numerous formulations of this conjecture. In line with [Sar17], part of them can be easily summarised in table 4.1.

4. 4.a Loopholes arguments: a Higgsing scenario

We now consider a scenario where a theory satisfies the WGC but the presence of a particle with the smallest mass can lead to an EFT which violetes it. This violation is carried by the higgsing of the theory. Recalling the conjecture 4.1, we consider a theory with two $U(1)$ gauge fields, A and B. For the sake of simplicity, let us consider the situation in which their gauge coupling is the same, call it g. Now we introduce two particles in the theory, each of which with charge under $U(1)_A$ and $U(1)_B$, let us call them P_A and P_B. Thus, P_A and P_B have

Conjecture form	Statement	Motivation
Electric WGC	In the theory there exists some particle with $\frac{q}{m} > 1$	The decay of a BH
Magnetic WGC	The $U(1)$ gauge theory's cutoff is bounded $\Lambda \lesssim gM_P$	Requirement the existence of a monopole which is not a BH
Strong electric WGC	The WGC particle is the only having the smallest mass	String theory arguments
Multi-field WGC	The spectrum must have particles whose charge to mass ratio defines a complex hull containing the unit ball in the charge space	The decay of a BH and geometrical interpretation

Table 4.1. Summary of the WGC forms in terms of their statements and motivations.

charges $(1, 0)$ and $(0, 1)$ under the two symmetries respectively. The mass of these particles is such that conjecture (4.1) is satisfied, hence surely less than the cutoff Λ of the theory. Let us now introduce a new scalar field in the theory, H, such that it has charges $(Z, 1)$ under the two symmetries, where $Z \gg 1$. At some point, we imagine that there will be a phase in the theory where H will acquire a *vev* v through Higgs mechanism. As a consequence, some combination of fields will acquire a mass. In particular, the following combination of fields A and B

$$V_H \triangleq A + \frac{B}{Z} \tag{4.67}$$

will acquire a mass $m_V^2 = (Zg)^2 v^2$. On the other hand, the other possible combination of fields

$$V_L \triangleq B - \frac{A}{Z} \tag{4.68}$$

will remain exactly massless. What are the consequences of such a Higgsing scenario? Firstly, there is now only one gauge field, V_L, and as a consequence the gauge coupling is slightly different. In fact, combination (4.68) is invariant under the maximal subgroup $U(1)_{V_L}$ which in turn implies a new gauge coupling for V_L

$$g_{\text{eff}} = \frac{g}{Z}\ , \tag{4.69}$$

the immediate consequence of it is a lowering of the cutoff of the theory. In fact if Z is sufficiently large then $\Lambda_{\text{apparent}} \sim gM_P/Z$. Now, particle P_A couples to this minimal charge while particle P_B becomes heavy. Actually, and here lies the loophole, the latter particle can live at energies *above* $\Lambda_{\text{apparent}}$ under the following circumstances (figure 4.5). It can happen that the value of the *vev* is approximately equal to Λ and one can adjust the theory such that $gZ \sim 1$. As a result, the cutoff of the low-energy $U(1)$ theory can be

$$m_V \approx \Lambda \approx gM_P\Lambda \lesssim \sqrt{g_{\text{eff}}}\ M_P\ . \tag{4.70}$$

Moreover, conjecture 1.1 is still satisfied by particle P_B, by the fact that $q_B \approx g$. In fact, it still has charge g under V_L in this Higgs phase. However, P_B could be so heavy that it actually desappears from the new EFT with cutoff $\Lambda_{\text{apparent}}$.
Nevertheless, other forms of the WGC can be violated in this EFT. This toy model is a blueprint for an important speculative question: the Swampland conjectures satisfied by an EFT of a given quantum gravity theory have to be satisfied by other EFTs emerging from

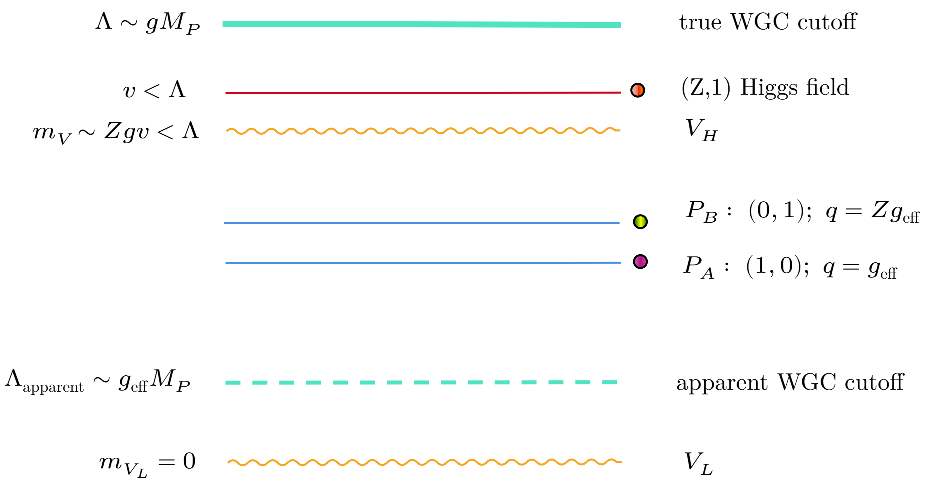


Figure 4.5. Spectrum of the toy model presented: charged particles are represented with blue straight lines while the gauge fields with the orange wavy ones. In the un-Higgs phase the theory satisfies the WGC ((4.21) upper sky-blue line). In the Higgs phase the cutoff is lowered, expressed by the sky-blue dotted line.

the latter as well? It is legit posing this question. The toy model presented is in some sense paradoxical: in the un-Higgs phase the WGC is satisfied in all its statements but in the Higgs phase no. However, conjecture 4.1 can come to rescue. In fact, the generalization of the previous case to a theory with N $U(1)$s all with the same coupling g and $N-1$ Higgs fields with charges $(Z, 1, \dots, 0)$, $(0, Z, 1, \dots, 0)$, $(0, 0, Z, 1, \dots, 0)$ and so far so on, leads to an upper bound which is always satisfied. To be precise, in accordance with (4.69), the remaining massless $U(1)$ symmetry has a new coupling constant of the form

$$g_{\text{eff}} = \frac{g}{Z^{N-1}} \ . \tag{4.71}$$

In line with the previous arguments, choosing $gZ \sim 1$, the effective coupling is such that $g_{\text{eff}} \sim g^N$. However, the multi-field WGC assures that the cutoff of the EFT is lowered by a factor $\propto \sqrt{N}$. Hence, the number of gauge groups can be chosen in order to maximize Λ:

$$(4.69) \xrightarrow{N\ U(1)s} \Lambda \lesssim g_{\text{eff}}^{\frac{1}{N}} M_P \quad 4.1 \implies \Lambda \lesssim \frac{g_{\text{eff}}^{\frac{1}{N}}}{\sqrt{N}} M_P \xrightarrow{\partial_N \Lambda = 0} \Lambda \lesssim \left(\log \frac{1}{g_{\text{eff}}}\right)^{-\frac{1}{2}} M_P \ . \tag{4.72}$$

Therefore, by the fact that $0 < g_{\text{eff}} < 1$ and that $\sqrt{x} < [\log(1/x)]^{-\frac{1}{2}}$ is satisfied for $0 < x < 1$, a theory satisfying the WGC in the UV assures that the Higgsing procedure *cannot* develop an EFT with gauge coupling violating the (4.72) [Sar17].
This result can be analogously presented by a bottom-up argument: BH physics. Imagine to be able to treat BHs with different (gauge) symmetries as separate thermodynamic ensambles. This can actually be achieved if all of them are such that their mass is of the same order of their radius and exceeds the BH temperature. The total entropy of the system acquires a logarithmical correction which accounts for the number of BHs satisfying the previous requirements. In formulas

$$S_{\text{tot}} = S_{BH} + \log\left(\frac{R}{g_{\text{eff}}}\right) \ , \tag{4.73}$$

so that if g_{eff} is extremely small, the logarithmic term in (4.73) becomes dominant to such an extent that we can approximate

$$S_{BH} \sim R^2 \lesssim \log\left(\frac{R}{g_{\text{eff}}}\right) \approx \log\left(\frac{1}{g_{\text{eff}}}\right) \ . \tag{4.74}$$

Recalling the CEB arguments exhibited previously, in order to preserve the validity of the entropy bound, it is necessary the introduction of a new physics. Actually, the introduction of a new cutoff (4.72).

4. 5 WGC and axions

In order to extend the conjecture to axions, it is necessary the introduction of the conjecture for general branes. Subsequently, it will suffice to consider the general expression valued for a 0-form. So in this section we will consider general p-forms, number of dimensions d and a dilaton coupling $\alpha_{p,d}$. Following the discussion in chapter 2, consider the action for General Relativity coupled to a dilaton field and a p-form gauge field in d dimensions to be

$$S = \frac{1}{2k_d^2} \int \mathrm{d}^d x \, \sqrt{-g} \, \left(R_d - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2e_{p;d}^2} \, e^{-\alpha_{p;d}\phi} \, F_{p+1}^2 \right) \tag{4.75}$$

and, following what already introduced in the Conventions, we will consider the electromagnetic duality given by

$$-\frac{1}{2e_{p;d}^2} \int \mathrm{d}^d x \, \sqrt{-g} \, e^{-\alpha_{p;d}\phi} \, F_{p+1}^2 \to -\frac{1}{2g_{p;d}^2} \int \mathrm{d}^d x \, \sqrt{-g} \, e^{\alpha_{p;d}\phi} \, G_{d-p-1}^2 \, . \tag{4.76}$$

The aim is to find a magnetically charged black brane solution for action (4.75), then using duality (4.76), finding the dual electric solution. Therefore, following [Lu93], by computing the magnetic charge and the ADM tension, one can formulate the extremality bound and as a result, the general formulation of the Weak Gravity conjecture for p-branes. Specifically, we will search for charged static black hole solutions with translational and rotational invariance in the $n = d-p-3$ spatial world-volume coordinates y^i so that one can dimensionally reduce to an $m = d - n = p+3$-dimensional theory with metric ansatz being

$$\mathrm{d}s^2 = e^{\frac{n}{m-2}\lambda} \, \mathrm{d}\hat{s}^2 + e^{-\lambda}\mathrm{d}y_i \, \mathrm{d}y^i \, . \tag{4.77}$$

Hence, following the discussion of chapter 2, the dimensionally reduced action is

$$\frac{1}{k^2} \int \mathrm{d}^m x \, \sqrt{-\hat{g}} \, \left(\hat{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{n(d-2)}{4(m-2)}\partial_\mu\lambda\partial^\mu\lambda - \frac{1}{2e_p^2} \, e^{-\alpha\phi - \frac{n(m-3)}{m-2}\lambda} \, F_{p+1}^2 \right) \tag{4.78}$$

from which one can considerably simplify computation by introducing two additional fields, σ and ρ, such that the seeking solutions become spherical symmetric-$SO(p+2)$ solutions to the equations of motion of

$$\int \mathrm{d}^m x \, \sqrt{-g} \left(\hat{R} - \frac{1}{2}(\nabla\rho)^2 - \frac{1}{2}(\nabla\sigma)^2 - \frac{1}{2} \, e^{B\rho} \, \hat{F}_{p+1}^2 \right) \quad \text{with} \quad B = -\sqrt{\frac{2(p+2)}{p+1}} \, . \tag{4.79}$$

Said r_+ and r_- the outer and inner horizons, the solution is

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1 - \frac{r_+^p}{r^p}\right)\left(1 - \frac{r_-^p}{r^p}\right)^{\frac{2p}{d-2}\gamma - 1} \mathrm{d}t^2 + \left(1 - \frac{r_-^p}{r^p}\right)^{\frac{2p}{d-2}\gamma} \mathrm{d}y_i \, \mathrm{d}y^i \\ &\quad \frac{\left(1 - \frac{r_-^p}{r^p}\right)^{\frac{\alpha^2}{p}\gamma - 1}}{\left(1 - \frac{r_+^p}{r^p}\right)} \mathrm{d}r^2 + r^2 \left(1 - \frac{r_-^p}{r^p}\right)^{\frac{\alpha^2}{p}\gamma} \mathrm{d}\Omega_{p+1}^2 \\ e^{-\alpha\phi} &= \left(1 - \frac{r_-^p}{r^p}\right)^{\alpha^2\gamma} \\ Q &= p(r_+ r_-)^{\frac{p}{2}}\sqrt{\gamma} \implies F_{p+1} = \frac{e_{p;d}}{k_d} \, Q\omega_{p+1} \end{aligned} \tag{4.80}$$

from which, according to [Lu93], one can compute the magnetic charge $\tilde{Q}$ and the ADM tension and then impose the extremality bound $r_+ > r_-$

$$\tilde{Q} = \frac{V_{p+1}}{g_{p;d}k_d}Q, \quad T = \frac{V_{p+1}}{2k_d^2}\left[(p+1)(r_+^p - r_-^p) + 2p\gamma r_-^p\right] \implies \gamma g_{p;d}^2\tilde{Q}^2 < k^2T^2 \ . \tag{4.81}$$

Using electromagnetic duality, the dilaton and flux become

$$e^{\alpha\phi} = \left(1 - \left(\frac{r_-}{r}\right)^{d-p-2}\right)^{\alpha^2\gamma}, \quad F_{p+1} = \frac{e_{p;d}}{k_d}(d-p-2)\frac{\sqrt{\gamma}(r_+r_-)^{\frac{d-p-2}{2}}}{r^{d-p-1}}\mathrm{d}t\wedge\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^{p-1}\wedge\mathrm{d}r \tag{4.82}$$

so that the extremality bound is

$$\gamma e_{p;d}^2 Q^2 < k^2T^2 \tag{4.83}$$

which for $d = 4, p = 1, \alpha = 0$ leads one to recover the result for a Reissner-Nordstrom BH (4.26). Now, 2.1 gives a condition for a BH to decay and in order to allow a black brane to decay too, there should exist a charged object of tension T_p and quantized charge q such that (4.83) is violated, namely

Conjecture 5.1: General Weak Gravity conjecture

In a d-dimensional theory characterised by a p-form gauge field symmetry, described by action (4.75), there should exist a charged object of tension T_p and quantized charge q such that

$$\left[\frac{\alpha^2}{2} + \frac{p(d-p-2)}{d-2}\right]T_p^2 \leq e_{p;d}^2 q^2 M_d^{d-2} \ , \tag{4.84}$$

allowing a black brane to decay.

Therefore, for $d = 4, p = 0$, the 0-form is an axion and thus the tension of the object charged under it is just the action of an instanton coupled to the axion (see Appendix E, subsection E. 2.b in particular), whose gauge coupling is inversely proportional to its decay constant. Therefore, 5.1 becomes

Conjecture 5.2: Weak Gravity conjecture for axions

In a four-dimensional theory, an axion with decay constant f_a must couple to instantons described by action S_{inst} such that

$$S_{\text{inst}} \leq \frac{M_P}{f_a} \qquad \text{ELECTRIC FORMULATION.} \tag{4.85}$$

Moreover, a magnetic Weak Gravity conjecture formulation can be addressed to axions too. In fact, from Appendix E. 2.b it is emphasised that the axion is a compact field and as a consequence, it admits a Hodge dual massless two-form gauge field $B = \frac{1}{2}B_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$. Its field strength $H = \mathrm{d}B$ is given by

$$\frac{H}{f} = f_a \star \mathrm{d}\theta \xrightarrow{\text{so that}} L_B = \frac{1}{2f^2}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} \tag{4.86}$$

with $f = 2\pi f_a$, accounting for Dirac quantization. According to 5.1 there exists a low-tension string charged under B such that $T \lesssim fM_P$. Now, imagine having a black hole with axionic

charge $b = \int_\Sigma B$, with Σ a two-sphere homotopic to the BH horizon.
The BH can evaporate through Hawking radiation, but, in line with previous chapters, it is necessary the presence of charged strings (in the sense of gauge symmetry) to avoid the remnant problem. In general, this kind of argument should apply also for the axions. Nevertheless, in literature one distinguishes between fundamental axions and ordinary Goldstone bosons. The latter provide semi-classical string solutions where the potential (spontaneously breaking the $U(1)_{PQ}$ symmetry) has a tension proportional to f^2. It happens that the *vev* at issue goes to zero at the core of these semi-classical strings while for the case of fundamental axions the core is singular due to the absence of a symmetry-restoring point at finite distance in field space (see figure 4.6).
In this sense $\sqrt{T}$ is interpreted as an ultraviolet cutoff on local quantum field theory. Therefore,

CONJECTURE 5.3: WEAK GRAVITY CONJECTURE FOR AXIONS

In a four-dimensional theory, the local quantum field theory is characterised by an UV cutoff

$$\Lambda_{UV} \lesssim \sqrt{f M_P} \qquad \text{MAGNETIC FORMULATION} \tag{4.87}$$

when an axion with decay constant $f_a = 2\pi f$ is present.

Hence, combining together (4.85) and (4.87), the Weak Gravity conjecture for axions can indeed be summarised by saying that

$$S_{\text{inst}} \lesssim \frac{M_P^2}{\Lambda^2} \,. \tag{4.88}$$

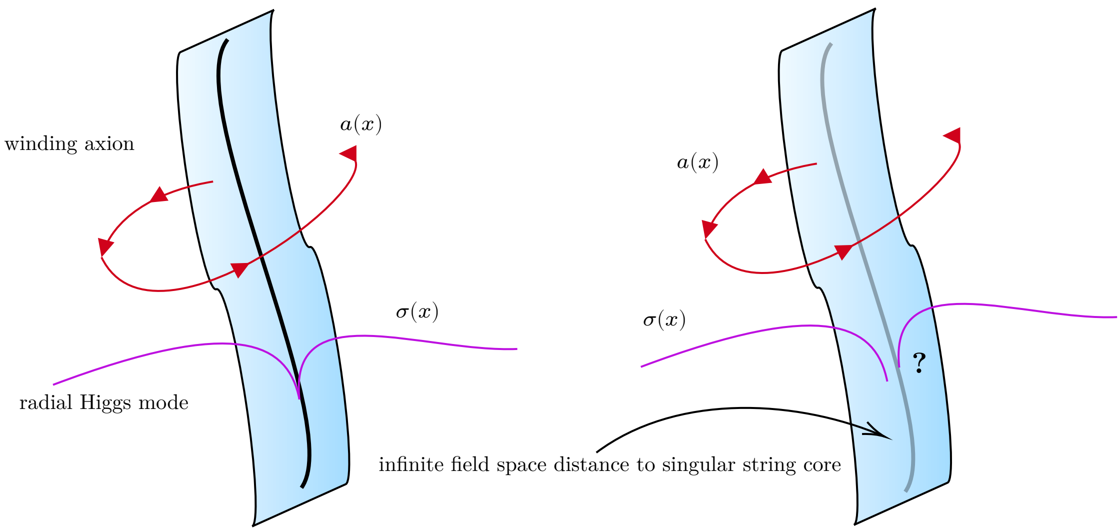


Figure 4.6. Illustration of the difference between a Goldstone boson-type axion (on the left) and a fundamental axion (on the right).

