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Impact of Quantum Gravity on Particle Physics and Cosmology

Relatori:
prof. Francesco Sannino
Correlatore:
prof. Salvatore Capozziello

Candidato:
Pellegrino Piantadosi
Matr. $N94000575$

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Chapter 1

Introduction

This thesis work is the result of the research activity carried out by the group set up by my supervisor and focused on the study of *Quantum Gravity* (QG). The work is motivated by the observation that the various gravity quantization models are not directly testable given the energy scales inaccessible to current technological possibilities. The idea is therefore to use these theories to formulate principles which are assumed to have general validity even at lower energy scales and which therefore lend themselves to reproducing the effects of Quantum Gravity in the infrared regime. Such principles are known as *Swampland conjectures*. The effect of the conjectures, depending on whether they are satisfied or not, is to divide the set of low-energy theories into two disjoint classes. On the one hand there are the theories of *Swampland* in which they are not verified on the other hand there are the theories of the *Landscape* which instead are consistent with a quantum description of gravity in the ultraviolet. The resulting operating mode is called *Swampland approach* and has as its main purpose the study of the phenomenology of the QG. In particular, we are interested in obtaining results related to Particle Physics and Cosmology for which it is possible to have an experimental confirmation.

In order to develop the Swampland approach the thesis will be articulated through the following structure. Chapter 2 begins with a brief presentation of the concept of effective theories and highlights the way in which these are reconciled with fundamental theories such as QG ones. The importance of these theories for the purposes of the thesis lies in the fact that the Swampland conjectures refer precisely to effective descriptions of gravity. At that point the Swampland program is defined, explaining its philosophy and the current research context. The main conjectures on which the applications are based will be stated here.
Chapter 3 is devoted to a quick recall of the description of Quantum Gravity provided by the String Theory. The reason for this choice lies in the fact that in

literature the theory of Quantum Gravity used as a reference for the Swampland program is precisely String Theory. The proposed conjectures draw their inspiration from the properties of this theory in which indeed they find their validity. The formalism of the bosonic string theory is then presented concisely and the main results of the theory of interest for the purposes of the thesis are retraced. Particular attention will be paid to highlighting how String Theory naturally provides a consistent description of Quantum Gravity through unusual mathematical properties such as the presence of extra dimensions.
In order to reconcile this last aspect of the theory with the observations, Chapter 4 proposes to provide a detailed description of the *compactification mechanism* for extra dimensions. The treatment in this case will take a more general approach in which we will first discuss the usual *Kaluza Klein theory* and then deal with some its extensions less known in the literature. Only at the end, the compactification mechanism will be applied to the case of a bosonic string by investigating the physical effects that this has on the spectrum of the system. The importance of the compactification process is also linked to the fact that it provides the formal mechanism of construction of effective theories of gravity described by String Theory. It is therefore essential to fully understand the meaning of Swampland's conjectures and their possible implications.
The following chapters are devoted to an in-depth discussion of conjectures and above all to the study of their applications. In particular, Chapter 5 focuses on the *No Global Symmetry conjecture*. For it, in addition to the usual properties and motivations known in the literature, a topological reformulation will be given in terms of *cobordism classes*. The aim of this discussion is to make the link with other secondary conjectures more evident and to arrive at a characterization of the instabilities of the vacua associated with the effective theories. The idea is that the imposition of this instability can be applied in the context of Particle Physics and, specifically, to have theoretical predictions concerning neutrino masses.
Finally, Chapter 6 is entirely dedicated to the *Swampland Distance conjecture*. In this case the mathematics of the language of its statement will be explained first and the close link that this conjecture has with the compactification of the bosonic string and the duality of the theory will be made explicit. Subsequently, it will be seen how also in this case it is possible to formulate conjectures concerning the instability of de Sitter-type vacua. In this way, application possibilities concerning Cosmology are opened. In particular, the applications will concern *inflationary* and *dynamic dark energy models*. The Swampland approach shows itself capable of imposing limits on cosmological models in order to be consistent with QG by inspecting two different evolution phases of the universe.

As will be seen, the applications can be various and lend themselves to formulating theoretical predictions on physical observables. Due to the nature of the Swampland approach, predictions are generally expressed in terms of bounds which, however, when combined with known theoretical results derived independently from the QG or with experimental bounds, lend themselves to strongly constrain the real-world physics also at low energies. It seems somewhat astonishing that this approach effectively works and it is an indication that research in the field of Quantum Gravity is worth pursuing. In fact, the conclusions will discuss possible developments in view of the experiments planned for the near future concerning both Particle Physics and Cosmology.

Chapter 2

The Swampland approach

2.1 Effective Field Theories

Effective theories provide descriptive models of physics valid in a fixed range of macroscopic variables values which characterize the system to be described. Typically, in the field of physics, they are considered energy scales fixed by an ultraviolet cut-off Λ below the which physics is correctly described by the effective theory (Figure 2.1). Above the cut-off, a consistent description requires a substantial modification of the theory by including additional degrees of freedom.
On the basis of empirical observations, in fact, it can be said that depending on the energy scale there can be different dynamics and physical phenomena whose description requires different mechanisms reconstructed precisely through different effective theories. The basic idea of this approach is that, for the purposes of describing physics at a certain energy scale, physics at significantly different energies has negligible effects or in any case which are perturbatively treatable.
The construction of an effective theory therefore requires the isolation of the degrees of freedom really relevant to the scale of interest and the identification of exact or approximate symmetries enjoyed by the system that one is interested in describing. On the other hand, the reduction of the degrees of freedom can in itself give rise to the formation of new symmetries that are explicitly broken by the degrees of freedom not considered. The resulting theory can therefore be regarded, in a certain sense, as a perturbative approximation of more fundamental effective theories valid at larger energy scales. The advantage of using effective theories instead of fundamental ones lies in the fact that the effective description is clearly simpler and therefore it typically allows a more immediate understanding of physics as well as offering a considerable simplification in the calculations and therefore in obtaining of theoretical predictions.
From what has been said we understand the relevance that effective theories have

for theoretical and experimental physics. Through the identification of reference scales, a priori, every physical theory can be considered as an effective model of a more fundamental theory.
The mathematical implementation of an effective theory of interest for Particle Physics and Fundamental Interactions refers to the conceptual framework of quantum field theories that combine the need to give a description of microscopic physics compatible with the principles of Special Relativity. The result of this implementation goes by the name of Effective Field Theories (EFT) [1, 2, 3]. An EFT is therefore described by a Lagrangian density that can decompose into the sum of operators of increasing mass size

$$\mathcal{L}_{eff}(x) = \sum_i C_i \mathcal{O}_i(x) \; . \tag{2.1}$$

The idea is therefore to consider a series expansion of the Lagrangian density controlled by powers of a small expansion parameter E/Λ, with E characteristic energy scale of effective theory. On the other hand, from power counting analysis it is known that a d-dimensional theory is renormalizable, and therefore predictive in the ultraviolet, only if the Lagrangian has terms with operators of mass dimension

$$[\mathcal{O}_i] \equiv \mathcal{D} \leq d \; . \tag{2.2}$$

Consequently, the series expansion can thus be rewritten as

$$\mathcal{L}_{eff}(x) = \sum_{i,\mathcal{D}} \frac{c_i^{(\mathcal{D})} \mathcal{O}^{(\mathcal{D})}}{\Lambda^{\mathcal{D}-d}} \; . \tag{2.3}$$

In the case of a $(d = 4)$-dimensional theory, we have

$$\mathcal{L}_{eff}(x) = \mathcal{L}_{\mathcal{D}\leq 4} + \frac{\mathcal{L}_5}{\Lambda} + \frac{\mathcal{L}_6}{\Lambda^2} + ... \tag{2.4}$$

For the purposes of the effective description, only a number of finite terms of dimension $\mathcal{D} \geq d$ are considered and the Λ coefficient that appears in the expansion parameter will be the ultraviolet cut-off of the EFT. Higher-order operators lend themselves to describing the new physics with respect to the renormalizable theory but in the low-energy regime. The determination of the expansion coefficients can be done either by knowing the fundamental theory or, more commonly, in empirically way.

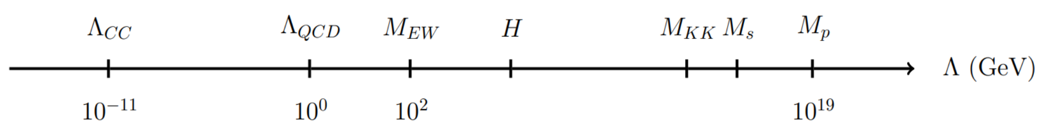

Figure 2.1: Main energy scales and cut-off for effective models of Physics [4]

2.2 The Swampland conjectures

Regarding the gravitational interaction, it is classically described by General Relativity (GR). On the other hand, the field theory constructed with the Einstein-Hilbert action is non-renormalizable. This has led to the hypothesis that General Relativity can provide an effective description of gravity valid at low energies and that it should therefore be replaced by a quantum description of gravity at the Planck scale. Starting from Einstein's theory it is possible to construct EFTs of matter weakly coupled to gravity that are internally self-consistent. The set of such theories is in principle very large (if not infinite) and therefore the problem arises of identifying the theories really suitable for describing the physics of the real world. A possible way to satisfy this need is to select the EFTs in light of the consistency with their possible extensions above their cut-off. In this context the Swampland approach [5, 4, 6, 7] intervenes. It aims to identify the EFTs consistent with an ultraviolet completion in a quantum gravity theory. EFTs compatible with a quantum description of gravity are said to be in the *Landscape*; EFTs that are incompatible are said to be in *Swampland*. The idea of the Swampland approach is to identify sufficient selection criteria to implement this classification and therefore to outline the Landscape. The proposed criteria will have more stringent effects as energy increases and indicate how an EFT should be modified above the cut-off in order to preserve its validity. This means that starting from a very large Landscape at low energies, there will be gradual restrictions as the scale increases until an ideal convergence in Quantum Gravity Theory is achieved at the Planck scale.

Note that, a priori, the Swampland program assumes a totally general approach in which the nature of the Quantum Gravity theory is not specified. For each theory of Quantum Gravity there can be a different Swampland program with its own selection criteria and with its own Landscape. However, it is expected that the general nature of the approach means that there is an overlap, albeit partial, of the Landscapes and therefore a certain sharing of the selective methods for the various theories of gravity. In literature, however, the Swampland approach was born and implemented mainly in the context of String Theory and it draws many inspirations and motivations from it.

The selection criteria proposed by the Swampland program are formulated as conjectures, that is, as statements for which there are no rigorous formal proofs but multiple evidences and heuristic proofs. In general, motivations inspired by String Theory are distinguished from motivations deriving from qualitative arguments of Quantum Gravity. In the first case we are dealing with proofs of the validity of the conjectures, even formally rigorous, but relegated to particular string models. In the second case we are dealing with qualitative justifications typically presented in the context of black hole physics.

There are many proposed conjectures but not independent of each other. In fact, it is still a matter of discussion to understand which conjectures are actually essential for the construction of the Swampland and which conjectures can be considered corollaries of the main ones. Topic of debate is also the formulation of the individual conjectures. Depending on the authors and the context in which they are presented, in fact, in literature there are different statements of the same Swampland criteria that take on different degrees of generality and therefore of formal abstraction. In the light of what has been said, this section introduces the main conjectures historically and physically considered most relevant [5] and binding to the next chapters the presentation of some secondary conjectures.
The first historically proposed conjecture, and perhaps the simplest and most intuitive one, is the *No Global Symmetry* (NGS) conjecture

Conjecture 1. *A theory with a finite number of states, coupled to gravity, can not have any exact global symmetry.*

This absolutely general conjecture simply summarizes the requirement that there should be no global symmetries in a quantum description of gravity. This request valid at the QG level also has repercussions at the EFT level. This means that any apparent low-energy global symmetry at a certain energy scale must be explicitly broken or gauged.

The next conjecture is the *Swampland Distance* conjecture (SDC), which refers to EFTs containing scalar fields

Conjecture 2. *Consider a theory, coupled to gravity, with a moduli space $\mathcal{M}$. Starting from any point $P \in \mathcal{M}$ there exists another point $Q \in \mathcal{M}$ such that the geodesic distance between P and Q, denoted by $d(P,Q)$, is infinite. Furthermore, there is an infinite tower of states with an associated mass scale M such that*

$$M(Q) \approx M(P)e^{-\alpha d(P,Q)} \tag{2.5}$$

where α is some positive constant.

The SDC then describes the effects of Quantum Gravity on EFT when moving through moduli space.

Finally, there is the *Weak Gravity* conjecture (WGC) which aims to set limits to the characteristic quantities of an EFT. For it two statements can be given depending on the application context.

Conjecture 3. *Given an EFT with a gauge group $U(1)$, there exists an electrically charged state with mass m and charge q under the gauge theory satisfying the relation*

$$\frac{qg}{m} \geq \frac{Q}{M} \sim \frac{\mathcal{O}(1)}{M_P} \tag{2.6}$$

where g is the gauge coupling constant, Q and M are respectively the charge and mass of an extremal black hole.

The name of the conjecture is due to the fact that the interpretation that can be given to this statement is that the gravitational interaction must be weaker than any other gauge interaction.
The conjecture thus formulated is known as electric version of the WGC to distinguish it from the magnetic version obtained by considering the magnetic monopole of mass $m \sim \Lambda/g^2$ and charge q/g for which the (2.6) becomes

$$\Lambda \leq g\, M_p \,. \tag{2.7}$$

It can be said that the current research work under the Swampland program is twofold. On the one hand there is the interest, as mentioned, in identifying the necessary selection criteria and their more general and restrictive formulations. This also includes the work done to study the links between the various criteria as well as the strong commitment to obtaining proofs of the conjectures of general validity. On the other hand, the main aim of Swampland approach is to extract from the various conjectures the physics useful for delineating the Landscape. So, ultimately, the goal is to obtain phenomenological implications that have applications, in particular, in Particle Physics and Cosmology.
In the next chapters, attention will be focused on the first two conjectures, the No Global Symmetry conjecture and the Swampland Distance conjecture, trying to reproduce both lines of research for them.

Chapter 3

Quantum Gravity in String Theory

In order to prepare the theoretical background necessary for the subsequent discussion and to understand the motivations behind the conjectures, this chapter retraces the main results of the String Theory. The discussion will be limited to the case of the bosonic string alone, first giving the most qualitative description and then reproducing the application of the conformal formalism.

3.1 The Bosonic string

Consider a bosonic string as a 1-dimensional object of length ℓ moving freely in spacetime. During its motion, the string describes a 2-dimensional surface, known as *worldsheet*, parametrized by local coordinates (τ, σ). The motion of the string is then described by functions $X(\tau, \sigma)$.
In general, a string can be open or closed depending on whether the parameter σ covers a finite interval, $\sigma \in [0, \ell]$, or is a periodic parameter, $\sigma = \sigma + \ell$ (Figure 3.1).

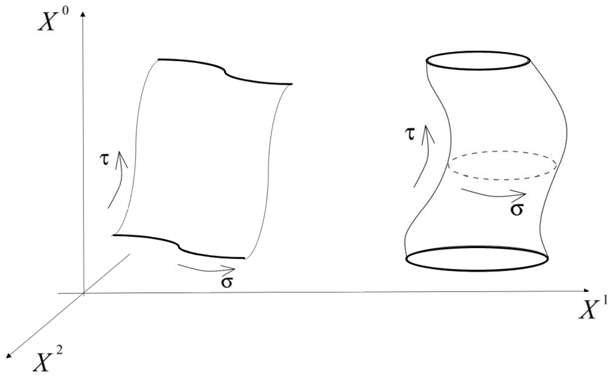

Figure 3.1: String worldsheet for both open and closed string

The simplest action of a bosonic string, obtained as a natural generalization of the relativistic point particle one, is provided by the *Nambu-Goto action*

$$S_{NG} = -T \int d\sigma d\tau \sqrt{(\dot{X}_\mu \partial_\sigma X^\mu)^2 - \dot{X}^2 (\partial_\sigma X)^2} \tag{3.1}$$

where $\dot{X}^\mu \equiv \partial_\tau X^\mu$, with $T \equiv (2\pi\alpha')^{-1}$ string tension and α' *Regge* parameter. On the other hand, the action that most appropriately describes the motion of the string in view of its quantization, is provided by the *sigma model* in which an auxiliary metric is introduced, $h_{\alpha\beta}(\sigma,\tau)$ with signature $(-,+)$, which gives rise to the so-called *Polyakov action*

$$S_P = -\frac{T}{2} \int \mathrm{d}^2\sigma \sqrt{-h}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \ . \tag{3.2}$$

The action thus written exhibits the following symmetries

- Poincaré invariance

$$X^\mu(\tau,\sigma) \to X'^\mu(\tau,\sigma) = \Lambda^\mu_\nu X^\nu(\tau,\sigma) + a^\mu \tag{3.3}$$

$$h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma) = h_{\alpha\beta}(\tau,\sigma) \tag{3.4}$$

- Diffeomorphism invariance

$$X^\mu(\tau,\sigma) \to X'^\mu(\tau',\sigma') = X^\mu(\tau,\sigma) \tag{3.5}$$

$$h_{\alpha\beta}(\tau,\sigma) \to h_{\alpha\beta}(\tau',\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta} h_{\gamma\delta}(\tau,\sigma) \tag{3.6}$$

- Weyl invariance

$$X^\mu(\tau,\sigma) \to X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma) \tag{3.7}$$

$$h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma) = \Omega^2(\tau,\sigma) h_{\alpha\beta}(\tau,\sigma) \tag{3.8}$$

of which the Poincaré invariance is a global symmetry of the worldsheet while the invariance under diffeomorphisms and Weyl invariance are local symmetries.
Once the action is known, the equations of motion are determined through the variational principle. For variations of the Polyakov action, with respect to X^μ and from the stationarity request

$$\delta_X S_P = 0 \tag{3.9}$$

we have the equations of motion

$$\partial_\alpha[\sqrt{-h}\, h^{\alpha\beta} \partial_\beta X^\mu] = \sqrt{-h}\, \nabla^2 X^\mu = 0 \ . \tag{3.10}$$

The boundary term is cancelled by assigning Neumann boundary conditions for the open string or periodic boundary conditions for the closed string

- Open string:

$$\partial_\sigma X^\mu(\tau, \sigma = 0) = \partial_\sigma X^\mu(\tau, \sigma = \ell) = 0 \tag{3.11}$$

- Closed string:

$$X^\mu(\tau, \sigma = 0) = X^\mu(\tau, \sigma = \ell) \tag{3.12}$$

$$h_{\alpha\beta}(\tau, \sigma = 0) = h_{\alpha\beta}(\tau, \sigma = \ell)\ . \tag{3.13}$$

These boundary conditions are the only ones consistent with the D-dimensional Poincaré invariance and with the equations of motion. The variation of the action with respect to the metric defines the energy-momentum tensor

$$T^{\alpha\beta}(\tau, \sigma) = -4\pi\sqrt{-h}\,\frac{\delta S_P}{\delta h_{\alpha\beta}} = -T(\partial^\alpha X^\mu \partial^\beta X_\mu - \frac{1}{2}h^{\alpha\beta}\partial_\gamma X^\mu \partial^\gamma X_\mu) \tag{3.14}$$

which, by virtue of the invariance under diffeomorphisms and the Weyl invariance, appears to be covariant conserved and traceless

$$\nabla_\alpha T^{\alpha\beta} = 0 \qquad\qquad T^\alpha_\alpha = 0\ . \tag{3.15}$$

Finally, if we require the invariance of the Polyakov action with respect to variations of the auxiliary metric, we obtain the equations of motion

$$T_{\alpha\beta} = 0\ . \tag{3.16}$$

These are sufficient to fully describe the classical dynamics of the bosonic string.

3.2 Quantization

What has been said so far concerns the classical description of the motion of a bosonic string. We are now interested in seeing what happens when we quantize the system. There are several quantization formalisms: in this section we will focus on *light-cone quantization* while in Section 3.3.1 we will discuss the alternative and more consistent *BRST quantization*.

Open string

We intend to quantize the open string by fixing the light-cone gauge in order to eliminate the redundancies due to the invariances under diffeomorphisms and

Weyl transformations, breaking however the covariance of the theory. The *light-cone* coordinates are then introduced into spacetime

$$x^{\pm} \equiv \frac{(x^0 \pm x^1)}{\sqrt{2}}, \qquad x^i \quad i = 2, .., D-1 \tag{3.17}$$

and the following gauge is fixed

$$X^+ = \tau \qquad\qquad \partial_\sigma h_{\sigma\sigma} = 0 \qquad\qquad \det(h_{\alpha\beta}) = -1 \tag{3.18}$$

which imposes three conditions for the three local symmetries (reparameterization of the two coordinates of the worldsheet and Weyl scaling).
In this gauge the string Hamiltonian becomes

$$H = \frac{T}{2}\frac{\ell}{p^+}\int_0^\ell \mathrm{d}\sigma \left(\frac{1}{T}\Pi^i\Pi^i + T\partial_\sigma X^i \partial_\sigma X^i\right) \tag{3.19}$$

where are defined the variables

$$x^-(\tau) = \frac{1}{\ell}\int_0^\ell \mathrm{d}\sigma X^-(\tau, \sigma) \tag{3.20}$$

$$p_- = -p^+ = \frac{\partial L}{\partial(\partial_\tau x^-)} = -T\ell h_{\sigma\sigma} \tag{3.21}$$

with L string Lagrangian.
Starting from Hamilton equations, free-wave type equations are found as equations of motion

$$\partial_\tau^2 X^i = c\, \partial_\sigma^2 X^i \tag{3.22}$$

with velocity $c \equiv T\ell/p^+$. The general solutions of such equations are provided by the following modes expansions

$$X^i(\tau, \sigma) = x^i + \frac{p^i}{p^+}\tau + \frac{i}{\sqrt{\pi T}}\sum_n \frac{\alpha_n^i}{n} e^{-i\pi n\tau/\ell} cos\left(\frac{\pi n \sigma}{\ell}\right) \tag{3.23}$$

where the expansion coefficients α_n^i are the string oscillators, such that $\alpha_{-n}^i = (\alpha_n^i)^\dagger$, and where center of mass variables for the string have been introduced

$$x^i(\tau) \equiv \frac{1}{\ell}\int_0^\ell \mathrm{d}\sigma X^i(\tau, \sigma) \tag{3.24}$$

$$p^i(\tau) \equiv \int_0^\ell \mathrm{d}\sigma \Pi^i(\tau, \sigma) = \frac{p^+}{\ell}\int_0^\ell \mathrm{d}\sigma\, \partial_\tau X^i(\tau, \sigma)\,. \tag{3.25}$$

At this point the system is quantized by imposing the canonical commutation rules which, in terms of Fourier components, provide the algebras

$$[x^-, p^+] = -i \qquad\qquad [x^i, p^j] = i\delta^{ij} \tag{3.26}$$

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n} \ . \tag{3.27}$$

In order to construct the Hilbert space associated with a bosonic string, the fundamental level states $|0;k\rangle$, with $k = (k^+, k^i)$, are defined as states annihilated by the operators α^i_m.
The states thus defined are then eigenstates of the center of mass momentum

$$\begin{aligned} &\alpha^i_m \, |0;k\rangle = 0 && \forall m \leq 0 \\ &p^+ \, |0;k\rangle = k^+ \, |0;k\rangle && p^i \, |0;k\rangle = k^i \, |0;k\rangle \ . \end{aligned} \tag{3.28}$$

The Hilbert space is then constructed by acting on $|0;k\rangle$ with the operators $\alpha^i_m{}^\dagger$, where $m > 0$, and the generic string state is then expressed as

$$|N;k\rangle = \left[\prod_i \prod_n \frac{(\alpha^i_{-n})^{N_{in}}}{\sqrt{n^{N_{in}} N_{in}!}}\right] |0;k\rangle \tag{3.29}$$

with N_{in} occupation numbers and $N = \sum_{i,n} nN_{in}$ string level. The momenta k of the center of mass constitute degrees of freedom common to those of a point particle while the occupation numbers N_{in} are additional internal string degrees of freedom.
If we insert the modes expansion in the expression for the Hamiltonian we find

$$H = p^- = \frac{p_i p^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{i,n>0} : \alpha^i_{-n}\alpha^i_n : - a\right) \tag{3.30}$$

with "::" normal ordering operation and a ordering constant. At this point, by requesting the Lorentz invariance hidden by the light-cone gauge, that is, by fixing the algebra of the Lorentz generators

$$[M^{\mu\nu}; M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}, \tag{3.31}$$

we obtain the critical bosonic string conditions

$$D = 26 \qquad\qquad a = 1 \ . \tag{3.32}$$

The Lorentz invariance, and therefore the consistency of the bosonic string quantum theory, fixes the dimension of spacetime to be $D = 26$.
In light of these critical conditions, the mass-shell relation for the open string becomes

$$\alpha' m^2 = N - 1 \tag{3.33}$$

and it is therefore possible to reconstruct the entire string spectrum.

- $N = 0$: There is *tachyon states* of negative mass

$$|0; k\rangle \qquad \alpha' m^2 = -1 \ . \tag{3.34}$$

  They indicates an internal inconsistency of the theory which can be interpreted as an instability (see also Section 4.2) in the sense of QFT. For this reason it is common opinion to regard the bosonic string theory as a minimal model to understand its extension in the more physical superstring theory [8].

- $N = 1$: We have bosonic states of zero mass

$$\alpha^i_{-1} |0; k\rangle \qquad \alpha' m^2 = 0 \tag{3.35}$$

  in vector representation of $SO(D-2)$. They can then be interpreted as the states associated with the photon.

- $N = 2$: We have the first two states of positive mass

$$\alpha^i_{-2} |0; k\rangle \ , \ \alpha^i_{-1}\alpha^j_{-1} |0; k\rangle \qquad \alpha' m^2 = 1 \tag{3.36}$$

  in tensor representation of $SO(D-2)$. Note that the mass is of the order of the Planck scale, therefore they are associated with practically unobservable particles.

As the levels increase there are other states with growing positive mass and in higher representations (*Kaluza Klein tower*).