Part III

Implication for Particle Physics

Chapter 5

Approximate Global Symmetries with the Weak Gravity Conjecture

Introduction

It has been stressed throughout this thesis how exact global symmetries are inconsistent with quantum gravity. Nevertheless, when Particle Physics comes to play, it is the Weak Gravity conjecture itself that can provide approximate global symmetries in some cases. In this chapter, definition of approximate global symmetries will be explained and an example will be provided for the sake of clarification.

5. 1 Approximate global symmetries?

So far, conjecture 1.1 rules the landscape of questioning the presence of exact global symmetries in quantum field theories coupled to gravity. If on the one hand literature, and I hope also this thesis, is pretty clear in providing supporting arguments towards the validity of the No-Global symmetry conjecture, on the other, quantum field theories present beautiful mechanisms that actually may deeply shake the already established certainty about the validity of the conjecture. To be precise, a global symmetry can *arise* after a non-linearly realization of a gauge symmetry that becomes a global symmetry for those original charged particles surviving at the low-energy theory. It is important at this level to clarify what is meant by non-linearly realised gauge symmetries. In general, in field theory symmetries are just a realization of field transformations under which the theory has to be invariant. These field transformations lead to conservation laws of energy-momentum, of total angular momentum, of charge, of isotopic spin and so far so on. Additionally, all fields and state vectors have to be classified according to representations of the corresponding groups. In terms of allowed processes, they give restrictions on the S-matrix elements between states with fixed number of particles. However, there are also field transformations realised non-homogeneously and non-linearly but they do not give new conservation laws and do not even give a further classification of

particle states. Instead, they can determine the dynamics of these particles.
At this point, however, what is the role of global symmetries? To be precise, when and how do these arise in this scenario? First of all, these global symmetries can only emerge as *approximate* ones, namely in the sense of the following definition.

Definition 5. 1.1

> *An effective field theory has an approximate global symmetry*[1] *if among all processes allowed by the theory, starting from a set of initial multi-particle states* $\{i\}$ *and ending to that of final multi-particle states* $\{j\}$, *there exists a subset of these having rates parametrically smaller.*

Now, one can divide an EFT action describing a theory with approximate global symmetries into classes [Dau+20]. Among other possibilities, the main subdivision is between

1. accidental approximate global symmetry: there are global symmetries preserved in an EFT lagrangian because it is not possible to write down those operators explicitly breaking the symmetry. A famous example is the Standard Model $B-L$-symmetry, there are no relevant or marginal operators, consistent with spacetime and gauge symmetries and Lorentz invariance, that can actually forbid this symmetry

2. gauge-derived approximate global symmetry: these follows with non-linearly realization of gauge symmetries. For example, consider a theory $U(1)$ gauge-invariant. One can Higgs the gauge field with an axion, ending up with both fields becoming massive. Nevertheless, if there are other fields in the theory that remain light, then they will be invariant under the $U(1)$ global symmetry.

At the end of the day, we are now questioning ourselves to what extent global symmetries are prohibited, namely: is there any cutoff telling us how far a symmetry-violating operator is suppressed in the lagrangian? Are we able to suggest conditions for an approximate global symmetry to arise?
However, what is actually meant by condition 2? So, consider a theory with a generic p-form and gauge the theory with a $p+1$-form, equivalently this is the Higgsing of the $p+1$-form by the p-form, namely

$$L = \frac{1}{g_p^2}\left|\mathrm{d}A_p\right|^2 + \frac{1}{g_{p+1}^2}\left|\mathrm{d}A_{p+1}\right|^2 \to L = \frac{1}{g_p^2}\left|\mathrm{d}A_p + A_{p+1}\right|^2 + \frac{1}{g_{p+1}^2}\left|\mathrm{d}A_{p+1}\right|^2 \tag{5.1}$$

so that

$$\int_{B_{p+1}} A_{p+1} + \int_{\partial B_{p+1}} A_p \subset \; S \tag{5.2}$$

is the action, gauge-invariant only if $\delta A_{p+1} = \mathrm{d}\chi_p$ and $\delta A_p = -\chi_p$. For the particular case of the Higgsing of a 1-form by a 0-form (an axion), the last term on the left-hand side (LHS) is indeed an instanton:

$$\int_{B_1(x_*)} A_1 + \phi(x_*) \subset \; S\;. \tag{5.3}$$

Hence, instanton is located at a point x_* on a worldline B_1, see figure 5.1. Note also that the first term on the LHS represents a charged particle, labelled with Φ for example, moving along the worldline. Recalling Appendix E, in the EFT lagrangian a potential defining the coupling with an axion and instantons will appear. This operator is proportional to the exponentiation

[1] Let me stress that by global symmetry it is meant the definition given in chapter 3.

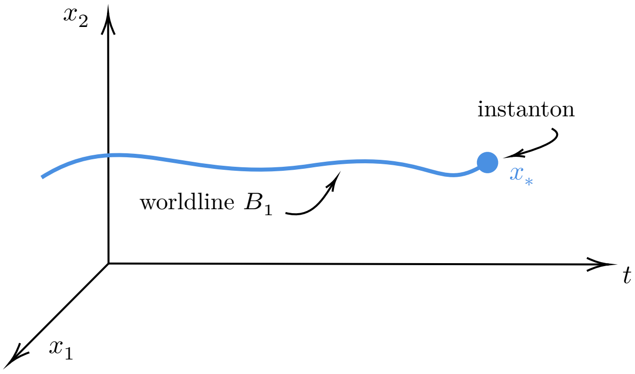


Figure 5.1. Illustration of a worldline of a charged particle ending on an instanton.

of the action describing the instanton. However, by the fact that the charged $p-1$-branes of the p-form gauge theory are gauge-invariant no more within the Higgsing scenario, this operator has a non-trivial changing in order to preserve gauge invariance, namely if $\delta\Phi = \Phi e^{i\chi}$:

$$e^{-S_{\text{inst}}+i\phi} \to \Phi \; e^{-S_{\text{inst}}+i\phi} \; . \tag{5.4}$$

Being gauge invariant, we are free to choose a gauge for ϕ, for example let $\phi = 0$. Nevertheless, after the gauge fixing, the RHS of (5.4) is a global $U(1)$-violating operator. Furthermore, using (4.88) of the WGC for axions, a constraint on the suppression of the operator coupling the axion with the instanton in the EFT lagragian will emerge. Hence, the WGC actually constrains the strength of the $U(1)$-violation in terms of the cutoff of the four-dimensional theory. In this case the suppression is exponential and it involved just one instanton; if there were k instantons, the suppression term would have been $\exp\{-kM_P^2/\Lambda^2\}$.

5. 1.a Four-dimensional example

We want to consider an axion gauged with a theory including fermions. Hence, consider a $U(1)$ gauge theory with fermions (just as the case of QED), whose action is

$$S_1 = \int \mathrm{d}^4x \, \left(i\bar{\psi}\not{D}\psi - \frac{1}{e^2}F^2\right) \tag{5.5}$$

and also a $SU(2)$ gauge theory, whose gauge fields are coupled with an axion, with action given by

$$S_2 = \int \mathrm{d}^4x \, \left(-f_a^2(\partial a)^2 - \frac{1}{g^2}\operatorname{Tr}\{G^2\} + \frac{a\operatorname{Tr}\{G\tilde{G}\}}{8\pi^2}\right) \; . \tag{5.6}$$

Now, gauging the axion with the gauge fields G^a is not possible in a trivial manner due to the gauge non-invariance of the last term in (5.6) under $\delta a = \chi$. Instead, we should write (5.5) in terms of the left-handed and right-handed fermions, translating $U(1) \to SU(2)$ gauge theory. So, gauging the axion, the overall action is

$$S = \int \mathrm{d}^4x \, \left(-\frac{1}{e^2}F^2 - \frac{1}{g^2}\operatorname{Tr}\{G^2\} - f_a^2(Da)^2 + i\bar{\psi}_L\not{D}\psi_L + i\bar{\psi}_R\not{D}\psi_R + \frac{a\operatorname{Tr}\{G\tilde{G}\}}{8\pi^2}\right) \; , \tag{5.7}$$

where the extended lagrangian is

$$L = -\frac{1}{e^2}F^2 - \frac{1}{g^2}\,\mathrm{Tr}\{G^2\} - f_a^2\Big[(\partial_\mu - iA_\mu)a\Big]^2 + i\bar{\psi}_L\gamma^\mu(\partial_\mu - iq_L A_\mu - iG_\mu)\psi_L + \\ + i\bar{\psi}_R\gamma^\mu(\partial_\mu - iq_R A_\mu - iG_\mu)\psi_R + \frac{a\,\mathrm{Tr}\Big\{G\tilde{G}\Big\}}{8\pi^2}\ . \tag{5.8}$$

The fermionic term in equation (5.8) could likely arise an $U(1)SU(N)^2$-anomaly. The situation is avoided by fixing the coupling of the axion when considering the Feynman diagram associated to the last term of (5.8), see figure 5.2.

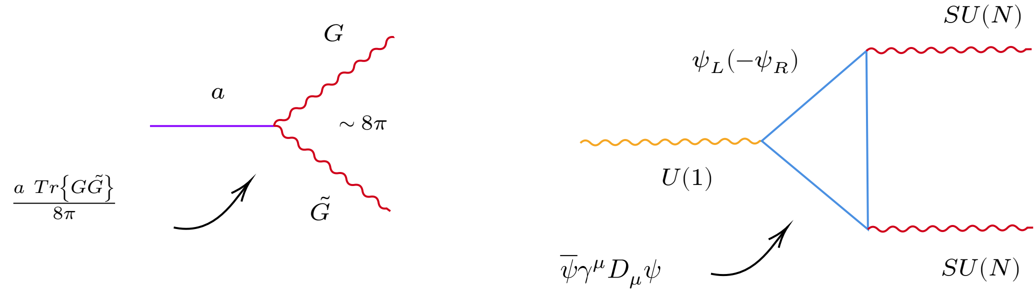


Figure 5.2. Illustration of the Feynman diagram corresponding to two terms in the lagrangian (5.8). The action (5.7) is anomaly-free and $U(1) \times SU(2)$-gauge invariant if $q_L - q_R = \mathrm{Tr}\{T^a T^b\}\delta_{ab}$.

At the energy scale of f_a, from the term $f_a^2(Da)^2$ the photon field will acquire a mass through the Higgs mechanism, and the axion will become its longitudinal degree of freedom. Therefore, under this energy scale, the effective degrees of freedom will be only those associated to the fermions and the gauge fields G^a, see figure 5.3.

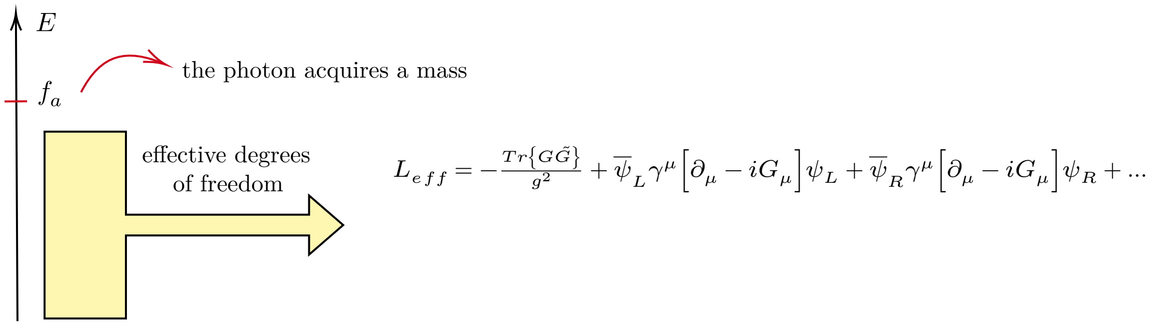


Figure 5.3. Effective action describing the degrees of freedom under the energy scale at which the photon acquires a mass.

Following the discussion in Appendix E, in particular equation (E.9), in the effective action the last term in (5.7) will give rise to a coupling between the instanton action and $\bar{\psi}_L\psi_R$ of the form

$$\cdots = e^{-S_{\text{inst}}}\bar{\psi}_L\psi_R + h.c. \xrightarrow{(5.4)} e^{-S_{\text{inst}}}\bar{\psi}_L\psi_R\, e^{i\phi} + h.c. \tag{5.9}$$

which according to the meaning of (5.4) is gauge invariant. Fixing the gauge $\phi = 0$, (5.9) explicitly violates the global $U(1)$.

Chapter 6

Conclusions and Outlooks

By homely gifts and hindered words
The human heart is told
Of nothing -
"Nothing" is the force
That renovates the World
— Emily Dickinson

6.1 Discussion

Nowadays, quantum gravity represents a compelling question among physicists. Partly because of our lack of knowledge about high-energy physics and partly because of the richness in mathematical models proposed. So far, the overall framework is thus a little bit chaotic and messy. String Theory is perhaps considered the leading theory in this scenario and it actually represents the birthplace for the so-called Swampland Program. Instead of approaching the study of quantum gravity at the energy of Planck scale, one can try to analyze the low-energy behaviour through the quantum field theory-eye. Therefore, the quantum gravity problem is grabbed by its infrared behaviour and dissected by questioning about correct behaviour that such an infrared theory has to satisfy in order to be a good quantum gravity theory in the ultraviolet.

In this thesis it was first introduced String Theory in an entry-level way by providing the basic knowledge in order to understand how the Swampland conjectures arose. So that, chapters 1 and 2 served as an introduction to the String Theory world and to the main characteristics that define the originality and the roots of the Swampland Program. Hence, the starting point is the closed bosonic string, the study of its dynamics and spectrum and then the discussion of a low-energy treatment. Furthermore, the very starting point is represented by the Kaluza-Klein reduction field theory, where it happens that, in the particular case of the compactification over a circle, a tower of states emerges, the Kaluza-Klein modes and the winding modes. String Theory not only provides the presence of particles as vibrational

modes of the string itself but also clarifies how energy is actually spent for a string to be wrapped around the compactified dimension, for example. However, it is the radius of the compactified dimension which is part of the fabric of this town. At the end of the day, this tower of states depends on it in the sense that the smaller it is the light are the Kaluza-Klein modes and the massive are the winding modes, on the contrary, the higher it is and the lighter are the winding modes while the Kaluza-Klein ones are more massive. Moreover, when the spacetime dimensions are reduced, the behaviour of the fields actually change. To be precise, the fields themselves adapt to the reduced world by becoming dimensional-reduced fields. Hence, the reduction of the metric tensor leads to the presence of a $U(1)$ gauge field as well as the reduced Kalb-Ramond field. The fields charged under these two gauge symmetries are the Kaluza-Klein and the winding modes, respectively.
Nevertheless, this tower of states is based on the periodicity number n of the quantized charge of the $U(1)$ gauge symmetry coming from the reduction of the metric tensor and w, the winding number telling how many times the string wraps around the compactified dimension. Therefore, at a given couple of values of n and w, we are frozen to a given energy. The duality between the Kaluza-Klein modes and winding modes, together with the existence of an energy cutoff, led to the first formulation of the Weak Gravity conjecture. Throughout all the discussion, a remarkable role is played by conformal field theory. Actually, as emphasised in chapter 3 and Appendix A, its formalism is the key for the emerging of the necessity of a natural absence of global symmetries in nature, leading to the first formulation of the No-Global symmetry conjecture.
Moreover, through the mathematical framework behind the AdS/CFT correspondence, proofs of the latter can be given in a independent way from String Theory. Precisely, a global symmetry on the boundary is not dual to a global symmetry in the bulk but to a long-range gauge symmetry one. Additionally, the problem of the forbidden global symmetries can be addressed in a completely different manner by using group field theory. The important hypothesis at the basis of all the discussion is the requirement of the compactness of the Lie manifold of the theory. The notion of global symmetry in quantum field theory then is strictly connected to that of splittability. Precisely, the Noether theorem crowning conservation principle, is actually reformulated via splittability. If one is able to demonstrate that Noether currents lead to a splittable theory then global symmetries are always present. However, it happens to not be the case.
Furthermore, what does "no global symmetries are allowed" really means in the context of an effective field theory? The Standard Model actually provides a very useful example for this through the approximate global symmetry given by the difference between the barion and lepton numbers. Precisely, approximate global symmetries can also be viewed as symmetry suppressed or at least controlled by some parameters. Interesting studies show that the Weak Gravity conjecture, as formulated for the axions, can really lead to a condition for a global symmetry of this kind, an example of which was presented in chapter 5.
Nevertheless, if on the one hand, String Theory itself seems to predict the behaviours dictated by the Weak Gravity and the No-Global symmetry conjectures, the very natural question concerns to what extend these seem reasonable. On the other hand, in fact, if in theory there is such a prediction, in some way it *must* be detectable in practise. For this exactly reason, the gaze is on black hole physics. Moreover, arguments coming from the latter are in fact not considered as very predictions but as signposts instead. However, both the Weak Gravity conjecture and the No-Global symmetry one are supported by black hole arguments. The first by conditions for a black hole to decay while the latter with the impossibility for a black

hole to shed its global information stored.
What is really noteworthy is how quantum gravity is actually intertwined with particle dynamics and how quantum gravity arguments have another reading key in conformal field theory. To be precise, as emphasised in chapter 4, the statement of the Weak Gravity conjecture which prohibits bound states has an elegant counterpart in terms of CFTs. The binding energy is defined as that energy spent by two one-particle states to become a bound state and by definition it is negative. Therefore, in accordance to the Weak Gravity conjecture, the binding energy must be positive. Nevertheless, the AdS/CFT actually furnishes a straightforward map between energy of states in the bulk theory with the scaling dimension of their dual operators on the boundary. It really appears that for unitary CFTs the Weak Gravity conjecture is expressed via conditions on anomalous dimensions.

6. 2 Outlooks

A good start is half of the job, would you not agree with me? So far, the long-road taken is slowly branching out leading to new frontiers. Starting from quantum gravity we could investigate new Physics independent from it. Actually, in the to-do list there is, at the present, a clarification of the Weak Gravity conjecture expressed as a convexity statement also for non abelian groups such as $O(n)$. Therefore, the first task is a deeper understanding and extension of tests for this conjecture. Is this a consequence, or if you want the influence, of gravity on CFT gauge theories? Or is it only a matter of change characterising these theories?
Moreover, my colleagues of our research group are at the moment working on other possible tests for the conjecture for every limit already studied by them. In fact, they are actually among the world-leaders of conformal field theories together with large-charge physics. These powerful instruments can actually lead us to new immediate questions: are we able to utter a word on black hole physics by talking the CFT language? Are there black hole's microstates mapped into CFT ones? What about holography?
All in all, is almost like we waited these months for the carousel of new possible Physics and now we are near to go on rides. We are full of excitement and ready to scream out loud. At the end of the day, we have already bought the ticket, don't you think?