Closed string

What is done for the quantization of open string in light-cone gauge can be repeated analogously for the closed string. The substantial difference is that in this case the periodicity condition must be set in the σ parameter. In this way we find the equations of motion

$$\partial^2_\tau X^i = c\, \partial^2_\sigma X^i \qquad X^i(\sigma) = X^i(\sigma + n\ell) \tag{3.37}$$

which are solved by the modes expansion

$$X^i(\tau, \sigma) = x^i + \frac{p^i}{p^+}\tau + \frac{i}{\sqrt{4\pi T}} \sum_n \left[\frac{\alpha^i_n}{n} e^{-2\pi i n(\sigma + c\tau)/\ell} + \frac{\tilde{\alpha}^i_n}{n} e^{2\pi i n(\sigma - c\tau)/\ell} \right] \tag{3.38}$$

where two independent sets of oscillators appear, α^i_n and $\tilde{\alpha}^i_n$, called left- and right-movers corresponding to oscillating waves moving in opposite directions along

the closed string.
Quantization is achieved by means of commutation rules

$$[x^-, p^+] = -i \qquad [x^i, p^j] = i\delta^{ij} \tag{3.39}$$

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n} \qquad [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m\delta^{ij}\delta_{m,-n} \,. \tag{3.40}$$

Starting from the ground level state $|0, 0; k\rangle$ defined as the state such that

$$\begin{aligned} &\alpha^i_n \,|0, 0; k\rangle = \tilde{\alpha}^i_n \,|0, 0; k\rangle = 0 \\ &p^- \,|0, 0; k\rangle = k^i \,|0, 0 : k\rangle \qquad p^i \,|0, 0; k\rangle = k^i \,|0, 0; k\rangle \,, \end{aligned} \tag{3.41}$$

the entire Hilbert space of closed string states is reconstructed

$$|N, \tilde{N}; k\rangle = \left[\prod_i \prod_n \frac{(\alpha^i_{-n})^{N_{in}} (\tilde{\alpha}^i_{-n})^{\tilde{N}_{in}}}{\sqrt{n^{N_{in}} N_{in}! n^{\tilde{N}_{in}} \tilde{N}_{in}!}}\right] |0, 0; k\rangle \;. \tag{3.42}$$

The mass-shell condition in this case becomes

$$\alpha' m^2 = 2(N + \tilde{N} - a - \tilde{a}) \tag{3.43}$$

and from the Lorentz invariance and periodicity request we find the critical conditions

$$D = 26 \qquad a = \tilde{a} = 1 \tag{3.44}$$

and the level matching condition

$$N = \tilde{N} \,. \tag{3.45}$$

Consequently it turns out

$$\alpha' m^2 = 4(N - 1) = 4(\tilde{N} - 1) \tag{3.46}$$

from which it is possible to reconstruct the closed bosonic string spectrum.

- $N = 0$: There is again a tachyon state of negative mass

$$|0; 0; k\rangle \qquad \alpha' m^2 = -4 \tag{3.47}$$

- $N = 1$: We have a bosonic massless state

$$|\Omega^{ij}\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1} \,|0, 0 : k\rangle \qquad \alpha' m^2 = 0 \tag{3.48}$$

  in rank-2 tensor representation of $SO(D-2)$ decomposable as

$$\Omega^{ij} = \frac{1}{2}\left(\Omega^{ij} + \Omega^{ji} - \frac{2}{D-2}\delta^{ij}\Omega^{kk}\right) + \frac{1}{2}(\Omega^{ij} - \Omega^{ji}) + \frac{1}{D-2}\delta^{ij}\Omega^{kk} \tag{3.49}$$

  that is, in the sum of a symmetrical part with zero trace associated with a spin-2 bosonic particle identified with the *graviton*; an antisymmetric part associated with a 2-form $B_{\mu\nu}$ known as *Kalb-Ramond field*; a scalar part associated with a scalar particle Φ called *dilaton*.

3.3 CFT in bosonic string theory

Since the worldsheet of a string is a 2-dimensional surface its vibrational modes can be studied by conformal field theories (CFT) in $d = 2$ dimensions [9, 10, 11]. In the Polyakov action (3.2) appear D free bosonic fields X^μ whose propagator in complex coordinates is

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\ln|z-w|^2 \,. \tag{3.50}$$

The energy-momentum tensor for such CFTs is given in the from

$$T(z) = -\frac{1}{2} :\partial_z X\, \partial_z X: \tag{3.51}$$

and then by carrying out the following OPEs

$$T(z)\partial_w X \sim \frac{1}{(z-w)^2}\,\partial_w X + \frac{1}{(z-w)}\partial_w(\partial_w X) + ... \tag{3.52}$$

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} - \frac{1}{(z-w)^2}\partial_w X\,\partial_w X - \frac{1/2}{(z-w)}\partial_w(\partial_w X \partial_w X) + ... \tag{3.53}$$

we find that the fields ∂X^μ are primary fields with weights $h = 1$ and that the central charge for each CFT is given by $c = 1$. Therefore, the total central charge of the CFT for the bosonic string, obtained from the contributions of the bosonic fields X^μ, is equal to

$$c_X = D \,. \tag{3.54}$$

The description of the string in terms of conformal field theory allows to develop an alternative quantization procedure to the light-cone quantization presented in Section 3.2, which appears to be formally more rigorous and suitable for studying string interaction theory.
Given the gauge symmetries of the Polyakov action, the path integral for the bosonic string can be evaluated by the *Faddev-Popov procedure* [12] according to which it can be written as

$$Z[g] = \int \mathcal{D}X\; \Delta_{FP}[g]\; e^{S[g,X]} \,. \tag{3.55}$$

The evaluation of the Faddev-Popov determinant $\Delta_{FP}[g]$ provides an expression of the type [12, 13]

$$\Delta_{FP}[g] = \int \mathcal{D}b\, \mathcal{D}c\; e^{iS_{ghost}} \tag{3.56}$$

$$S_{ghost} = \frac{1}{2\pi} \int d^2\sigma \sqrt{|g|}\, b_{\alpha\beta} \nabla^\alpha c^\beta \tag{3.57}$$

where the ghost fields $b_{\alpha\beta}, c^\alpha$ appear as Grassmann fields. The path integral (3.55) is therefore reduced to

$$Z[g] = \int \mathcal{D}X\, \mathcal{D}b\, \mathcal{D}c\, e^{-S[g,X]-S_{gh}[g,b,c]} \tag{3.58}$$

where the dynamic fields are now the X fields and the ghost fields b, c. The introduction of ghost variables is a direct consequence of the gauge-fixing procedure operated by the Faddev-Popov procedure which, unlike the light-cone gauge, preserves the Lorentz invariance.
If working in conformal gauge,

$$g_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}\ , \tag{3.59}$$

in complex coordinates the metric becomes $g_{zz} = g_{\bar{z}\bar{z}} = 0, g_{z\bar{z}} = g_{\bar{z}z} = e^{2\omega}$ and if we use the notation $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$, $b \equiv b_{zz}$, $c \equiv c^z$, the ghost action takes the simple form

$$S_{gh} = \frac{1}{2\pi} \int d^2z (b\bar{\partial}c + \bar{b}\partial\bar{c})\ . \tag{3.60}$$

The equations of motion associated with such action are given by

$$\bar{\partial}b = \partial\bar{b} = \bar{\partial}c = \partial\bar{c} = 0 \tag{3.61}$$

from which we deduce that the fields b, c are holomorphic and the fields $\bar{b}, \bar{c}$ are the associated anti-holomorphic fields.
In these terms it appears that the ghost fields give rise to a bc-CFT, that is a 2-dimensional conformal field theory in which the propagator is provided by

$$\langle c(z)b(w)\rangle = \langle b(z)c(w)\rangle = \frac{1}{z-w}\ . \tag{3.62}$$

The energy-momentum tensor T_{gh} of the theory takes the form

$$T_{gh} = -2b\partial c - (\partial b)c \tag{3.63}$$

and, making use of the (3.50), from the contractions with this tensor we obtain the OPEs

$$T_{gh}(z)b(w) \sim \frac{2}{(z-w)^2}b(w) + \frac{1}{(z-w)}\partial b(w) + ... \tag{3.64}$$

$$T_{gh}(z)c(w) \sim -\frac{1}{(z-w)^2}c(w) + \frac{1}{(z-w)}\partial c(w) + ... \tag{3.65}$$

$$T_{gh}(z)T_{gh}(w) \sim -\frac{13}{(z-w)^4} + \frac{2}{(z-w)^2}T_{gh}(w) + \frac{1}{(z-w)}\partial T_{gh}(w) + ... \quad (3.66)$$

from which we see that the ghost fields b and c are primary fields with weights $h = 2$ and $h = -1$ respectively and that the central charge of the bc-CFT for the bosonic string is equal to $c = -26$.

Now consider the whole theory action obtained from (3.58)

$$S = S_P + S_{gh} = \frac{1}{2\pi}\int d^2z \left[\partial X\bar{\partial}X + (b\bar{\partial}c + \bar{b}\partial\bar{c})\right] \quad (3.67)$$

the momentum energy tensor is in this case provided by

$$T(z) = T_X(z) + T_{gh}(z) = -\frac{1}{2} : \partial X\bar{\partial}X : -2b\partial c - (\partial b)c \quad (3.68)$$

and the central charge is accordingly

$$c = c_X + c_{gh} = D - 26\ . \quad (3.69)$$

For this purpose it should be remembered that, according to what has been said in Section 3.1, the Weyl invariance imposes that the energy-momentum tensor has no trace at the classical level. This result is generally valid for any classical CFT. At the quantum level, however, the situation is different. In this case, in fact, the expectation value $\langle T^\alpha_\alpha\rangle$ must be considered, which turns out to be null only in flat spaces. In [13] starting from the conservation of the energy-momentum it is found that at least for quasi-flat spaces the fallowing relation holds

$$\langle T^\alpha_\alpha\rangle = -\frac{c}{12}R \quad (3.70)$$

where R is the Ricci scalar. It is therefore concluded that for curved spaces with $R \neq 0$, in order to avoid the Weyl anomaly, a CFT theory must have zero central charge, $c = 0$. In the case of the bosonic string this translates into the request to have a space with critical dimension $D = 26$ in accordance with what was found in Section 3.2.

3.3.1 The BRST quantization

The action (3.67) obtained from the path integral turns out to be invariant under *BRST transformation* defined by

$$\delta_B X^\mu = -i\kappa\left(c\partial X^\mu + \bar{c}\bar{\partial}X^\mu\right) \quad (3.71)$$

$$\delta_B b = -i\kappa\, T \qquad \delta_B \bar{b} = -i\kappa\, \bar{T} \tag{3.72}$$

$$\delta_B c = -i\kappa\, c\, \partial c \qquad \delta_B \bar{c} = -i\kappa\, \bar{c}\, \bar{\partial}\bar{c}\,. \tag{3.73}$$

From Noether's theorem the conserved charge associated with such symmetry is

$$Q_B = \frac{1}{2\pi i}\oint dz : c(z)\,(T_X(z) + T_{gh}(z)) : \tag{3.74}$$

which, by virtue of the condition (3.65), turns out to be nilpotent, $Q^2 = 0$.
At this point it should be observed that not all the states of the Hilbert space associated with the theory (3.67) constitute physical states. In the context of BRST quantization, physical states $|\psi\rangle$ must satisfy the necessary (and not sufficient) condition

$$Q_B\,|\psi\rangle = 0\,. \tag{3.75}$$

States that satisfy (3.75) are called BRST-closed states. By virtue of the nilpotence of Q_B it can be deduced that the same relation is also satisfied by all states that can be written in the form

$$|\phi\rangle = Q_B\,|\chi\rangle \tag{3.76}$$

which are called BRST-exact states. On the other hand, if $|\psi\rangle$ is a physical state and $|\phi\rangle$ is a BRST-exact, we immediately see they are orthogonal to each other

$$\langle\psi|\,|\phi\rangle = \langle\psi|\,Q_B\,|\chi\rangle = (\langle\psi|\,Q_B)\,|\chi\rangle = 0\,. \tag{3.77}$$

This means that the BRST-exact states do not constitute physical states of the theory since they do not contribute to the transition amplitudes. Furthermore, an additional consequence of (3.75) is that all physical states are defined up to an exact BRST state, i.e. $|\psi\rangle$ and $|\psi\rangle + Q_B\,|\chi\rangle$ are physically equivalent. Therefore the Hilbert space of physical states is obtained as the quotient space

$$\mathcal{H}_{phy} = \mathcal{H}_{closed}/\mathcal{H}_{exact}\,. \tag{3.78}$$

Since the physical states are in equivalence classes, for each class it is possible to choose a ghost-free representative state for which the condition (3.75) is equivalent to requiring that the physical states have primary states of weight $(h, \tilde{h}) = (1, 1)$. In the light of the state-operator map, each physical state of string is associated with an operator $V(z, \bar{z})$, called *vertex operator*, such as to satisfy the aforementioned condition and such that the integral $\int d^2 z V$ is a Weyl invariant. Since the integration $\int d^2 z$ has weight $(-1, -1)$, the Weyl invariance is consistent with the requirement that physical states have weight $(+1, +1)$. The search for physical states and the construction of the string spectrum is then led back to the construction of vertex operators.

In the case of the closed string, the above operators will be integrated on the worldsheet. The simplest operator is provided by the tachyon operator

$$V_t =: e^{ik\dot{X}(z,\bar{z})} : \tag{3.79}$$

for which the physicality request $h = \tilde{h} = 1$ provides

$$m_t^2 = -k^2 = -\frac{4}{\alpha'} \tag{3.80}$$

that is, the state associated with it is actually a negative mass state according to what we saw in Section 3.2.
The vertex operator for the first excited state is obtained by adding factors $\partial X, \bar{\partial} X$

$$V_g = \xi_{\mu\nu} : e^{ik\dot{X}} \partial X^\mu \bar{\partial} X^\nu : \ . \tag{3.81}$$

In this case we have $h = \tilde{h} = 1 + \alpha' p^2/4$ and then the request for physicality imposes the mass relation

$$m_g^2 = 0 \ . \tag{3.82}$$

On the other hand, carrying out the OPE with the energy-momentum tensor we find

$$T(z)V_g(w,\bar{w}) \sim \frac{ik^\sigma \xi_{\sigma\nu}}{(z-w)^3} \partial_{\bar{w}} X^\nu e^{ik\dot{X}(w,\bar{w})} + \frac{1 + k^2/2}{(z-w)^2} V_g(w,\bar{w}) + ... \tag{3.83}$$

so that, in order V_g to be primary, we find the condition of transversal polarization

$$k^\mu \xi_{\mu\nu} = k^\nu \xi_{\mu\nu} = 0 \ . \tag{3.84}$$

We therefore find massless states for which the decomposition (3.49) is valid.
In the case of the open string, the integration of the vertex operators must be carried out along the boundary of the worldsheet, $\int dsV$. The fundamental operator is still the tachyon operator $V_t =: e^{ik\dot{X}(z,\bar{z})}$: which also in this case is associated with tachyon states but of mass

$$m_t^2 = -k^2 = -\frac{1}{\alpha'} \ . \tag{3.85}$$

The massless state associated with photons is obtained starting from the vertex operator build by adding a factor ∂X to the tachyon operator

$$V_{ph} = \xi_\alpha : e^{ik\dot{X}} \ \partial X^\alpha : \tag{3.86}$$

with α index of longitudinal directions. For it the physical claim of Weyl invariance provides again $m^2_{ph} = 0$, while the requirement that V_{ph} is primary gives the condition of photon polarization

$$\xi_\alpha \, k^\alpha = 0 \, . \tag{3.87}$$

In both cases of closed and open string, the higher excited states are obtained by acting with vertex operators constructed by adding further factors ∂X, $\bar{\partial} X$ and, as already seen, they appear to have increasing positive masses of the order of the Planck scale.

3.4 The Effective Theory of Gravity

The motion of a bosonic string in a curved spacetime (target space) with metric $G_{\mu\nu}$ is described by the so called *non-linear sigma model* in which the action is obtained as a natural generalization of the Polyakov action (3.2)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{|g|} \, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \, G_{\mu\nu} \, . \tag{3.88}$$

In the context of string theory the curvature of spacetime is produced by the graviton obtained by the symmetric part of the tensor (3.49) associated to the massless states of the string spectrum. In fact, if the spacetime metric tensor is decomposed as a perturbation around the metric of a flat space

$$G_{\mu\nu}(X) = \delta_{\mu\nu} + h_{\mu\nu}(X) \tag{3.89}$$

we see that the action associated with quantum fluctuation is expressed by the term

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{|g|} \, g^{\alpha\beta} \, \partial_\alpha X^\mu \partial_\beta X^\nu \, h_{\mu\nu}(X) \, . \tag{3.90}$$

In terms of CFT this operator, under the identification $h_{\mu\nu} = \xi_{\mu\nu} e^{ik\dot{X}}$, corresponds exactly to the vertex operator of the closed bosonic string associated with the graviton state (3.49). This means that the single graviton contributes to a perturbation of the metric of the target space and the generic fluctuation $h_{\mu\nu}$ is obtained as a superposition of contributions of multiple graviton states with different polarization.

The theory resulting from non-linear sigma model in conformal gauge has worldsheet action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, G_{\mu\nu}(X) \partial_\alpha X^\mu \partial^\alpha X^\nu \tag{3.91}$$

which describes a theory of interaction that can be treated perturbatively through an expansion in $\sqrt{\alpha'}/r_c$ where r_c is the characteristic curvature radius of the target space [12, 13] . The interaction theory turns out to be a conformal theory at the classical level. On the other hand, at the quantum level the perturbative study of the β-function $\beta_{\mu\nu}(G) \equiv \mu\partial G_{\mu\nu}(X;\mu)/\partial\mu$, with μ scale of the theory, in the 1-loop approximation provides [12, 13]

$$\beta_{\mu\nu}(G) = \alpha' \, R_{\mu\nu} \; . \tag{3.92}$$

We see then that the conformal invariance is satisfied for Ricci-flat target spaces, i.e. solutions of Einstein's equation in vacuum

$$R_{\mu\nu} = 0 \; . \tag{3.93}$$

The same result is obtained by requiring the Weyl invariance and therefore by traceless condition for the energy-momentum tensor associated with the theory on a curved worldsheet. In this sense we see that String Theory contains General Relativity as a 1-loop approximation. On the other hand the computation of 2-loop contributions provides [13]

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + \frac{1}{2}\alpha'^2 R_{\mu\lambda\rho\sigma} R_{\nu}^{\lambda\rho\sigma} \tag{3.94}$$

from which we obviously obtain the lowest order correction to Einstein's equation. At this point, remember that the closed string spectrum at the level of massless states, in addition to the graviton, contains the Kalb-Ramond field and the dilaton. The theory that accounts for the presence of these additional fields is described by the action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{|g|} \big[G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} + i B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} + \\ + \alpha' \Phi(X) R^{(2)} \big] \; . \tag{3.95}$$

As regards the additional terms with respect to the non-linear sigma model, it is possible to give them physical interpretations. It is understood that the second term constitutes the generalization of the electromagnetic interaction term in the QED action for a particle in the case of a string. In fact, being the worldsheet a 2-dimensional space, the analogue of the potential A_μ for the worldline is covered by the tensorial potential $B_{\mu\nu}$. In other words, this term indicates that the bosonic string is charged under the Kalb-Ramond field which is invariant under the gauge transformation

$$B_{\mu\nu} \quad \to \quad B'_{\mu\nu} = B_{\mu\nu} + \partial_\mu C_\nu + \partial_\nu C_\mu \; . \tag{3.96}$$

In this case the analogue of the Maxwell tensor $F_{\mu\nu}$, that is the strength field such that $H = dB$, is provided by the 3-form

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{3.97}$$

which can be geometrically assimilated to the torsion tensor which would imply a modification of General Relativity through the introduction of an antisymmetric correction to the affine connection $\Gamma^\lambda_{\mu\nu}$.
As far as the dilatonic term is concerned, since $R^{(2)}$ is the curvature scalar of the worldsheet, it is seen to give zero contribution for flat worldsheets. Furthermore, in the approximation in which the dilaton is constant, $\Phi \sim \langle\Phi\rangle \equiv \Phi_0$, it can be seen that this term is proportional to the Euler characteristic $\chi = \frac{1}{4\pi}\int d^2\sigma\, \sqrt{g}\, R^{(2)}$. On the other hand, in the path-integral approach for the study of the string interaction, it is necessary to add up on all the topologies that are classified by $\chi = 2 - 2h - b$ with h number of handles, b number of boundaries. Consequently, each term will be weighted by a factor $e^{\chi\Phi_0}$. It is concluded that the dilatonic term in (3.95) plays a crucial role in the definition of the perturbative expansion in which the string coupling constant is precisely $g_s = e^{\Phi_0}$ for the closed string and $g_O = \sqrt{g_s}$ for the open string. These constants therefore have a dynamical nature and do not constitute arbitrary constants of the theory.
The theory described by action (3.95) must be clearly Weyl invariant. On the basis of what has been said in Section 3.1 it is understood that the study of this invariance is traced back to the study of the traceless of the energy-momentum tensor $\langle T^\alpha_\alpha\rangle$ which will receive a contribution from each term of the action identified by the relative beta functions

$$\begin{aligned}\langle T^\alpha_\alpha\rangle = &-\frac{1}{2\alpha'}\,\beta_{\mu\nu}(G)g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{i}{2\alpha'}\,\beta_{\mu\nu}(B)\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \\ &-\frac{1}{2}\beta(\Phi)R^{(2)}\,.\end{aligned} \tag{3.98}$$

From an explicit calculation of these (see [13]) we find the following 1-loop expressions

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu^{\lambda\kappa} \tag{3.99}$$

$$\beta_{\mu\nu}(B) = -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi H_{\lambda\mu\nu} \tag{3.100}$$

$$\beta(\Phi) = -\frac{\alpha'}{2}\nabla^2\Phi + \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\,. \tag{3.101}$$

Note that the β-function of one field is affected by the presence of the other fields. In particular, the first term of (3.99) coincides with (3.92) and is corrected by the presence of $H_{\mu\nu\rho}$ and Φ. The condition of traceless $\langle T^{\alpha}_{\alpha}\rangle$ and therefore the Weyl invariance is obtained by requesting

$$\beta_{\mu\nu}(G) \;=\; \beta_{\mu\nu}(B) \;=\; \beta(\Phi) \;=\; 0\,. \tag{3.102}$$

The beta functions are calculated in the 1-loop approximation, consequently looking for the action associated with these equations of motion we find an effective low energy action

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{|G|}\,e^{-2\Phi}\left(R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi\right) \tag{3.103}$$

where κ is an interaction constant such that $\kappa_0 \sim (\sqrt{\alpha'})^{24}$.
In order to restore the classical Einstein-Hilbert term and thus make evident the presence of General Relativity, a conformal transformation of the metric can be performed

$$\tilde{G}_{\mu\nu}(X) = e^{-4\tilde{\Phi}/(D-2)}G_{\mu\nu}(X) \tag{3.104}$$

with $\tilde{\Phi} \equiv \Phi - \Phi_0$. The associated curvature scalar takes on the expression

$$\begin{aligned}\tilde{R} = e^{-4\tilde{\Phi}/(D-2)}\Bigg[&R - 2(D-1)\nabla^2\left(\frac{-2\tilde{\Phi}}{(D-2)}\right) - \\ &- (D-2)(D-1)\partial_\mu\left(\frac{-2\tilde{\Phi}}{(D-2)}\right)\partial^\mu\left(\frac{-2\tilde{\Phi}}{(D-2)}\right)\Bigg]\end{aligned} \tag{3.105}$$

and the action (3.95) becomes

$$S = \frac{1}{16\pi G}\int d^{26}X\sqrt{|\tilde{G}|}\left[\tilde{R} - \frac{1}{12}e^{-\tilde{\Phi}/3}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{6}\partial_\mu\tilde{\Phi}\,\partial^\mu\tilde{\Phi}\right] \tag{3.106}$$

in which Newton's constant is identified through the relation $8\pi G = \kappa e^{2\Phi_0}$. The action (3.106) is therefore said to be expressed in the Einstein frame as opposed to (3.95) which is instead expressed in the string frame.

3.5 Dp-branes and Open Strings

The Nambu-Goto action (3.1) can be immediately generalized and adapted to the description of the dynamics of p-dimensional objects, called p-brane, by means of the so-called *Dirac action*

$$S = -T_p\int d\mu_p \tag{3.107}$$

where $d\mu_p = \sqrt{-det(G_{\alpha\beta})}\, d^{p+1}\sigma$ is the infinitesimal $(p+1)$-dimensional volume element, with $G_{\alpha\beta}$ induced metric on the wordvolume, and T_p that can be interpreted as the tension of the brane. If the auxiliary metric $h_{\mu\nu}$ is introduced, then the analogue of the Polyakov action (3.2) becomes

$$S_P = \frac{T_p}{2} \int d^{p+1}\sigma \, \sqrt{|h|} \left[h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - (p-1) \right] \, . \tag{3.108}$$

It is understood that the case $p = 0$ coincides with the action of a relativistic point-like particle; the case $p = 1$ coincides with that of a bosonic string; the case $p = 2$ fits to describe a 2-dimensional membrane. The importance of such objects lies in the fact that they appear naturally in the context of String Theory. In fact, it is now well established that string theory is not only a theory of strings. The presence of 1-dimensional objects as the fundamental elements gives rise to objects of different dimensions (particles and multidimensional membranes). To understand how the presence of p-branes manifests itself in the context of bosonic string theory, consider again the open string discussed in Section 3.1. It has been said that the differentiation with respect to closed strings occurs due to the boundary conditions. It has also been pointed out that the only boundary conditions compatible with the Poincaré invariance are the Neumann boundary conditions. Actually this is true by restricting the study to strings only. If we admit the possibility of having $(p > 1)$-dimensional objects then the situation changes. In general, one can think of having a mix of Neumann and Dirichlet conditions. Consider the following situation of Neumann conditions for first $p+1$ coordinates and Dirichlet ones for remaining $D-p-1$ coordinates

$$\partial_\sigma X^\alpha(\sigma = 0) = \partial_\sigma X^\alpha(\sigma = \pi) = 0 \qquad\qquad \alpha = 0, ...p \tag{3.109}$$

$$X^a(\sigma = 0) = c_0^a \quad \text{and} \quad X^a(\sigma = \pi) = c_\pi^a \qquad a = p+1, ...D \, . \tag{3.110}$$

The Dirichlet conditions ensure that the extreme points of the open string are fixed along a directions and are therefore free to move on a $(p+1)$-dimensional hypersurface. For the spatial part of such hypersurface we speak of *Dirichlet brane* or synthetically of *Dp-brane*. A D-brane is therefore definable as the p-dimensional object on which open strings can end. In situation (3.109, 3.110) the Lorentz group then undergoes the following breaking

$$SO(1, D-1) \quad \rightarrow \quad SO(1, p) \times SO(D-p-1) \, . \tag{3.111}$$

Note that the limiting case considered in Section 3.1 corresponds to when the ends of the open string are free to move in all directions or, in other words, to the case in which the whole D-dimensional space is a D-brane. The particular case in which the Dirichlet condition is imposed for the coordinate X^0 corresponds to the

localization in time, that is to the formation of an instanton (D(-1)-brane) [13].
In order for the Poincaré invariance to be preserved, however, the D-branes must be dynamical objects described by the Dirac action (3.107), that is, they must be an integrant part of the theory. Note that the formation of such branes is associated with the imposition of conditions at the Dirichlet boundary. On the other hand, these conditions, and the consequent D-brane formation, come out naturally in the context of string compactification which will be discussed in Section 4.2.
Consider the case where we have $\mathcal{N}$ Dp-brane. Each end point of an open string can then lie on one of these $\mathcal{N}$ branes. If these are labelled by $i = 1, ..\mathcal{N}$ it is possible to associate additional open string degrees of freedom (i, j) known as Chan-Paton factors or charges. The generic string state will then be represented by

$$|N, k; \lambda\rangle = \sum_{i,j=1}^{N} \lambda_{ij} |N, k; ij\rangle \tag{3.112}$$

where λ_{ij} are Hermitian $\mathcal{N} \times \mathcal{N}$ matrices called Chan-Paton matrices. The usefulness of these factors lies in the fact that they allow the construction of non-Abelian gauge groups. If the Chan-Paton factors introduce $\mathcal{N}$ degrees of freedom then the oriented open strings describe a gauge group $U(\mathcal{N})$: the ends $\sigma = 0$ are associated with fundamental representation **N**, the ends $\sigma = \pi$ will be associated with anti-fundamental representation $\bar{\mathbf{N}}$. In fact, the resulting string spectrum of the lower levels than that found in Section 3.2, 3.3.1 will now be characterized by tachyon state and massless states with the following components

$$(T)^i_j \qquad (\phi^\alpha)^i_j \qquad (A_a)^i_j \, . \tag{3.113}$$

In particular, the field A_a lends itself to describe the connection of a gauge group $U(\mathcal{N})$. In the following chapter, this aspect will be addressed in more detail. The charges associated with the gauge fields are located on the D-branes which therefore act as their source.
In the case of non-oriented strings, there is symmetry by reversal orientation and the two ends are indistinguishable, therefore they are associated with the same fundamental real representation. The group described by non-oriented open strings can then be orthogonal or symplectic [14].

Chapter 4

The Compactification mechanism

The possibility of having extra dimensions with respect to the four observed dimensions of spacetime is not a novelty introduced by String Theory. This situation was previously studied by Kaluza [15] and Klein [16] and the theory developed by them is the basis of the compactification process which, in String Theory, allows to theoretically solve the problem of treating and interpreting the additional dimensions.

4.1 The Kaluza-Klein theory

The original idea pursued by Kaluza was that of tracing the fundamental interactions as a consequence of the geometry of spacetime. To this end he studied the possibility of having a $(D = 5)$-dimensional spacetime in the context of General Relativity. Consider therefore a 5-dimensional Riemann spacetime $(\mathcal{M}^{(5)}, g_{MN})$ on which coordinate systems of the type $(x^M) = (x^\mu, x^4)$ are fixed, with $M, N = 0, ...4$, $\mu = 0, 1, 2, 3$. On this space it is possible to introduce an affine connection Γ^C_{AB} and, similarly to a 4-dimensional Riemann space, the curvature tensor is defined as

$$R^D_{ABC} = \partial_B \Gamma^D_{CA} - \partial_C \Gamma_{BA} + \Gamma^D_{BE} \Gamma^E_{CA} - \Gamma^D_{CE} \Gamma^E_{BC} \tag{4.1}$$

from which the 5-dimensional Ricci scalar $R^{(5)}$ is obtained by contraction.
The action that describes the Kaluza theory is therefore provided by the Einstein-Hilbert action in the absence of matter but defined on $\mathcal{M}^{(5)}$, i.e. by

$$S = -\frac{M_P^{(5)3}}{2} \int d^5x \sqrt{-det(g_{MN})}\, R^{(5)} \tag{4.2}$$

where $M_P^{(5)} = (8\pi G^{(5)})^{-1/3}$ with $G^{(5)}$ gravitational constant associated to the 5-dimensional space.

Starting from the action (4.2), through the variational principle, the field equations are obtained

$$R_{AB} = 0 \tag{4.3}$$

which correspond to the usual Einstein's field equations in vacuum for a 5-dimensional geometry.
The metric tensor g_{MN} is a symmetric tensor with $D(D+1)/2 = 15$ independent components and then can be decomposed into the following block matrix

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} - \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & -\phi \end{pmatrix} \tag{4.4}$$

where $g_{\mu\nu}$ is the 4×4 symmetrical part with 10 independent components, A_μ is the vector part with 4 components, and ϕ is the scalar part with 1 component. These elements can be interpreted respectively as the metric content of the 4-dimensional geometry, a gauge vector field and an additional dilatonic scalar field. In fact, under transformation of coordinates of the type

$$x^M = (x^\mu, x^4) \quad \rightarrow \quad x'^M = (x^\mu, x^5 + f(x)), \tag{4.5}$$

we see that the components of g_{MN} are transformed according to the laws

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) \qquad\qquad \phi'(x') = \phi(x) \tag{4.6}$$

$$A'_\mu(x') = A_\mu(x) + \partial_\mu f(x) \tag{4.7}$$

that is with A_μ actually behaving as an Abelian gauge field.
In order to study physics from the point of view of 4-dimensional spacetime, it is assumed that there is no dependence on the additional dimension x^4, i.e. the Kaluza's cylindrical conditions are imposed

$$\partial_4 g_{\mu\nu}(x) = 0 \qquad \partial_4 A_\mu(x) = 0 \qquad \partial_4 \phi(x) = 0 \tag{4.8}$$

with $\partial_4 \equiv \partial/\partial x^4$.
If we indicate with $\mathcal{R}$ the linear size of the extra dimension, the action of the theory can be rewritten in form

$$S = -\frac{M_P^{(5)3}}{2} \int_0^{\mathcal{R}} dx^4 \int d^4x \sqrt{-det(g_{\mu\nu})}\, \phi^{1/2} R^{(5)} = \tag{4.9}$$

$$= -\frac{M_P^2}{2} \int d^4x \sqrt{-det(g_{\mu\nu})}\, \phi^{1/2} \left[R + \frac{1}{4}\phi\, F_{\mu\nu}F^{\mu\nu} + \frac{2}{3}\frac{\partial_\mu\phi\partial^\mu\phi}{\phi} \right] \tag{4.10}$$

where R is the 4-dimensional curvature scalar and where the strength field is introduced

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ . \tag{4.11}$$

Furthermore, to bring the gravity term back to the usual Einstein-Hilbert action, the following identification is made for the Planck scale

$$M_P^2 = \frac{1}{8\pi G} = \mathcal{R}\, M_P^{(5)3} \tag{4.12}$$

which indicates that a priori the 5-dimensional Planck scale may differ from the 4-dimensional one depending on the geometry of the extra dimension. Note that for the scale to remain the same it must be $\mathcal{R} \sim M_P^{-1}$.

From the obtained expression, it is concluded that starting from a pure 5-dimensional curved geometry it is possible to obtain a 4-dimensional physics with a gauge field A_μ and a scalar field ϕ.

In fact, if we derive the equations of motion by means of the variational principle, we find

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = \frac{8\pi G}{2}\, \phi^2\, T_{\mu\nu}^{em} - \frac{1}{\phi}\, [\nabla_\mu(\partial_\nu \phi) - g_{\mu\nu}\partial_\alpha \partial^\alpha \phi] \tag{4.13}$$

$$\nabla^\mu F_{\mu\nu} = -3 \frac{\partial^\mu \phi}{\phi} F_{\mu\nu} \tag{4.14}$$

$$\partial^\mu \partial_\mu \phi = \frac{8\pi G}{4} \phi\, F_{\mu\nu} F^{\mu\nu}\ . \tag{4.15}$$

If the scalar field is assumed to be constant and fixed at $\phi = 1$, they become

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}^{em} \tag{4.16}$$

$$\nabla^\mu F_{\mu\nu} = 0 \tag{4.17}$$

where the first is the usual Einstein field equation with an electromagnetic energy-momentum tensor

$$T_{\mu\nu}^{em} = \frac{1}{4}\, g_{\mu\nu}\, F_{\alpha\beta} F^{\alpha\beta} - F_\mu^\alpha F_{\nu\alpha} \tag{4.18}$$

and the second is Maxwell equation in curved geometry.

Conformal scaling

If we perform a conformal transformation on the metric (4.4)

$$g_{MN} \quad \to \quad g'_{MN} = \Omega(\phi)\, g_{MN}\ , \tag{4.19}$$

the determinant of the metric becomes $\sqrt{-det(g'_{MN})} = \Omega^{5/2}(\phi)\sqrt{-det(g_{\mu\nu})}$ and the 5-dimensional curvature scalar can be explained in the form

$$R'^{(5)}(g') = \Omega^{-1}[R^{(5)}(g) - 4\nabla_M \nabla^M \ln \Omega - 3(\nabla_M \ln \Omega)(\nabla^M \ln \Omega)]\ . \tag{4.20}$$

The action of the theory in this case becomes

$$S = -\frac{M_P^{(5)3}}{2}\int dx^4 \int d^4x\sqrt{-g}\,\phi^{1/2}\Omega^{3/2}\big[R^{(5)}(g) - 4\nabla_M\nabla^M \ln\Omega - \\ - 3(\nabla_M \ln\Omega)(\nabla^M \ln\Omega)\big] \tag{4.21}$$

and if the conformal scaling factor is chosen as $\Omega(\phi) = \phi^{-1/3}$ then it boils down to

$$S = -\frac{M_P^{(5)3}}{2}\int dx^4 \int d^4x\sqrt{-g}\left[R^{(5)}(g) + \frac{1}{3}(\partial_\mu \ln\phi)(\partial^\mu \ln\phi)\right] . \tag{4.22}$$

At this point, repeating the procedure followed previously and introducing the radionic field

$$\Phi = -\frac{1}{\sqrt{3}}\ln\phi \tag{4.23}$$

the 4-dimensional action is concluded

$$S = -\frac{M_P^2}{2}\int d^4x\sqrt{-g}\left[R + \frac{e^{-\sqrt{3}\,\Phi}}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi\right] \tag{4.24}$$

in which, in addition to the Einstein-Hilbert and the Maxwell terms, a massless scalar field term appears in minimal coupling due to the presence of the radion Φ. It is also noted that due to this field the gauge term is instead not minimal coupled. This situation can be avoided if, as before, $\phi = const = 1$ is chosen. Additionally the introduction of a scale factor Ω in the metric allows us to report the gravitational term that appears at the 4-dimensional level in its canonical expression without necessarily requiring $\phi = 1$.

4.1.1 Dimensional reduction

To reconcile Kaluza's theory with observations, that is, to give a theoretical justification to the cylindrical conditions (4.8), Klein introduced the *compactification process* as a mechanism of dimensional reduction. The idea is that the non-observation of the extra-dimensions is due to the fact that they have such a geometry as to be strongly curved and therefore extremely small in size compared to the usual macroscopic dimensions. It is said precisely that the extra dimensions are *"compactified"*. In particular, toroidal compactifications are usually considered in which the D-dimensional space is supposed to be realized according to the decomposition

$$\mathcal{M}^D = \mathcal{M}^{(D-d)} \times \mathcal{K}^d \tag{4.25}$$

where the d-dimensional subspace $\mathcal{K}^d$ has the topology of a torus. In the simple case of Kaluza's 5-dimensional model, Klein assumed that the fifth dimension has the topology of a circle of radius $\mathcal{R}$, i.e. that $\mathcal{M}^D = \mathcal{M}^{(D-d)} \times S^1$.
In this case for the characterizing fields of the 5d metric it is possible to make explicit the periodic dependence on the x^4 coordinate and then write them according to the Fourier series

$$g_{\mu\nu}(x) = \sum_n g_{\mu\nu}^{(n)}(x^\mu)e^{inx^4/\mathcal{R}} \tag{4.26}$$

$$A_\mu(x) = \sum_n A_\mu^{(n)}(x^\mu)e^{inx^4/\mathcal{R}} \tag{4.27}$$

$$\phi(x) = \sum_n \phi^{(n)}(x^\mu)e^{inx^4/\mathcal{R}} \, . \tag{4.28}$$

Each of the fields is then determined by assigning its Fourier modes which, by virtue of the periodic dependence, have quantized momenta along x^4.
It can be seen then that in the compactification limit $\mathcal{R} \to 0$ only the mode $n = 0$ independent of the coordinate x^4 contributes in accordance with the cylindrical condition (4.8) assumed by Kaluza. It is then concluded that the dimensional reduction operated by the compactification mechanism allows a theoretical explanation for why the extra dimensions are not directly observable. The 5-dimensional theory with dimensional reduction provided by the compactification on the circle takes the name of *Kaluza-Klein theory*.
We intend to study in detail the compactification process for the fields $g_{\mu\nu}$, A_μ, ϕ.

Scalar field compactification

Consider a massless scalar field ϕ in 5d spacetime with extra dimension compacted on the circle. From the periodicity condition for the x^4 coordinate follows the Fourier expansion

$$\phi(x^M) = \sum_n \phi_n(x^\mu)e^{inx^d/R} \tag{4.29}$$

and the momentum along the x^4 coordinate is quantized according to the eigenvalues

$$p_d = \frac{n}{R} \, . \tag{4.30}$$

The action describing the dynamics of the scalar field in 5-dimensional space can therefore be made explicit using the (4.29) [17, 18, 19]

$$\begin{aligned} S &= \frac{1}{2}\int d^5x \partial_M \phi(x^\mu, x^4)\, \partial^M \phi(x^\mu, x^4) = \\ &= \frac{1}{2}\int d^4x \left[\sum_n \partial_\mu \phi^{(-n)}\, \partial^\mu \phi^{(n)} - \frac{n^2}{\mathcal{R}}\phi^{(-n)}\, \phi^{(n)}\right] = \\ &= \int d^4x \left[\frac{1}{2}\partial_\mu \phi^{(0)}\, \partial^\mu \phi^{(0)} + \sum_{n=1}^{\infty}\left(\partial_\mu \phi^{(n)\dagger}\, \partial^\mu \phi^{(n)} - \frac{n^2}{\mathcal{R}}\phi^{(n)\dagger}\, \phi^{(n)}\right)\right] . \end{aligned} \tag{4.31}$$

From the expression of the action it is therefore deduced that, at the level of 4-dimensional physics, the modes $\phi^{(n)}$ provide an infinite set of d-dimensional fields, known as the Kaluza-Klein tower, with a mass-shell condition

$$m^2 = -p_\mu p^\mu = \frac{n^2}{R^2} . \tag{4.32}$$

At energies $E << R^{-1}$ there are only x^d-independent fields and the physics is therefore d-dimensional. In order to observe the effects of compactification, and then to observe the K-K tower, one should reach energy of order $E \approx R^{-1}$.

Gauge field compactification

In the case of a vector field $A_M(x^M)$ in the same 5-dimensional space with x^4 coordinate compactified on a circle, we have the Fourier expansion

$$A_\mu(x) = \sum_n A_\mu^{(n)}(x^\mu) e^{inx^4/\mathcal{R}} . \tag{4.33}$$

Analogously to what has been done for the scalar field, the study of 4-dimensional physics can be made explicit by introducing this expansion in the action for A_M. Introducing the strength field $F_{MN} = \partial_M A_N - \partial_N A_M$ we then have [17, 18, 19]

$$\begin{aligned} S &= -\frac{1}{4}\int d^5x F^{MN} F_{MN} = \\ &= \int d^5x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu A_5 - \partial_5 A_\mu\right)\left(\partial^\mu A^5 - \partial^5 A^\mu\right)\right] = \\ &= \int d^4x \sum_n \left[\frac{1}{4}F_{\mu\nu}^{(-n)} F^{(n)\mu\nu} + \right. \\ &\left. + \frac{1}{2}\left(\partial_\mu A_5^{(-n)} + i\frac{n}{\mathcal{R}}A_\mu^{(-n)}\right)\left(\partial^\mu A^{5(-n)} - i\frac{n}{\mathcal{R}}A^{\mu(-n)}\right)\right] . \end{aligned} \tag{4.34}$$

If it is performed the gauge transformation

$$A_\mu^{(n)} \quad \rightarrow \quad A_\mu^{(n)} - i\frac{\mathcal{R}}{n}\partial_\mu A_5^{(n)} \qquad\qquad A_5^{(n)} \quad \rightarrow \quad 0 \quad \forall n \neq 0 \tag{4.35}$$

the action becomes

$$\begin{aligned} S = \int d^4x \Bigg[&\left(-\frac{1}{4}F_{\mu\nu}^{(0)}F^{(0)\mu\nu} + \frac{1}{2}\partial_\mu A_5^{(0)}\partial^\mu A^{(0)5} \right) + \\ &+ 2\sum_{n\geq 1}\left(-\frac{1}{4}F_{\mu\nu}^{(-n)}F^{(n)\mu\nu} + \frac{1}{2}\frac{n^2}{\mathcal{R}^2}A_\mu^{(-n)}A_\nu^{(n)} \right) \Bigg] . \end{aligned} \tag{4.36}$$

From this expression we deduce that in 4-dimensions the zero-modes of the field A_M provide a 4-dimensional gauge field A_μ and a real scalar field, while the non-zero modes give rise to massive vector fields.

4.1.2 Non-Abelian gauge field

The original Kaluza-Klein theory allows to derive a gauge field A_μ and a scalar field ϕ in 4-dimensional geometry starting from a 5-dimensional space without sources. This allows a geometric origin to be given to the electromagnetic interaction. To study what happens for the strong and electroweak interactions, it is necessary to investigate the possibility of deriving non-Abelian gauge fields in analogy to what has been done for the Abelian ones. For this purpose we are interested in a generalization of the theory capable of contemplating this possibility [20]. Consider a space with n compact dimensions, $(\mathcal{M}^{(4+n)}, g_{MN})$, on which it is possible to identify charts with coordinates $(x^M) = (x^\mu, x^m)$, where $M, N = 0, 1, ..4 + n - 1$, $\mu = 0, 1, 2, 3$, $m = 4, ..4 + n - 1$. Suppose that the $(4+n)$-dimensional space can be decomposed into $\mathcal{M}^{(4+n)} = \mathcal{M}^4 \times K^n$ and that the subspace of extra-dimensions admits a group of isometries G whose generators are provided by the Killing vectors (K_m^a) with $a = 1, ..N$ group index. The G group in general has a non-Abelian structure for which, defined $K^a = K_m^a\partial^m$, it results

$$[K^a; K^b] = f_c^{ab}\ K^c \tag{4.37}$$

with f_c^{ab} group structure constants.

In this case the decomposition of the metric g_{MN}, which has $D(D+1)/2$ independent degrees of freedom, can be achieved by introducing a tensor ϕ_{mn} and vector fields A_μ^a associated with each Killing vector of $\mathcal{M}^n$

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} - \phi_{mn}A_\mu^a K_a^m\ A_\nu^b K_b^n & \phi_{mp}A_\mu^a K_a^m \\ \phi_{np}A_\nu^a K_a^p & -\phi_{mn} \end{pmatrix} . \tag{4.38}$$

For the components of the metric, according to compactification limit, the following dependence on space coordinates is assumed

$$g_{\mu\nu}(x) = g_{\mu\nu}(x^\mu) \qquad \phi_{mn}(x) = \phi_{mn}(x^m) \tag{4.39}$$

$$A^a_\mu(x) = A^a_\mu(x^\mu) \qquad K^m_a(x) = K^m_a(x^m)\ . \tag{4.40}$$

In order to study the transformation laws of the various fields, we consider an infinitesimal coordinates transformation of the type

$$x^M \quad \to \quad x'^M = x^M + \xi^M \qquad \xi^M = (0, \epsilon^a(x) K^m_a) \tag{4.41}$$

The mixed components of the metric under this transformation have the following variation [21]

$$\delta g_{\mu m} = -g_{mn}\partial_\nu \xi^n - g_{\mu m}\partial_m \xi^n - \xi^n \partial_n g_{\mu m} \tag{4.42}$$

for which it is located

$$\begin{aligned} \delta(A^a_\mu K_{am}) &= K_{am}\partial_\mu \epsilon^a - A^a_\mu K_{an}(\partial_m K^n_b)\epsilon^b - \epsilon^b K^n_b(\partial_n K_{am})A^a_\mu = \\ &= K_{am}(\partial_\mu \epsilon^a - f^a_{bc}\epsilon^b A^c_\mu) - A^a_\mu \epsilon^b (K^n_a \partial_n K_{bm} + K_{an}\partial_m K^n_b) = \\ &= K_{am}(\partial_\mu \epsilon^a - f^a_{bc}\epsilon^b A^c_\mu) \end{aligned} \tag{4.43}$$

where the algebra of the Killing vectors and their properties are used. Consequently we find the following variation for the vector fields at K_a fixed

$$\delta A^a_\mu(x) = \partial_\mu \epsilon^a(x) - f^a_{bc}\epsilon^b(x) A^c_\mu \tag{4.44}$$

which coincides with the transformation law of non-Abelian gauge fields.
On the other hand, if the procedure of the previous section is repeated, i.e. the D-dimensional action is rewritten by inserting the decomposition of the metric (4.38), it is found an effective 4-dimensional action with Yang-Mills term

$$S = -\frac{M_P^2}{2}\int d^4x\sqrt{-g}\left[R - \frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu}_a\right] \tag{4.45}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc}A^b_\mu A^c_\nu$.
In this case we have introduced the volume of the extra-dimension subspace $\mathcal{K}^n$

$$\mathcal{V}_n = \int d^n x\sqrt{det(\phi_{mn})} \tag{4.46}$$

and the following identification was made for the Plank scale

$$M_P^{(D)\,D-2} = \frac{M_P^2}{\mathcal{V}_n}\ . \tag{4.47}$$

It is concluded that the Kaluza-Klein theory admits in principle the possibility of deriving from a pure geometry with extra-dimensions all the matter fields of the Standard Model (SM) as long as we consider a space with a suitable isometries group G.
In terms of orders of magnitude, if $\mathcal{R}$ is the linear dimension of the compact space, (4.47) gives

$$\mathcal{R}^{D-4} \sim \frac{M_P^2}{M_P^{(D)\,D-2}} \,. \tag{4.48}$$

Also in this case the Planck scale in multidimensional space can a priori be different from M_P. The empirical observations fix at least the upper limit $\mathcal{R} \leq 10^{-4}m$. The possibility of having a scale $M_P^{(D)}$ at lower energies, such as for example of the order of TeV and thus making the effects of Quantum Gravity observable, therefore requires a sufficient number n of extra dimensions. For $M_P^{(D)} \sim TeV$ must be $n \geq 2$. On the other hand, from some Standard Model test, experiments ([22, 23]) would seem to further lower the upper limit for $\mathcal{R}$ thus requiring a much greater number of extra dimensions.
In light of this, the complications that occur in $D > 5$ should be mentioned at this point. If one is looking for D-dimensional spaces of the type $\mathcal{M}_D = \mathcal{M}_4 \times \mathcal{M}_n$ with $\mathcal{M}_4$ Minkowski spacetime, the absence of sources in $\mathcal{M}_D$ requires that

$$R_{mn}(\phi) = 0 \tag{4.49}$$

where notation $m, n = 4, ...D-4$ is used.
However, this condition is incompatible with the presence of a non-Abelian isometry group. In fact, it is found $f_c^{ab} = 0$, that is we have N Abelian fields. In these cases, the presence of non-Abelian gauge fields and therefore of Yang-Mills theories in 4-dimensional space would seem to require the abandonment of Kaluza's idea of producing matter starting from a purely geometric theory.

4.1.3 Spontaneous compactification

If we admit the possibility of having material sources also in D-dimensions, an interesting mechanism known as spontaneous compactification occurs. To understand how this mechanism works, consider the action of a D-dimensional theory of gravity with a minimally coupled matter term.

$$S = \frac{1}{2}\int d^Dx\sqrt{-g}\,R(g) \;+\; S_m \,. \tag{4.50}$$

The matter action is generally defined in terms of an antisymmetric tensor of rank r, $F_{M_1..M_r}$, as

$$S_m \propto \int d^Dx\sqrt{-g}\,F_{M_1..M_r}F^{M_1..M_r} \tag{4.51}$$

which is associated with the energy-momentum tensor

$$T_{MN} \equiv \frac{2}{\sqrt{-g}} \frac{\delta\sqrt{-g}\,\mathcal{L}_m}{\delta g^{MN}} \propto r \left(F_{MM_2..M_r} F_N^{M_2..M_r} - \frac{1}{2r} g_{MN}\, F_{M_1..M_r} F^{M_1..M_r} \right) . \tag{4.52}$$

From the variation of the action (4.50) and from the stationarity request we find the equations of motion

$$R_{MN} + \frac{1}{2} g_{MN} R = T_{MN} \tag{4.53}$$

$$\partial_M (\sqrt{-g}\, F^{MN_2..N_r}) = 0 . \tag{4.54}$$

In particular, we are interested in geometries factorizable as the product of maximally symmetrical spaces with metrics

$$g_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu) & 0 \\ 0 & g_{mn}(x^m) \end{pmatrix} \tag{4.55}$$

that is, described by equations

$$R_{\mu\nu} + g_{\mu\nu}\Lambda_x = 0 \qquad R_{mn} + g_{mn}\Lambda_y = 0 \qquad R_{\mu m} = 0 . \tag{4.56}$$

In this case the energy-momentum tensor must then be able to decompose itself into form

$$T_{MN} = \begin{pmatrix} g_{\mu\nu}T_x & 0 \\ 0 & g_{mn}T_y \end{pmatrix} \tag{4.57}$$

which is compatible with (4.52) provided that the tensor $F_{i_1..i_r}$ satisfies the following relations

$$F_{\mu M_2..M_r} F_\nu^{M_2..M_r} \propto a_1 T_x + b_1 T_y \tag{4.58}$$

$$F_{m M_2..M_r} F_n^{M_2..M_r} \propto a_2 T_x + b_2 T_y . \tag{4.59}$$

The curvature scalar is given by

$$R(g) \equiv g^{MN} R_{MN} = g^{\mu\nu} R_{\mu\nu} + g^{mn} R_{mn} = -4\Lambda_x + (4-D)\Lambda_y \tag{4.60}$$

therefore, explicating the field equation (4.56) we find the following equations that relate Λ_x, Λ_y to T_x, T_y

$$\Lambda_x + \frac{D-4}{2}\Lambda_y = T_x \qquad 2\Lambda_x + \frac{D-6}{2}\Lambda_y = T_y . \tag{4.61}$$

In particular, if we set the relation[1] $T_x + T_y = 0$, it results

$$\Lambda_x = \frac{D-5}{3}\Lambda_y \tag{4.62}$$

[1] It can be motivated by argoument from supersymmetry (see [24])

that is, to have a compact space $\mathcal{M}_n$ with $\Lambda_y > 0$, the macroscopic spacetime must be of anti-de Sitter type ($\Lambda_x < 0$). The mechanism that allows the formation of such space with compact dimensions through the source term S_m constitutes the spontaneous compactification process.