Appendix A

Conformal Field Theory

Introduction

This appendix aims to give an overview of conformal field theory (CFT) in order to appreciate and learn the basis for the study of string interactions. In fact, the language of CFT provides a quantum field theory of strings where operators create and destroy entire strings. The appendix will review the algebra of the conformal group and will introduce the vertex operators.

A. 1 The conformal group in D dimensions

Let us consider a D-dimensional manifold $\mathcal{M}$ which can be either Riemannian or pseudo-Riemannian. In the first case we will consider the lorentzian metric $\eta_{\mu\nu}$ while in the second the euclidean one, namely the Kronecker delta $\delta_{\mu\nu}$.

Definition A. 1.1

A D-dimensional manifold is called conformally flat if the invariant line element can be written as

$$\mathrm{d}s^2 = \underbrace{e^{w(x)}}_{\triangleq\ \Omega(x)} \,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \begin{cases} \eta_{\mu\nu} & \textit{lorentzian signature} \\ \delta_{\mu\nu} & \textit{euclidean signature} \end{cases} \tag{A.1}$$

$w(x)$ is called the conformal factor, allowed to be x-dependent.

A metric satisfying definition A. 1.1 is called *conformally flat*. As a result

Definition A. 1.2

The conformal group is a subgroup of the group of general coordinate transformations, in general diffeomorphisms, preserving the conformal flatness of the metric.

The conformal transformations have the property of preserving angles between vectors while distorcing the lengths. Therefore, under a change of coordinates $x^\mu \mapsto x'^\mu \equiv y^\mu = x^\mu + \epsilon^\mu$, said $g_{\mu\nu}$ the metric tensor of a given D-dimensional manifold, then

$$g'_{\mu\nu}(x') = \Omega(x)\ g_{\mu\nu}(x) \iff g_{\mu\nu}(x^\mu + \epsilon^\mu)\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} = \Omega(x)\ g_{\alpha\beta}(x)\ . \tag{A.2}$$

We are interested in finding the generators of the infinitesimal conformal transformations ϵ^μ. Hence, let us start with

$$\left(g_{\mu\nu}(x) + \epsilon^\rho\partial_\rho g_{\mu\nu}(x)\right)\left(\delta^\mu{}_\alpha + \partial_\alpha\epsilon^\mu\right)\left(\delta^\nu{}_\beta + \partial_\beta\epsilon^\nu\right) = \Omega(x)\ g_{\alpha\beta}(x) \tag{A.3}$$

$$\xrightarrow{\mathcal{O}(\epsilon^2)} \underbrace{\epsilon^\rho\partial_\rho g_{\alpha\beta}(x) + g_{\alpha\nu}(x)\partial_\beta\epsilon^\nu + g_{\mu\beta}(x)\partial_\alpha\epsilon^\mu}_{\mathcal{L}_{\epsilon^\rho\partial_\rho}\ g_{\alpha\beta}(x)\quad \text{Lie derivative of}\quad g_{\alpha\beta}(x)} = (\Omega(x) - 1)\, g_{\alpha\beta}(x) \tag{A.4}$$

then, recalling the definition of affine connection ∇ and said $\Gamma^\alpha{}_{\beta\gamma}$ its coefficients, we have

$$\nabla:\quad \mathfrak{X}(\mathcal{M})\times\mathfrak{X}(\mathcal{M})\to\mathfrak{X}(\mathcal{M}) \implies \left(\nabla_\mu\epsilon\right)_\lambda = \partial_\mu\epsilon_\lambda - \Gamma^\sigma{}_{\mu\lambda}\epsilon_\sigma\ , \tag{A.5}$$

so equation (A.4) becomes

$$\nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha = (\Omega(x) - 1)\, g_{\alpha\beta}(x)\ . \tag{A.6}$$

Now, considering a constant metric $\nabla \equiv \partial$, by contracting (A.6) with $g^{\alpha\beta}$ one ends up with

$$\partial_\alpha\epsilon_\beta + \partial_\beta\epsilon_\alpha = \frac{2}{D}\nabla\cdot\epsilon\ g_{\alpha\beta} \implies \left(g_{\sigma\beta}\,\Box + (D-2)\partial_\sigma\partial_\beta\right)\nabla\cdot\epsilon = 0\ . \tag{A.7}$$

Therefore, depending on the dimensionality of $\mathcal{M}$ the cases are two. Let us now consider the case $D > 2$. From equation (A.7) ϵ^μ can be at most quadratic in x[1]. Hence, the conformal group contains the following transformations:

1. translations

$$\epsilon^\mu = a^\mu \triangleq P^\mu \quad \text{whose infinitesimal generators are}\quad a^\mu\partial_\mu \quad \text{in number}\quad D \tag{A.8}$$

2. Lorentz transformations

$$\epsilon^\mu = \omega^\mu{}_\nu\ x^\nu \triangleq J^\mu{}_\nu \quad \text{whose infinitesimal generators are}\quad \omega^\mu{}_\nu\ x^\nu\partial_\mu \quad \text{in number}\quad \frac{D(D-1)}{2} \tag{A.9}$$

3. dilatations

$$\epsilon^\mu = \lambda x^\mu \triangleq D \quad \text{whose infinitesimal generators are}\quad \lambda x^\mu\partial_\mu \quad \text{in number}\quad 1 \tag{A.10}$$

4. special conformal transormations:
   they are a sequence of transformations, inversion→translation→inversion, therefore

$$\epsilon^\mu = b^\mu x^2 - 2x^\mu b\cdot x \triangleq K^\mu \quad \text{whose infinitesimal generators are}\quad \left(b^\mu x^2 - 2x^\mu b\cdot x\right)\partial_\mu \quad \text{in number}\quad D\ . \tag{A.11}$$

[1]Actually, from (A.7) it does not immediately follow that ϵ has to be at most quadratic in x. However, for $D > 2$ one can show the stronger relation $\partial_\mu\partial_\nu\partial_\rho\epsilon_\lambda = 0$ for any indices μ, ν, λ, ρ, which then indeed implies that ϵ is at most bilinear in x and which leads to classification explained in the text.

To summarize, the following infinitesimal transformations are conformal

$$\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x \tag{A.12}$$

and the total number of generators is $\frac{(D+2)(D+1)}{2}$, which satisfy the following $SO(2,D)$ algebra

$$\begin{aligned}
\left[J_{\mu\nu}, J_{\rho\sigma}\right] &= i\eta_{\mu\rho}J_{\nu\sigma} + \quad \text{permutation} && \text{(A.13)}\\
\left[J_{\mu\nu}, P_\rho\right] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) && \text{(A.14)}\\
\left[J_{\mu\nu}, K_\rho\right] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) && \text{(A.15)}\\
\left[J_{\mu\nu}, D\right] &= 0 && \text{(A.16)}\\
\left[D, P_\mu\right] &= iP_\mu && \text{(A.17)}\\
\left[D, K_\mu\right] &= -iK_\mu && \text{(A.18)}\\
\left[K_\mu, P_\nu\right] &= -2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D\ , && \text{(A.19)}
\end{aligned}$$

which in turn gives us the following conventions: those fields annihilated by the lowering operator K_μ are called *primary fields* while those obtained by successive application of P_μ are said *descendants*.

A. 2 The conformal group in two dimensions

In this case, equation (A.7) becomes the Cauchy-Riemann equation

$$\begin{aligned}
\partial_1\epsilon_1 &= \partial_2\epsilon_2 && \text{(A.20)}\\
\partial_1\epsilon_2 &= -\partial_2\epsilon_1 && \text{(A.21)}
\end{aligned}$$

therefore, writing $\epsilon(z) = \epsilon^1 + i\epsilon^2$ and $\bar\epsilon(\bar z) = \epsilon^1 - i\epsilon^2$ in terms of the complex coordinates $z, \bar z = x^1 \pm ix^2$, it is clear that conformal transformations coincide with the analytic coordinate transformations

$$z \to f(z) \qquad \bar z \to \bar f(\bar z) \tag{A.22}$$

whose local algebra is infinite dimensional. In fact, labelling with $\mathbb{C} = (\mathbb{R}, \delta)$ it is easy to recover the algebra of $Conf(\mathbb{C})$. Let $\gamma \in Conf(\mathbb{C})$, $g \in \mathcal{F}(\mathbb{C})$ and $\rho(\gamma)$ the representation of γ on $\mathcal{F}(\mathbb{C})$. Therefore

$$\rho(\gamma)g(z) \triangleq g(\gamma^{-1}(z)) \quad \text{writing} \quad \gamma(z) = e^{-i\epsilon(z)}\ z \implies \gamma^{-1}(z) \simeq (1 + i\epsilon(z))z \tag{A.23}$$

$$\rho(\gamma)g(z) \simeq g((1+i\epsilon(z))z) \simeq g(z) + i\epsilon z\,\frac{\mathrm{d}g(z)}{\mathrm{d}z} \tag{A.24}$$

but $\epsilon(z)$ can be expandend in its Laurent series being a holomorphic function, hence

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^n \implies \rho(\gamma)g(z) \simeq g(z) + i\sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}\,\frac{\mathrm{d}g(z)}{\mathrm{d}z} \tag{A.25}$$

$$\implies \rho(\gamma)g(z) \simeq g(z) + i\sum_{n\in\mathbb{Z}} \epsilon_n l_n g(z) \quad \text{with} \quad l_n \triangleq z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z} \quad \text{the generators of} \quad Conf(\mathbb{C})\ . \tag{A.26}$$

Introducing the notation $\partial_z = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$, then the algebra satisfied by l_n and $\bar{l}_n$ is the classical Virasoro algebra[2]

$$\begin{cases} l_n &= z^{n+1}\partial \\ \bar{l}_n &= \bar{z}^{n+1}\bar{\partial} \end{cases} \Longleftrightarrow \quad \text{Virasoro algebras} \quad [l_m, l_n] = (m-n)l_{m+n}, \; [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}$$
$$\text{and} \quad [l_m, \bar{l}_n] = 0 \; . \tag{A.27}$$

In the quantum case the Virasoro algebra is affected by a *conformal anomaly* because of the presence of quantum mechanical breaking of the classical conformal symmetry. However, $Conf(\mathbb{C})$ contains a finite-dimensional subgroup, called *restricted conformal group*, formed by

$$l_{-1} : z \to z + \epsilon \quad \text{translations} \tag{A.28}$$
$$l_0 : z \to z + \epsilon z \quad l_0 + \bar{l}_0 \quad \text{scalings}, \quad i(l_0 - \bar{l}_0) \quad \text{rotations} \tag{A.29}$$
$$l_1 : z \to z + \epsilon z^2 \qquad \text{special conformal transformations.} \tag{A.30}$$

In chapter 2 it was shown that constraints (2.25) were a consequence of the Weyl invariance, moreover, by the fact that the worldsheet is invariant under translations, the energy-momentum tensor satisfies $\partial^a T_{ab} = 0$. It is useful to perform a Wick rotation $\tau \mapsto -i\tau$ in order to work with euclidean signature on the worldsheet instead of lorentzian one. Therefore, introducing

$$z = e^{\tau - i\sigma} \quad \text{and} \quad \bar{z} = e^{\tau + i\sigma} \; , \tag{A.31}$$

the worldsheet becomes the complex plane where the Euclidean time becomes the radial distance from the origin of the plane (representing the infinite past) to a circle. Furthermore, in chapter 2 it was also shown the residual symmetries of the conformal gauge $\tau \pm \sigma \to f_{\pm}(\tau \pm \sigma)$, that now become $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$. Henceforth, we will consider conformally invariant two-dimensional field theory in String Theory, having performed a conformal mapping of an infinitesimally long cylinder (the complex plane minus the origin) onto a plane, see figure A.1. In this language, the non-null components of the energy-momentum tensor are $T_{zz} = T(z)$

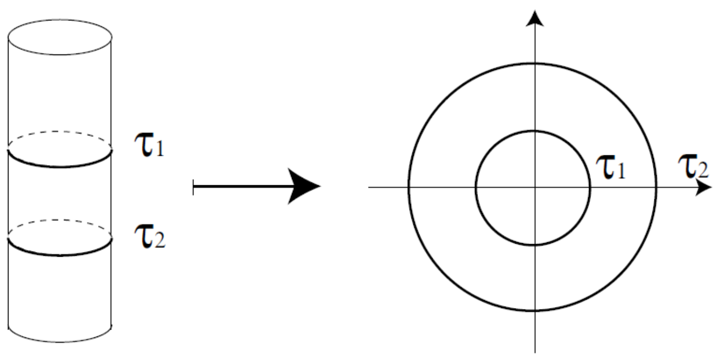


Figure A.1. Illustration of the conformal mapping explained in the text.

and $T_{\bar{z}\bar{z}} = \tilde{T}(\bar{z})$ and the conservation conditions become

$$\bar{\partial}\tilde{T}(\bar{z}) = 0 \quad \text{and} \quad \partial T(z) = 0 \; . \tag{A.32}$$

[2]Actually, it is called the *Witt algebra* whose central extension is indeed the Virasoro algebra, as explained later.

For what said, the Virasoro generators can be interpreted as the modes of the energy-momentum tensor, hence, the latter is indeed the generator of two-dimensional conformal transformations. As a result, expressions (2.27), (2.28), and (2.29) with (2.30) become

$$X_R^\mu(\tau,\sigma) \to X_R^\mu(z) = \frac{1}{2}x^\mu - \frac{i}{4}p^\mu \log z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\ z^{-n} \tag{A.33}$$

$$\implies \partial X^\mu(z,\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\ \alpha_n^\mu\ z^{-n-1} \tag{A.34}$$

$$X_L^\mu(\tau,\sigma) \to X_L^\mu(\bar{z}) = \frac{1}{2}x^\mu - \frac{i}{4}p^\mu \log \bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}\ \bar{z}^{-n} \tag{A.35}$$

$$\implies \partial X^\mu(z,\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\ \tilde{\alpha}_n^\mu\ \bar{z}^{-n-1}\ . \tag{A.36}$$

Hence, using (2.25) it easy to compute the holomorphic component of the energy-momentum tensor

$$T(z) = \sum_{n=-\infty}^{\infty}\ \frac{L_n}{z^{n+2}} \tag{A.37}$$

$$\tilde{T}(\bar{z}) = \sum_{n=-\infty}^{\infty}\ \frac{\tilde{L}_n}{\bar{z}^{n+2}}\ . \tag{A.38}$$

A. 3 Conformal fields and string interaction formalism

In field theory, if $\mathcal{M}$ is a spacetime manifold considered limited along the time direction from two spatial hypersurfaces σ_1 and σ_2, the Schwinger postulate ensures that

$$\delta S = F[\sigma_2] - F[\sigma_1]\ , \tag{A.39}$$

where S is the action of a given theory while F is the generator of a canonical transformation. It can be proven that

$$F[\sigma] \triangleq \int_\sigma J^\mu\ \mathrm{d}\sigma_\mu \tag{A.40}$$

with J^μ being the current associated to the transformation performed on the system. Hence, a canonical transformation is said to be a symmetry if (A.39) is null, namely if F is conserved. It is named the *conserved charge* associated to the current J^μ. If $\Omega[\sigma]$ is an observable associated to the system then its variation under a canonical transformation is expressed through

$$\delta\Omega[\sigma] = \{\Omega[\sigma], F[\sigma]\}\ . \tag{A.41}$$

Therefore, switching to quantum formulation, for infinitesimal conformal transformation one has:

$$\begin{cases}\delta z &= \epsilon(z)\\ \delta\bar{z} &= \tilde{\epsilon}(\bar{z})\end{cases} \implies Q = \frac{1}{2\pi i}\oint\left[\underbrace{T(z)\epsilon(z)}_{\triangleq J(z)}\mathrm{d}z + \tilde{T}(\bar{z})\tilde{\epsilon}(\bar{z})\mathrm{d}\bar{z}\right] \implies \begin{cases}\delta_\epsilon\Phi(z,\bar{z}) &= [Q_\epsilon, \Phi(z,\bar{z})]\\ \delta_{\tilde{\epsilon}}\Phi(z,\bar{z}) &= [Q_{\tilde{\epsilon}}, \Phi(z,\bar{z})]\ .\end{cases} \tag{A.42}$$

We are almost near to approach the formalism through which string interactions are studied. There is the need to introduce the definition of *conformal field*.

Definition A. 3.1

A conformal field is a field which under finite conformal transformations $z \to f(z)$ transforms as

$$\Phi(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\tilde{h}} \Phi(f,\bar{f}) \tag{A.43}$$

where $(h,\tilde{h})$ are called the weights of $\Phi(z,\tilde{z})$ and specify how the latter transforms under scale transformations.

Hence, equation (A.42) leads to

$$\delta_\epsilon \Phi(f,\bar{f}) = \oint \epsilon(z) \; \left[T(z)\Phi(f,\bar{f}) - \Phi(f,\bar{f})T(z)\right] \mathrm{d}z \tag{A.44}$$

which has to be evaluated in a radial ordering. The final result is the integral along the contour shown in figure A.2, which is actually computed by using the operator product expansion (OPE).

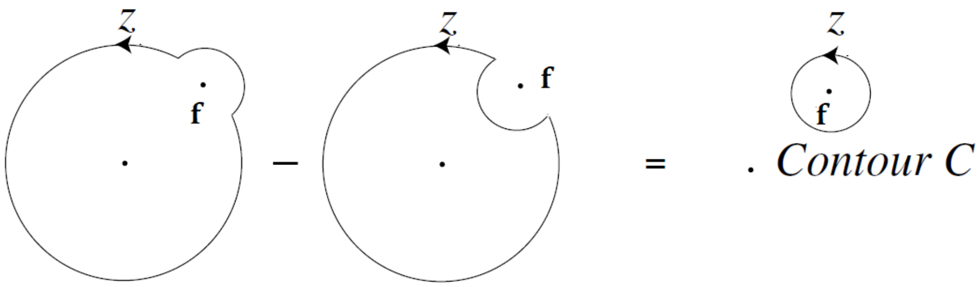


Figure A.2. Illustration of the contour along which the integral (A.44) is performed.

A. 3.a Operator Product Expansion (OPE)

In CFT language, a *field* is a *local operator* $\mathcal{O}$ and it represents any local expression that can be written down. In most of the cases, one can be interested in studying the behaviour of near local operators. Usually, one locally approximates local operators inserted at nearby points by a string of operators at these points. This approximation is the so-called *operator product expansion (OPE)*, namely

$$\mathcal{O}_i(z,\bar{z}) \; \mathcal{O}_j(w,\bar{w}) = \sum_k \; C_{ij}{}^k(z-w,\bar{z}-\bar{w}) \; \mathcal{O}_k(w,\bar{w}) \; , \tag{A.45}$$

which has singular behaviour as $z \to w$. The OPE is usually used in operator insertions inside time-ordered correlation functions [Ton09]:

$$\langle \mathcal{O}_i(z,\bar{z}) \; \mathcal{O}_j(w,\bar{w}) \dots \rangle = \sum_k \; C_{ij}{}^k(z-w,\bar{z}-\bar{w}) \; \langle \mathcal{O}_k(w,\bar{w}) \dots \rangle \; . \tag{A.46}$$

The OPE furnishes powerful tools in conformal theory and gives straightforward conditions regarding the existence of a symmetry in a theory. In order to understand it, it is necessary to introduce some basic notions about Ward identities.
If ϕs label the fields defined in a given theory, a transformation $\phi \mapsto \phi + \epsilon(z)\delta\phi$ is said to be a quantum symmetry if it leaves invariant both the action and the measure of the path integral,

namely[3]

$$
\begin{aligned}
Z = \int \mathcal{D}\phi\ e^{-S[\phi]} \mapsto \int \mathcal{D}\phi'\ e^{-S[\phi']} = \int \mathcal{D}\phi\ e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int J^a \partial_a \epsilon\right) \\
\xrightarrow[\text{one has a symmetry if}]{Z \text{ must be invariant}} \int \mathcal{D}\phi\ e^{-S[\phi]} \left(\frac{1}{2\pi} \int J^a \partial_a \epsilon\right) = 0
\end{aligned}
\tag{A.47}
$$

which gives the quantum version of the Noether theorem: $\langle \partial_a J^a \rangle = 0$; if this is not satisfied one says that there is an *anomaly*. However, one can extrapolate an analogue condition when having insertions in the path integral. For example, let us consider

$$
\langle \mathcal{O}_i(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\phi\ e^{-S[\phi]}\ \mathcal{O}_i(x_1) \dots \mathcal{O}_n(x_n) \tag{A.48}
$$

where, under the same transformation as before, each operator changes as $\mathcal{O}_i \mapsto \mathcal{O}_i + \epsilon(z)\delta\mathcal{O}_i$. Now two are the main possibilities:

1. $\epsilon(\vec{x})$ has support away from the operator insertions: in this case the condition for a symmetry is the same
2. $\epsilon(\vec{x})$ has support in some region near, for example, x_1: in this case one has

$$
\int \mathcal{D}\phi\ e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int J^a \partial_a \epsilon\right)\ (\mathcal{O}_1 + \epsilon\delta\mathcal{O}_1)\ \mathcal{O}_2 \dots \mathcal{O}_n \tag{A.49}
$$

which, up to $\mathcal{O}(\epsilon^2)$ gives the so-called Ward identity

$$
-\frac{1}{2\pi} \int_\epsilon \partial_a \langle J^a(\vec{x})\ \mathcal{O}_1(x_1) \dots \rangle = \langle \delta\mathcal{O}_1(x_1) \dots \rangle\ . \tag{A.50}
$$

Condition (A.50) is valid for every type of transformation but for conformal ones things simplify a lot. In fact, moving to complex coordinates and using Stokes' theorem, (A.50) reads as

$$
\frac{1}{2\pi} \oint_{\partial\epsilon} \mathrm{d}z\, \langle J_z(z,\bar{z})\ \mathcal{O}_1(x_1) \dots \rangle - \frac{1}{2\pi} \oint_{\partial\epsilon} \mathrm{d}\bar{z}\, \langle J_{\bar{z}}(z,\bar{z})\ \mathcal{O}_1(x_1) \dots \rangle = \langle \delta\mathcal{O}_1(x_1) \dots \rangle\ . \tag{A.51}
$$

and by the fact that J_z and $J_{\bar{z}}$ are holomorphic and anti-holomorphic functions, one can use the residues' theorem, so that (A.51) becomes

$$
\begin{aligned}
\frac{1}{2\pi} \oint_{\partial\epsilon} \mathrm{d}z\ \ J_z(z,\bar{z})\ \mathcal{O}_1(x_1) = -\operatorname{Res}[J_z \mathcal{O}_1] \overset{\text{(A.42)}}{=} -\operatorname{Res}[\epsilon(z) T(z) \mathcal{O}_1] = \delta\mathcal{O}_1 \\
\implies J_z(z)\mathcal{O}_1(w,\bar{w}) \approx \frac{\operatorname{Res}[J_z \mathcal{O}_1(w,\bar{w})]}{z-w}
\end{aligned}
\tag{A.52}
$$

where z and $\bar{z}$ have been considered as independent variables. A straightforward example can be made with translations (A.28):

$$
\mathcal{O}(z-\epsilon) = \mathcal{O}(z) - \epsilon\partial\mathcal{O}(z) + \mathcal{O}(\epsilon^2) \implies T(z)\mathcal{O}(w,\bar{w}) \approx \frac{\partial\mathcal{O}(w,\bar{w})}{z-w}\ . \tag{A.53}
$$

In CFT, one considers good operators those having good behaviour under dilatations. In fact, differently from translations, not all operators have good transformations under rotations

[3]The factor $\frac{1}{2\pi}$ has been considered for later convenience.

and scalings (A.29). Nevertheless, one can choose a basis of local operators with these characteristics. Generalising definition A. 3.1, we will say that under a transformation $\delta z = \epsilon z$ and $\delta \bar{z} = \bar{\epsilon}\bar{z}$, an operator has weights $(h, \tilde{h})$ if it transforms as

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}(\tilde{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}) \ , \tag{A.54}$$

where the weights are real positive numbers and $h\mathcal{O}$ and $\tilde{h}\mathcal{O}$ are for those operators being eigenstates of dilatations and rotations. Therefore, identity (A.51) reads as

$$T(z)\mathcal{O}(w,\bar{w}) \approx h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} \ . \tag{A.55}$$

This leads us to the following definition

Definition A. 3.2

> *A local operator having an OPE truncating at order $(z-w)^{-2}$ and $(\bar{z}-\bar{w})^{-2}$ is called a primary operator and its OPE does not have higher singularities.*

Moreover

Definition A. 3.3

> *The scaling dimension Δ of an operator is*
>
> $$\Delta = h + \tilde{h} \ . \tag{A.56}$$

Additionally, Ward identities constrain the Green function. Furthermore, if $\mathcal{O}_i$ is a primary operator with fixed scaling dimension Δ_i, then the set of $(\mathcal{O}_i, \Delta_i)$ gives the spectrum of the CFT and the two-point function is

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{A\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}} \ . \tag{A.57}$$

In CFTs one is usually interested in computing the spectrum of the weights by the fact that it is equivalent to compute the mass spectrum for a common QFT, due to the isomorphism between states and dual operators. In a slightly different notation, the weights of primary operators are the critical exponents in statistical mechanics. In fact, in a more general theory with gauge fields, fermions and scalars, all dimensionless couplings run with the energy scale μ [Zaf00]. If g labels the coupling of the theory, then the classical dimension d of a field will be corrected by the *anomalous dimension* γ, given by

$$\Delta = d + \gamma(g) \quad \text{where} \quad \gamma(g) = \frac{1}{2}\mu\frac{\mathrm{d}\log Z}{\mathrm{d}\mu} \ . \tag{A.58}$$

In most of the cases, QFTs have a conformally invariant behaviour in the correspondence of a fixed point, where the β-function of the theory is null and as a consequence the energy-momentum tensor is traceless. In this case, the renormalization group equation to solve is exactly the Ward identity for dilatations.

In order to introduce the so-called vertex operators, let us now consider

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \ \partial_a X\partial^a X \tag{A.59}$$

for the case of a free scalar field. Within the path integral formalism, solving the following

$$0 = \int \mathcal{D}X\frac{\delta(e^{-S}\,X(\sigma'))}{\delta X(\sigma)} = \int \mathcal{D}Xe^{-S}\left[\frac{1}{2\pi\alpha'}\,\partial^2 X(\sigma)X(\sigma') + \delta(\sigma-\sigma')\right] \tag{A.60}$$

with the help of the identity $\partial^2 \log(\sigma - \sigma')^2 = 4\pi\delta(\sigma - \sigma')$, we end up with

$$\langle \partial^2 X(\sigma) X(\sigma') \rangle = -2\pi\alpha' \delta(\sigma - \sigma') \implies \langle X(\sigma) X(\sigma') \rangle = -\frac{\alpha'}{2} \log(\sigma - \sigma')^2 \; , \tag{A.61}$$

that in complex coordinates reads as

$$\langle X(z) X(w) \rangle = -\frac{\alpha'}{2} \log(z - w)^2 \quad \implies \quad \partial X(z) \partial X(w) = -\frac{\alpha'}{2} \, \frac{1}{(z-w)^2} + \text{non singular.} \tag{A.62}$$

Additionally, (A.62) helps us to further compute a current $J(z)$ by focusing on the energy-momentum tensor T. Nevertheless, in QFT one has to pay attention to normal ordering, so, by virtue of the *Wick's theorem*, we can relate the temporal ordering to the normal one by computing all possible contractions. So that, computing T from (A.59)

$$T = -\frac{1}{\alpha'} \, \partial X \partial X \implies T(z) \partial X(w) \approx \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z - w} \iff \partial X(z) \partial X(w) = -\frac{\alpha'}{2} \, \frac{1}{(z-w)^2} \tag{A.63}$$

we understand that ∂X is indeed a primary operator with weights $(1, 0)$. Moreover, it can be demonstrated that $\exp\{ikX\}$ is a primary operator with weights $(\alpha' k^2/4, \alpha' k^2/4)$. However, T itself is not a primary operator but instead has weights $(2, 0)$

$$T(z) T(w) \approx \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(w)}{z - w} \tag{A.64}$$

where c is the so-called central charge which is related to the CFT degrees of freedom via the *c-theorem*.

A. 3.b State-operator correspondence and vertex operators

Recalling the (A.31), also called the radial quantization, we can map states living on the spatial slides of the cylinder whose temporal evolution is governed by the hamiltonian to those living in the complex plane, with D determining their temporal evolution. Additionally, while on the cylinder we can decompose the energy-momentum tensor in Fourier modes, on the complex plane we can decompose it through a Laurent expansion, namely

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \implies L_n = \frac{1}{2\pi i} \oint \mathrm{d}z \; z^{n+1} T(z) \tag{A.65}$$

which enables us to understand that $J(z) = z^{n+1} T(z)$ for conformal transformations $\delta z = z^{n+1}$, hence, L_n is the associated conserved charge. Moving to QFT, conserved charges become generators [Ton09], in this case they are the Virasoro generators satisfying

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} \, m(m^2 - 1) \delta_{m+n,0} \; . \tag{A.66}$$

Now, if $\Phi(z)$ denotes a conformal field of weight h, then it is possible to associate to it a state $|\Phi\rangle$, called *highest-weight state*, such that

$$L_0 \, |\Phi\rangle = h \, |\Phi\rangle \quad \text{and} \quad L_n \, |\Phi\rangle = 0, \quad n > 0 \; . \tag{A.67}$$

More precisely, one talks about the so-called *state-operator correspondence* by the fact that starting from the worldsheet, so starting from taking in consideration the oscillation states of the string, one ends up in studying the complex plane and using the two-dimensional conformal language these states are indeed *associated* with operators. To be even more precise, recalling figure A.1, the path integral on the disk in the complex plane with an operator at the origin maps to the path integral on the cylinder with a specified initial state. The formal definition of the state-operator correspondence is given by

$$|\Phi\rangle = \lim_{z\to 0} \Phi(z)\ \ |0\rangle \tag{A.68}$$

where $|0\rangle$ is the conformal vacuum. Therefore, the distant past in the cylinder is mapped to a single point $z = 0$ in the complex plane: a state in the cylinder in the far past is equivalent to specify a local disturbance at the origin [Ton09]. These states are in one-to-one correspondence with *local operators*. The state-operator map relates primary fields to primary operators. (A.68) leads us to an important definition:

Definition A. 3.4

A vertex operator V_ϕ is a worldsheet operator representing the emission or absorption of a given physical on-shell string mode $|\Phi\rangle$, on a specific point on the string worldsheet. There is indeed a one-to-one correspondence between string on-shell states and vertex operators.

In literature, it is shown that the $N = 0$ string level leads to a tachionic particle state, label it with $|0;k\rangle$. Therefore, the easiest vertex operator that can be written is that associated with a tachion. In fact, observing that, in order to have the state $|0;k\rangle$, the action of the creation operators $\alpha_n^{\dagger\,\mu} = \alpha^\mu_{-n}$ has to be null, by the fact that $X^\mu(z,\bar z) = X^\mu_R(z) + X^\mu_L(\bar z)$, kinematic information of a particle state is encoded in the conformal operator

$$1 + ik\cdot X \approx e^{ik\cdot X}\ . \tag{A.69}$$

As a result, by the fact that a physical Fock state is written as

$$|\Phi\rangle = \Pi_i \alpha^{\mu_i}_{-m_i}\ \Pi_j \tilde\alpha^{\nu_j}_{-n_j}\,|0;k\rangle \tag{A.70}$$

by using the identities

$$\alpha^\mu_{-m} = \sqrt{\frac{2}{\alpha'}}\oint \frac{\mathrm{d}z}{2\pi}\ z^{-m}\partial X^\mu(z) \implies \sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\partial^m\ X^\mu(0) \tag{A.71}$$

$$\tilde\alpha^\mu_{-m} = \sqrt{\frac{2}{\alpha'}}\oint \frac{\mathrm{d}z}{2\pi}\ \bar z^{-m}\bar\partial X^\mu(\bar z) \implies \sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}\bar\partial^m\ X^\mu(0) \tag{A.72}$$

a general close-string vertex operator is of the form

$$V_\Phi(z,\bar z) = \ \ :\ \Pi_i \partial^{m_i} X^{\mu_i}(z)\ \Pi_j \bar\partial^{n_j} X^{\nu_j}(\bar z)\ e^{ik\cdot X(z,\bar z)}\ :\ . \tag{A.73}$$

Appendix B

AdS_5/CFT_4 correspondence and semi-classical methods

Introduction

This appendix aims to present the basic notions about the AdS/CFT correspondence, mainly based on the work [Zaf00]. It is supposed known the properties of an anti-de Sitter space, otherwise I recommend [Car19] for a full overview. Additionally, the semi-classical approximation of *Large Charge* is introduced.

B. 1 Charting the 5D anti-de Sitter space

First of all, the $5D$ anti-de Sitter space can be described as the embedding of an hyperboloid in a flat $\mathbb{R}^{2,4}$, by imposing constraints via the Ricci scalar and the cosmological constant

$$x_0^2 + x_5^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2 \quad \text{and} \quad \frac{1}{R^2} = \frac{\Lambda(<0)}{12} \ , \tag{B.1}$$

whose line element is $\mathrm{d}s^2 = -\mathrm{d}x_0^2 + \mathrm{d}x_5^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2 - \mathrm{d}x_4^2$. Hence, it immediately appears the fact that AdS_5 has isometry group $O(2,4)$ identical to the conformal group in four dimensions (see Appendix A). Now, there is a number of possible suitable set of coordinates satisfying (B.1), most notably of which are

1. global coordinates:

$$\begin{aligned} x_0 &= R\cosh\rho\cos\tau \\ x_1 &= R\cosh\rho\sin\tau \\ x_i &= R\sinh\rho\hat{x}_i \quad \text{with} \quad \sum_{i=1}^{4} \hat{x}_i^2 = 1 \end{aligned} \implies \mathrm{d}s^2 = R^2\left(-\cosh^2\rho\ \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho \mathrm{d}\Omega_3\right) \tag{B.2}$$

   where the time is periodic and hence there are close time-like curves.

2. Poincaré coordinates: by introducing a four-Lorentz vector x_μ and a fifth coordinate $u > 0$

$$\left.\begin{aligned} x_0 &= \frac{1}{2u}\left(1 + u^2(R^2 + \vec{x}^2 - t^2)\right) \\ x_5 &= Rut \\ x_{1,2,3} &= Rux_{1,2,3} \\ x_4 &= \frac{1}{2u}\left(1 - u^2(R^2 - \vec{x}^2 + t^2)\right) \end{aligned}\right. \implies \mathrm{d}s^2 = R^2\left(\frac{\mathrm{d}u^2}{u^2} + u^2 \mathrm{d}x_\mu \mathrm{d}x^\mu\right) \tag{B.3}$$

where the metric has slices isomorphic to four-dimensional Minkowski spacetime. A useful redefinition of the metric is made by

$$u = \frac{1}{z} \implies R^2\left(\frac{\mathrm{d}z^2 + \mathrm{d}x_\mu \mathrm{d}x^\mu}{z^2}\right) . \tag{B.4}$$

From Appendix A and the properties of AdS_5, we have seen that both apparently realize theories with $O(2,4)$ symmetries. Therefore the two discussions should be related, there should exist a correspondence between them. In fact, it happens that the dynamics in AdS_5 can actually be reformulated as a boundary effect described by a four-dimensional local conformal field theory. Hence, a correspondence between CFT in four dimensions and gravitational theories in AdS_5 exists.

B. 2 The correspondence

Let us first introduce some conventions. The fields in five dimensions are called *bulk fields* while the CFT's are named as *boundary fields*. In CFT, the spectrum is specified by a basis of primary operators. Therefore, the correspondence between AdS_5 and CFT is carried by a correspondence between a AdS-field Φ and an operator in the CFT having the same quantum numbers. They know each other via boundary couplings. So that, if L_{CFT} labels the four-dimensional CFT lagrangian, every operator $\mathcal{O}$ can be associated to Φ through

$$L_{\text{CFT}} + \int \mathrm{d}^4x \ \Phi\mathcal{O} . \tag{B.5}$$

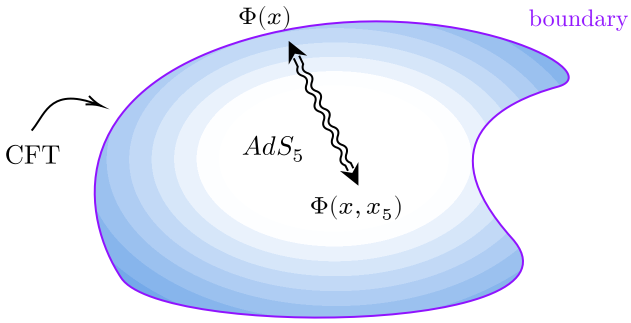
In (B.5), $\Phi(x)$ is a four-dimensional field and hence it can be considered as the boundary value of a five-dimensional bulk field $\Phi(x, x_5)$. Clearly, as a result, there is an additional condition that must be imposed. Therefore, if S_{AdS} is the AdS-action describing the interaction between bulk fields, we can demand that $\Phi(x, x_5)$ solve the five-dimensional equations of motion arising from S_{AdS}. This leads us to state that for every source configuration $\Phi(x)$ there is a five-dimensional field configuration $\Phi(x) \to \hat{\Phi}(x, x_5)$ such that[1]

$$\underbrace{e^{W(\Phi)}}_{\text{arbitrary 4-dim. off-shell configuration } \Phi(x)} = \left\langle e^{\int \Phi\mathcal{O}} \right\rangle_{\text{QFT}} = e^{S_{\text{AdS}}(\hat{\Phi})} , \tag{B.6}$$

where $W(\Phi)$ is the functional generator for connected correlation functions of the operator $\mathcal{O}$, namely

$$\langle \mathcal{O} \dots \mathcal{O} \rangle_{\text{connected}} = \frac{\delta^n W}{\delta \Phi^n}\Big|_{\Phi=0} . \tag{B.7}$$

[1] The right condition on the boundary to impose should be of the form $\hat{\Phi}(x, x_5) \sim f(x_5)\Phi(x)$.