An alternative mechanism that provides a flat spacetime, free from fine tuning problems, i.e. with $\Lambda_x = 0$, is obtained by introducing a scalar field ϕ (a dilaton) in non-minimal coupling, with interaction potential $V(\phi)$. The D-dimensional action is in this case given by

$$S = -\int d^D x\sqrt{-g}\,\left[\frac{e^{-\phi}}{2}\left(R(g) + \partial_M\phi\,\partial^M\phi\right) + V(\phi)\right] + S_m \qquad (4.63)$$

where the matter term is the same of (4.51). The equations of motion, obtained from variations with respect to the metric, the gauge field and the scalar field, turn out to be

$$R_{MN} - \frac{1}{2}g_{MN}R + \nabla_M(\partial_N\phi) + \frac{1}{2}g_{MN}\partial_A\phi\partial^A\phi - g_{MN}\nabla_A(\partial^A\phi) = e^{\phi}(T_{MN} + g_{MN}V(\phi)) \qquad (4.64)$$

$$R(g) + \nabla_M(\partial^M\phi) - \partial_M\phi\,\partial^M\phi = 2e^{\phi}\partial_\phi V \qquad (4.65)$$

$$\partial_M(\sqrt{-g}\,F^{MN_2..N_r}) = 0\,. \qquad (4.66)$$

If we make the simplifying hypothesis of constant scalar field, $\phi = \phi_0 = const$, the field equations for the factored geometry and the equation for the dilaton become

$$\frac{R(g)}{2} + \Lambda_x = -e^{\phi_0}(T_x + V(\phi_0)) \qquad (4.67)$$

$$\frac{R(g)}{2} + \Lambda_y = -e^{\phi_0}(T_y + V(\phi_0)) \qquad (4.68)$$

$$R(g) = 2e^{\phi_0}\partial_\phi V|_{\phi_0}\,. \qquad (4.69)$$

At this point, considering again the condition $T_x + T_y = 0$, we find

$$\Lambda_x + \Lambda_y + R(g) = -2e^{\phi_0}V(\phi_0) \qquad (4.70)$$

Ultimately, since $\mathcal{M}_4$ is required to be a Minkowski spacetime, from (4.69) we conclude that

$$\Lambda_x = 0 \qquad \Lambda_y = -\frac{R(g)}{D-4} \qquad (4.71)$$

if and only if the following constraint on the dilatonic potential is verified

$$\left(\frac{\partial_\phi V}{V}\right)_{\phi_0} = -\frac{D-4}{D-5} \qquad (4.72)$$

which is satisfied for example by an exponential potential of the type $V \sim e^{-\phi(D-4)/(D-5)}$.

4.2 The String compactification

Now we can specify the compactification mechanism in the case of String Theory to solve the extra dimension problem. As we will see, this mechanism has interesting repercussions on the interpretation of the string dynamics and on its spectrum.

Closed String

Consider a closed bosonic string in a $(D = 26)$-dimensional spacetime with a compacted dimension on a circle S^1 of radius $\mathcal{R}$ with periodic condition

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi\mathcal{R}\omega \qquad \omega \in \mathbb{Z} \, . \tag{4.73}$$

ω is called *winding number* and indicates the number of windings of the string around the compacted dimension with a positive or negative sign depending on the winding direction (Figure 4.1). Consider then the mode expansion of the closed string coordinates with winding number ω. The expansion of the X^μ coordinates, with $\mu = 0, ..24$, it will remain unchanged while the expansion of X^{25} must be modified by adding a linear term in σ necessary to satisfy the periodic boundary condition

$$X^{25}(\sigma, \tau) = x^{25} + 2\alpha' p^{25}\tau + 2\mathcal{R}\omega\sigma + \ldots \, . \tag{4.74}$$

Also in this case, the compactification on the circle causes the p^{25} component of the momentum to be quantized

$$p^{25} = \frac{\kappa}{\mathcal{R}} \qquad \kappa \in \mathbb{Z} \tag{4.75}$$

with κ *Kaluza-Klein excitation number*. By decomposing the compactified coordinate into left- and right-movers contributions we have the following expansions

$$X^{25}(\sigma, \tau) = X^{25}_L(\tau + \sigma) + X^{25}_R(\tau - \sigma) \tag{4.76}$$

$$X^{25}_L(\tau + \sigma) = \frac{1}{2}(x^{25} + \tilde{x}^{25}) + \left(\alpha' \frac{\kappa}{\mathcal{R}} + \omega\mathcal{R}\right)(\tau + \sigma) + .. \tag{4.77}$$

$$X^{25}_R(\tau - \sigma) = \frac{1}{2}(x^{25} - \tilde{x}^{25}) + \left(\alpha' \frac{\kappa}{\mathcal{R}} - \omega\mathcal{R}\right)(\tau - \sigma) + \ldots \, . \tag{4.78}$$

At this point, going to write the on-shell relation in terms of ω and κ we find

$$\alpha' m^2 = 2\alpha' \left[\left(\frac{\kappa}{\mathcal{R}}\right)^2 + \left(\frac{\omega\,\mathcal{R}}{\alpha'}\right)^2 \right] + 2N_L + 2N_R - 4 \tag{4.79}$$

with level matching condition

$$N_R - N_L = \omega\,\kappa\,. \tag{4.80}$$

It can be noted that both the on-shell and level matching conditions are invariants by exchanging the winding and the Kaluza-Klein numbers. But such operation is equivalent to carrying out the inversion of the compactification radius

$$\mathrm{T}: \quad \omega \;\leftrightarrow\; \kappa \quad \Rightarrow \quad \mathcal{R} \;\leftrightarrow\; \tilde{\mathcal{R}} \equiv \frac{\alpha'}{\mathcal{R}}\,. \tag{4.81}$$

This symmetry of the bosonic string is known as *T-duality* and indicates that compactification on a circle of radius $\mathcal{R}$ is physically equivalent to compactification on a circle of radius $\tilde{\mathcal{R}}$. In terms of oscillators, the T-duality transformation corresponds to making the substitutions $\alpha_0^{25} \to -\alpha_0^{25}$, $\tilde{\alpha}_0^{25} \to \tilde{\alpha}_0^{25}$ and therefore to inverting the sign of the right component of the compacted string position leaving unchanged the left one

$$\begin{aligned} \mathrm{T}: \quad X^{25}(\sigma,\tau) &= X_L^{25}(\tau,\sigma) + X_R^{25}(\tau-\sigma) \quad \to \\ \to \qquad & \tilde{X}^{25}(\sigma,\tau) = X_L^{25}(\tau+\sigma) - X^{25}(\tau-\sigma) \end{aligned} \tag{4.82}$$

where $\tilde{X}^{25}$ is the dual coordinate with expansion

$$\tilde{X}^{25}(\sigma,\tau) = \tilde{x}^{25} + 2\alpha'\frac{\kappa}{\mathcal{R}}\sigma + 2\omega\mathcal{R}\tau + .. \tag{4.83}$$

which is associated to a momentum component $\tilde{p}^{25} = \mathcal{R}\omega/\alpha' = \omega/\tilde{\mathcal{R}}$.

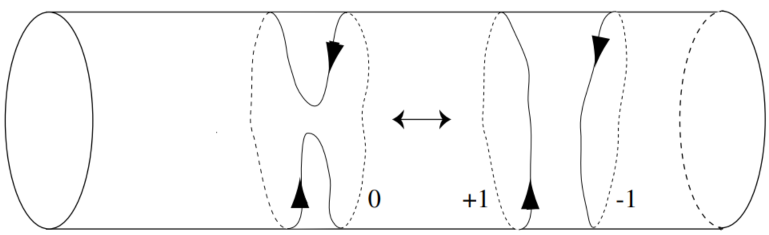

Figure 4.1: Closed string windings along compact dimension [14]

Open String

The compactification of an open string provides the mechanism that makes the appearance of the D-branes discussed in Section 3.5 natural.
Consider the case of an open string with only Neumann boundary conditions for

which the modes expansion (3.23) holds; its decomposition into right and left-movers components gives[2]

$$X_R^\mu(\tau-\sigma) = \frac{x^\mu - \tilde{x}^\mu}{2} + \frac{1}{2}p^\mu(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)} \tag{4.84}$$

$$X_L^\mu(\tau-\sigma) = \frac{x^\mu + \tilde{x}^\mu}{2} + \frac{1}{2}p^\mu(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)} \;. \tag{4.85}$$

If as before we compact the coordinate X^{25} on a circle of radius $\mathcal{R}$, the T-duality transformation gives

$$\mathrm{T}\;: \qquad X_R^{25} \;\rightarrow\; -X_R^{25} \qquad X_L^{25} \;\rightarrow\; X_L^{25} \;. \tag{4.86}$$

Consequently the dual coordinate $\tilde{X}^{25}$ becomes

$$\tilde{X}^{25}(\tau,\sigma) = X_L^{25} - X_R^{25} = \tilde{x}^{25} + p^{25}\sigma + \sum_{n\neq 0}\frac{\alpha_n^{25}}{n}e^{-in\tau}\sin(n\sigma) \;. \tag{4.87}$$

Since this coordinate has no linear terms in τ it is concluded that the dual open string has no momentum in x^{25} direction along which it can therefore only oscillate. Furthermore we see that (4.87) satisfies Dirichlet boundary conditions as the ends points in $\sigma = 0, \pi$ are now fixed

$$\tilde{X}^{25}(\tau,0) = \tilde{x}^{25} \qquad\qquad \tilde{X}^{25}(\tau,\pi) = \tilde{x}^{25} + 2\pi\kappa\tilde{\mathcal{R}} \;. \tag{4.88}$$

In this way the numbers κ are associated with invariant winding numbers which otherwise would not be stable since an open string can be topologically contracted in a point.

T-duality thus allows to pass from an open bosonic string with Neumann boundary conditions on a circle of radius $\mathcal{R}$ to a dual open bosonic string with Dirichlet boundary conditions on a circle of radius $\tilde{\mathcal{R}} = \alpha'/\mathcal{R}$. On the other hand, in Section 3.5 it is discussed how the presence of such boundary conditions requires the existence of D-branes as dynamical physical objects of the theory. In this specific case of compactification along the x^{25} direction there is the presence of a D24-brane. In general for a compactification on an n-dimensional torus T^n we will have a D$(25-n)$-brane.

In Section 3.5 we also discussed how multiple D-branes allow us to construct gauge groups. In the context of compactification on a circle, a potential A, if suitably diagonalized so that

$$A = -\frac{1}{2\pi\mathcal{R}}diag(\theta_1, \theta_2, .., \theta_{\mathcal{N}}) \tag{4.89}$$

[2]For this context the string length is expressed in units such that $\sigma \in [0,\pi]$.

makes it possible to have non-trivial holonomies associated with Wilson's lines

$$U = e^{i\int_0^{2\pi\mathcal{R}} dx\ A} \ . \tag{4.90}$$

In the presence of a potential (4.89), the canonical conjugate moment for an open string $|N,k,ij\rangle$ is given by $\Pi_{ij} = p + \frac{\theta_i - \theta_j}{2\pi\mathcal{R}}$ so that the quantization $\Pi^{25} = \kappa/\mathcal{R}$, with $\kappa \in \mathbb{Z}$, gives the kinetic momentum

$$p^{25} = \frac{\kappa}{\mathcal{R}} + \frac{\theta_j - \theta_i}{2\pi\mathcal{R}} \qquad k \in \mathbb{Z} \ . \tag{4.91}$$

The mode expansion of the dual coordinate $\tilde{X}^{25}_{ij}$ then becomes

$$\tilde{X}^{25}_{ij} = \tilde{x}^{25}_0 + \theta_i \tilde{R} + 2\tilde{\mathcal{R}}\sigma\left(\kappa + \frac{\theta_j - \theta_i}{2\pi}\right) + \ldots \ . \tag{4.92}$$

It can be seen that the two ends points of the open string in $\sigma = 0, \pi$ are located respectively in $\tilde{x}^{25}_0 + \theta_i\tilde{\mathcal{R}}$, $\tilde{x}^{25}_0 + \theta_j\tilde{\mathcal{R}}$. By virtue of the T-duality, in fact, the parameters θ_i identify the positions of the D24-branes along the dual circle (Figure 4.2).
The mass spectrum for an open bosonic string with Chan-Paton factors is thus modified

$$m^2_{ij} = \frac{1}{\alpha'}(N-1) + \left(\frac{\kappa}{\mathcal{R}} + \frac{\theta_j - \theta_i}{2\pi\mathcal{R}}\right)^2 \ . \tag{4.93}$$

It is interesting to investigate the ground state and first massless states with $k = 0$, i.e. the string states that do not wrap along the circle.

- $N = 0$, $\mathcal{N} = 1$: The ground state is again a tachyon state with negative mass

  $$m^2 = -\frac{1}{\alpha'} \ . \tag{4.94}$$

  In this case its presence indicates an instability of the D-brane which tends to decay into closed string radiation [13, 14].

- $N = 1, \mathcal{N} = 1$: In the case of only one D-brane we have $i = j$ and then we can have the following massless states

  $$\alpha^{25}_{-1}\,|0,k\rangle \qquad \alpha^{\mu}_{-1}\,|0;k\rangle \quad \mu = 0,..,24 \tag{4.95}$$

  of which the first describes oscillation in the compact direction and corresponds to a scalar field A_{25}, the second describes oscillations tangential to the D24-brane and correspond to a gauge vector field A_μ with symmetry group $U(1)$.

- $N = 1, \mathcal{N} > 1$: In the case of $\mathcal{N}$ branes the mass-shell condition takes the general form

$$m_{ij}^2 = +\frac{1}{\alpha'}(N-1) + \left(\frac{\theta_j - \theta_i}{2\pi\mathcal{R}}\right)^2 \qquad i,j = 1,..,\mathcal{N} \,. \tag{4.96}$$

If the D-branes are all coincident, that is, if $\theta_i = \theta_j \quad \forall\, i,j$, there are additional massless states with respect to the previous case and the states (4.95) are now associated with scalar fields $(A_{25})_{ij}$ and vector fields $(A_\mu)_{ij}$ giving rise to a non-Abelian symmetry group $U(\mathcal{N})$. If one or more D-branes change position, the aforementioned group is broken. For example, if a D-brane changes position on the dual circle, we see the following symmetry breaking

$$U(\mathcal{N}) \quad \rightarrow \quad U(\mathcal{N}-1) \,\times\, U(1) \,. \tag{4.97}$$

In general, for $\mathcal{N}_0$ coincident branes we have a non-Abelian gauge group $U(\mathcal{N}_0)$ and many abelian gauge groups $U(1)$ for each remaining D-brane. If $\theta_i \neq \theta_j \quad \forall\, i,j$ then there are $\mathcal{N}$ Abelian gauge groups

$$U(\mathcal{N}) \quad \rightarrow \quad U(1)^{\mathcal{N}} \,. \tag{4.98}$$

This is the peculiar way of describing gauge groups in the context of String Theory and it constitutes the method with which to investigate the matter theories such as the Standard Model and its extensions outlining the Landscape of the theory.

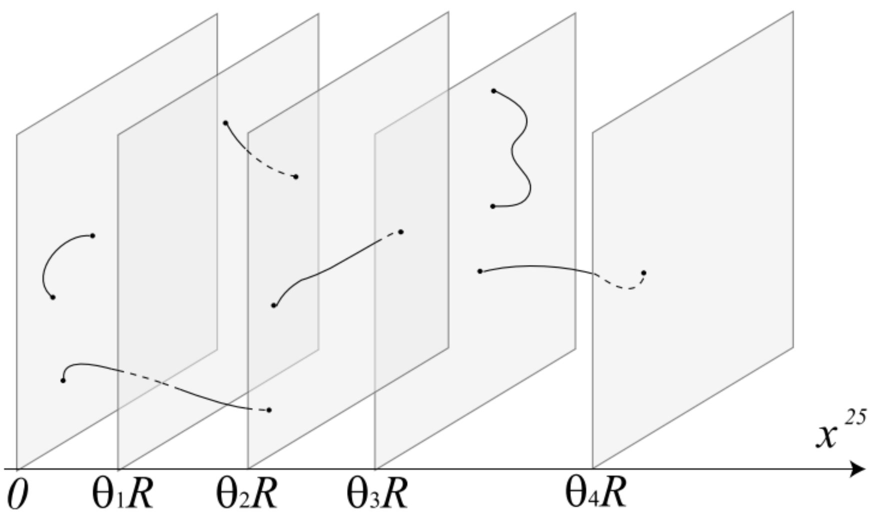

Figure 4.2: Open strings ending on D-branes [14]

Chapter 5

The No Global Symmetry conjecture

The first conjecture introduced in the context of the Swampland program was the No Global symmetry conjecture (NGS). It is in fact a formally very simple and general conjecture but which, precisely because of these characteristics, provides important restrictions on real world physics. The conjecture, without making particular limiting hypotheses, requires that a consistent QG theory be devoid of global symmetries

> **No Global Symmetry conjecture.** A theory with a finite number of states, coupled to gravity, can not have any exact global symmetry.

The conjecture thus exposed also applies to EFTs so that it can be concluded that the apparent global symmetries observed in low energy regimes must actually be broken or gauged at a certain energy scale. In the first case the symmetry observed in the IR is an approximate symmetry and it must then happen that at higher orders of approximation of the EFT there must be an operator who explicitly breaks the symmetry. It should be noted, in fact, that the existence of a mechanism capable of reproducing a spontaneous symmetry breaking is not sufficient to satisfy the conjecture. With a spontaneous break there is no elimination of symmetry but rather a different manifestation of it. In the second case, however, evidently, high-energy physics must have additional degrees of freedom with respect to EFT such as to reproduce the gauge theory associated with the symmetry. In addition to enjoying the evidence of validity in the context of String Theory presented in Chapter 2, No Global Symmetry has evidence from black hole physics. From the study of black holes in the context of QFT, it is known that a black hole actually emits via Hawking radiation. Since Hawking radiation has a purely thermal spectrum, this means that a black hole charged under a global symmetry, once it reaches the dimensions of the Planck scale, will cease its emission reducing itself to a stable charged remnant. The result is therefore the obtaining of an infinite

number of remnants (one for each charged black hole) below a finite energy scale with consequent entropy problems.
These problems can also be addressed in the light of the No-Hair theorem according to which a stable black hole is completely described by its mass, its angular momentum and its gauge charge. Any global charges would in no way be observable, giving rise to infinite uncertainty and therefore again to entropy problems.

In String Theory the idea of not being able to have global symmetries derives from the connection between the worldsheet of the string and the target space. In general, already at the bosonic string level it can be seen that a continuous global symmetry in the worldsheet always corresponds to a gauge symmetry of the target space. In fact, in perturbative string theory, to each global symmetry of the worldsheet it is possible to associate some Noether currents $j_z, j_{\bar{z}}$ to which a conserved charge is associated

$$Q = \frac{1}{2\pi i} \oint (dz j_z - d\bar{z} j_{\bar{z}}) \ . \tag{5.1}$$

Conformal invariance ensures that the currents $j_z, j_{\bar{z}}$ are transformed respectively as tensors of type $(1,0), (0,1)$. Starting from the two currents it is then possible to construct two vertex operators, $j_z \bar{\partial} X^\mu e^{ikX}$ and $\partial X^\mu j_{\bar{z}} e^{ikX}$, which create massless gauge vectors in the target space associated with the two components of the charge Q. This means that the global symmetry of the worldsheet becomes a gauge symmetry from the point view of the target space. It is concluded that in String Theory there is no way to construct global symmetries in accordance with the NGS conjecture.

5.1 Cobordism Classes

The NGS conjecture in its simplest formulation, although general, has the drawback that it is not very predictive in quantitative terms. In fact, if a global symmetry is actually an approximate symmetry, the conjecture, while foreseeing its explicit break, does not give any information about the energy scale at which this break must take place. In other words, it is not possible to establish a priori the degree of approximation of the symmetry. The resulting problem is that it is then difficult to draw phenomenological implications in a low-energy regime.
In order to remedy this limitation, it is useful to investigate possible reformulations of the conjecture in terms of topological quantities and, in particular, through a classification based on the concept of *cobordism classes*.
Cobordism is defined as an equivalence relationship between compact manifolds of the same dimension. To study the nature of this equivalence, let us consider

for simplicity oriented manifolds M such that $\bar{M}$ is the manifold with opposite orientation. The following definition will then be given

Definition 1. *Two smooth, closed and oriented k-dimensional manifolds M and N are said to belong to the same cobordism class if the disjoint union of M and $\bar{N}$, constitutes the boundary of a closed oriented $(k+1)$-dimensional manifold W, i.e. if*

$$M \sqcup \bar{N} = \partial W \ . \tag{5.2}$$

The manifolds belonging to the same cobordism class are said to be *"cobordant"* and the manifold W is called the cobordism of M and N (Figure 5.1). Note that connectivity is not required for W. It is therefore easy to verify that the cobordism thus constructed effectively defines an equivalence class.

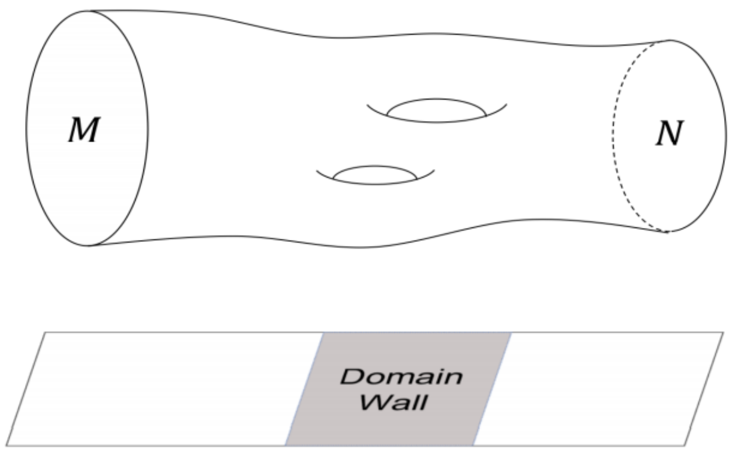

Figure 5.1: Cobordism between two manifolds [25]

The idea behind the notion of cobordism is to generalize the concept of diffeomorphism by defining a sort of equivalence of manifolds at the topological level. A manifold can be obtained from a manifold of the same cobordism class adjoining it by means of an appropriate[1] topological deformation process.
Given a set of cobordism classes of closed and compact k-dimensional manifolds, denoted with Ω_k, it can be seen that Ω_k acquires an Abelian group structure. For this purpose, the disjoint union operation is considered as a group operation. If

[1]For the purposes of physics, only topological deformations compatible with the dynamics of the theory can be considered.

$[M]$ is the cobordism class of the manifold M, then it is defined

$$[M_1] + [M_2] = [M_1 \sqcup M_2] \ . \tag{5.3}$$

The above operation satisfies all group properties, since $\forall\ [M], [M_i] \in \Omega_k$:

- closure : $[M_1] + [M_2] \in \Omega_k$
- associativity : $([M_1] + [M_2]) + [M_3] = [M_1] + ([M_2] + [M_3])$
- neutral element : $\exists\ \ [\oslash] \equiv 0\ :\ [M] + 0 = [M]$
- inverse element : $\forall [M]\ \exists [\bar{M}]\ :\ [M] + [\bar{M}] = 0$

where, in particular, the neutral element indicates a cobordism class with an empty manifold or simply the set of k-dimensional manifolds $[\oslash]$ which are boundary of a $(k+1)$-dimensional manifold W, $[\oslash] = [\partial W]$. The inverse element $[\bar{M}]$ can be interpreted as a class of suitable manifolds of opposite orientation to those of $[M]$. In fact, their disjoint union by definition of cobordism gives rise to a boundary and is therefore an element of the trivial class. For the group Ω_k the commutativity property is also valid since the disjoint union operation is trivially invariant due to the exchange of the two manifolds since there is no notion of ordering.

- commutivity : $[M_1] + [M_2] = [M_2] + [M_1]$.

It is concluded that the group Ω_k it actually constitutes an Abelian group. Furthermore it is possible to give a ring structure to Ω_k under the Cartesian product operation. Indeed, if M_1 is cobordant with M_2 and N_1 is cobordant with N_2, then $M_1 \times N_1$ is cobordant with $M_2 \times N_2$.
For instance, the low-dimensional cobordism groups for compact oriented manifolds are

- $\Omega_0 = \mathbb{Z}$
- $\Omega_1 = 0$
- $\Omega_2 = 0$
- $\Omega_3 = 0$
- $\Omega_4 = \mathbb{Z}$.