However, how CFT operators are actually mapped to the fields in the bulk? By the fact that they have the same $O(2,4)$ quantum numbers, Φ can be found through symmetries. In fact, in the case of conserved currents, one can introduce a background gauge field, ending up with the following action

$$L_{\mathrm{CFT}} + \int \mathrm{d}^4x \ \sqrt{-g} \ \left(g_{\mu\nu}T^{\mu\nu} + A_\mu J^\mu + \phi F_{\mu\nu}F^{\mu\nu}\right) \ , \tag{B.8}$$

which in turn suggests that the energy-momentum tensor is the operator associated with the graviton while a current is the operator associated to a gauge field in AdS. The latter supports chapter 3 arguments: global symmetries in the CFT correspond to gauge symmetries in AdS.

B. 2.a An example

Let us introduce an example in order to better understand the usefulness of this correspondence. Let us choose the Poincaré coordinates in the form (B.4) ($R = 1$ for simplicity) and let us observe that a dilatation in the spacetime coordinates in the CFT corresponds to a $SO(2,4)$ isometry in AdS. So, let us consider the case of a massive scalar field whose AdS_5-action is

$$S \sim \int \mathrm{d}^5x \ \sqrt{-g} \ \left(g^{MN}\partial_M\Phi\partial_N\Phi + m^2\Phi^2\right) \overset{(\mathrm{B.4})}{=} \int \mathrm{d}z\mathrm{d}x \ \frac{1}{5}\left(z^2(\partial_z\Phi)^2 + z^2(\partial_\mu\Phi)^2 + m^2\Phi^2\right) \tag{B.9}$$

whose equation of motion is

$$\partial_z\left(\frac{\partial_z\Phi}{z^3}\right) + \partial_\mu\left(\frac{\partial_\mu\Phi}{z^3}\right) = \frac{m^2\Phi^2}{z^5} \ . \tag{B.10}$$

When considering only the z behaviour in (B.10), there are two solutions

$$\phi \sim z^{\Delta_\pm}, \ \Delta_\pm \quad \text{are the square roots of} \quad m^2 = \Delta(\Delta-4) \iff \Phi \sim \Phi_0 z^{\Delta_-} + \Phi_1 z^{\Delta_+} \ , \tag{B.11}$$

where it is clear from the power-law behaviour that Δ is actually the scaling dimension for the field. Including now the x_μ dependence, the general solution reads as

$$\Phi(z,x_\mu) \sim \left(\Phi_0(x)z^{\Delta_-} + \mathcal{O}(z)\right)\left(\Phi_1(x)z^{\Delta_+} + \mathcal{O}(z)\right) \xrightarrow[m^2 \ \text{positive definite}]{\text{boundary conditions} \ z=0} z^{\Delta_-}\Phi_0(x) \equiv \hat{\Phi}(x) \ . \tag{B.12}$$

Additionally, another way to find a solution of (B.10) is via the Green function, namely that function connecting the boundary-valued field $\hat{\Phi}$ with the bulk solution $\Phi(z,x_\mu)$ with the following properties

$$\begin{aligned} \Phi(z,x_\mu) &= \int \mathrm{d}x'_\mu G(z, x'_\mu - x_\mu) \ \hat{\Phi}(x'_\mu) \\ (\Box - m^2)G &= 0 \\ G &\xrightarrow{z\to 0} z^{\Delta_-}\delta(x'_\mu - x_\mu) \end{aligned} \iff G(z,x_\mu) = c\frac{z^{\Delta_+}}{(z^2+x_\mu^2)^{\Delta_+}} \tag{B.13}$$

so that

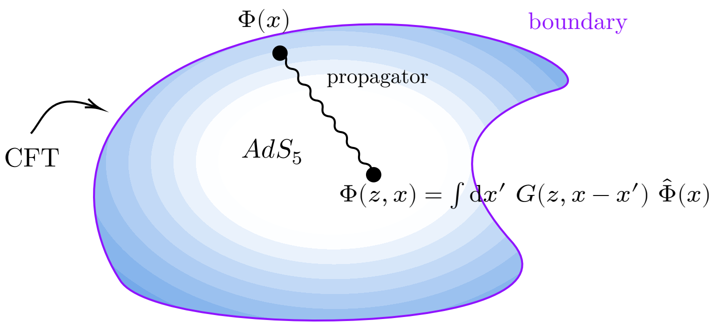


Figure B.1. Sketch of the boundary-bulk propagator.

$$\Phi(z, x_\mu) \sim \Phi_0(x)(z^{\Delta_-} + \mathcal{O}(z)) + c \int \mathrm{d}x' \frac{\Phi_0(x')}{|x'-x|^{2\Delta_+}} (z^{\Delta_+} + \mathcal{O}(z)) \ . \tag{B.14}$$

By substituting (B.14) into (B.9), one ends up with

$$S \sim \int_{\text{boundary}} \sqrt{-g}\Phi\partial^n\Phi + \underbrace{\int \sqrt{-g}\Phi(\Box - m^2)\Phi}_{=0} \sim \frac{1}{z^3}\Phi\partial_z\Phi\Big|_{\text{boundary}} \xrightarrow[\text{contrib.}\sim z^3]{\text{finite}} \int \mathrm{d}x\mathrm{d}x' \frac{\hat{\Phi}(x)\hat{\Phi}(x')}{|x'-x|^{2\Delta_+}} \tag{B.15}$$

from which the two-point function is given by (B.6)-(B.7)

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{\delta^2 S}{\delta\hat{\Phi}(x)\hat{\Phi}(x')^2}\Big|_{\hat{\Phi}=0} = \frac{1}{(x-x')^{2\Delta_+}} \ , \tag{B.16}$$

so Δ_+ is actually the conformal dimension of the dual operator of Φ. Generalising to fields with arbitrary spin we have:

$$\begin{array}{rll}
\text{scalar } \Phi & (j_1, j_2) = (0,0) & m^2 = R^2\Delta(\Delta-4) \\
\text{vector } A_\mu & (j_1, j_2) = \left(\frac{1}{2}, \frac{1}{2}\right) & m^2 = R^2(\Delta-1)(\Delta-3) \\
\text{symm.tensor } g_{\mu\nu} & (j_1, j_2) = (1,1) & m^2 = R^2\Delta(\Delta-4) \\
\text{anti-symm.tensor } B_{\mu\nu} & (j_1, j_2) = (1,0)+(0,1) & m^2 = R^2(\Delta-2)^2 \\
\text{spin}\frac{1}{2}\ \psi & (j_1, j_2) = \left(\frac{1}{2}, 0\right) + \left(0, \frac{1}{2}\right) & m = R(\Delta-2) \\
\text{spin}\frac{3}{2}\ \psi & (j_1, j_2) = \left(\frac{1}{2}, 1\right) + \left(1, \frac{1}{2}\right) & m = R(\Delta-2)
\end{array}$$

B. 3 Fundamental Interactions

Conformal field theory crowns fundamental interactions. That is because, at a classical level, the only two operators that violate scale invariance in the Standard Model lagrangian are the Higgs mass and the cosmological constant. For this reason, SM is studied around its conformal limit [Ant+20]. In particular, one aims to study conformal theories with continuous global symmetries via a semi-classical approach, in order to get access to non-perturbative insights of the theory. Furthermore, given a theory displaying conformal invariance, one studies the scaling dimension of particular operators [Ant+21b]. Moreover, there are different ways to proceed, depending on the properties of the path integral formulation. Basically, one divides it into two categories of theories [Bad+19]:

1. Weakly Coupled (WC): the contribution to a physical observable $\mathcal{O}$ is the sum of a classical one $\mathcal{O}_c$ with one determined by a quantum fluctuation around the classical trajectory, $\mathcal{O}_q$ (i.e. the harmonic oscillator)

2. Strongly Coupled (SC): all observables are quantum mechanical.

So, recalling Appendix A, the simplest object to study is the scaling dimension of the operator Φ^n in WC-$U(1)$ theories, at the fixed point where there is conformal invariance. Hence, starting from the lagrangian in the euclidean spacetime

$$L = \partial\bar{\Phi}\partial\Phi + \frac{\lambda_0}{4}\left(\bar{\Phi}\Phi\right)^2 \tag{B.17}$$

it can be demonstrated that when a small parameter, ϵ, is introduced to modify the number of spacetime dimensions, the theory admits a Wilson-Fisher (WF) fixed point in $d = 4-\epsilon$ dimensions, occurring at[2]

$$\beta(\lambda) = 0 \iff \lambda = \lambda_* \implies \frac{\lambda_*}{(4\pi)^2} = \frac{\epsilon}{5} + \frac{3}{25}\epsilon^2 + \mathcal{O}(\epsilon^3)\ , \tag{B.18}$$

for $\epsilon \ll 1$ the theory is WC. One wants to compute the anomalous dimension of Φ^n, γ_{Φ^n}, and then its scaling dimension

$$\Delta_{\Phi^n} = n\left(\frac{d}{2} - 1\right) + \gamma_{\Phi^n} \tag{B.19}$$

when n is large, so in the limit of large charge or large number of legs in the Feynman diagrams. If l labels the number of loops occurring in a given Feynman diagram, when computing the renormalization function, one has to account for the so-called *daisy diagrams* that scale with $\lambda^l n^{2l}$ and the connected diagrams, which scale as $\lambda^l n^{l+1}$, see figure B.2.

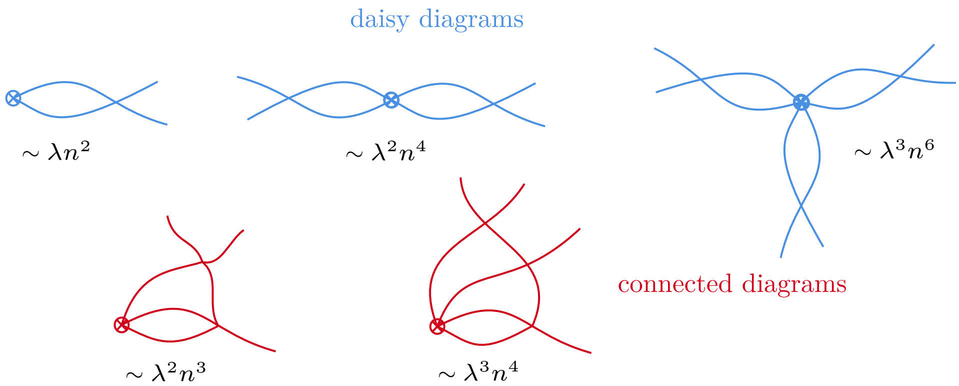


Figure B.2. Example of daisy diagrams and connected diagrams.

If one performs all the computations, one will be aware of that the leading contribution at order k is made by the connected diagrams. This leads us to write the anomalous dimension as [Bad+19]

$$\gamma_{\Phi^n} = n\sum_{l=1} \lambda^l\ P_l(n) \quad \iff \quad \gamma_{\Phi^n} = n\sum_{k=0} \lambda^k\ F_k(\lambda n)\ . \tag{B.20}$$

[2] Here λ is the renormalized value of λ_0.

Instead of tedious and limited perturbative approach, (B.20) is formally proved via semi-classical approximations. In fact, rescaling the fields $\Phi \mapsto \Phi/\lambda_0$, the two-point function is written as

$$\langle\bar{\Phi}^n(x_f)\Phi(x_i)\rangle \triangleq Z_{\Phi^n}^2 \mapsto Z_{\Phi^n}^2\lambda_0^2 = \frac{\int \mathcal{D}\Phi\mathcal{D}\bar{\Phi}\ e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\Phi}\partial\Phi+\frac{1}{4}(\bar{\Phi}\Phi)^2-\lambda_0 n\left(\log\bar{\Phi}(x_f)+\log\Phi(x_i)\right)\right]}}{\int \mathcal{D}\Phi\mathcal{D}\bar{\Phi}\ e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\Phi}\partial\Phi+\frac{1}{4}(\bar{\Phi}\Phi)^2\right]}} \tag{B.21}$$

and it can be solved by using a *saddle point expansion*, where λ_0 is small and $\lambda_0 n$ is fixed. Things simplify a lot when a WF fixed point is present. In fact, one can get advantage of conformal invariance and map the theory from the plane to the cylinder

$$\mathbb{R}^d \mapsto \mathbb{R}\times\mathbb{S}^{d-1} \implies (r,\Omega_{d-1})_{\mathbb{R}^d} \mapsto (\tau,\Omega_{d-1})_{\mathbb{R}\times\mathbb{S}^{d-1}}\Big|_{r=Re^{\frac{\tau}{R}}} \implies \mathrm{d}s^2_{\text{cyl}} = \mathrm{d}\tau^2 + R^2\mathrm{d}\Omega^2_{d-1}\ , \tag{B.22}$$

so mapping dilatations to time-translations. Therefore, in this new world the action becomes

$$S_{\text{cyl}} = \int \mathrm{d}^d x\ \sqrt{-g}\left[g^{\mu\nu}\partial_\mu\bar{\Phi}\partial_\nu\Phi + m^2\bar{\Phi}\Phi + \frac{\lambda_0}{4}(\bar{\Phi}\Phi)^2\right] \tag{B.23}$$

and the two-point function of a primary operator $\mathcal{O}$ with its conjugate is

$$\langle\mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\text{cyl}} = |x_f|^{\Delta_\mathcal{O}}\ |x_i|^{\Delta_\mathcal{O}}\langle\mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\mathbb{R}^d} = \frac{|x_f|^{\Delta_\mathcal{O}}\ |x_i|^{\Delta_\mathcal{O}}}{|x_f-x_i|^{2\Delta_\mathcal{O}}} \tag{B.24}$$

so that, the limit $x_i \to 0$ on the plane equals $\tau_i \to -\infty$ on the cylinder (see also Appendix A), hence

$$\langle\mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\text{cyl}} \overset{\tau_i\to-\infty}{=} e^{-E_\mathcal{O}(\tau_f-\tau_i)} \quad \text{where} \quad E_\mathcal{O} = \frac{\Delta_\mathcal{O}}{R}\ . \tag{B.25}$$

Therefore, on the plane the eigenvalues of the dilatation operator D that compound the spectrum of operator dimension, are now mapped to the energy eigenvalues on the cylinder. Hence, if one is interested in computing the lowest scaling dimension of a primary operator on $\mathbb{R}^d$, this is equivalent to compute the ground energy on the cylinder. For the case of $\mathcal{O} = \Phi^n$, defining $\tau_f - \tau_i \triangleq \pm T/2$ one has

$$\langle\bar{\Phi}^n(x_f)\Phi^n(x_i)\rangle_{\text{cyl}} \overset{T\to\infty}{=} e^{-E_{\Phi^n}T} \tag{B.26}$$

which in turn leads to [Bad+19]

$$\Delta_{\Phi^n} = \frac{1}{\lambda_*}\underbrace{\Delta_{-1}(\lambda_* n)}_{\text{0-loop}} + \underbrace{\Delta_0(\lambda_* n)}_{\text{1-loop}} + \lambda_*\underbrace{\Delta_1(\lambda_* n)}_{\text{2-loops}} + \ldots \tag{B.27}$$

that can equivalently be seen as the sum of the semi-classical contribution Δ_{-1} with the quantum corrections.

Appendix C

't Hooft anomaly and global symmetries

Introduction

This appendix aims to provide a large landscape of theories whose global symmetry is not splittable. In particular it will be emphasized how 't Hooft anomalies can be responsible for not providing the existence of Noether currents nor the property of splittability. Therefore, in the following sections the definition of 't Hooft anomaly together with a couple of example theories will be introduced.

C. 1 Background gauge fields and Principal Bundles

First of all, when talking about global symmetries, the first further step is to consider to turn on a background gauge field for that global symmetry. The fundamental requirement is the property of splittability due to the need of the presence of a Noether current for the continuous global symmetry under examination. One way to introduce a background gauge field for a continuous global symmetry group G with a set of Noether currents being J^μ_a is made by adding to the action the following term

$$\delta S = \int_{\mathcal{M}} \mathrm{d}^4x \ \sqrt{-g}\ A^a_\mu(x)(J^\mu_a(x) + \dots) = \int_{\mathcal{M}} A^a \wedge (\star J_a + \dots) \tag{C.1}$$

namely, with the introduction of a coupling term between the Noether currents and an arbitrary one-form A^a. Let us recall the meaning of this one-form in differential geometry. Therefore [Nak03],

Definition C. 1.1

A differential fibre bundle (E, π, M, F, G) consists of

1. *A differential manifold E, called the total space.*

2. *A differential manifold M, called the base space.*

3. *A differential manifold F, called the fibre.*

4. *A surjective map $\pi: \quad E \to M$ called projection map whose inverse valued at a point p of M is the fibre at point p, $\pi^{-1}(p) = F_p$.*

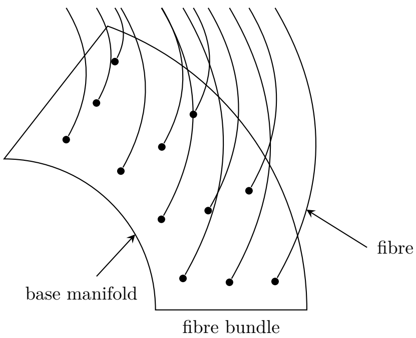
5. *A Lie group G which acts on the left on the fibre F and it is called the structure group.*

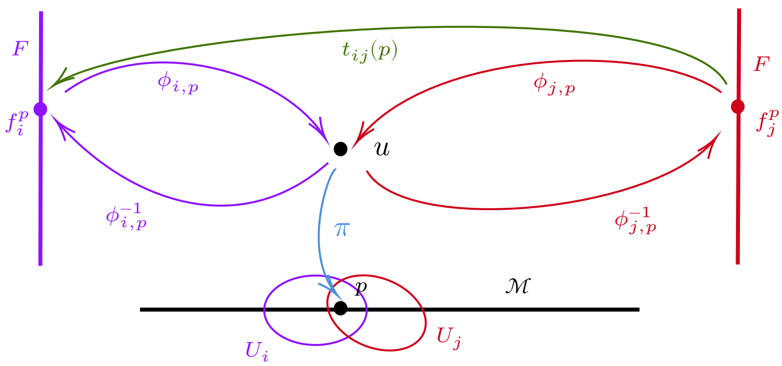
6. *If $\{U_i\}$ labels a set of open covering of M and ϕ_i is a diffeomorphism defined as $\phi_i: \quad U_i \times F \to \pi^{-1}(U_i) \;/\; \pi \circ \phi_i(p, f) \triangleq \phi_{i,p}(f) = p \in U_i$, then ϕ_i is called the local trivialization. This because a point in an open set of M is written as direct product of the open set itself and the fibre in that point.*

7. *There can be exist different local trivialization through the so-called transition functions. If $U_i \cap U_j \neq \emptyset$ then $t_{ij}(p) \triangleq \phi_{i,p}^{-1} \circ \phi_{j,p}: \quad F \to F \in G$, so $t_{ij}: U_i \cap U_j \to G$, the transition function enable to move from a set of coordinate of a fibre at point $p \in U_i \cap U_j$ to another set. Moreover, one can require the following consistency conditions*

$$t_{ii}(p) = \textit{identity map}, \quad p \in U_i \tag{C.2}$$
$$t_{ij}(p) = t_{ji}^{-1}(p), \quad p \in U_i \cap U_j \tag{C.3}$$
$$t_{ij}(p) \cdot t_{jk}(p) = t_{ik}(p) \quad p \in U_i \cap U_j \cap U_k \;. \tag{C.4}$$

If these conditions are all satisfied then one can glue together local pieces of a fibre bundle and the latter is called a trivial bundle and can be written as $M \times F$.





Additionally, one defines a section to be a smooth map $\sigma : \quad M \to E$ such that $\pi \circ \sigma = id|_M$. When $F \equiv G$ one talks about a G-bundle over M or, equivalently, it is called a principal bundle $P(M, G)$. The transition functions are still defined through the left-action of the structure group on the fibre but now one can also define a right-action on the fibre $P \times G \to P$ which is both *transitive* and *free*:

$$\phi_i : \quad U_i \times G \to \pi^{-1}(U_i) \;/\; \phi_i^{-1}(u \in \pi^{-1}(U_i)) = (p, g_i) \implies ua = \phi_i(p, g_i a),\ a \in G, u \in \pi^{-1}(p) \tag{C.5}$$

which enable us to find a relation between two sections $\sigma_i(p)$ and $\sigma_j(p)$ with $p \in U_i \cap U_j$ as $\sigma_i(p) = \sigma_j(p) t_{ji}(p)$, see figure C.1.

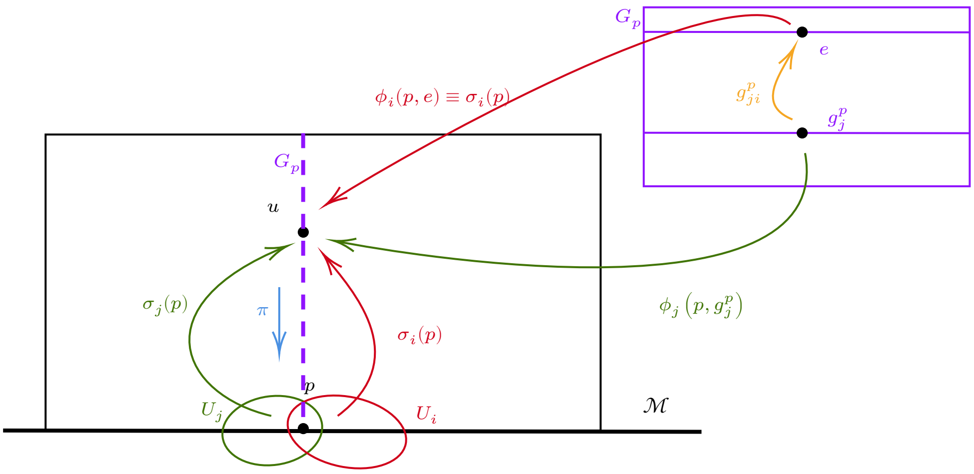


Figure C.1. A sketch of a principle bundle.

Now, a connection on a fibre bundle is defined as a $\mathfrak{g}$-valued one-form[1] that satisfies certain axioms and has a precise geometrical definition. In fact, it is defined from the unique separation of the tangent space $T_u P$ into the vertical subspace $V_u P$ and the horizontal one $H_u P$. This separation is performed in a systematic way by the introduction of a Lie algebra-valued one-form $\omega \in \mathfrak{g} \otimes T^* P$, called the Ehresmann connection. Finally, if $\{U_i\}$ is an open set of M and σ_i is a local section on each of these, it is possible to introduce a *local* Lie algebra-valued one-form A_i on U_i as the pullback of the Ehresmann connection by σ_i, $A_i \triangleq \sigma_i^* \,\omega \in \mathfrak{g} \otimes \Omega^1(U_i)$. Additionally, in order to let the Ehresmann connection be uniquely defined on P, the local connections A_i and A_j are related by the *gauge transformations* $A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} \mathrm{d} t_{ij}$.
Having introduced (C.1) in the action, Noether currents are naturally extended as follows [HO19]

$$\tilde{J}^\mu_a(x) \triangleq \frac{\delta(\delta S)}{\delta A^a_\mu(x)} = J^\mu_a(x) + \ldots \tag{C.6}$$

while in a non-lagragian framework, in QFT one simply defines a new set of *unnormalised expectation values in the presence of* A^a_μ, namely

$$\langle T \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_A \triangleq \langle T \mathcal{O}_1 \ldots \mathcal{O}_n e^{i\delta S} \rangle \quad \text{with } T \text{ the time-ordering operator.} \tag{C.7}$$

Moreover, one can define the partition function in the presence of a background gauge field A as

$$Z[A] \triangleq \langle 1 \rangle_A = \langle T e^{i \int_{\mathcal{M}} \mathrm{d}^d x \ \sqrt{-g} \ A^a_\mu(x) \left(J^\mu_a(x) + \ldots \right)} \rangle \ . \tag{C.8}$$

[1] $\mathfrak{g}$ is the Lie algebra of the Lie group G.

C. 2 'T Hooft anomaly and examples

A quantum field theory is said to have an 't Hooft anomaly [CDI19]-[HO19] if there is no way in providing the gauge invariance of (C.8) for all global symmetries. Clearly, the presence of an 't Hooft anomaly is not an inconsistency of a given QFT but they have important implications:

1. it is not possible to make all the background gauge fields dynamical
2. a global symmetry can be broken even though the Noether currents for the background gauge fields are neutral under the symmetry.

C. 2.a Chiral anomaly in $1+1$-dimensional Minkowski space

In particular, implication 2 can be understood with a straightforward example: chiral anomaly of a free complex Dirac fermion in $1+1$-dimensional Minkowski space. The action of the theory is

$$S = i\int \mathrm{d}^2x\ \bar{\psi}\partial\!\!\!/\ \psi \tag{C.9}$$

which is invariant under two $U(1)$ global symmetries with currents

$$\psi \mapsto e^{i\theta}\psi \qquad J^\mu_v = \bar{\psi}\gamma^\mu\ \psi \tag{C.10}$$
$$\psi \mapsto e^{i\theta\gamma^3}\psi \qquad J^\mu_p = \bar{\psi}\gamma^\mu\gamma^3\ \psi \tag{C.11}$$

hence, (C.1) becomes

$$\delta S = i\int \mathrm{d}^2x\ \bar{\psi}\gamma^\mu\left(A^v_\mu + A^p_\mu\gamma^3\right)\ \psi \tag{C.12}$$

according to which,

$$\partial_\mu J^\mu_v = \partial\tilde{J}^\mu_v = 0 \tag{C.13}$$
$$\partial_\mu J^\mu_p = \partial_\mu\tilde{J}^\mu_p = -\frac{\epsilon^{\mu\nu}}{2\pi}\ F^v_{\mu\nu}\ . \tag{C.14}$$

As a result, the two currents are not both conserved. Nevertheless one can try to add other terms, proportional to $A^v_\mu A^p_\mu$, such as

$$\delta S = \int \mathrm{d}^2x\ \frac{\epsilon^{\mu\nu}}{\pi}\ A^v_\mu A^p_\nu \tag{C.15}$$

so that

$$\partial_\mu\tilde{J}^\mu_v = \partial_\mu\left(J^\mu_v - \frac{\epsilon^{\mu\nu}}{\pi}\ A^p_\mu\right) = -\frac{\epsilon^{\mu\nu}}{2\pi}\ F^p_{\mu\nu} \tag{C.16}$$
$$\partial_\mu\tilde{J}^\mu_p = \partial_\mu\left(J^\mu_p + \frac{\epsilon^{\mu\nu}}{\pi}\ A^v_\mu\right) = 0 \tag{C.17}$$

but now is J^μ_p conserved. Therefore, the theory has an 't Hooft anomaly. Additionally, (C.17) leads to the charge

$$\tilde{Q}_p = \int_\Sigma\left(\star J_p - \frac{1}{\pi}\ A^v\right) \tag{C.18}$$

which *appears* to be conserved. Nevertheless, recalling that the requirement of gauge fields going to zero at infinity at $\mathbb{R}^2$ leads to consider the spacetime to be topologically $\mathbb{S}^2$, the gauge fields configurations are non-trivial anymore. In fact, one can consider a principal bundle P with fibre $U(1) = \mathbb{S}^1$ with base space $\mathbb{S}^2$ [Nak03]. Hence, P represents the topological setting of the magnetic monopole. The northern and southern hemispheres can be two open covering of the base space $\{U_N, U_S\}$ whose intersection is just a strip: the equator. Furthermore, if ϕ_N and ϕ_S label the local trivializations of P, one can choose $g_{N/S} = \exp\left\{i\alpha_{N/S}\right\}$ as fibre coordinates, so that

$$\phi_N^{-1} = (p, e^{i\alpha_N}) \tag{C.19}$$
$$\phi_S^{-1} = (p, e^{i\alpha_S}) \tag{C.20}$$

while the transition functions $t_{NS}: \quad \mathbb{S}^1 \to U(1)$ can be chosen as

$$t_{NS} = e^{in\phi} \ , \quad n \in \mathbb{Z} \quad \text{number of magnetic flux unit through the base space.} \tag{C.21}$$

On this principal bundle, a family of connection is given by the Wu-Yang monopoles

$$A_N = \frac{n}{2}(1 - \cos\theta)\ \mathrm{d}\phi \quad 0 \leq \theta \leq \frac{\pi}{2} \tag{C.22}$$
$$A_S = -\frac{n}{2}(1 + \cos\theta)\ \mathrm{d}\phi \quad \frac{\pi}{2} \leq \theta \leq \pi \ , \tag{C.23}$$

that can be the background gauge field for A^v and as a consequence, the charge (C.18) has to be defined separately on the northern and southern hemisphere. Additionally, condition (C.21) leads to non-conservation

$$\tilde{Q}_{p\ N} = \tilde{Q}_{p\ S} - 2n \implies U(e^{i\theta}, \mathbb{S}^2) \triangleq e^{i\theta\tilde{Q}_p} \quad \text{as symmetry operator} \tag{C.24}$$

but U violates condition 3. 2.1.3.6, therefore the pseudo-vector $U(1)$ symmetry is explicitly broken by the background gauge field for the vector $U(1)$ symmetry.

C. 3 Anomalies and splittability

In this section the connection between anomalies and splittability will be emphasised. The first example will be the chiral anomaly in $1+3$-dimensional Minkowski space which in turn is also an example for implication 1.

C. 3.a Chiral anomaly in $1+3$-dimensional Minkowski space

Let us start with a $U(N)$-invariant lagrangian of massless left-handed Weyl fermions

$$L = i\sum_{i=i}^{N} \bar{\psi}_i \partial\!\!\!/ \left(\frac{1+\gamma^5}{2}\right) \psi_i \tag{C.25}$$

whose currents are

$$J_a^\mu = \sum_{ij} \bar{\psi}_i \left[\gamma^\mu \left(\frac{1+\gamma^5}{2}\right) \otimes (T)_{ij}\right] \psi_j \tag{C.26}$$

where $(T)_{ij}$ are the Lie algebra matrices of $U(N)$. Hence, introducing the background gauge fields A_μ^a, the current conservation equation is [HO19]

$$D_\mu J_a^\mu = \frac{D_{abc}}{24\pi^2}\epsilon^{\mu\rho\nu\sigma}\ \partial_\mu A_\rho^b \partial_\nu A_\sigma^c \quad \text{where} \quad D_{abc} \triangleq \frac{1}{2}\mathrm{Tr}\{\{T_a, T_b\}\ T_c\} \tag{C.27}$$

where the latter can be arranged such that, for every triple of distinct currents, only one of them has an anomalous contribution to its conservation equation from background gauge fields of the other two. Without background gauge fields, the two currents of the theory are

$$J_v^\mu = \bar{\psi}\gamma^\mu\ \psi \tag{C.28}$$
$$J_p^\mu = \bar{\psi}\gamma^\mu\gamma^5\ \psi \tag{C.29}$$

which in these hypothesis are clearly conserved. Turning to background gauge fields, in order to make these dynamical, we should preserve the vectorial current at the expanse of the pseudo-vectorial one ('t Hooft anomaly). This leads to the famous ABJ anomaly

$$\partial_\mu J_p^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{16\pi^2}\ F^v_{\mu\nu}\ F^v_{\rho\sigma} \qquad \Longleftrightarrow \qquad \mathrm{d}\star J_p = \frac{1}{4\pi^2}\ F^v\wedge F^v\ , \tag{C.30}$$

which in turn leads to the improved charge

$$\tilde{Q}_p = \int_{\mathbb{R}^3}\ \left(\star J_p - \frac{1}{4\pi^2}\ A^v\wedge F^v\right)\ . \tag{C.31}$$

The main difference with the case in subsection C. 2.a lays in the fact that on $\mathbb{S}^4$, unlike the case of $\mathbb{S}^2$, there are no non-trivial configurations for background gauge fields. Therefore, even in the presence of an 't Hooft anomaly, chiral anomaly is still preserved at this stage. However, once A^v is made dynamical, things change. In fact, if one builds symmetry operators from (C.31), all conditions of definition 3. 2.1 of a global symmetry are preserved, except condition 3.5. This is because, in addition to Wilson lines built from A^v and the field strength F^v, there are the so-called 't Hooft loops that must be preserved. If $C = \partial D$ represents a contractible curve, an 't Hoopf loop is represented as [HRR15]-[HO19]-[HRR17]

$$T_n(C) \triangleq e^{\frac{i\ 2\pi n}{q^2}\ \int_D\ \star F} \tag{C.32}$$

which is nothing else than a narrow tube around the closed curve C. So, how to study the action of the charge (C.31) on 't Hooft lines? Following [HO21] and [Gai+15], symmetries can equivalently be defined as path integral insertions instead of operators on a given Hilbert space. This insertion can be assembled by using the operators $U(g,\Sigma)$ to surround whatever the surface encloses, in this case the (C.32). So, we are in $\mathbb{S}^4$ and we want to assemble this kind of insertion which has to be realised without the tube surrounding the 't Hooft line. This means that we have to remove the tube $B^3\times\mathbb{S}^1$, obtaining spaces with topology $\mathbb{S}^2\times B^2$ that can be glued together along their boundary $\mathbb{S}^2\times\mathbb{S}^1$. However, from $\mathbb{S}^4$ we have arrived to $\mathbb{S}^2$, somehow, which allows non-trivial $U(1)$ bundles. Therefore, $\mathbb{S}^2\times B^2$ has to be divided in northern and southern regions with topology $B^2\times B^2$ and the connections are that of the Wu-Yang monopoles, which differ by $A^v_N = A^v_S + n\mathrm{d}\phi$, with ϕ the angular coordinate on $\mathbb{S}^1$. Hence, the difference in charge from above to below has a non local factor and thus the symmetry transformed operator includes it

$$e^{i\frac{nN_f\ \theta}{4\pi^2}}\ \int_{\mathbb{S}^1\times B^2}\ \mathrm{d}\phi\wedge F^v\ , \tag{C.33}$$

so the chiral symmetry is preserved only by the $\mathbb{Z}_{N_f}$ subgroup because if $\frac{nN_f\ \theta}{2\pi}$ is an integer, (C.33) becomes a Wilson loop on C. Nevertheless, this subgroup is not splittable, precisely the symmetry is not splittable on $\mathbb{S}^2\times\mathbb{S}^1$ by the fact that (C.31) is not gauge invariant but the following is

$$\tilde{Q}_p(R)\mapsto\tilde{Q}_p(R) - \frac{N_f}{4\pi^2}\ \int_{\partial R}\ \lambda^v\ F^v\ . \tag{C.34}$$

Therefore, (3.19) is exactly as (C.34) with $\star F$ replaced by F and the situation is exactly as for the $\mathbb{R} \times \mathbb{R}$ theory but the problem is that now the unbreakable lines are not Wilson lines but 't Hooft lines instead.

Appendix D

Convex charge conjectures

Introduction

This appendix aims to provide the reader a deeper knowledge about the recent formulation of the Weak Gravity conjecture given in [AP21]. A mathematical description is thus introduced following the recent work [Ant+21a].

D. 1 Mathematical preliminaries

Conjecture as stated in chapter 4 in the charge convexity-formulation has a precise mathematical prescription. Hence, for a full understanding of the latter, let us introduce some notation and then several formulations of the same conjecture.
First of all, let us assume that our conformal field theory is invariant under a continuous internal global symmetry group G. Now, let us denote with V the real vector space representing the charge space of G, namely it is spanned by all possible fundamental weights w of the symmetry group. This thus means that V is composed by any possible weight associated to an irreducible representation of G.
It can be demonstrated that it is always possible to associate to any weight w a multiplet of operators transforming under an irreducible representation of G, such that the latter contains the weight w as well as the lowest possible scaling dimensions among all multiplets of this kind. Let us denote this representation with $r_L[w]$. On the contrary, let us label with $r_H[w]$ the irreducible representation having some highest weight w. Moreover, given an irreducible representation r, $w_h[r]$ labels its highest weight.
In general, one can visualise V as a lattice whose points are indeed the fundamental weights.

Definition D. 1.1

> *The rational direction is defined as that ray in the charge space V connecting the origin with another lattice point.*

Therefore, if d_0 represents the distance from the origin to the first fundamental weight found on the direction of a given rational direction, any other weight along the latter can be said to sit at a distance d from the origin such that $d = nd_0$; n is called its distance index.

Furthermore, taking in consideration a reducible representation r_R of G such that $r_R = \oplus_i\, m_i r_i$, namely such that it can be decomposed into a direct sum of irreducible representations r_i of G, let us introduce

$$\bar{\Delta}(r_R; w) \triangleq \min\{\Delta(r_i)/w \text{ belongs to } r_i\} \ . \tag{D.1}$$

In order to understand the word *convexity* in the statement of the conjecture, it is important to stress some mathematical definitions.

Definition D. 1.2

A function $f: \mathbb{R} \to \mathbb{R}$ is called superadditive if for $x, y \in \mathbb{R}$

$$f(x+y) \geq f(x) + f(y) \tag{D.2}$$

is satisfied.

Additionally,

Definition D. 1.3

A continuous function f is said to be convex if for all x, y belonging to a convex subset in $\mathbb{R}$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{D.3}$$

is valid, with $\lambda \in [0,1]$.

Definitions D. 1.2 and D. 1.3 are related through the Petrovic theorem. Moreover, the latter definition implies the former, while the contrary is not always true.

D. 1.a The conjectures

We are ready to properly enunciate the convex charge conjecture in different formulations. So, the conjecture as stated in the text in chapter 4 is properly formulated as follows.