In the case $k = 0$, the only oriented manifolds are oriented points $\bullet_+, \bullet_-$, and their disjoint unions. Note that $\bullet_+ \sqcup \bullet_-$ is the the 0-dimensional circle S^0 which is the boundary of 1-dimensional disk D^1. Therefore the neutral element will be

the class of all disjoint unions of points or sets of points with opposite orientations. For each non-coupled point we will have different cobordism classes so that $\Omega_0 = \mathbb{Z}$.
In the case $k = 1$ the simplest closed 1-dimensional manifold is the oriented circle S^1 but itself is an element of $[\emptyset]$ since it is the boundary of the disk D^2. Consequently, every disjoint union of circles will be an element of $[\emptyset]$ and then the cobordism group $\Omega_1$is trivial. The same argument applies to $k = 2$ since in this case the oriented closed manifolds are spheres S^2, tori T^2 and generic handle surfaces which in turn are respectively boundary of the balls D^3, solid tori and generic handlebodies.
The case $k = 4$ differs from the lower dimensional cases because the Poincaré duality theorem holds, according to which for each $4n$-dimensional manifold M it is possible to associate a symmetrical quadratic form induced by the cup-product [26]. Consequently, the *manifold signature* $\sigma(M)$ can be defined as the sum of the number of positive eigenvalues minus the sum of the number of negative eigenvalues. The signature is therefore an integer that classifies the $4k$-dimensional topologies. In the context of cobordism classes it appears that this signature has the following properties [27]

- $\sigma(M_1 \sqcup M_2) = \sigma(M_1) + \sigma(M_2)$
- $[M_1] = [M_2] \qquad \Rightarrow \qquad \sigma(M_1) = \sigma(M_2)$
- $\sigma(\partial W) = 0 \qquad \forall\ \partial W = M_1 \sqcup M_2$
- $\sigma(M_1 \# M_2) = \sigma(M_1) + \sigma(M_2)$

where "#" denotes the connected sum. Consequently, given that the complex projective plane $\mathbb{CP}^2$ has $\sigma(\mathbb{CP}^2) = 1$ [26], from disjoint union of $\mathbb{CP}^2$s we obtain non-trivial cobordism classes such that $\Omega_4 = \mathbb{Z}$.

5.2 Cobordism in Quantum Gravity

The mathematical formalism of the cobordism classes can be used and specialized for the study of Quantum Gravity. For this purpose, consider a D-dimensional theory with k compacted dimensions. An EFT can be associated with each k-dimensional manifold. In this context, two manifolds will be said to be cobordant if obtainable from each other through topological processes consistent with the QG theory. EFTs associated with two cobordant manifolds are connected by a domain wall. With this prescription, then consider the cobordism group obtained as the set of compactification manifolds classes Ω_k^{QG}. To understand the meaning of group structure, let take the disjoint union of two compactification manifolds. If this union is cobordant with a third compactification manifold N, then it results

$$[M_1] + [M_2] = [M_1 \sqcup M_2] = [N] \ . \tag{5.4}$$

This means that the group structure allows a topological description of the dynamic process that links the union of two EFTs to a third EFT through a domain wall (Figure 5.2).

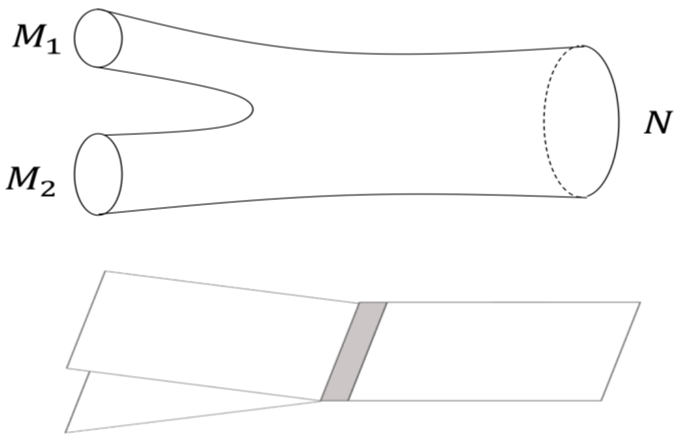

Figure 5.2: Cobordism with the disjoint union of two manifold [25]

The neutral element $[\varnothing] = 0$ will correspond to the class of manifold that are boundaries. EFTs compactified on such manifolds have domain walls that act as ends of their vacua (Figure 5.3).

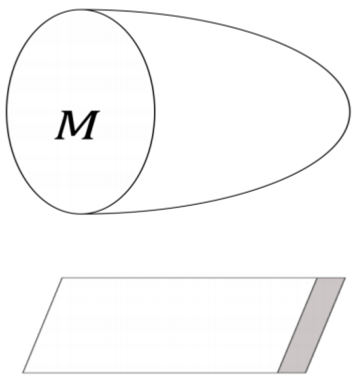

Figure 5.3: Neutral element of cobordism group [25]

Given this definition, it is therefore interesting to rewrite the No Global symmetry conjecture in terms of cobordism classes. The link between the physics of QG and the topology of such classes is given by the fact that the cobordism classes of k-dimensional manifolds give rise to the formation of a global topological charge. It is therefore reasonable to rewrite No Global Symmetry in the following form.

Triviality of the Coborsism Classes. Consider some D-dimensional QG theory compactified on a k-dimensional internal manifold. All cobordism classes must vanish

$$\Omega_k^{QG} = 0. \tag{5.5}$$

In fact, the non-triviality of the cobordism classes $\Omega_k^{QG} \neq 0$ implies that there would exist non-cobordant k-dimensional manifolds. In the interpretation in which an EFT is associated with a k-dimensional manifold, this would mean that non-triviality should be interpreted as the absence of a domain wall between the two EFTs. On the other hand, the lack of interpolation of two EFTs would give rise to a global topological charge $[M] \in \Omega_d^{QG}$ associated with a global $(D-k-1)$-form symmetry (see Appendix A) in contradiction with the No Global Symmetry conjecture.
Cobordism groups form an homology $H_k(pt)$ which can be used to make a classification of topological spaces for every dimensions. Given a single homology group, its class elements are associated with topological invariants given by generalized winding numbers. In the light of explicit group expressions as cyclic groups, we see that such winding numbers assume only integer numbers and then define topological charges as quantum numbers.
To better understand the mechanism of production of a global charge, consider a flat D-dimensional theory and a compactification manifold M^k. Make the connected sum $\mathbb{R}^k \times M^k$ obtained by cutting a k-dimensional sphere from M^k and $\mathbb{R}^k$ and gluing together the boundaries thus formed. The resulting space is almost flat except for a small region. This procedure gives rise to the formation of a gravitational soliton [25, 28] (see next subsection) which acts as a $(D-k-1)$-dimensional defect. Therefore, each set of closed and compact k-dimensional manifolds can be mapped to a set of $(D-k-1)$-dimensional defects. If the cobordism group is not trivial, $\Omega_k^{QG} \neq 0$, since for example $[M] \neq 0$, then in the presence of the gravitational soliton there is a topological invariant that defines a generalized global charge in contradiction with the No Global Symmetry conjecture. In fact, the same arguments made in the case of an ordinary global symmetry

apply to it if the topological defect falls into a black brane.
The charge obtained has a topological nature. On the other hand, in many cases a topological invariance can be interpreted as a physical global symmetry of the system. This is the case, for example of Particle Physics where the baryon number is reproduced as winding number in the skyrmion model [29].

The topological formulation of the conjecture, in addition to the higher level of formality, appears to be in some ways more general but above all it allows to draw important topological conclusions about the physics of the QG. If we study the EFTs of Swampland in terms of cobordism classes, the non-triviality of Ω_k^{QG} in the low energy regime competent for EFTs, in order to have a compatibility with the QG in the ultraviolet, it must be interpreted as ignorance of the degrees of freedom necessary to obtain triviality. In accordance with what has already been said, for the NGS conjecture to be valid, a global symmetry approximated at low energies must be explicitly broken or gauged. In the first case, the presence of topological defects is required (Figure 5.4) while in the second the presence of gauge fields.
In this sense, the No Global symmetry conjecture, expressed as the triviality of the cobordism classes, allows us to predict the existence of new objects otherwise deducible only from direct observation at high energies.

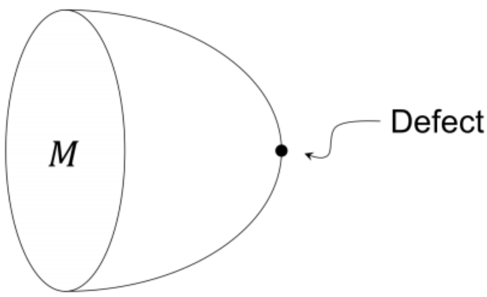

Figure 5.4: Triviality of the cobordism through a topological defect

5.2.1 Gravitational instantons

To make more clear the scope of the cobordism conjecture and the related arguments, consider gravitational instantons [30]. Note in fact that a topological instanton can be regarded as a topological soliton in higher dimensions. In particular, consider the case of compact theories on 4-dimensional manifolds $(k = 4)$. It has been seen that in this case the cobordism group for oriented manifolds is $\Omega_4 = \mathbb{Z}$. On the other hand we know that, wanting to apply the cobordism to gravity, only the compactification manifolds consistent with the gravitational dynamics must be considered. The idea is to introduce the analog of Yang-Mills instanton in the case of the theory of gravity. A gravitational instanton is then defined as a classical solution of Einstein's equations with finite energy. The geometries that can be interpreted as gravitational instantons are then classified as hyperkahler 4-dimensional manifolds, i.e. Ricci-flat Riemann manifolds (solutions of Einstein's equation in vacuum) with holonomy group contained in Sp (1). Such solutions are automatically dual or antidual, that is, for them it results

$$R_{\mu\nu} = 0 \qquad\qquad \tilde{R}_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\rho} R^{\lambda\rho} = \pm R_{\mu\nu} \ . \tag{5.6}$$

Each 4-dimensional compact hyperkahler manifold can be regarded as a $K3$ surface [31] or a compact torus T^4. In the case of Riemannian manifolds $\mathcal{M}$ the signature introduced in Section 5.1 can be analytically expressed as [32]

$$\sigma(\mathcal{M}) = -\frac{1}{96\pi^2} \int d^4x \sqrt{|g|}\, \epsilon^{\mu\nu\lambda\rho} R^{\alpha\beta}_{\mu\nu} R_{\lambda\rho\alpha\beta} \ . \tag{5.7}$$

It is a topological term that constitutes the gravitational analogue of the θ-term of the QCD. It indeed contributes to the CP violation [33]. The signatures (5.7) for hyperkahler manifolds are respectively given by $\sigma(K3) = -16$ and $\sigma(T^4) = 0$ [32, 25].

Given a gravitational instanton with $K3$ geometry, it can be interpreted as a fluctuation of spacetime at the quantum level if the connected sum operation is carried out between a flat D-dimensional spacetime and the gravitational instanton (Figure 5.5).

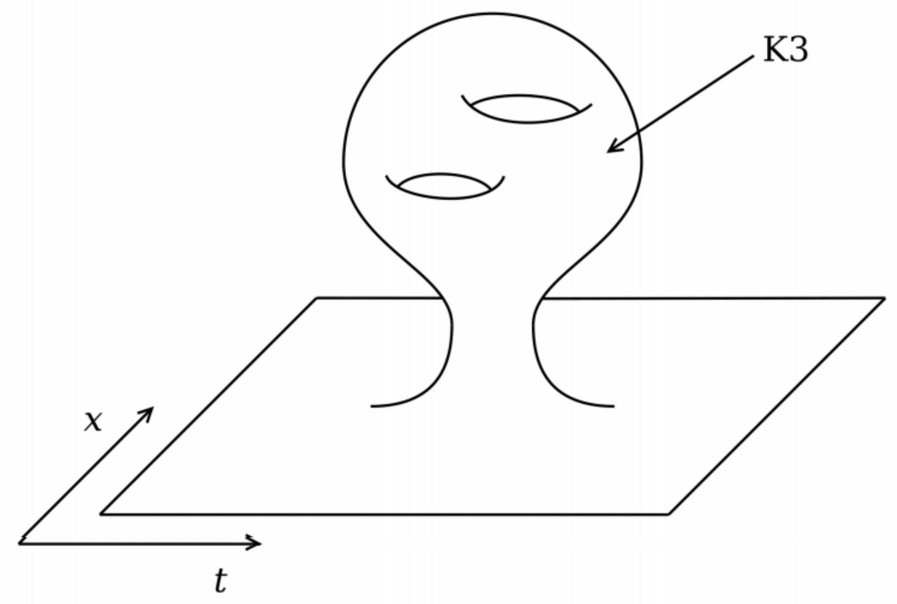

Figure 5.5: Gravitational instanton obtained through the connected sum of $K3$ with the flat space [33]

The resulting geometry will no longer be a solution of Einstein's equations [34, 33] but, by virtue of the signature properties, this procedure leaves the geometry invariant under cobordism. In fact it turns out

$$\sigma(K3 \,\#\, \mathbb{R}^4) = \sigma(K3) = \ -16 \ . \tag{5.8}$$

It is then explicitly seen how the non-triviality of the cobordism group for $k = 4$ is expressed in the context of QG. In accordance with the reformulation of the NGS, there must be a topological defect associated with this group such as to ensure $\Omega_k^{QG} = 0$. To this end, it should be noted that the physical properties of compactification require that the manifolds have a spin structure. The definition of hyperkahler 4-manifold ensures the presence of this structure by virtue of the holonomy group [31], therefore the formally most suitable context in which to study the compactification processes is that of the Superstring Theory. In this case, in fact, we have $D = 10$ and it is known that there are 5-dimensional branes called $NS5$-branes which can therefore be regarded as the topological $(D-k-1)$-dimensional objects associated with the cobordism group Ω_4^{QG}.

5.3 The Completeness Hypothesis

The No Global Symmetry conjecture expresses the incompatibility of QG with global symmetries. The discourse is clearly different for the gauge symmetries which, however, are affected by the impact of the conjecture described above. In fact, global and local symmetries are not completely disconnected from each other. Qualitatively a global symmetry can be regarded as the limit of a gauge symmetry with coupling constant $g \to 0$.
To clarify this link, consider for example the particular case of an EFT with gauge symmetry $U(1)$ weakly coupled to gravity through the introduction of the Einstein-Hilbert action

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{M_P^2}{2} R - \frac{1}{4g^2} F^2 \right] . \tag{5.9}$$

Locally the action of the gauge symmetry is equivalently expressed by the following transformations of the vector field $A_\mu(x)$

$$A_\mu \to A_\mu + \partial_\mu \lambda \qquad A_\mu \to A_\mu + \sigma_\mu \tag{5.10}$$

with $\lambda(x)$ scalar parameter and σ field such that $\partial_{[\mu}\sigma_{\nu]} = 0$.
From the global point of view the situation is crucially different [5]. The condition $\partial_{[\mu}\sigma_{\nu]} = 0$ for the σ field it is in fact incompatible with the gauging of the transformation and gives rise to a generalized global symmetry in contradiction with the NGS conjecture. The problem of the presence of generalized global symmetries within the EFTs can however be solved by introducing charged gauge fields. A gauge theory is in fact invariant under transformations $A_\mu \to A_\mu + \partial_\mu \lambda$ but not under generalized global transformations $A_\mu \to A_\mu + \sigma_\mu$.
What said in the case of a $U(1)$ gauge symmetry can then be repeated also for a generic discrete symmetry. In this case, the elimination of global symmetries will require the presence of all charged states predicted by the Dirac quantization of the charge.
This observation then leads to express a further swampland criterion that EFTs must satisfy in order to be consistent with UV completion in a QG theory known as completeness conjecture

Completeness Conjecture. A theory with a gauge symmetry, coupled to gravity, must have states of all possible charges under the gauge symmetry.

Note that, even in this case, although the conjecture has a general character, it does not make any quantitative reference about the mass of the charged states.

This means that there is no a priori information regarding the energy scale to which the conjecture can give observable effects.

The completeness of the spectrum can also be revisited in the light of the notion of cobordism. In particular, it can be seen that it is a consequence of the triviality of the cobordism group. To this end, consider the more general case of a theory coupled to a p-form gauge field A_p associated with a p-form symmetry group $U(1)$ (see Appendix A). Introduced the field strength $F_{p+1} = dA_p$, the resulting theory has an action of the form

$$S_{eff} = S_{EH} - \frac{1}{2} \int d^4x \sqrt{-g}\, F \wedge \star F \tag{5.11}$$

which is invariant under gauge symmetry transformations

$$A_p \to A_p + \lambda_p \tag{5.12}$$

with λ_p a closed p-form. In this case the cobordism group is given by the set of classes formed by closed and compact manifolds with p-form gauge field A_p. The topological invariant is indeed associated with the charge

$$Q(M^{(D-p-1)}, A_p) = \int_M^{(D-p-1)} \star F_{p+1} \tag{5.13}$$

which is conserved by virtue of the equation of motion

$$d \star F_{p+1} = 0 \, . \tag{5.14}$$

This situation indicates the presence of an internal flow that gives rise to a non-trivial cobordism group

$$\Omega^{QG}_{p+1} = \mathbb{Z} \, . \tag{5.15}$$

In this case, the demand for triviality of the cobordism group can be satisfied by introducing charged objects under the gauge group that act as a source for the flow. In doing so, in fact it results

$$d \star F_{p+1} = J \tag{5.16}$$

and therefore we have the explicit breaking of global symmetry [35, 36, 37]. In particular, in order for the breakdown of the group to be completed, i.e. it concerns all the topological invariants of Ω^{QG}, it is necessary that the spectrum of the gauge field is complete, that is it must contain all source charges compatible with Dirac's quantization. What said for the field A_p can be repeated for the dual field $\tilde{A}_{D-p-2}$.

In fact, the theory with p-form symmetry $U(1)$ also enjoys a $(D-p-2)$-form global symmetry associated with the gauge transformation

$$\tilde{A}_{D-p-2} \rightarrow \tilde{A}_{D-p-2} + \tilde{\lambda}_{D-p-2} \tag{5.17}$$

with $\tilde{\lambda}_{D-p-2}$ a closed $(D-p-2)$-form. The equation of motion for this field is given by

$$dF_{p+1} = 0 \; . \tag{5.18}$$

Then, in this case the associated defects charges are

$$Q(M^{(p+1)}, A_p) = \int_{M^{(p+1)}} F_{p+1} \; . \tag{5.19}$$

To have $\Omega^{QG} = 0$ it is therefore necessary to introduce $(D-p-3)$-dimensional charged objects that act as magnetic monopoles for the A_p field.
If what has been said were specialized to the case $D = 4$, one concludes that the redefined NGS conjecture has as a consequence the completeness of the electric and magnetic spectrum of the coupled gauge theory [38, 39].

5.4 Relation with Weak Gravity conjecture

The NGS conjecture has an interesting link with the weak gravity conjecture stated in Chapter 2. To clarify their connection, it should be noted that the evidence and the reasons that typically come forward for the WGC, such as those reported in [5, 4], are analogous to the arguments proposed for the justification of the validity of the NGS conjecture. The reasons supporting the conjecture in fact again refer to the black holes physics but in the presence of a gauge symmetry rather than a global one. It has been said that a priori in a QG theory there is clearly no obstruction to the presence of gauge symmetries. On the other hand, as already observed, a global symmetry can be regarded as the limiting case of a gauge symmetry with coupling constant $g \rightarrow 0$. This would seem to suggest the possibility of forming a global symmetry by such procedure. This possibility, however, is prevented precisely by the WGC. In fact, remember that the magnetic version of the conjecture requires that for an EFT with a cut-off Λ and coupled to a gauge theory with coupling constant g, we must have

$$\Lambda \leq g M_P \; . \tag{5.20}$$

This means that if we tried to form a global symmetry in the limit $g \rightarrow 0$ the cut-off of the theory would go to zero. In other words, the effective description of the theory would lose its validity and would therefore require the introduction of

additional degrees of freedom.
From a macroscopic point of view, we see that in the presence of a gauge symmetry $U(1)$ the number of black holes N_{BH} that can form below the energy scale Λ is finite with a trend

$$N_{BH} \sim \frac{\Lambda}{gM_P} \, . \tag{5.21}$$

It is clearly seen that the attempt to form a global symmetry would meet the entropy problems already discussed in the previous section being $N_{BH} \to \infty$.
The connection between NGS conjecture and the WGC, in addition to provide a support for the correctness of the two conjectures, highlights a certain internal consistency of the Swampland program which confirms its validity.

5.5 The Instability of AdS vacua

The triviality of the cobordism group has further consequences. The fact that k-dimensional compactification manifolds constitute the edges of $(k+1)$-dimensional manifolds, $M_k = \partial W$, implies that such manifolds can approach to have zero size. This means that in QG the formation of so-called bubbles of nothing is topologically allowed. Here *"bubble of nothing"* means a non-perturbative instability obtained when the extra dimensions collapse [40]. This process can take place already in the case of a 5-dimensional space with a compacted dimension as it happens for the Kaluza-Klein model in which the compactification manifold is S^1 with compactification radius $\mathcal{R}$. If we schematize the bubble of nothing as a 3d ball B^3 of radius R, such that the space is given by $(\mathbb{R}^4 - B^3) \times S^1$, the space-time metric is then described by

$$ds^2 = \left(1 - \frac{R^2}{r^2}\right)^{-1} dr^2 + r^2 \, d\Omega_3 + \mathcal{R}^2 \left(1 - \frac{R}{r^2}\right) d\theta^2 \, . \tag{5.22}$$

This is a solution a priori allowed by the equations of motion. The effective compacting radius is expressed by

$$\mathcal{R}_{eff}(r) = \mathcal{R} \sqrt{1 - \frac{R^2}{r^2}} \tag{5.23}$$

which, in accordance with what has been said, tends to collapse into the limit $r \to R$. This configuration can also be found in the case of compactifications in several dimensions.
The instability caused by the presence of a bubble of nothing occurs when the conditions are created for this bubble to expand and cause the annihilation of space. The satisfaction of these conditions will clearly depend on the dynamics

of the system in which the bubble is formed. At this point we are interested in establishing the conditions for the topological defects allowed by the cobordism conjecture to produce or not instability of the space. To this end, it is useful to study the case of instability supported by the presence of flows. In [41] it is studied a particular process known as fragmentation of the anti-de Sitter (AdS) space. Consider two EFTs associated with vacua stabilized by the presence of internal flows, f and $f+Q$, generated by a $(p-1)$-form with field strength F_p

$$\int_{\Sigma_p} F_p = f\ . \tag{5.24}$$

The two vacua are interpolated by a domain wall provided by a $(D-p-1)$-brane of charge Q under the p-form. Consider then the metric of the AdS space

$$ds^2 = R^2\left(\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2_{d-2}\right) \tag{5.25}$$

with $\Omega_{d-2} = \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)}$ volume of $(d-2)$-dimensional unit sphere. Suppose that the brane has spherical symmetry and is therefore described by action

$$S = R^{d-1}\Omega_{d-2}\int d\tau\left(T\sinh^{d-2}\rho\sqrt{\cosh^2\rho + \left(\frac{d\rho}{d\tau}\right)^2} - M_P^2 Q\sinh^{d-1}\rho\right) \tag{5.26}$$

where T is the brane tension. In the brane-test approximation, i.e. for negligible brane charge with respect to the flow, the solutions of the equation of motion associated with the action (5.26) provide the expression for the radius ρ of the brane

$$\tanh\ \rho\ =\ \frac{T}{M_P^2 Q}\ . \tag{5.27}$$

We then see that (5.26) describes the nucleation of a bubble which tends to expand as long as its tension is smaller with respect to its charge. This occurs because the contraction produced by the tension T fails to contrast the repulsion of the charged regions of the brane. The result is the decay of the AdS vacuum into a vacuum with lower flow and a charged p-brane. The limiting case $\tanh\rho = 1$ corresponds to a supersymmetric AdS vacuum [41] which prevents the bubble from expanding and thus maintains its stability. On the other hand the WGC, opportunely reformulated to the case of a p-brane, requires the condition

$$T \leq QM_P^2 \tag{5.28}$$

so that at least limited to the case of vacua supported by flows, an intrinsic instability of the non-supersymmetric AdS spaces is achieved. For generalization, the following conjecture is formulated

> **Non-SUSY Anti-de Sitter Instability conjecture.** Any non-supersymmetric anti-de Sitter vacuum is unstable or at best metastable.

The usefulness of this conjecture is that, by placing dynamic conditions on the vacua of the EFT, it lends itself to setting limits on the parameters of the theory so that the instability of the AdS space is ensured.