Conjecture 1.1: Convex Charge Conjecture

When considering any CFT with a continuous internal global symmetry group G, chosen any rational direction, there exists a weight w_0 on the latter such that

$$\tilde{\Delta}(Q_1 + Q_2) \geq \tilde{\Delta}(Q_1) + \tilde{\Delta}(Q_2) \ , \tag{D.4}$$

where $\tilde{\Delta} \triangleq \Delta\Big(r_H\Big[Qw_h[r_L[w_0]]\Big]\Big) \in \mathbb{R}$ and Q_1, Q_2 are integers such that $Q_1, Q_2 \geq 0$.

This can be formulated in terms of OPE properties (see Appendix A) as follows.

Conjecture 1.2: Convex Charge Conjecture (OPE)

When considering any CFT with a continuous internal global symmetry group G, chosen any rational direction, there exists a weight w_0 on the latter such that

$$\bar{\Delta}(Q_1 + Q_2) \geq \bar{\Delta}(Q_1) + \bar{\Delta}(Q_2) \ , \tag{D.5}$$

where $\bar{\Delta} \triangleq \bar{\Delta}\Big(\mathrm{Sym}^Q(r_L[w_0]; Qw_0\Big) \in \mathbb{R}$ and Q_1, Q_2 are integers such that $Q_1, Q_2 \geq 0$.

Additionally, when formulated in terms of weights, the statement of the conjecture is the following.

CONJECTURE 1.3: CONVEX CHARGE CONJECTURE (WEIGHT)

When considering any CFT with a continuous internal global symmetry group G, chosen any rational direction, there exists a weight w_0 on the latter such that

$$\hat{\Delta}(Q_1 + Q_2) \geq \hat{\Delta}(Q_1) + \hat{\Delta}(Q_2) \ , \tag{D.6}$$

where $\hat{\Delta} \triangleq \Delta\Big(r_L[Qw_0]\Big) \in \mathbb{R}$ and Q_1, Q_2 are integers such that $Q_1, Q_2 \geq 0$.

Connection between these three formulations can be appreciated in figure D.1

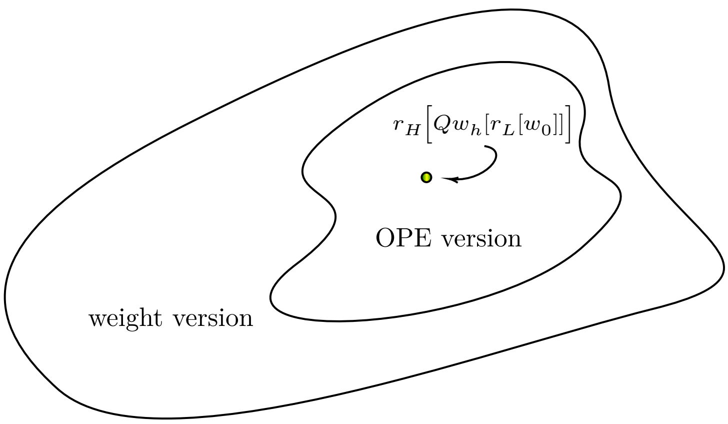


Figure D.1. Visual sketch of the statements 1.1, 1.2 and 1.3. When the symmetry group G is abelian, the three regions coincide.

Appendix E

U(1) problem, Θ-angle and the axion

Contents: **E. 1 Symmetries of QCD lagrangian.** **E. 2 True vacuum of QCD.** E. 2.a The ϑ-angle – E. 2.b Peccei-Quinn symmetry and the axion. **E. 3 Axion in inflationary models.** E. 3.a Axion from extra dimensions.

Introduction

This appendix aims to inform the reader about the θ-angle and all the story developed (which actually is still developing) around it. Therefore, the main insights will be provided and summarised here and the axion will be introduced at the end.

E. 1 Symmetries of QCD lagrangian

First of all let us recap the main properties of the QCD lagrangian. Starting from the work of Gell-Mann and Ne'eman in 1961, it was clear that hadronic particles group together in octects and decuplets according to $SU(3)$ representations, with same value of spin-parity [GM61]. Nevertheless, it soon appeared that the isospin symmetry $SU(2)$ and the latter $SU(3)$ symmetry were not exact. However it was also suggestive that the model was not completely wrong. This led Gell-Mann and Zweig in 1964 to propose the existence of elementary particles, fermions having spin 1/2, characterised by a $SU(3)$ global symmetry by the fact that they were supposed to show up in three different *flavours*: up, down and strange. Therefore, by direct sum of irreducible representation of $SU(3)$ it was possible to explain the hadronic spectrum, both mesonic and barionic [DGH14]. If on the one hand this was enlightening, on the other it soon ran into paradoxes regarding charge conservation, spin-statistics and also puzzling about the kind of hadrons that in nature we actually are led to observe. It was first Greenberg and then Han and Nambu that came up with a straightforward solution [PS95]: an additional degree of freedom, the colour. Hence, this set the tone for the development of the Yang-Mills theories: strong interactions are described by non-abelian $SU(3)$ gauge theory of colour with the additional global $SU(3)$ flavour symmetry.
Needless to say, the overall theory is relativistic-invariant and as a result, in order to describe a fermion belonging to the whole Lorentz group, one has to consider the Dirac representation,

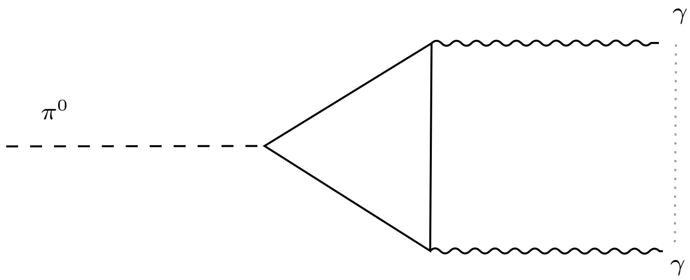
given by the direct sum of of the two Weyl representations, the left-handed $(\frac{1}{2}, 0)$ and the right-handed one $(0, \frac{1}{2})$. By the fact that the two Weyl fermions, ψ_L and ψ_R, belong to two different representations they will independently transform under two different global transformations, $U(1)_L$ and $U(1)_R$. However, if α_L and α_R are the two parameters of the transformation, it is possible to write $U(1)_L \times U(1)_R = U(1)_V \times U(1)_A$ due to the γ_5 matrix

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \alpha_{V/A} \triangleq \frac{\alpha_L \pm \alpha_R}{2} \implies \begin{cases} \psi_D & \xrightarrow{U(1)_V} \; e^{i\alpha_V}\psi_D \\ \psi & \xrightarrow{U(1)_A} \; e^{i\alpha_A\gamma_5}\psi \end{cases} . \tag{E.1}$$

The two currents related to (E.1) are $V^\mu = \bar{\psi}\gamma^\mu\psi$ and $A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$, called the vector and axial currents, and are classically preserved, namely the QDC massless lagrangian is invariant under the group

$$U_L(N_f) \times U_R(N_f) \approx SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A \ . \tag{E.2}$$

Nevertheless, when performing the path integral, namely when going to the quantum level, a classical-preserved symmetry could not be preserved anymore and it is said that an anomaly appears. This is actually what happens for the axial symmetry $U(1)_A$. To be more precise, it was studied the process $\pi^0 \to \gamma\gamma$



according to which new quantum contributes emerge and the Noether equation reads as

$$\partial_\mu A^\mu = 2N_f Q + 2\sum_{i=1}^{N_f} m_i \bar{\psi}_i\gamma_5\psi_i \quad \text{with} \quad Q = \frac{g^2}{64\pi^2}\, F^a_{\mu\nu}\, \epsilon^{\mu\nu\rho\sigma}\, F^a_{\rho\sigma} \ . \tag{E.3}$$

Therefore, by the fact that the axial symmetry is broken, according to the Goldstone theorem there should be a massless particle, the Goldstone boson of the $U(1)_A$ symmetry. Where and what is this particle? The question was addressed as the *U(1) problem* by 't Hooft [Hoo86].

E. 2 True vacuum of QCD

If one performs a chiral transformation, $\psi \mapsto \exp\{i\alpha\gamma_5\}\psi$, it seems that there are no *apparent* problems, in fact in the path integral formulation

$$\delta S = \alpha \int \mathrm{d}^4x \ \partial_\mu A^\mu = \alpha\frac{g^2 \ N_f}{32\pi^2}\int \mathrm{d}^4x \, F^{\mu\nu}_a \ \tilde{F}_{a \ \mu\nu} = \alpha\frac{g^2 \ N_f}{32\pi^2}\int \mathrm{d}^4x \ \partial_\mu K^\mu \tag{E.4}$$

the chiral anomaly can be written in terms of boundary terms. For this reason it did not arouse interest. Nevertheless, 't Hooft soon realised that the QCD vacuum state has a more complicated structure responsible for the fact that the $U(1)_A$ is not a symmetry on a quantum level.

It is important to stress that every gauge field is said to carry a *topological charge*, the so-called *winding number* n (or, equivalently, the QCD *instanton* number) [DGH14]:

$$n \triangleq i\,\frac{g^2}{32\pi^2}\,\frac{4}{3}\int \mathrm{d}^3x\ \mathrm{Tr}\left\{\epsilon_{ijk}\ A_n^i A_n^j A_n^k\right\} \Longrightarrow \frac{g^2\ N_f}{32\pi^2}\int \mathrm{d}^4x\ \partial_\mu K^\mu = \frac{g^2\ N_f}{32\pi^2}\int \mathrm{d}^3x\ K^\mu\Big|_{t=-\infty}^{t=+\infty} = n_+ - n_- \ . \tag{E.5}$$

Therefore, the vacuum state can be visualised as a collection of gluon fields grouped together in equivalence classes labelled by winding numbers. If and only if the difference between asymptotic configurations of winding numbers is such that (E.5) is null, $U(1)_A$ is a symmetry: 't Hooft showed that this is not possible.

E. 2.a The ϑ-angle

Recalling transformation properties of gauge fields, see Appendix C, for the vacuum state to be gauge invariant it is necessary that it can be written as a coherent superposition of all topological classes, namely as a Block function. Hence, the true vacuum state of QCD is said θ-vacuum and it is invariant up to a global phase

$$|\theta\rangle = \sum_n\ e^{-in\theta}\,|n\rangle \implies U\,|\theta\rangle = e^{i\theta}\,|n\rangle \quad \text{with U generator of a gauge transformation.} \tag{E.6}$$

If X is an arbitrary operator, its expectation value on $|\theta\rangle$ is

$$\begin{aligned}\langle 0|X|0\rangle &= \int\ [\mathcal{D}A_\mu]\ [\mathcal{D}\psi]\ [\mathcal{D}\bar{\psi}]\ Xe^{iS_{\mathrm{QCD}}} = \sum_{n_+,n_-}\ \langle n_+|X|n_-\rangle \implies \\ \langle\theta|X|\theta\rangle &= \sum_{n_+,n_-}\ e^{i(n_+-n_-)\theta}\,\langle n_+|X|n_-\rangle \\ &\implies \int\ [\mathcal{D}A_\mu]\ [\mathcal{D}\psi]\ [\mathcal{D}\bar{\psi}]\ Xe^{iS_{\mathrm{QCD}}+i\frac{g^2}{32\pi^2}\int \mathrm{d}^4x\ F_a^{\mu\nu}\ \tilde{F}_{a\ \mu\nu}}\end{aligned} \tag{E.7}$$

so that the QCD lagrangian is modified including a new term, the so-called θ-*term*

$$L_{\mathrm{QCD}} = L_{\mathrm{QCD}}^{\theta=0} + \theta\frac{g^2}{32\pi^2}\int\ \mathrm{d}^4x\ F_a^{\mu\nu}\ \tilde{F}_{a\ \mu\nu}\ . \tag{E.8}$$

Additionally, in the original paper [Hoo86] by 't Hooft, it was also stressed out a relation between the last term of (E.8) with the corresponding EFT formulation. In fact, introducing the complex meson field matrices with the quantum numbers of the quark-anti-quark composite operator $\bar{q}_{Rj}q_{Li} = \phi_{ji}$, he claimed that

$$\Im(e^{i\theta}\det\phi) \approx\ F_a^{\mu\nu}\ \tilde{F}_{a\ \mu\nu} \iff i\frac{128\pi^2}{N_f\ g^2}\,\Im(S\det\phi) = F\tilde{F} \tag{E.9}$$

where S is the so-called *schizon field*. The identification (E.9) leads to anomalous Ward identities (see figure E.1) by postulating that $\left[Q_{5,\mathrm{sym}}, S\right] = -2N_f S$, with $Q_{5,\mathrm{sym}} \triangleq \int \tilde{A}^0\ \mathrm{d}^3x$, with $\tilde{A}^\mu \triangleq A^\mu - 2N_f K^\mu$, $\partial_\mu\tilde{A}^\mu = 0$.

It was demonstrated by Jackiw and Rebbi that a chiral rotation changes the θ-vacuum. In fact, let Ω_n be the gauge matrices associated to the gauge transformation operator U. They can be obtained by compounding, namely $\Omega_n = [\Omega_1]n$, so that on the vacuum state one has

$$\Omega_1\,|n\rangle = |n+1\rangle\ . \tag{E.10}$$

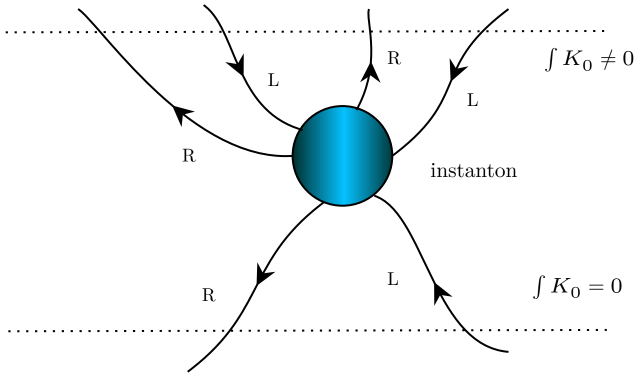


Figure E.1. Illustration of effective instanton action and its charge symmetry properties.

Hence, considering the time independent chiral charge Q_A associated to (E.3), under a gauge transformation changing the n-vacua it shifts as

$$\Omega_1 Q_A \Omega_1^{-1} = Q_A + N_f \tag{E.11}$$

so that

$$\Omega_1 e^{i\frac{\alpha}{N_f}\ Q_A} |\theta\rangle = \Omega_1 e^{i\frac{\alpha}{N_f}\ Q_A}\ \Omega_1^{-1}\Omega_1 |\theta\rangle = e^{i(\alpha+\theta)}\ e^{i\frac{\alpha}{N_f}\ Q_A} |\theta\rangle \tag{E.12}$$

giving

$$e^{i\frac{\alpha}{N_f}\ Q_A} |\theta\rangle = |\theta+\alpha\rangle\ \ . \tag{E.13}$$

However, the last term on the right hand side of (E.8) violets the CP symmetry. Therefore, including in the lagrangian also the Yukawa sector, there is a two level of CP violation: a weak violation coming from the Yukawa interactions and a strong violation coming from (E.8). Nevertheless, the overall framework needs to give an explanation to the $U(1)$ problem and in the $SU(3)$ theory, it is the η^0 meson the chosen particle to be the pseudo-Goldstone boson. Hence, in addition to the previous violations, it should be included also the *vev* of the η^0 meson when $U(1)_A$ is broken.
The very next question is: how do we see the degrees of freedom associated to these CP violation in the QCD lagrangian? This can be seen through a redefinition of fields in the lagrangian. Firstly, one introduces the pseudo-Goldstone fields associated to the chiral symmetry and axial symmetry breaking [KRB08], namely

$$\theta_0 = \frac{\eta^0(x^\mu)}{f_0} \quad \text{and} \quad \vec{\theta}_\pi = \frac{\vec{\pi}(x^\mu)}{f_\pi} \quad \text{with } f_0 \text{ and } f_\pi \text{ energy scales related to } \Lambda_{\mathrm{QCD}}. \tag{E.14}$$

Now, redefining the quarks

$$q_R = e^{i\frac{(\theta_0+\vec{\theta}_\pi\cdot\sigma)}{2}}\tilde{q}_R \quad \text{and} \quad q_L = e^{-i\frac{(\theta_0+\vec{\theta}_\pi\cdot\sigma)}{2}}\tilde{q}_L \quad \text{namely} \quad q = e^{i\gamma_5\frac{(\theta_0+\vec{\theta}_\pi\cdot\sigma)}{2}}\tilde{q} \tag{E.15}$$

the quark mass contribution to the lagrangian becomes

$$\bar{q}_L m_q q_r + h.c. \mapsto -m_u v^3 \cos(\theta_0+\theta_3) - m_d v^3 \cos(\theta_0-\theta_3) \tag{E.16}$$

where the factor $-v^3$ originates from the formation of quark condensates $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle$. Including the effect of the *vev* of the η^0 meson, the previous contribution reads as

$$V \sim -m_u v^3 \cos(\theta_0+\theta_3) - m_d v^3 \cos(\theta_0-\theta_3) - \Lambda^4 \cos(2\theta_0-\theta)\ , \tag{E.17}$$

with Λ an energy scale related to the η^0 mass. The cosine form is chosen in order to preserve the periodicity in $2\theta_0 - \theta$.

E. 2.b Peccei-Quinn symmetry and the axion

Let us go back to the $U(1)$ problem. A solution was given by Peccei and Quinn [PQ77b]-[PQ77a] by adding to the Standard Model a global chiral symmetry $U(1)_{PQ}$ which is spontaneously broken giving rise to a pseudo-Goldstone boson, a pseudo-scalar particle, a dynamical field that responds to the QCD potential adjusting its *vev* to cancel θ. This dynamical field is the so-called *axion*, $a(x)$. Under the $U(1)_{PQ}$ the axion transforms as $a(x) \mapsto a(x) + \alpha f_a$ while the anomaly current associated to the symmetry is

$$\partial_\mu A^\mu_{PQ} = \xi \frac{g^2}{32\pi^2}\ F^{\mu\nu}_a \tilde{F}^a_{\mu\nu} \tag{E.18}$$

so that one can include in the lagrangian an effective potential for the axion (actually it is the coupling of the axion with the QCD instantons) together with a kinetic term

$$L = L_{\rm SM} + \theta \frac{g^2}{32\pi^2}\ F^{\mu\nu}_a \tilde{F}^a_{\mu\nu} - \frac{1}{2}\partial_\mu a \partial^\mu a + \xi \frac{a}{f_a}\frac{g^2}{32\pi^2}\ F^{\mu\nu}_a \tilde{F}^a_{\mu\nu} + L_{\rm int}\left[\frac{\partial^\mu a}{f_a};\psi\right]. \tag{E.19}$$

The Peccei-Quinn solution for the axion is easily found by minimazing the effective potential with respect to $\langle a\rangle$, ending up with [KRB08]

$$\langle a\rangle = -\frac{f_a}{\xi}\ \theta \tag{E.20}$$

which in turn lead us to express the effective vacuum angle in a periodic potential for the axion field

$$V_{\rm eff} \sim \cos\left(\theta + \xi\frac{\langle a\rangle}{f_a}\right) \tag{E.21}$$

or equivalently, following the notation in the previous subsection, defining $\theta_a(x) = a(x)/f_a$, at low energies the axion appears in the instanton contribution to the potential

$$V \sim -m_u v^3 \cos\left(\theta_0 + \theta_3\right) - m_d v^3 \cos\left(\theta_0 - \theta_3\right) - \Lambda^4 \cos\left(2\theta_0 + \theta_a - \theta\right)\ , \tag{E.22}$$

so that the minimum of the potential is given by $\partial_{\theta_0} V = \partial_{\theta_3} V = \partial_{\theta_a} V = 0$, namely:

$$\theta_0 + \theta_3 = 0 \tag{E.23}$$

$$\theta_0 - \theta_3 = 0 \tag{E.24}$$

$$2\theta_0 + \theta_a - \theta = 0 \tag{E.25}$$

so for example, $\theta_0 = \theta_3 = 0$ and $\theta_a = \theta$. In conclusion, the axion *vev* is adjusted by the QCD potential to cancel any possible value of θ, so any effect of CP violation [Redpt].
Besides QCD axions, in String Theory [SW06] there exist other type of axions that violate the $U(1)_{PQ}$ symmetry. This kind of axion still couples to instantons with a mass scale M and the instanton will generate a potential whose general form is

$$V(a) \sim -M^4\ e^{-S_{\rm inst}}\ e^{i(a+\alpha)} \approx -2M^4\ e^{-S_{\rm inst}}\ \cos\left(a+\alpha\right) \tag{E.26}$$

where $S_{\rm inst}$ is the action describing the instanton while α is a phase.