5.6 Neutrino masses

We now intend to study an application of the AdS instability conjecture in the case of the Standard Model. In 4d the nature of an AdS space is quite exotic as it is known that the vacuum associated with the SM coupled to gravity via an Einstein-Hilbert term is unique and is not expected to be of AdS type. However, the situation changes when SM compactifications are studied [42, 43, 44]. In this case, we find a Landscape of vacua which also includes AdS ones. In particular, if we consider the SM coupled to gravity (SM + GR) as a 4d theory and we compact only one dimension on a circle, it is legitimate the formation of a vacuum $AdS_3 \times S^1$. The justification for considering this type of geometry derives from the fact that they are used to describe exotic scenarios such as, in particular, Reissner-Nordstrom black holes. For a discussion of the interpolation of such geometries with ordinary spacetime see [42]. Furthermore, it is expected that what happens in the compactification limit can then also be reflected in more physical decompactification scenarios.
If $\mathcal{R}$ is the radion and r is an arbitrary distance scale, in accordance with what was said in the Chapter 4, the metric of the compact theory for large $\mathcal{R}$ can take the form

$$ds^2 = \frac{r^2}{\mathcal{R}^2} ds^2_{(3)} + \mathcal{R}^2 \left(d\theta - \frac{\sqrt{2}}{M_P r} A_\mu dx^\mu \right) \tag{5.29}$$

where A_μ is the graviphoton field. The effective action associated with the theory is given by

$$S = \int d^3x \sqrt{|g|}\, (2\pi r) \left[\frac{1}{2} M_P^2 R - \frac{1}{4} \left(\frac{\mathcal{R}}{r} \right)^4 F_{\mu\nu} F^{\mu\nu} - M_P^2 \left(\frac{\partial \mathcal{R}}{\mathcal{R}} \right)^2 - \left(\frac{r}{\mathcal{R}} \right)^2 \Lambda_{c.c} \right] \tag{5.30}$$

in which it is taken into account a contribution from a non-zero cosmological constant $\Lambda_{c.c}$ which provides a potential term for the radionic field. On the other hand, the small observed value of $\Lambda_{c.c}$ means that the contributions associated with

1-loop quantum corrections are decisive. In this case the quantum corrections are identified with the contributions to the Casimir energy $\mathcal{C}$ (see Appendix B). If i denotes a state index for the spectrum of $SM + GR$, we find the following correct expression for the radionic potential

$$V(\mathcal{R}) = \frac{2\pi r^3 \Lambda_{c.c}}{\mathcal{R}^2} + \sum_i (2\pi\mathcal{R})(-1)^{s_i} n_i \mathcal{C}_i(\mathcal{R}) \tag{5.31}$$

where n_i are the degrees of freedom of the various states of the spectrum, $s_i = 0, 1$ respectively for fermions and bosons, and $\mathcal{C}_i$ precisely denote the Casimir potential for i-th state provided by the general formula (B.14)

$$\mathcal{C}_i(\mathcal{R}) = (-1)^{s_i} \sum_{j=1}^{\infty} \frac{2m_i^4}{(2\pi)^2} \frac{\mathcal{K}_2(2\pi j \mathcal{R} m_i)}{(2\pi j \mathcal{R} m_i)^2} \tag{5.32}$$

with $\mathcal{K}_2$ Bessel function.
Given a particle of mass m in the spectrum of the SM, the Casimir energy associated with this particle has an exponential trend of the type

$$\mathcal{C} \propto e^{-2\pi m \mathcal{R}} \qquad \text{for} \qquad \mathcal{R} \gg 1/m\ . \tag{5.33}$$

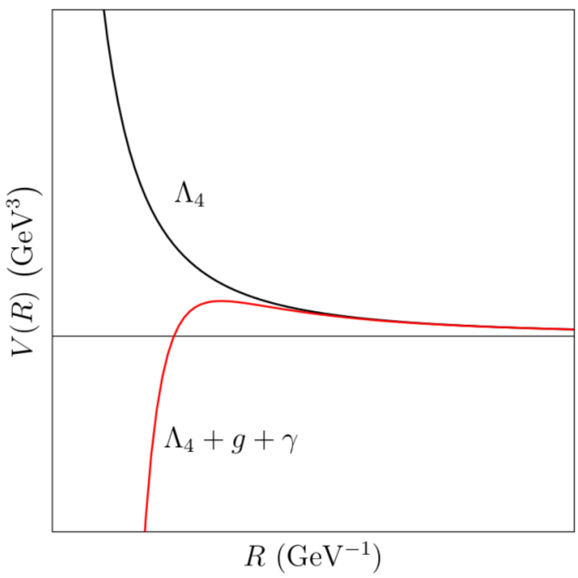

Figure 5.6: Radionic potential trend due to the contributions from cosmological constant, graviton and photon [44]

It is therefore concluded that the relevant contributions will come from the light components of the $SM + GR$ spectrum and specifically from states with mass less than $1/\mathcal{R}$. The massless states of the spectrum are those associated with photon and graviton, for which the Casimir potential is reduced in both cases to

$$\mathcal{C}_g = \mathcal{C}_{ph} = -\frac{2}{720\pi}\frac{r^3}{\mathcal{R}^6} \ . \tag{5.34}$$

The potential is negative since we are dealing with bosons and the equal number of degrees of freedom $(n = 2)$ has been taken into account. It is then seen that the contribution of the massless states is dominant with respect to the cosmological constant term for small values of the radionic field for which the overall potential $V(\mathcal{R}) = V_\Lambda + \mathcal{C}_g + \mathcal{C}_{ph}$ is therefore negative, returning the trend shown in Figure 5.6.

The maximum point is found for radion value

$$\mathcal{R}_{max} = \left(\frac{1}{120\pi^2\Lambda_{c.c}}\right)^{1/4} \simeq 7.55 \cdot 10^{10} GeV^{-1} \tag{5.35}$$

where we used the cosmological constant value [45] $\Lambda_{c.c} = (2.4 \cdot 10^{-3} eV)^4$ in natural units $(\Lambda_{c.c} = \rho_{\Lambda_{c.c}})$. We see that the mass scale associated with this maximum point is of the order of $10^{-3}eV$, that is, of the order of magnitude of the mass expected for light neutrinos [23]. It is therefore important for $\mathcal{R} < \mathcal{R}_{max}$ to take into account the contributions associated with neutrinos as particles associated with the lighter massive states of the SM. The Casimir potential due to neutrinos is expressed by the general formula (B.14) which can be expanded for small values of $m\mathcal{R}$ as

$$\mathcal{C}_\nu \approx +\frac{\pi^2}{(2\pi\mathcal{R})^4}\left[\frac{1}{90} - \frac{1}{6}(m_\nu\mathcal{R})^2 + \frac{1}{48}(m_\nu\mathcal{R})^4 + \mathcal{O}(m_\nu\mathcal{R})^6\right] \ . \tag{5.36}$$

Ultimately we therefore find the following expression for the radionic potential corrected by the Casmir energy due to gravitons, photons and neutrinos

$$V(\mathcal{R}) \approx \frac{2\pi r^3\Lambda_{c.c}}{\mathcal{R}} - 4\left(\frac{r}{720\pi\mathcal{R}^6}\right) + \frac{\pi^2}{(2\pi)^3}\frac{r^3}{\mathcal{R}^6}\sum_\nu n_\nu\left[\frac{1}{90} - \frac{1}{6}(m_\nu\mathcal{R})^2 + \frac{1}{48}(m_\nu\mathcal{R})^4\right] \ . \tag{5.37}$$

It can be seen that the presence of neutrinos, as fermions, brings positive contributions that can compensate for the bosonic ones and give rise to the formation of stable minima even for negative values of the potential depending on the numerical values of the masses m_ν (Figure 5.7). At this point it is clear in what sense the no-SUSY instability conjecture can be applied. In order to ensure the instability of the AdS vacua, it is possible to set upper limits to the neutrino masses.

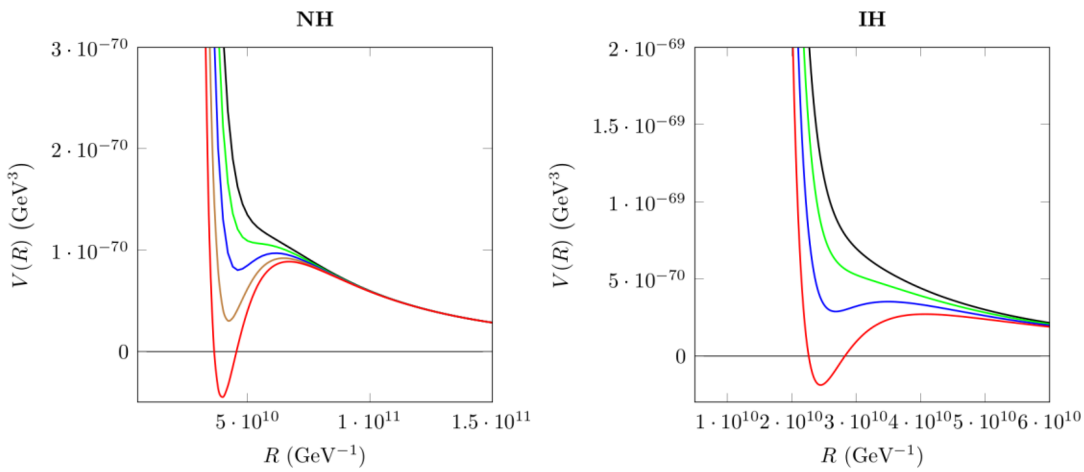

Figure 5.7: Radionic potential trend for different values of neutrino mass for both NH and IH [44]

In order to study these limits, reference is made to the experimental results obtained from the study of the oscillation processes for atmospheric and solar neutrinos that fix respectively the differences in mass

$$\Delta m^2_{atm} \simeq (7.53 \pm 0.18) \cdot 10^{-5} eV^2 \tag{5.38}$$

$$\Delta m^2_{\odot} \simeq (2.44 \pm 0.06) \cdot 10^{-3} eV^2 \, . \tag{5.39}$$

From these data, if m_i with $i = 1, 2, 3$ denote the eigenstates masses, we have $\Delta m^2_{12} = \Delta m^2_{\odot}$, $\Delta m^2_{23} = \Delta m^2_{atm}$ for normal hierarchy (NH) and $\Delta m^2_{12} = \Delta m_{\odot}$, $\Delta m^2_{31} = \Delta m^2_{atm}$ for inverted hierarchy (IH). At this point, from the study of the radionic potential, fixed the values (5.38, 5.39), we find the conditions on the mass m_1 for NH and on m_3 for IH required for the formation of AdS, dS vacua and for the absence of minima. In the case of Majorana neutrinos the number of degrees of freedom useful for the compensation of the bosonic contributions is insufficient to give rise to the formation of minima with positive potential. It is therefore concluded that this analysis tends to exclude the possibility of having Majorana type neutrinos at least in the case of the particle content of the SM.
In the case of Dirac neutrinos, on the other hand, the greater number of fermionic degrees of freedom (twice) allows, in addition to the realization of minimum AdS, also the realization of the conditions for having dS vacua or even the absence of stationary points for the radionic potential. The results found from the study of the exact expression for potential (5.37) are summarized in Figure 5.8.

	NH	IH
No vacuum	$m_{\nu_1} < 6.7$ meV	$m_{\nu_3} < 2.1$ meV
dS_3 vacuum	6.7 meV $< m_{\nu_1} <$ 7.7 meV	2.1 meV $< m_{\nu_3} <$ 2.56 meV
AdS_3 vacuum	$m_{\nu_1} > 7.7$ meV	$m_{\nu_3} > 2.56$ meV

Figure 5.8: Neutrino mass bounds for have dS_3, AdS_3 or for have no vacuum in both NH and IH [44]

In particular, it is concluded that the absence of stable AdS vacua is ensured for lightest neutrino mass thus constrained

$$m_\nu \leq 7.7 \cdot 10^{-3}\, eV^2 \qquad \text{(NH)} \tag{5.40}$$

$$m_\nu \leq 2.6 \cdot 10^{-3}\, eV^2 \qquad \text{(IH)}\,. \tag{5.41}$$

These results are obviously crucially linked to the value of the cosmological constant (see Figure 5.9 and 5.10, 5.11) for which the experimental value was assumed ([45]). If, on the other hand, $\Lambda_{c.c.}$ is left as a parameter and minimizes the potential (5.37), the positive minimum condition in approximation $m\mathcal{R} \sim 1$ is translated by

$$\Lambda_{c.c} \geq \frac{n_\nu(30n_\nu(\sum_i m_i^2)^2 + (4-3n_\nu)\sum_i m_i^4)}{(-3072+2304n_\nu)\pi^2}\,. \tag{5.42}$$

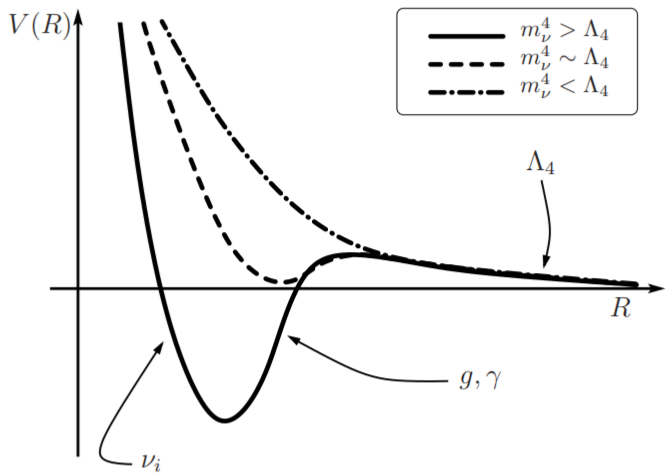

Figure 5.9: Radionic potential trend for different neutrino mass/cosmological constant relations [42]

We see then that in terms of orders of magnitude there is the following interesting relationship between the cosmological constant and the mass of the lightest neutrino

$$m_\nu \lesssim \Lambda_{c.c}^{1/4} . \tag{5.43}$$

This is a limit that according to the data of *Plank 2018* appears to be experimentally verified. It constitutes an interesting link between a cosmological parameter and a parameter of Particle Physics that indicates how QG is able to relate two different areas of physics. The phenomenon of neutrino oscillation, in this sense, confirms the idea of a non-zero value of the cosmological constant regardless of the presence of dark energy.

Note that the analysis stopped to consider the contribution of neutrinos while neglecting the other states of the SM spectrum. This approximation is motivated by the fact that the lightest mass state after that of neutrinos is the state associated with electrons for which the Casimir potential is suppressed by a factor $e^{-2\pi m_e \mathcal{R}} \sim e^{-m_e/m_\nu}$.

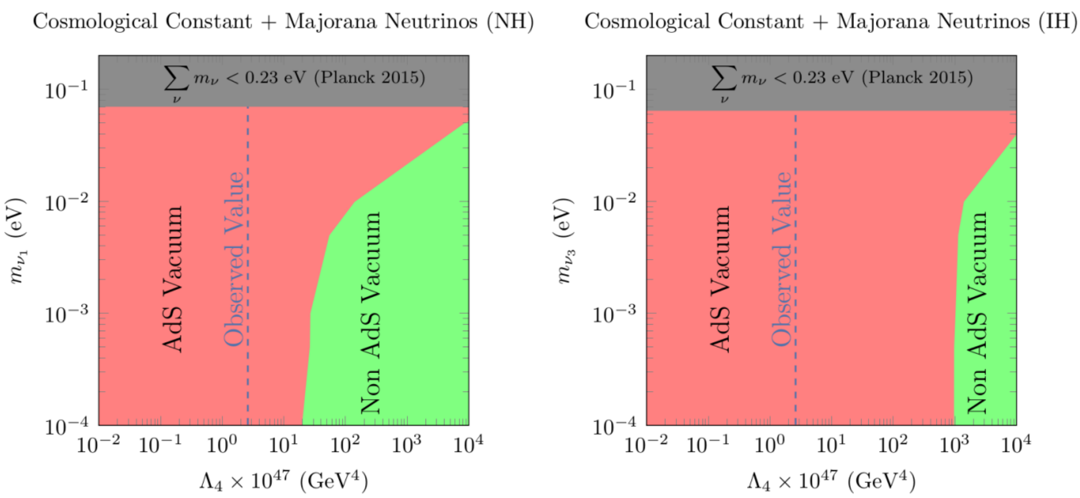

Figure 5.10: Allowed and non allowed regions for Majorana neutrino masses depending on cosmological constant value for both NH and IH [44]

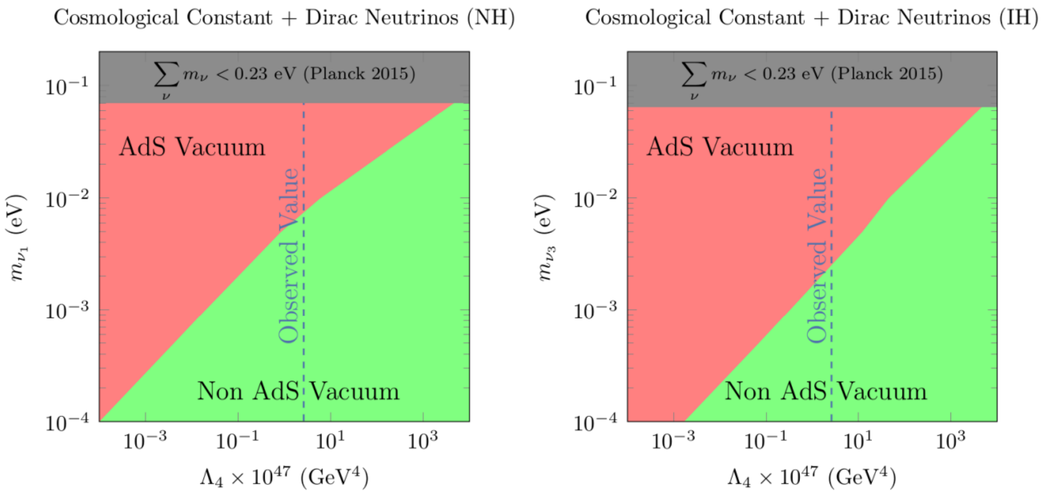

Figure 5.11: Allowed and non allowed regions for Dirac neutrino masses depending on cosmological constant value for both NH and IH [44]

Chapter 6

The Swampland Distance conjecture

Let's take into consideration the Swampland Distance conjecture (SDC) which, more then No Global symmetry conjecture, as seen in detail in Chapter 3, draws its major motivation in String Theory. The purpose of this chapter is to study in detail the mathematical formalism on which the statement of the conjecture is based in order to make the appropriate generalizations with respect to what has already been said for the compactification of the bosonic string and above all in order to investigate the possible phenomenological consequences. In the context of String Theory, in fact, we saw the motivation that gave impetus to the formulation of the conjecture but in reference to the theory of QG for which the validity of the conjecture is somewhat expected and does not constitute an element of novelty. Instead, the goal is to study the effects of the conjecture at the EFT level.
To this end, let's start by recalling the statement of the conjecture already given in Chapter 2 in its simplest version.

> **Distance conjecture**. Consider a theory, coupled to gravity, with a moduli space $\mathcal{M}$. Starting from any point $P \in \mathcal{M}$ there exists another point $Q \in \mathcal{M}$ such that the geodesic distance between P and Q, denoted by $d(P, Q)$, is infinite.
> Furthermore, there is an infinite tower of states with an associated mass scale M such that
> $$M(Q) \approx M(P)e^{-\alpha d(P,Q)} \tag{6.1}$$
> where α is some positive constant.

The first step to extrapolate physics from the SDC is to clarify the mathematics on which the formulation of the statement is based and therefore, in particular, to provide the definition of moduli space and of the concept of geodesic distance on this space.

6.1 The Moduli space

In general, in differential geometry, the moduli space is defined as the space of generic geometric classes and can intuitively be thought of as the space of the characteristic parameters of a given geometry. In physics the concept of moduli space is used in Quantum Field Theory to indicate the space of the possible vacuum states of the theory. On the other hand, typically, the possible vacua of a QFT are labeled by the expectation values of some scalar fields. If a scalar field has continuous degeneration of minimum points then there can be a manifold $\mathcal{M}$ called precisely *vacuum* or *moduli space*. In the context of Quantum Gravity, and in particular in string theory, the notion of moduli space in addition to characterizing the string background serves to control the trend of the physical parameters of interest of the theory. In this sense, in the view of Swampland approach, it is useful to investigate the behaviour of the theory by moving along the moduli space and, in particular, to study its behaviour when approaching asymptotic limits. The knowledge of QG behaviour as the parameters of the theory change can in fact prove useful to test the limits of validity of the EFTs and deduce the necessary changes in order to preserve compatibility with the QG. This is what we intend to do with the Distance conjecture. Given a k-dimensional moduli space $\mathcal{M}$ it can in general be parametrized by the expectation values of real scalar field ϕ^i, with $i = 0, ..k$, which defines a coordinate system on the moduli manifold. The same field is also used for the definition of the metric of the manifold starting from its kinetic term. If we consider an EFT weakly coupled to gravity and with the ϕ scalar field, it will be described by the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2} - h_{ij}(\phi^i)\partial\phi^i\partial\phi^j + ...\right] . \tag{6.2}$$

The kinetic term that appears for the field ϕ implicitly defines the metric h_{ij} of the moduli space $\mathcal{M}$ which characterizes the degree of complexity of the space. $(\mathcal{M}, h_{ij})$ will then be a Riemann manifold on which it is possible to define a notion of distance and, in particular, of geodesic distance. For this purpose, given two points of the manifold, P and Q, and given a geodetic curve γ connecting the two points, the geodetic distance is defined

$$d(P,Q) \equiv \int_\gamma \mathrm{d}s\left(h_{ij}\frac{\partial\phi^i}{\partial s}\frac{\partial\phi^j}{\partial s}\right) . \tag{6.3}$$

Note at this point that the conjecture imposes limits on the possible space of the modules compatible with the QG. It is in fact required that for every point P there exists a point Q at an infinite geodetic distance. This request is not satisfied by all the spaces parametrized by scalar fields as in the case of fields with periodic expectation values $\phi \sim \phi + 2\pi$. In this case, for example, the compatibility of such a field with the QG is allowed only if the moduli space associated to it can be framed in a larger moduli space compatible with the conjecture. It should also be noted that the evidence given in Chapter 3 about the compactification process of a bosonic string shows the validity of the conjecture in the simplest case in which the moduli space is the real semi-axis, $\mathcal{M} = \Re^+$, and the parameter that acts as modulus is the radion r. It is therefore a somewhat restrictive example that does not account for the vast applicability of the conjecture to spaces of the more complicated moduli.

Consider, in the light of the formalism introduced, the relationship between the mass scales of the infinite tower of states evaluated at two different points of the manifold

$$M(Q) \approx M(P)e^{-\alpha d(P,Q)} \ . \tag{6.4}$$

It is clear that in the asymptotic limit of infinite geodesic distance there is a massless tower of states and that the approach to this limit is of a decreasing exponential type. On the other hand, the conjecture does not refer to the quantitative value of the positive constant α that regulates the decay rate. However, it can be said that, on the basis of the reasons presented in the context of String Theory, a rate of the order of unity is expected.

In any case, the presence of an infinite tower of zero mass states expresses the breakdown of the effective description which can be better formalized in terms of cut-off through the analogous relation

$$\Lambda \approx M_P e^{-\alpha \Delta\phi} \tag{6.5}$$

in which the trend of the validity of an EFT is evident as the geodetic distance on the moduli space increases, i.e. as the parameters of the theory change. In particular, it is concluded that the EFTs are correctly defined only for finite geodetic distances and, by inverting the above relation, for the scalar field we find

$$\Delta\phi \approx \frac{1}{\alpha} \ln \frac{M_P}{\Lambda} \tag{6.6}$$

which expresses the variability of the field that can be described by EFT with a cut-off Λ.

The presence of an infinite tower of states that frustrates the effective theory gives rise to the cut-off (6.5). This consequence of the SDC is consistent with a more general idea according to which the formation of a large number N_s of weakly coupled states makes gravity strongly coupled and therefore identifies a cut-off scale called species bound cut-off. This result is contained in the following conjecture [5]

Species Bound cut-off. Given a D-dimensional effective field theory coupled to gravity with N_s species states below a cut-off scale Λ, there is a species bound cut-off

$$\Lambda_s = \frac{M_P}{N^{\frac{1}{D-2}}} > \Lambda \ . \tag{6.7}$$

The consequence of this statement, according to the SDC, is that the effects of Quantum Gravity can be observed even before the Planck scale. On the other hand, it is an expected and desirable result on which the entire study of the Swampland approach is based.

6.1.1 Relation with others conjectures

The relationship of the SDC with the Weak Gravity conjecture is twofold. A first way to relate the two conjectures is through the species bound cut-off presented in the previous section. If we consider a tower of states with a mass variation Δm, the number of species N_s can be written as

$$N_s = \frac{\Lambda}{\Delta m} \ . \tag{6.8}$$

By virtue of the species bound it is therefore

$$\Lambda_s = M_P^{\frac{D-2}{D-1}} \Delta m^{1/D-1} \ . \tag{6.9}$$

On the other hand, the electric version of the WGC expressed in terms of mass gap gives $\Delta m \sim g$ which replaced in (6.9) provides

$$\Lambda_s \sim g^{1/D-1} M_P^{\frac{D-2}{D-1}} \ . \tag{6.10}$$

It is then concluded that according to the SDC when the gauge coupling constant g goes to zero we have $\Lambda_s \to 0$ and therefore the breaking of the effective description.
For the magnetic version, the link appears to be in some ways more direct. Indeed, as the SDC does, this version of the conjecture imposes a restriction on the cut-off of the theory but in terms of the coupling constant g.

$$\Lambda \leq g M_P^{\frac{D-2}{2}} \ . \tag{6.11}$$

In [5, 4] the scale imposed by the WGC is associated with the formation scale of an infinite tower of states. This tower is then identified with the state tower foreseen by the SDC. This identification provides that there is the following correspondence between the gauge coupling constant g and the scalar field ϕ associated with the moduli space of the EFT

$$g \to e^{-\phi} \, . \tag{6.12}$$

Also note that this relationship with the electric and magnetic version of the WGC highlights an interesting link that the SDC has with the No Global Symmetry conjecture. The fact that for $g \to 0$ there is the formation of an infinite tower of states and that at the scale (6.11) the EFT breaks, means that, according to what has been said in section 5.4, the SDC as well as the WGC provide an obstruction to the formation of a global symmetry justifying in a certain sense what is required by the NGS conjecture.

The different way of relating the SDC to the WGC is to refer to an alternative formulation of the second. Depending on the context, the WGC has in fact multiple formulations. In particular, in the case of the presence of a scalar field, such as that of the SDC, the following scalar version is proposed

Scalar Weak Gravity conjecture. Given a D-dimensional EFT weakly coupled to gravity with some massless scalar fields, there must exist a state with mass m satisfying the following condition

$$g^{ij}\partial_i m \, \partial_j m > \frac{D-3}{D-2} \, m^2 \tag{6.13}$$

The idea of such version of the WGC is to specify the condition for the gravitational interaction to be weaker than any scalar interaction. It can be seen then that in the case of a single scalar field ϕ it is reduced to imposing the following condition on the course of the mass of the field

$$|\partial_\phi m| \; > \; \frac{1}{\sqrt{2}} \, m \tag{6.14}$$

It is then noted that the scalar WGC is certainly satisfied for masses that decrease exponentially as the distance in the moduli space increases, i.e. for $m \sim e^{-\alpha\phi}$, in accordance with the SDC (6.1).

6.1.2 The Distance conjecture in String Theory

Consider the effective D-dimensional theory associated with the compactification of a bosonic string on a circle. If $\mathcal{R}$ is the compactification radius, the scalar field

ϕ associated with the radion of the theory is introduced

$$\phi \equiv \sqrt{\frac{D-1}{D-2}} \ln \mathcal{R} \, . \tag{6.15}$$

The action will then be obtained as the sum of the Einstein-Hilbert action with the kinetic term of the field ϕ

$$S_{eff} = M_P^{D-2} \int \mathrm{d}^D x \sqrt{-g} \left[\frac{R}{2} - \frac{1}{2} \left(\partial \phi \right)^2 \right] \, . \tag{6.16}$$

The radionic scalar field defines a moduli space consisting of the real axis, $\mathcal{M} = \mathbb{R}$. In this space, two asymptotic regions associated with compactification and decompactification limits are observed

- $\mathcal{R} \rightarrow 0 \quad \Rightarrow \quad \phi \rightarrow -\infty$
- $\mathcal{R} \rightarrow +\infty \quad \Rightarrow \quad \phi \rightarrow +\infty$.

In the first case there is the formation of an infinite tower of states due to the string winding modes around the compact size. The mass of these modes, in terms of the scalar field, has the exponential trend

$$m_w \sim e^{-\sqrt{\frac{D-1}{D-2}}\phi} \, . \tag{6.17}$$

Similarly, in the decompactification limit it has been seen that in this case the tower of states is provided by Kaluza Klein states. The mass m_{KK} of these states maintains the same exponential trend in the opposite limit (Figure 6.1).

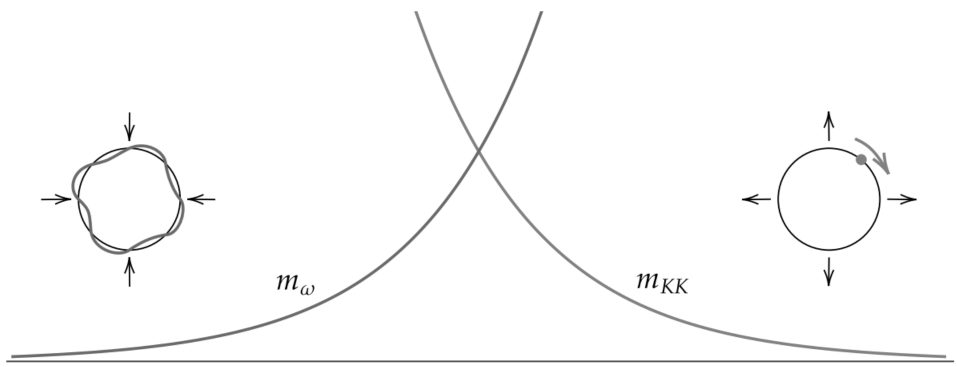

Figure 6.1: Mass scale exponential decadence for both winding and Kaluza-Klein modes respectively in compactification and decompactification limits [4]

It is concluded that in the simple compactification model of a bosonic string the Swampland Distance Conjecture is verified if we take the radion as scalar potential, $\phi(\mathcal{R})$, and the real axis $\mathbb{R}$ as the moduli space. Furthermore, the direct observation of the conjecture in this specific model allows to explicitly determine the decrease rate

$$\alpha = \sqrt{\frac{D-1}{D-2}} \tag{6.18}$$

which is therefore consistent with the expectation $\alpha \sim \mathcal{O}(1)$.
Also note that the realization of the conjecture is associated with the presence of T-duality within the theory

$$\mathrm{T}: \quad \omega \leftrightarrow \kappa \quad \Rightarrow \quad \mathcal{R} \leftrightarrow \tilde{\mathcal{R}} \sim \frac{1}{\mathcal{R}}\,. \tag{6.19}$$

The relationship is associated with the fact that the two asymptotic bounds in the module space provide two dual descriptions of compactification as already discussed in the Section 4.2.
This feature, at least in the context of String Theory, highlights a relationship between distance conjecture and duality which is supposed to have general validity. The SDC therefore is supposed to formally summarize the observation that the theory of Quantum Gravity provided by String Theory has dualities. On the other hand, it is no coincidence that it is now well established that duality transformations play a crucial role in String Theory as they provide a unified picture of the various possible string theories.

6.2 The de Sitter vacuum

The expression provided for the SDC refers to a moduli space that can be parametrized through a scalar field ϕ which is supposed to be potential-free. This restriction was dictated by the desire to give an essential version of the conjecture sufficient to contain what is needed to account for the evidence that one has in String Theory and to carry out natural generalizations. However, if we want to deeper address the application of the conjecture in order to quantitatively test the EFTs, it is necessary to extend the conjecture to the case of fields with scalar potential $V(\phi)$ and therefore to the case of fields not suitable for parametrizing moduli spaces. A refined Swampland Distance conjecture is therefore proposed

Refined Distance conjecture. Consider a theory, coupled to gravity, in which there is a scalar field ϕ with a potential $V(\phi)$. Given two points in the field space,$P, Q \in \mathcal{M}$, there exists an infinite tower of states with an associated mass scale M such that

$$M(Q) \approx M(P)e^{-\frac{\alpha d(P,Q)}{M_P}} \tag{6.20}$$

if $d(P, Q) \geq M_P$.

So we will no longer talk about moduli spaces but, more generally, about *field* or *pseudo-moduli space*. It should be noted that the proposed reformulation, in addition to operating the desired generalization, resolves the already highlighted ambiguity of quantitatively estimating the decadence rate α contextually to the variation of the geodetic distance under consideration, $d(P, Q)$. The exponential behavior of the mass scale is now supposed to be manifest for distances $d(P, Q) \geq M_P$. In other words, we deduce that in the refined SDC we explicitly include the requirement that the decadence rate should be $\alpha \approx \mathcal{O}(1)$ at least within the limits of asymptotic distances.
The goal of generalizing the SDC lies in the fact that it is now possible to adapt the statement of the conjecture to a large class of EFTs with a generic scalar potential $V(\phi)$ and use it as a selection criterion to narrow the set of EFTs compatible with an embedding in QG. To see how this procedure works, it is necessary to extract the physics contained in the SDC.
The fact that at asymptotic geodetic distances in $\mathcal{M}$ there is the formation of an infinite tower of massless states can be formalized directly in terms of the number of states $N(\phi)$ within the EFT, i.e. the number of states at energy below the cut-off Λ, as

$$N(\phi) \approx n(\phi)e^{\beta\phi} \tag{6.21}$$

where β is a positive constant, generally different from α, depending on the constituent objects of the tower and $n(\phi)$ refers to the number of towers.
At this point, consider a space with a curvature radius R and study the trend of entropy as a consequence of the formation of the towers of states. It will be found that the aforesaid entropy S_{tower} will depend on the number of states N and on the geometric properties of the space, $S = S(N, R)$, with R curvature radius of the space. In particular, it can be assumed that $S(N, R)$ has a polynomial trend in the two variables N, R of the type

$$S_{tower} \approx N^{\gamma} R^{\delta} \ . \tag{6.22}$$

Moreover, according to what Bousso found [46], it can be said that for a generic *quasi-de Sitter* space the upper bound to the entropy set by the Gibbons-Hawking entropy [47] must hold [1]

$$S_{tower}(N,R) \leq S_{GH} = R^2 \; . \tag{6.23}$$

In the case of a quasi-de Sitter space, from effective cosmology models, we find (see Appendix D)

$$V(\phi) \approx R^{-2} \; . \tag{6.24}$$

Then, assuming that the tower of states provides a preponderant contribution so that the Bousso bound can be thought to be saturated, we conclude

$$V(\phi) \approx N^{-\frac{2\gamma}{2-\delta}} \; . \tag{6.25}$$

The result thus obtained can be used to relate the scalar potential with its gradient, obtaining what goes under the name of Asymptotic de Sitter conjecture

Asymptotic de Sitter conjecture. A scalar potential $V(\phi_i)$ of an EFT weakly coupled to gravity presents a runaway behavior when approaching an infinite field distance point

$$\frac{|\nabla V|}{V} \geq \frac{c}{M_P} \tag{6.26}$$

with c parameter such that $c \sim \mathcal{O}(1)$.

In the light of the derivation presented starting from the SDC, the parameter c that appears in the conjecture can be explained in terms of the parameters of the power laws of entropy S and of the growth rate of the number of states N as

$$c = \frac{2\gamma\beta}{2-\delta} \; . \tag{6.27}$$

The name of the conjecture is due to the fact that the relation obtained between potential and its gradient excludes the possibility of having de Sitter vacua at least in the asymptotic limit of large geodetic distances in the scalar field space. In other words, de Sitter vacua would be incompatible with QG in accordance with the multiple evidences provided by String Theory.

[1] Actually, the inequality used constitutes a less restrictive version of the Bousso bound $S_{matter} \leq S_{GH} = R_g R_c$, in which R_g and R_c are respectively the gravitational and cosmological radius.

Actually, the result obtained starting from the refined version of the SDC constitutes a particular and widely accepted case of a secondary conjecture proposed within the Swampland program. This conjecture, known as de Sitter conjecture, attempts to generalize what was found by assuming that the exclusion of de Sitter vacua is valid in all regimes of the pseudo-moduli space.

de Sitter conjecture. The scalar potential $V(\phi)$ of a theory coupled to gravity must satisfy either

$$\frac{|\nabla V|}{V} \geq \frac{c}{M_P} \tag{6.28}$$

or

$$\frac{min(\nabla_i \nabla_j V)}{V} \leq -\frac{c'}{M_P^2} \tag{6.29}$$

with c, $c' \geq 0$ order one constants.

Typically, in the literature the condition (6.28) is known as the ordinary de Sitter conjecture while the addition of the condition (6.29) gives rise to its refined version [48, 49, 50].
This is clearly a very stringent request for which, however, there is not enough evidence to attest its goodness. The restriction imposed on the second derivatives of the potential is made necessary by the fact that the first condition alone would give rise to counter-examples and incompatibility with consolidated predictions of the Standard Model (see Section 6.4).

6.3 Applications in Cosmology

Based on what has been said in the previous section, it is interesting to go to study the applications that the Swampland approach has for the purposes of cosmology. In particular, the implications of the SDC and its redefined version will be studied in detail. As already mentioned, the de Sitter conjecture, both in its asymptotic and in its generalized formulation, expresses an obstruction to the possibility of having de Sitter vacuum in certain regions of the moduli space. In fact, based on current observations, it would seem that our universe is in a phase of expansion for which there is a potential $V(\phi) > 0$. The de Sitter conjecture therefore requires that the potential cannot be in a minimum point and must rather to roll continuously. On the other hand, the derivation of the asymptotic de Sitter conjecture from the redefined SDC was carried out thinking of quasi-de Sitter cosmological models. The resulting theoretical framework is that of a consistent program that lends itself to formulating conclusions regarding the physics of the evolution of

the expanding universe. The applications concern both inflationary and dynamic dark energy models [48, 49, 51]. The interest lies in the possibility of formulating constraints on the characterizing parameters of these models to be compared with cosmological observations.
To study this type of implications, first of all introduce the Hubble constant H which is required to be less than the cut-off of the effective theory to be studied. The speech made in the previous section immediately fits into the study of Cosmology if the following correspondence is made

$$R^{-1} \quad \to \quad H \ < \ \Lambda \, . \tag{6.30}$$

It is a necessary condition for a consistent cosmological effective theory.
In the Appendix D it is shown that the inflationary models in slow-roll approximation are fixed by the following two parameters

$$\epsilon = \frac{M_P^2}{2}\left(\frac{\partial V}{V}\right)^2 \tag{6.31}$$

$$\eta = M_P^2 \frac{\partial^2 V}{V} \, . \tag{6.32}$$

In a perturbative treatment, given a generic perturbation ψ it is studied its spectral distribution in terms of powers

$$\Delta_\psi^2(k) = \frac{k^2}{2\pi^2}\left|\psi_k\right|^2 \tag{6.33}$$

with k mode index. To parametrize the dependence on the modes, the spectral index is introduced

$$n_s \equiv 1 + \frac{d \ln \Delta^2(k)}{d \ln k} \, . \tag{6.34}$$

Furthermore, since the scalar perturbations are flanked by the tensor perturbations, assuming that both originate from the inflationary phase, it makes sense to quantify the relative intensity of the two perturbation amplitudes (Δ_s and Δ_t respectively) by introducing the so-called tensor-to-scalar ratio

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_s^2(k)} \, . \tag{6.35}$$

For a quasi-de Sitter space we find the following expressions [52]

$$\Delta_s^2 = \frac{1}{8\pi^2}\frac{H^2}{\epsilon} \tag{6.36}$$

$$\Delta_t^2 = \frac{2}{\pi^2}\frac{H^2}{M_P^2} \, . \tag{6.37}$$

The two quantities n and r then allow to characterize a generic inflationary model and therefore also the particular case of a slow-roll approximation model. The parameters ϵ and η are thus related to the spectral index and the scalar-tensor ratio

$$n_s = 1 - 6\epsilon + 2\eta \qquad r = 16\epsilon \,. \tag{6.38}$$

At this point, note that the variation of the scalar field dictated by the SDC (6.6) can be re-expressed in terms of these quantities as

$$\Delta\phi \leq -\frac{1}{2\lambda}\left[\ln\left(\frac{\pi^2\Delta_s^2}{2}\right) + \ln r\right] \tag{6.39}$$

where where we used the result (6.36). On the other hand, in the context of inflationary cosmology there is a further constraint. In the hypothesis of short inflationary durations such that $r \sim const$, by virtue of (6.38) and (D.22, D.24), it holds the so-called Lyth bound for $\Delta\phi$ [53, 54]

$$\Delta\phi \geq \left(\frac{r}{0.002}\right)^{1/2} \,. \tag{6.40}$$

This means that it is possible to use the combination of the two constraints (6.39, 6.40) offered by Cosmology and Swampland conjectures to impose restrictions on possible inflationary models compatible with the quantum description of gravity (figure 6.2).

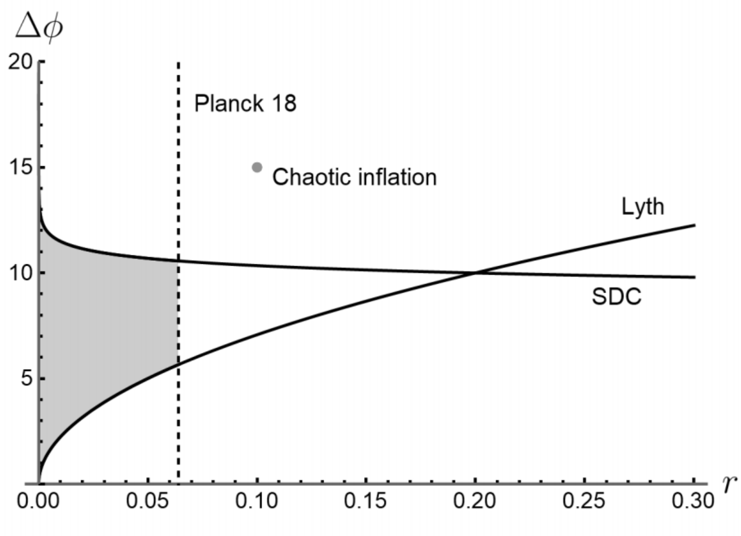

Figure 6.2: Allowed scalar field excursion as function of the tensor-to-scalar ratio obtained by the combination of SDC and Lyth bounds with Planck 2018 result. Same inflationary models are excluded like, for example, chaotic inflation types [4]

As far as the application of the de Sitter conjecture is concerned, the situation is more subtle. If it is true that the absence of de Sitter vacua is qualitatively in agreement with inflationary scenarios, the same cannot be said from a quantitative point of view. The constant c in (6.28) is in fact directly linked to the inflation parameter ϵ. The slow-roll conditions (D.18) would appear to conflict with the condition (6.28) required by the conjecture. There is therefore potentially a tension with inflation that is more or less stringent depending on how precise the expectation $c \sim \mathcal{O}(1)$ is.
On the basis of recent observations, the upper limit for the tensor-scalar ratio has been established [45]

$$r \; < \; 0.0063 \tag{6.41}$$

so that, by virtue of the second of relations (6.38), we find the following experimental result

$$\frac{|\nabla V|}{V} \lesssim \frac{0.02}{M_P} \, . \tag{6.42}$$

This limit is rather in conflict with the condition (6.28) unless we leave greater margins of variability to the unity constant c. A different way of solving the tension is to refer to the refined version of the de Sitter conjecture. The application of conditions (6.28, 6.29) to slow-roll inflation then reduces to the following alternatives

$$\epsilon \; \geq \; \frac{c^2}{2} \qquad \text{or} \qquad \eta \; \leq \; -c' \, . \tag{6.43}$$

The idea therefore is that during inflation for a certain number of e-folds condition (6.28) is met and for the remaining number of e-folds the inflation potential becomes sufficiently concave to satisfy condition (6.29).

The tensions of de Sitter conjecture with observations are reduced when applied to dark energy models [55]. The idea of a monotonic rolling potential fits into so-called dynamic dark energy scenarios. In these models the accelerated expansion of the current universe is attributed to a time-varying scalar field Q rather than to the cosmological constant. A fluid model also applies to it but the state equation $p = w\rho$ is now characterized by a parameter

$$w = \frac{\dot{Q}^2/2 - V_Q(Q)}{\dot{Q}^2/2 + V_Q(Q)} \, . \tag{6.44}$$

Among the most accredited models of dynamic dark energy there is the quintessential one in which the potential takes the exponential form

$$V_Q(Q) = V_0^{(Q)} e^{-\frac{c_Q}{M_P} Q} \, . \tag{6.45}$$

In this case the deviation from the parameter $w = -1$ predicted by the standard cosmological model is expressed by

$$1 + w = \frac{2(\partial_Q V)^2}{(\partial_Q V)^2 + 3H\frac{c_Q}{M_P}V_Q^2} \ . \tag{6.46}$$

In order for observation to be satisfied [45] we find that a consistent quintessential model must have

$$0.5 \ \leq \ c_Q \ \leq \ 0.9 \ . \tag{6.47}$$

On the other hand, if the de Sitter conjecture is applied, the lower bound is found

$$1 + w \ > \ \frac{2c^2}{6 + c^2} \tag{6.48}$$

which, according to observations, is satisfied if we have at least

$$c \ \sim \ 0.6 \ . \tag{6.49}$$

Such estimate is in good agreement with the expectation $c \sim \mathcal{O}(1)$, consequently it can be said that the Swampland approach is consistent with dark energy models. Otherwise, given that Swampland conjectures should in principle have general validity regardless of the application context, one could think of using the result (6.42) obtained from the study of inflation. In this case the limit (6.48) would become

$$1 + w \ > \ \frac{2(0.02)^2}{6 + (0.02)^2} \ = \ 1.3 \ \cdot \ 10^{-4} \ . \tag{6.50}$$

This means that the de Sitter conjecture would still make predictions for dynamic dark energy scenarios but these would not be observable given the current measurement sensitivity.

6.3.1 Modified Gravity models

To study in more detail the implications that Swampland approach may have on Cosmology, consider the case of a modified gravity model of type $f(R)$ [56] coupled to matter

$$S = \int d^4x\sqrt{-g} \ [f(R) + \mathcal{L}_m] \ . \tag{6.51}$$

To fit the $f(R)$ description to an effective string theory, consider the effective graviton-dilaton string action

$$S = \int d^Dx\sqrt{-g^{(D)}}e^{-2\phi}[R + 4\nabla_\mu\phi \ \nabla^\mu\phi + \Lambda] \ . \tag{6.52}$$

Note that this action, expressed for Friedmann-Roberston-Walker metrics (D.4), enjoys the invariance for *scale factor duality*, $a \to a^{-1}$ and $\phi \to \tilde{\phi} = \phi - (D - 1)\ln a$, indirectly connected to T-duality [57].
Let us perform a conformal transformation of the type

$$g_{\mu\nu} \;\; \to \;\; \tilde{g}_{\mu\nu} = e^{-2k\phi} g_{\mu\nu} \, . \tag{6.53}$$

The two theories can be related by means of the following correspondence [58]

$$\sqrt{-g}\, e^{-2\phi}[R + 4\nabla_\mu \phi \, \nabla^\mu \phi + \Lambda] = \sqrt{-\tilde{g}}\, f(\tilde{R}) \tag{6.54}$$

where

$$f(\tilde{R}) = 2\Lambda e^{2\phi} \, . \tag{6.55}$$

In the Einstein frame the (6.52) becomes [59, 60]

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2} + \frac{1}{2} \tilde{g}_{\mu\nu} \partial^\mu \phi \, \partial^\nu \phi - V(\phi) + \mathcal{L}_m \right] \tag{6.56}$$

where the scalar field is given by

$$\phi = \frac{1}{2k} \ln(\partial_R f) \tag{6.57}$$

while the potential associated with it has form

$$V = -\frac{1}{2} \left[\frac{f - R\, \partial_R f}{(\partial_R f)^2} \right] \, . \tag{6.58}$$

We now have an explicit expression of the scalar potential for which the conjectures can be implemented. Depending on the functional expression of $f(R)$ we will have a different model of modified gravity. Then we can at this point think of applying the Swampland conjectures to select the $f(R)$ models compatible with the quantum description of gravity provided by the string theory.
In particular, for the scalar field ϕ, the Distance conjecture is applied to narrow the range of variation consistent with the effective treatment it provides (in Planck units)

$$\Delta\phi = \frac{1}{2k} \frac{1}{\partial_R f} \Delta R \sim \frac{1}{\alpha} \ln \frac{M_P}{\Lambda} \, . \tag{6.59}$$

Furthermore, for the scalar potential one can use the de Sitter conjecture from which, in the case of $f(R)$ invertible function, it is found

$$\frac{|\nabla_\phi V|}{V} = \left| \left(\frac{4k}{f - R\partial_R \, f} \right) \left(\frac{R\, \partial_R f}{2} - f \right) \right| \; > \; c \, . \tag{6.60}$$

In [59] it is discussed what happens for a generic power law $f(R)$ model of the type

$$f(R) \propto R^{1+\lambda} \equiv R + \lambda\, R \ln R + \mathcal{O}(\lambda^2) \tag{6.61}$$

with $\lambda \ll 1$. Note that this parametrization has the usefulness of highlighting the perturbative deviations with respect to General Relativity which corresponds to the limit case $\lambda = 0$. The scalar potential (6.58) for such theories assumes the explicit form

$$V(\phi) = \frac{\lambda}{2(1+\lambda)^2} \left(\frac{e^{2k\phi}}{1+\lambda} \right) . \tag{6.62}$$

The analysis of such potential shows that models which are meaningful to slow-roll inflation generally tend to fit well with the SDC. In particular it turns out that the conjecture is satisfied for

$$0.39 \ \leq \ \lambda \ \leq \ 1.5 \, . \tag{6.63}$$

Otherwise, as already discussed above, greater tensions can be deduced with the de Sitter conjecture. In this case the class of compatible theories is restricted to models with power law such that

$$0 \ \leq \ \lambda \ \leq \ 0.62 \qquad \text{or} \qquad \lambda \ > \ 2 \tag{6.64}$$

of which, however, only the former are relevant for inflation. The combination of the two limits turns out to be particularly stringent and, in particular, admits the Liouville modified gravity theory $f(R) \sim R^{3/2}$ [61]. This is consistent with the fact that such theory exhibits the string duality, especially in light of SDC's relationship with T-duality presented in Section 6.1.2. Note that the analysis is conducted assuming a unit order of magnitude for the constant c. Therefore, the general considerations made at the beginning of the section apply to the interpretation of these results.

6.4 Connection with Particle Physics

In order to study the tensions of the dS conjecture (6.28) with Particle Physics it is necessary to consider the scalar fields of the SM [62, 63]. In particular, consider the Higgs field h for which the potential is perturbably expressed by

$$V_h(h) = \lambda(h^2 - v^2)^2 \tag{6.65}$$

it is clear that the potential has a minimum for $h = v$ and a maximum for $h = 0$ where it results

$$V_h(h=0) = \lambda v^4 \, . \tag{6.66}$$

In order to apply the conjecture, consider the combination with the quintessence potential (6.45), i.e. consider the EFT with scalar potential

$$V(Q,h) = V_Q(Q) + V_h(h) \; . \tag{6.67}$$

In this case, from what has been said in Section 6.3 and from the estimate (6.47), we find the following relationship between the potential (6.67) and its gradient (in Planck units)

$$\frac{|\nabla V|}{V} \approx \frac{|\nabla V_Q|}{V_h(0)} \sim \frac{10^{-120}}{10^{-65}} \sim 10^{-55} \; . \tag{6.68}$$

It is clearly seen that this estimate is strongly in conflict with the expectation highlighted by the de Sitter conjecture. A possible solution to the problem can be constructed by assuming a different expression for the Higgs potential suggested by the quintessence

$$V(Q,h) = V_Q(Q) + V_h'(h,Q) = e^{-\lambda Q}(V_h(h) + V_0) \; . \tag{6.69}$$

In this way the potential has no maximum points and the de Sitter conjecture is restored.

A second point of conflict arises from the extension of SM obtained by introduction of the axion scalar fied a. In such a model we have an additional scalar field

$$V_a(a) = \Lambda_{QCD}\left[1 - \cos\left(\frac{a}{f_a}\right)\right] \tag{6.70}$$

which similarly presents a maximum for $a = \pi f_a$ in which

$$V_a(a = \pi f_a) = 2\Lambda_{QCD} \; . \tag{6.71}$$

The evaluation of the relationship between the potential $V(Q,a) = V_Q(Q) + V_a(a)$ and its gradient gives in this case the following estimate (in Planck units)

$$\frac{|\nabla V|}{V} \sim \frac{\Lambda_Q}{\Lambda_{QCD}} \sim \frac{10^{-120}}{10^{-76}} \sim 10^{-44} \tag{6.72}$$

which again contradicts the constraint posed by the de Sitter conjecture. Also in this case the problem would be solved by ad hoc modifications of the axion potential. On the other hand, given the scarce evidence for the de Sitter conjecture, it is preferred to modify the result of the Swampland program and adapt it to Particle Physics. This can be done by referring to the refined version of the conjecture proposed in Section 6.2 by adding the condition (6.29) which is in fact certainly satisfied by the potentials (6.67, 6.70).

Chapter 7

Conclusions

In this thesis the Swampland approach was presented as a possible method of investigation to test the effects that Quantum Gravity can have on observable low-energy physics. To do this, it is made use of Swampland conjectures, i.e. principles that impose severe restrictions on effective descriptions of matter coupled to gravity.

In order to understand the origin of the conjectures it was necessary to recall the main results of String Theory in which EFTs are constructed by compactification. The compactification process was then discussed in detail by presenting the usual theory of Kaluza Klein to describe the mechanism by which it is possible to derive matter starting from a pure geometry with extra dimensions. Furthermore, a lesser known extension in the literature concerning the extension of the theory to the case of non-Abelian gauge fields has been dealt with. We have seen how in this situation there are difficulties in the realization of Kaluza's idea. On the other hand, the advantage of considering terms of matter also in extra-dimensional geometry is that of obtaining a spontaneous compactification. The application of the compactification mechanism in the specific context of String Theory explicitly realizes the T-duality of the theory, i.e. the equivalence of the compactification and decompactification limits, as an exotic property of geometry at the Planck scale. Furthermore, the compactification of the bosonic string naturally takes into account the possibility of having Dirichlet boundary conditions for the open bosonic string consistently with the Lorentz invariance. However, this possibility requires the existence of D-branes within the theory as dynamic objects which play an important role for the description of gauge symmetries.