E. 3 Axion in inflationary models

The axion ensured interest in Cosmology, in particular in the theoretically attractive idea of inflation. So firstly, let us briefly introduce the basic concept of inflation [Vauto]. The latter is a period of accelerated expansion of the universe and it was first introduced in order to solve some general problems related to the Big Bang hypothesis, such as the understanding of why the universe is homogeneous and isotropic nowadays. For the sake of simplicity, we now shift to natural units $8\pi G = M_P^{-2} = 1 = \hbar = c$. We consider the Einstein-Hilbert action in the presence of a scalar field ϕ

$$S = \mathrm{d}^4x\ \sqrt{-g}\ \left(\frac{1}{2}R + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \tag{E.27}$$

from which, according to variational principle, varying with respect to the metric one finds the equation of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}(\phi)\ . \tag{E.28}$$

We now consider the cosmological model according to which the universe is spatially homogeneous and isotropic, namely the metric is the Friedmann-Robertson-Walker (FRW) one

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2\right)\right] \tag{E.29}$$

where $a(t)$ is the scale function and k determines the curvature of the universe: positive means a closed universe, negative an open one while a null value means that the universe is spatially flat. We will consider the latter case. In literature the FRW equations are known [Car19], putting $H \triangleq \frac{\dot{a}}{a}$ the Hubble parameter, in our case they read as

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{k}{a^2} \tag{E.30}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 + \frac{k}{a^2} \tag{E.31}$$

while the equation of motion (with the inclusion of a friction term) for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0\ . \tag{E.32}$$

The equation describing the acceleration and expansion of the universe is the following

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2\left(1 + \underbrace{\frac{\dot{H}}{H^2}}_{\triangleq -\epsilon_H}\right) = H^2(1-\epsilon_H)\quad \text{acceleration}\qquad \Longleftrightarrow\quad \frac{\ddot{a}}{a} > 0 \iff 0 < \epsilon_H < 1. \tag{E.33}$$

However, what is the role of the scalar field in the inflation scenario? We will consider a particular approximation: the case of flat potential $V(\phi)$. This means that we can neglect the acceleration $\ddot{\phi}$ so that (E.32) and thus (E.30) become

$$\dot{\phi} = -\frac{1}{3H}\partial_\phi V \approx 0 \tag{E.34}$$

$$H^2 = \frac{V}{3} \approx \text{constant} \implies \epsilon_H \approx 0\ , \tag{E.35}$$

it is useful the introduction of the *slow-roll parameter* η_H, in terms of which the flat potential condition reads as $\eta_H \ll 1$

$$\eta_H \triangleq -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{E.36}$$

and also the definition of the slow-roll parameters in terms of the potential V

$$\epsilon_V \triangleq \frac{1}{2}\left(\frac{\partial_\phi V}{V}\right)^2 \quad \text{and} \quad \eta_V \triangleq \frac{\partial_\phi^2 V}{V} \ . \tag{E.37}$$

Now, there are a number of models of inflation. We will consider the so-called *natural inflation*. The latter assumes that the scalar field, the inflaton, is the axion $a(x)$, based on the following lagrangian[1]

$$L = \frac{f_a^2}{2}\,(\partial a)^2 - V_0(1-\cos a) \tag{E.38}$$

Turning back to SI units, (E.37) becomes

$$\epsilon = \frac{M_P^2}{2}\left(\frac{\partial_a V}{V}\right)^2 \sim \frac{M_P^2}{f_a^2} \quad \text{and} \quad \eta = M_P^2\frac{\partial_a^2 V}{V} \sim \frac{M_P^2}{f_a^2} \tag{E.39}$$

and so, the condition for acceleration $\epsilon \ll 1$ and $\eta \ll 1$, gives $M_P \ll f_a$. This means that the spontaneous breaking scale is above M_P, thus the effective field theory description is invalidated. However this problem can be overcome through a $5D$ model whose extra dimension is compactified on a circle of radius R [AH+03].

E. 3.a Axion from extra dimensions

Consider the case of a one-form gauge symmetry [HRR16]; it is possible to define an axion-like field as the Wilson line operator associated to the compact component of the field, namely as

$$a(x) = \oint_0^R \mathrm{d}^5x \ \ A_5(x, x^5) \ . \tag{E.40}$$

The gauge transformations of the field are indeed discrete shift symmetries which imply that the axion is compact $a \mapsto a + 2\pi n$ with $n \in \mathbb{Z}$. Therefore, starting form a $5D$ action

$$\int \mathrm{d}^5x \left[\frac{1}{2}M_5^3 R - \frac{1}{4g_5^2}\ F_{\mu\nu}^2\right] \implies M_4^2 \equiv 2\pi R M_5^3 \quad \text{and} \quad \frac{1}{g_4^2} \equiv \frac{2\pi R}{g_5^2} \tag{E.41}$$

so that $\frac{f_a^2}{2}2\pi R \equiv \frac{1}{4g_5^2}$, at low energies one has

$$L_{\text{axion}} = \frac{f_a^2}{2}\partial_\mu a\partial^\mu a - V(a) + \ldots \quad \text{with} \quad f_a^2 = \frac{1}{2\pi R g_5^2} = \left(\frac{1}{2\pi R g_4}\right)^2 \ , \tag{E.42}$$

where the potential is at one-loop [HIL98], [ABQ01], [GIQ02]

$$V(a) = -\frac{1}{R^4}\sum_I (-1)^{F_I}\ \frac{3}{64\pi^6}\sum_{n=1}^{\infty}\frac{\cos{(nQa)}}{n^5} \ . \tag{E.43}$$

with Q the charge of the massless fields coupled with A_5. Therefore, the slow-roll condition can be easily satisfied for sufficiently small values of g_4 by the fact that now it reads as $2\pi g_4 M_4 R \ll 1$.

[1] To be precise, the inflaton is the pseudo-Goldstone boson described by the lagrangian in the text where f_a is the scale of spontaneous breaking of axial symmetry but it is also present a potential which breaks explicitly the symmetry.

Appendix F

$\mathcal{N} = 2$ supersymmetry and supergravity

Introduction

This appendix provides an introduction to the basic concepts of supersymmetry and supergravity. Hence, only those aspects needed to understand the WGC formulation for scalar fields will be exposed. The overall appendix is mainly based on [Erb15].

F. 1 Hermitian and Kahler manifolds

For what comes next, it is useful the introduction of some definitions. This because in this theory, the vector scalars live on a special Kahler manifold and the following definitions help to define an introductory, but complete, framework for the understanding of the WGC for scalars.
So, consider a manifold $(\mathcal{M}, g)$ with complex dimension m with metric

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{F.1}$$

with a torsionless connection. Firstly

Definition F. 1.1

The manifold $\mathcal{M}$ is almost complex if it admits an almost complex structure ${J_a}^b(x)$ such that

$${J_a}^c\ {J_c}^b = -{\delta_a}^b \tag{F.2}$$

so that the eigenvalues of J are $\pm i$. If J is globally defined, $(\mathcal{M}, J)$ is called a complex manifold.

Definition F. 1.2

If J is compatible with the metric then $(\mathcal{M}, J)$ is a hermitian manifold, namely if

$${J_a}^c\ g_{cd}\ {J_b}^d = g_{ab} \quad \Longleftrightarrow \quad JgJ^T = g\ , \tag{F.3}$$

as a result $J_{ab} = -J_{ba}$.

Starting from J one can define the so-called *fundamental two-form of $\mathcal{M}$*, Ω, which is real and symplectic, as

$$\Omega = -J_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b \ . \tag{F.4}$$

Definition F. 1.3

Given a hermitian manifold, the latter is said a Kahler manifold if the fundamental two-form is closed

$$\mathrm{d}\Omega = 0 \ , \tag{F.5}$$

equivalently, if J is covariantly constant.

Additionally, if one locally expresses the x^a in complex coordinates $x^a = (\tau^i, \bar{\tau}^{\bar{i}})$, the metric becomes

$$\mathrm{d}s^2 = 2g_{i\bar{j}}\, \mathrm{d}\tau^i \mathrm{d}\bar{\tau}^{\bar{j}} \ , \tag{F.6}$$

moreover, in these coordinates if J is diagonal for a hermitian manifold, then the metric tensor can be expressed as

$$J_a{}^b = i \begin{pmatrix} \delta_i{}^j & 0 \\ 0 & -\delta_{\bar{i}}{}^{\bar{j}} \end{pmatrix} \implies g_{ab} = \begin{pmatrix} 0 & g_{i\bar{j}} \\ g_{j\bar{i}} & 0 \end{pmatrix} \implies \Omega = 2i\ g_{i\bar{j}} \mathrm{d}\tau^i \wedge \mathrm{d}\bar{\tau}^{\bar{j}} \tag{F.7}$$

and it happens that the fundamental two-form is closed if and only if there exists a real function called the Kahler potential $K(\tau, \bar{\tau})$ such that $g_{i\bar{j}} = \partial_i \partial_{\bar{j}}\, K$.

F. 2 Supersymmetry algebra

Firstly, supersymmetry was born in order to answer the following question: is it possible to unify internal gauge symmetries with spacetime symmetries? The ones answering the first were Coleman and Mandula [CM67], proposing that the symmetry group should necessarily be a direct product between conformal group and internal group. Nevertheless, it came out to be not sufficient and for this reason Haag and others [HLS75] proposed the so-called *superconformal group*, namely they added anticommuting operators. The supergroup then contains an automorphism sub-group, the $\mathcal{R}$-symmetry group which acts both on the fermionic generators and as an internal symmetry [Erb15]. Now, supersymmetry is generated by some fermionic generators Q that combine together bosons with fermions, namely

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle \quad \text{and} \quad Q\,|\text{boson}\rangle = |\text{fermion}\rangle \ , \tag{F.8}$$

and, in general, such a theory is defined by the number $\mathcal{N}$ of fermionic generators. The consequence is that there are constraints on the theory itself. Let us consider the case of $\mathcal{N} = 2$. Precisely, considering the Poincaré algebra with P_μ and $J_{\mu\nu}$ the generators of translation and Lorentz transformations, denoting by Q_α the fermionic generator of supersymmetry and labelling with $\mathcal{R}^A$ the generator of the $U(2)_\mathcal{R}$ $\mathcal{R}$-symmetry, they satisfy the following algebra

$$\left[J_{\mu\nu}, P_\sigma\right] = \eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu \tag{F.9}$$

$$\left[J_{\mu\nu}, J_{\rho\sigma}\right] = \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - (\mu \leftrightarrow \nu) \tag{F.10}$$

$$\left\{Q_\alpha, \bar{Q}^\beta\right\} = -i\frac{1}{2}\delta_\alpha^{\ \beta}\ P_L\gamma_\mu\ P^\mu, \qquad \left\{Q^\alpha, \bar{Q}_\beta\right\} = -i\frac{1}{2}\delta^\alpha_{\ \beta}\ P_R\gamma_\mu\ P^\mu \tag{F.11}$$

$$\left[P_\mu, Q_\alpha\right] = 0, \qquad \left[P_\mu, Q^\alpha\right] = 0 \tag{F.12}$$

$$\left[J_{\mu\nu}, Q_\alpha\right] = -i\frac{1}{2}\gamma_{\mu\nu}\ Q_\alpha, \qquad \left[J_{\mu\nu}, Q^\alpha\right] = -i\frac{1}{2}\gamma_{\mu\nu}\ Q^\alpha \tag{F.13}$$

$$\left\{Q_\alpha, Q_\beta\right\} = -\frac{1}{2}\epsilon_{\alpha\beta}P_L\ Z, \qquad \left\{Q^\alpha, Q^\beta\right\} = -\frac{1}{2}\epsilon^{\alpha\beta}P_R\ Z \tag{F.14}$$

$$\left[\mathcal{R}^A, Q_\alpha\right] = \left(U^A\right)_\alpha^{\ \beta}\ Q_\beta, \qquad \left[\mathcal{R}^A, Q^\alpha\right] = \left(U^A\right)^\alpha_{\ \beta}\ Q^\beta \tag{F.15}$$

$$\left[T^a, T^b\right] = f^{ab}_{\ \ c}\ T^c \tag{F.16}$$

where U^A are the matrices of the $\mathcal{R}$-symmetry and the T^as the generators of the internal symmetry, while Z is the central charge. Here an interesting consideration comes: local supersymmetry includes General Relativity. That is because the anticommutators of the fermionic generators (which actually are Weyl spinors) close on the momentum and, as a result, it is not possible to make supersymmetry local without making local the Poincaré group. One talks about *supergravity.* The latter theory is composed by three multiplets:

1. gravity multiplet, which contains the metric and the vector field, *the graviphoton*

$$\left\{g_{\mu\nu}, \psi_{\alpha\ \mu}, \psi^\alpha_\mu, A^0_\mu\right\} \quad \text{where} \quad \alpha = 1, 2 \tag{F.17}$$

2. n_v vector multiplets, containing a vector field and a complex scalar field

$$\left\{A^i_\mu, \lambda^{\alpha\ i}, \lambda^{\bar{i}}_\alpha, \tau^i\right\} \quad \text{with} \quad \tau^i \in \mathbb{C} \quad \text{and} \quad i = 1, \dots, n_v \tag{F.18}$$

3. n_h hypermultiplets, containing four real scalar fields

$$\left\{\zeta^\mathcal{A}, \zeta_\mathcal{A}, q^u\right\} \quad \text{with} \quad q^u \in \mathbb{R} \quad \text{and} \quad u = 1, \dots, 4n_h;\ \mathcal{A} = 1, \dots, 2n_h \tag{F.19}$$

where $\psi_{\alpha\ \mu}$ are the *gravitini*, $\lambda^{\alpha\ i}$ are the *gaugini* while $\zeta^\mathcal{A}$ the *hyperini.*

F. 2.a Lagrangian

We now gather the gauge fields as follows

$$A^I = \left(A^0, A^i\right) \quad \text{with} \quad I = 0, \dots, n_v \tag{F.20}$$

and we want to write down a bosonic lagrangian. As already anticipated, the latter live on a special Kahler manifold, so that, using the previous section and the definition in [Erb15]-[Nak03], it suffices to know that for such a manifold a given section is actually a section of the tensor bundle $\mathcal{L} \otimes \mathcal{SV}$, where $\mathcal{L} \to \mathcal{M}$ is the line bundle while $\mathcal{SV} \to \mathcal{M}$ is the vector bundle. The component of such a section can be denoted by

$$v = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \tag{F.21}$$

where X^I are the so-called *homogeneous coordinates* and are such that $\tau^i = X^i/X^0$, so they provide a projective parametrization of the manifold.
We now define a general symplectic vector in these coordinates

$$V = \begin{pmatrix} V^I \\ V_I \end{pmatrix} \quad \Longleftrightarrow \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{F.22}$$

with the latter defining the scalar product

$$\langle V, W \rangle \triangleq V^t \Omega W = V^I W_I - W^I V_I = V^M \Omega_{MN} W^N, \quad M, N = 1, \ldots, 2(n_v + 1) \ . \tag{F.23}$$

The first issue concerns how to couple the gauge fields with the vector scalars. This is achieved through the *period matrix*, a simplectic matrix, such that $F_I = N_{IJ}\, X^J$, where $F^I = \mathrm{d}A^I$. It is called "the period" matrix because the geometric structure of the field-space is determined by the periods $\{X^I, F_I\}$, holomorphic functions of the scalar fields. So that

$$L = \frac{R}{2} + \frac{1}{4}\, \mathfrak{I}(\mathcal{N}_{IJ})\ F^I_{\mu\nu}\ F^{J\ \mu\nu} - \frac{1}{8}\mathfrak{R}(\mathcal{N}_{IJ}) \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\ F^I_{\mu\nu}\ F^J_{\ \rho\sigma} - g_{i\bar{j}} \partial_\mu \tau^i \partial^\mu \bar{\tau}^{\bar{j}} \tag{F.24}$$

where the imaginary and real part of the period matrix correspond to the gauge couplings and the θ-term, respectively. Additionally, $\mathfrak{R}$ and $\mathfrak{I}$ define the real, symplectic matrix M of dimension $2(n_v + 1)$ as

$$M = \begin{pmatrix} 1 & -\mathfrak{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{I} & 0 \\ 0 & \mathfrak{I}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\mathfrak{R} & 1 \end{pmatrix} \Longleftrightarrow M^T \Omega M = \Omega \ . \tag{F.25}$$

Furthermore, if q_I and p^I label arbitrary constants, let us also define

$$\mathcal{Q} \triangleq \begin{pmatrix} p^I \\ q_I \end{pmatrix} \quad \text{and} \quad \mathcal{Q}^2 \triangleq -\frac{1}{2} \mathcal{Q}^T M \mathcal{Q} \tag{F.26}$$

where the latter is closely related to the central charge. In fact, in this setting, the Kahler potential is $K = -\log\left[i\left(\bar{X}^I F_I - X^I \bar{F}_I\right)\right]$ and the central charge [CDF96] is

$$Z = e^{\frac{K}{2}} \left(q_I X^I - p^I F_I\right) \tag{F.27}$$

so that the following identity is valid [CDF96]

$$\mathcal{Q}^2 = |Z|^2 + g^{ij} D_i Z \bar{D}_j \bar{Z} \ , \tag{F.28}$$

where the covariant derivative is defined as in [Erb15] and [Pal17].

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