Having clarified the theoretical background, we then moved on to an in-depth analysis of Swampland conjectures and their applications.

The approach used differs from the treatments commonly found in the literature. In fact, a minimal line of research was used aimed at reducing to a minimum the number of conjectures actually essential to extrapolate physics from the Swampland. In particular, only three conjectures were presented as fundamental selection criteria and, also for these, the various links were highlighted in the course of the thesis in order to formulate an internally consistent program. From these it has been shown how it is possible to derive secondary conjectures such as consequences or natural generalizations of them. To do this it was necessary to pay attention to the formulation of the single conjectures by making simple refinements or non-trivial mathematical abstractions. The latter was the case of the No Global Symmetry conjecture for which a topological version was proposed and justified in a very detailed way in order to make clear the physical meaning of the mathematics behind the cobordism classes. Cobordism has been shown to be a powerful tool for characterizing the physics of Quantum Gravity. It has in fact allowed, through the support of the WGC, to account for the instability of the AdS vacua which, as we have seen, is the main result that lends itself to direct application to Particle Physics. It has been studied in detail how the instability of the AdS vacua of the compacted Standard Model allows to derive constraints on the neutrino masses. This was done by considering the 1-loop corrections to the radionic potential and then carrying out the explicit calculation for the various contributions of the $SM + GR$ spectrum. It has been seen, in particular, that in addition to the massless states (graviton and photon), the contribution of the Casimir energy of neutrinos as particles with a smaller non-zero mass was also significant. We found an upper bound to the mass of neutrinos and an interesting relationship (in order of magnitude) of this with the cosmological constant consistent with experimental observations. The discussion concerned the simple case of SM coupled to gravity minimally extended to account for the non-zero mass of neutrinos in accordance with the oscillation phenomenon. The result, in this case, is that for a simple count of fermionic degrees of freedom the possibility of having Majorana neutrinos would seem to be excluded.

On the other hand, the discussion made would naturally fit into dealing with theories beyond SM. In fact, in these cases the addition of fermionic and/or bosonic degrees of freedom would change the stabilization of the radionic potential, allowing for the possibility of also having Majorana neutrinos. A possible development of the thesis work would therefore be to study the variability of the mass bound according to the specific extension of the model. In light of the results that would be obtained, the experiments on neutrino physics planned for the future would therefore prove to be decisive for fixing the BSM physics consistent with the quantum description of gravity in the ultraviolet. The same could be said by considering a greater number of compact dimensions. This possibility, even if already admitted in the literature, lacks a more general treatment that systematically takes into account all the main and accredited BSM models.

The Distance conjecture was instead used to implement applications in the field of Cosmology. For this purpose, the instability of the de Sitter vacua has been

derived from it at least in asymptotic regions of the moduli space to which the conjecture refers. Also in this case, the approach followed is axiomatic and the derivation of the instability has been explicitly reported through the study of the potential behaviour associated with the EFT moduli. At that point, the idea was to generalize the result using the de Sitter conjecture, that is, the idea that all de Sitter-type vacua should be at most metastable. The combined action of the two conjectures (SDC and dS conjecture) provided selection criteria applicable to both inflationary and dynamic dark energy models. For the inflationary models we have seen how the SDC directly provides a limit on the excursion of the inflatonic field as a function of the tensor-to-scalar ratio which, combined with the Lyth bound and the results of the Planck 2018 collaboration, provides a region of variability that allows to constrain slow-roll inflation parameters. The situation is more delicate in the case of the application of the de Sitter conjecture. For it, in fact, a direct application to inflationary models appears to be quantitatively in tension with the experimental observations. In order to make the application of the two Swampland criteria more explicit, models of modified gravity of the type $f(R)$ have been considered with an appropriate parametrization. The results obtained are that, as expected, the SDC fits well to the main slow-roll inflationary models while the de Sitter conjecture tends to be particularly stringent.
The tensions, on the other hand, are resolved if we consider the refined version of the conjecture motivated on the other hand by the application in the field of Particle Physics. In this case the tension is underlined through quantitative estimates referring to the scalar potentials of both the Higgs boson and the axion fields.

Appendix A

The p-form symmetries

Consider a theory in D-dimensional space and a generic group of global symmetry transformations. If the symmetry group is continuous, each generator can be associated with a Noether current j which give us a conserved charge

$$Q = \int_{M^{(D-1)}} j \ . \tag{A.1}$$

This charge is defined as the $(D-1)$-dimensional submanifold $M^{(D-1)}$ changes. The symmetry transformations form a group G and can be studied, at the quantum level, in terms of the action of operators associated with the submanifolds, $U_g(M)$, which, in the case of continuous transformations, are expressed by

$$U_g(M^{(D-1)}) = e^{ig\int_{M^{(D-1)}} j} \qquad \forall\, g \in G \ . \tag{A.2}$$

Given two elements of the group g, g' such that $gg' = g'' \in G$, the operators associated with them must satisfy the group composition law

$$U_g(M^{(D-1)})U_{g'}(M^{(D-1)}) = U_{g''}(M^{(D-1)}) \ . \tag{A.3}$$

The operators $U(M^{(D-1)})$ are topological in the sense that they are invariant under diffeomorphic modifications of the manifold $M^{(D-1)}$. In the case of a global symmetry group of the type $G = U(1)$, there is a phase acquisition if the deformation involves a charged local operator $V(x)$ with charge q. In this case we have

$$U_\alpha(M^{(D-1)})V(x) = e^{i\alpha q}V(x) \qquad \forall g = e^{i\alpha} \in U(1) \ . \tag{A.4}$$

In general, if $R(g)$ is a representation of the group G, the same relation becomes

$$U_g(M^{(D-1)})V_i(x) = R_i^j V_j(x) \ . \tag{A.5}$$

Having established this formalism, we intend to introduce a new notion of global symmetry suitable for a topological treatment [64, 65, 38]. Given a D-dimensional theory, a p-form global symmetry is defined as the symmetry group obtained from topological operators associated with $(D-p-1)$-dimensional manifold, $U(M^{(D-p-1)})$, which satisfy the composition law of group

$$U_g(M^{(D-p-1)})U_{g'}(M^{(D-p-1)}) = U_{g''}(M^{(D-p-1)}) \tag{A.6}$$

with $g'' = gg'$. These operators act on charged operators associated with p-dimensional manifolds $\mathcal{C}^{(p)}$. If $R_g(V)$ is a representation of g, then the action of the group operators on the charged operators is provided by the law

$$U_g(M^{(D-p-1)})V(\mathcal{C}^{(p)}) = R_g(V)V(\mathcal{C}^{(p)}) \; . \tag{A.7}$$

A conserved $(p+1)$-form current j can be associated with a continuous p-form symmetry. The conserved charge is given by

$$Q = \int_{M^{(D-p-1)}} j \; . \tag{A.8}$$

Note that the case $p = 0$ corresponds to an ordinary global symmetry.
Consider, as a simple example, the case of a theory with Abelian gauge symmetry $U(1)$. In this case the electric and magnetic current provide conserved 2-form and $(D-2)$-form currents, respectively

$$J_2^e = \frac{2}{g^2} \star F \qquad J_2^m = \frac{1}{2\pi} F \tag{A.9}$$

where g now is the coupling constant and for which, by virtue of the equations of motion, it results

$$d(\star F) = dF = 0 \; . \tag{A.10}$$

The two currents are associated respectively with 1-form and (D-3) -form global symmetries with charge operators

$$U_e(M^{(D-2)}) = e^{2i\frac{\alpha}{g^2}\int_{M^{(D-2)}} \star F} \qquad U_m(M^{(2)}) = e^{i\frac{\eta}{2\pi}\int_{M^{(2)}} F} \; . \tag{A.11}$$

In the case $D = 4$ there are therefore two 1-form global symmetries. The presence of charged particles breaks 1-form global symmetries conservations gauging an ordinary symmetry with current J.

Appendix B

The Casimir Energy

This appendix shows the derivation of the Casimir energy expression in the case of a space with periodic boundary conditions. The discussion follows the approach reported in [42].
Consider a massless scalar field ϕ with dynamics described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi \, \partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \; . \tag{B.1}$$

The energy-momentum tensor is defined by

$$T_{\mu\nu} = \mathcal{L} g_{\mu\nu} - 2 \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \; . \tag{B.2}$$

In the 1-loop approximation, the expectation value of $T_{\mu\nu}$ can be expressed in terms of the propagator of the theory $G(x - x') = \langle \phi(x) \phi(x') \rangle$ as

$$\langle T_{\mu\nu} \rangle = \lim_{x \to x'} \left[\frac{1}{2}(\partial_\mu \partial'_\nu + \partial_\nu \partial'_\mu) - \frac{1}{2} g_{\mu\nu} (\partial^{mu} \partial'_\mu + m^2) \right] G(x - x') \; . \tag{B.3}$$

Suppose that a spacetime dimension is compactified on a circle, i.e. that the usual periodic boundary conditions are imposed for its coordinate. If the scalar field ϕ is charged ($\phi \in \mathbb{C}$), the following boundary conditions occur

$$\phi(x, y + 2\pi\mathcal{R}) = e^{i\theta} \phi(x, y) \; . \tag{B.4}$$

The expectation value of the energy-momentum tensor can consequently be expanded by highlighting the contribution of the Casimir energy

$$\begin{aligned} \langle T_{\mu\nu} \rangle &= \frac{1}{2} \lim_{x \to x'} (\partial_\mu \partial'_\nu + \partial_\nu \partial'_\mu) \sum_n{}' G_\infty(x - x' + 2\pi n \mathcal{R} \widehat{y}) = \\ &= -\sum_n{}' \partial_\mu \partial_\nu G_\infty(2\pi n \mathcal{R} \widehat{y}) = \\ &= (\rho_{\mathcal{C}}(\mathcal{R}) \eta_{\mu\nu} + \mathcal{R} \rho'_{\mathcal{C}}(\mathcal{R}) \delta_\mu(y) \delta_\nu(y)) \end{aligned} \tag{B.5}$$

where the Casimir energy density appears

$$\rho_{\mathcal{C}}(\mathcal{R}) = 2\sum_n{}' \frac{\partial G_\infty((2\pi n\mathcal{R}\widehat{y})^2)}{\partial(2\pi n\mathcal{R}\widehat{y})^2} \, . \tag{B.6}$$

This result obtained for a charged scalar field is actually valid for a generic bosonic and fermionic field unless a minus sign for the latter.
In order to preserve the Null Energy Condition

$$T_{\mu\nu}n^\mu n^\nu = -2n_y^2\mathcal{R}\rho'_{\mathcal{C}}(\mathcal{R}) \geq 0 \qquad \forall\, n^\mu \quad : \quad n^2 = 0\, , \tag{B.7}$$

it is clear that negative Casimir energy densities are required

$$\rho_{\mathcal{C}}(\mathcal{R}) \leq\ 0\, . \tag{B.8}$$

The condition is then satisfied for fermionic fields but not for bosonic ones.
The determination of the Casimir energy can be traced back to the calculation of the Green function. In the general case, outside the cone of light it is analytically expressed in the form

$$G_\infty(x) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ikx}}{k^2+m^2} = \frac{m^{d-2}}{(2\pi)^{d/2}}\frac{\mathcal{K}_{d/2-1}(mx)}{(mx)^{d/2-1}} \tag{B.9}$$

where $\mathcal{K}_j(z)$ are Bessel functions defined by

$$\mathcal{K}_j(z) \equiv \frac{1}{2}\int_0^\infty d\beta\; \beta^{j-1}e^{-\frac{z}{2}\left(\beta+\frac{1}{\beta}\right)}\, . \tag{B.10}$$

For them, in particular, the following properties hold

$$\partial_z\left(\frac{\mathcal{K}_j(z)}{z^j}\right) = -\frac{\mathcal{K}_{j+1}(z)}{z^j} \tag{B.11}$$

$$z^j\mathcal{K}_j(z) \underset{z\to 0}{\longrightarrow} 2^{j-1}\Gamma(j)\left[1-\frac{z^2}{4(j-1)}+\mathcal{O}(z^4)\right] \tag{B.12}$$

$$z^j\mathcal{K}_j(z) \underset{z\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{2}}\; z^{j-1/2}e^{-z}\, . \tag{B.13}$$

From (B.11) we have the following exact expression for the Casimir energy density

$$\rho_{\mathcal{C}}(\mathcal{R}) = -\sum_{n=1}^\infty \frac{2m^d}{(2\pi)^{d/2}}\frac{\mathcal{K}_{d/2}(2\pi nm\mathcal{R})}{(2\pi nm\mathcal{R})^{d/2}}\cos(n\theta) \tag{B.14}$$

while in the limit (B.12) we find the approximate expression for small values of $m\mathcal{R}$

$$\rho_{\mathcal{C}}(\mathcal{R}) = -\frac{2}{(2\pi\mathcal{R})^d\Omega_{d-1}}\mathrm{Re}\left[\sum_k \frac{e^{ik\theta}}{k^d} - \frac{2\pi^2\sum_k(e^{ik\theta}/k^{d-2})}{d-2}(m\mathcal{R})^2 + \mathcal{O}(m\mathcal{R})^4\right] . \tag{B.15}$$

Finally, from the limit (B.13) we obtain instead the trend of the Casimir energy density associated to fields with mass $m \to \infty$ for which there is an exponential decay until $m\mathcal{R} > 1$

$$\rho_{\mathcal{C}}(\mathcal{R}) = -\frac{(m\mathcal{R})^{(d-1)/2}}{(2\pi\mathcal{R})^d}e^{-2\pi m\mathcal{R}}\cos\theta\ . \tag{B.16}$$

Note that in the particular case of a massless field, for $d = 4$, Green's function reduces to

$$G_\infty(2\pi n\mathcal{R}\widehat{y}) = \frac{1}{4\pi(2\pi n\mathcal{R})^2} \tag{B.17}$$

so that the Casimir energy density assumes the simple expression

$$\rho_{\mathcal{C}}(\mathcal{R}) = -\frac{4}{(2\pi)^6\mathcal{R}^4}\sum_n \frac{1}{n^4} = -\frac{\pi^2}{90}\frac{1}{(2\pi\mathcal{R})^4}\ . \tag{B.18}$$

In the context of the study of the SM compactification, we are interested in the contribution made to the radionic potential for which we consider the Casimir potential

$$\mathcal{C}(\mathcal{R}) = 2\pi\mathcal{R}\rho(\mathcal{R}) = -\frac{1}{720\pi\mathcal{R}^3} \tag{B.19}$$

appropriately rescaled.

Appendix C

Neutrino Physics

C.1 Neutrino oscillations

In the context of the Standard model as gauge theory $SU(3) \times SU(2) \times U(1)$, neutrinos are singlets of $SU(3)$, neutral under $U_{EM}(1)$, and if associated with charged leptons give rise to $SU(2)$ doublets. They are assumed to be massless and described as left-handed Weyl spinors. There are three neutrino flavours associated with the three lepton families

$$\left[\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \right]. \tag{C.1}$$

Each neutrino is associated with an antineutrino of opposite chirality $\bar{\nu}_\alpha$, with $\alpha = e, \mu, \tau$, and interacts weakly with leptons ℓ_α through the following lagrangian

$$\mathcal{L}_{SM}^{weak} = -\frac{g}{\sqrt{2}} \sum_{\alpha=e,\mu,\nu} \bar{\nu}_{\alpha,L} \gamma^\mu \ell_{\alpha,L} W_\mu - \frac{g}{2\cos\theta_W} \sum_{\alpha=e,\mu,\tau} \bar{\nu}_{\alpha,L} \gamma^\mu \nu_{\alpha,L} Z_\mu + h.c. \tag{C.2}$$

The relevant interaction vertices are indicated in Figure C.1. The fact that neutrinos have zero mass has as a consequence that the SM preserves the lepton flavour.

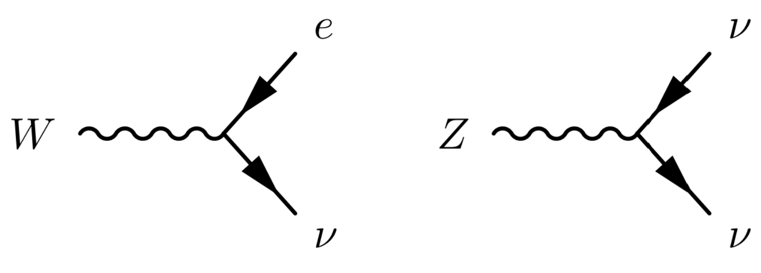

Figure C.1: Neutrinos interaction vertices

The description provided by the SM is however rather unsatisfactory. The need to go further in SM is highlighted by the experimental observation of neutrino oscillation. To describe this process, assume that neutrinos have a certain mass m_α. In this case we distinguish then the flavor or interaction eigenstates $|\nu_\alpha\rangle$, with $\alpha = e, \mu, \nu$, from the mass eigenstates $|m_i\rangle$, with $i = 1, 2, 3$, for which there will be a neutrino mixing process described by the superposition

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i} |m_i\rangle \ . \tag{C.3}$$

$U_{\alpha i}$ is the unitary 3×3 mixing matrix called the PMNS matrix (Pontecorvo, Maki, Nakagawa, Sakata). It can be parametrized by 3 angles and 3 phases as

$$U_{\alpha i} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix} \tag{C.4}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$, with $\theta_{ij} \in [0, \pi/2]$ and $i, j = 1, 2, 3$, δ and α_{ij} are respectively the Dirac CP violation phase and Majorana CP violation phases. The mass states $|m_i\rangle$ are orthonormal eigenstates of the free Hamiltonian, $\langle m_j| |m_i\rangle = \delta_{ij}$ and correspond to eigenvalues $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$. The neutrinos produced by weak interaction are found in an initial flavour state (C.3). In the approximation in which it can be regarded as plane wave state, its temporal evolution is then provided by

$$|\nu_\alpha, t\rangle = e^{-iHt} |\nu_\alpha\rangle = \sum_i U^\star_{\alpha i} e^{iE_i t} |m_i\rangle \ . \tag{C.5}$$

At this point it is then possible to calculate the probability of oscillation $P_{\alpha\beta}$ defined as the probability of obtaining a flavour state $|\nu_\beta\rangle$ at the time t starting from the flavour state $|\nu_\alpha\rangle$

$$P_{\alpha\beta}(t) = |\langle\nu_\beta| |\nu_\alpha, t\rangle|^2 = |\sum_i U^*_{\alpha i} U_{\beta i} e^{-iE_i t}|^2 = \sum_{ij} U^*_{\alpha j} U_{\beta i} U_{\alpha j} U^*_{\beta j} e^{-i(E_i - E_j)t} \ . \tag{C.6}$$

In all situations of experimental interest, neutrinos behave like ultra-relativistic particles and, if the detection is made at distance L from point production and at the time T, the following approximations are valid

$$E_i - E_j \approx \frac{m_i^2 - m_j^2}{2p} \qquad \text{and} \qquad L \approx T \tag{C.7}$$

in which it is assumed that all neutrinos have approximately the same momentum $p >> m_i$. The oscillation probability (C.6) therefore becomes

$$P_{\alpha\beta}(L) \approx \sum_{ij} U^*_{\alpha j} U_{\beta i} U_{\alpha j} U^*_{\beta j} e^{-i\frac{\Delta m^2_{ij} L}{2p}} \quad \text{(C.8)}$$

where $\Delta m^2_{ij} \equiv m^2_j - m^2_i$. It is clearly seen that the possibility of observing the phenomenon of oscillation requires that the neutrinos are not degenerate in mass (existence of non-zero masses) and that there is a lepton mixing $(U \neq 1)$. Also note that the oscillation is a consequence of the quantum mechanical nature of the neutrino states and in fact reflects the uncertainty that occurs in the measurement of the energy-momentum.

In the particular case of only two generations, the mixing matrix (C.4) is parameterized by a single angle θ

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \quad \text{(C.9)}$$

The oscillation probability will depend on the difference $\Delta m^2_{12} \equiv m^2_2 - m^2_1$ and takes the explicit form

$$P_{\alpha\beta} = \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 L}{4p} \right) . \quad \text{(C.10)}$$

The expression (C.10) has the following simple properties for $\alpha \neq \beta$

$$P_{\alpha\beta} = P_{\beta\alpha} \qquad\qquad P_{\alpha\alpha} = P_{\beta\beta} = 1 - P_{\alpha\beta} \quad \text{(C.11)}$$

where $P_{\alpha\alpha}$ defines the so-called *survival probability*. Note that in the dependence on the ratio $\Delta m^2_{ij} L/p$ the parameters p and L are variable according to the experimental context. In particular, for $P_{\alpha\beta}$ to be sensitive to small values Δm^2_{ij} it is necessary to have large L/p ratios. In the case of atmospheric neutrinos $p_{atm} \sim 1GeV$, $L_{atm} \sim 10^4 km$ and then there are appreciable mass differences $\Delta m^2_{atm} \sim 10^{-5} eV^2$; for solar neutrinos $p_\odot \sim 1MeV$, $L_\odot \sim 10^8 km$ and it is therefore possible to probe $\Delta m^2_\odot \sim 10^{-11} eV^2$. From experimental observation of the oscillation for these two classes of neutrinos, the following estimates are found [23]

$$\Delta m^2_{atm} \simeq (7.53 \pm 0.18) \cdot 10^{-5} eV^2 \quad \text{(C.12)}$$

$$\Delta m^2_\odot \simeq (2.44 \pm 0.06) \cdot 10^{-3} eV^2 . \quad \text{(C.13)}$$

Although this results account for non-zero masses, they do not establish the absolute mass of neutrinos and leave open the possibility of having two different orders for neutrinos mass eigenvalues: normal hierarchy (NH) or inverted hierarchy (IH) depending on whether it is respectively $m_3 > m_1, m_2$ or $m_3 < m_1, m_2$ (Figure C.2) [66].

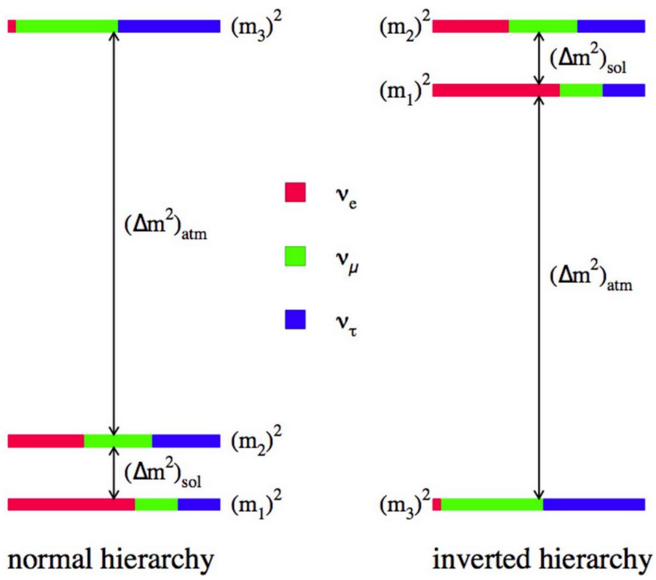

Figure C.2: Normal and inverted hierarchies for neutrino masses

C.2 The Standard Model extension

The oscillation phenomenon is indirect proof that neutrinos have non-zero masses. At this point it then becomes necessary to theoretically explain the origin of those masses into a consistent framework beyond the Standard Model. For this purpose, however, it is useful to discuss the fermionic nature of neutrinos.

C.2.1 The Dirac and Majorana neutrinos

A priori, fermions can have two different natures depending on how the terms of mass are constructed. On the one hand there are the *Dirac fermions*, that is the fermions normally considered in the contex of the Standard Model, whose mass is obtained by means of two independent 4-spinors of opposite chirality ψ_L, ψ_R, through a Lorentz invariant bilinear

$$\mathcal{L}_m^{Dirac} = -m\bar{\psi}_L\psi_R + h.c. = -m\bar{\psi}\,\psi \tag{C.14}$$

which acts as a mass term called Dirac mass term. Mass eigenstates are obtained from the combination

$$\psi = \psi_L + \psi_R \tag{C.15}$$

which goes on to describe Dirac fermions as particles associated with a respective antiparticle obtained by charge conjugation $\psi^c = C\gamma^0\psi^* = i\gamma^2\psi^*$.
On the other hand, there are the *Majorana fermions* whose mass is instead built using only one 4-spinor, for example ψ_L, combined with its charge-conjugated ψ_L^c in the Lorentz invariant

$$\mathcal{L}_m^{Majorana} = -m\bar{\psi}_L\psi_L^c + h.c. = -m\bar{\chi}\,\chi \tag{C.16}$$

called Majorana mass term. The eigenstates in this case are provided by

$$\chi = \psi_L + \psi_L^c \tag{C.17}$$

which turn out to be self-conjugated, $\chi^c = \chi$, so they describe Majorana fermions as particles that coincide with their own antiparticles. However, their dynamics are still described by the Dirac equation.
In general it is possible to construct the more general mass Lagrangian by considering a combination of Dirac and Majorana mass terms for both chiral components ψ_L, ψ_R

$$\begin{aligned}\mathcal{L}_m &= D\bar{\psi}_L\psi_R + A\bar{\psi}_L^c\psi_L + B\bar{\psi}_R^c\psi_R + h.c = \\ &= \frac{1}{2}D(\bar{\chi}\omega + \bar{\omega}\chi) + A\bar{\chi}\chi + B\bar{\omega}\omega = \\ &= \begin{pmatrix}\bar{\chi}, & \bar{\omega}\end{pmatrix}\begin{pmatrix}A & \frac{1}{2}D \\ \frac{1}{2}D & B\end{pmatrix}\begin{pmatrix}\chi \\ \omega\end{pmatrix}\end{aligned} \tag{C.18}$$

with $\omega = \psi_R + \psi_R^c$. The mass matrix can be diagonalized in order to obtain two Majorana eigenstates

$$\begin{aligned}\varphi_1 &= \cos\theta\,\chi - \sin\theta\,\omega \\ \varphi_2 &= \sin\theta\,\chi + \cos\theta\,\omega\end{aligned} \tag{C.19}$$

with $\tan 2\theta = D/(A-B)$, which correspond to the two eigenvalues

$$m_{1,2} = \frac{1}{2}\left[(A+B) \pm \sqrt{(A-B)^2 + D^2}\right] . \tag{C.20}$$

We conclude that the Lagrangian (C.18) actually describes two distinct Majorana particles. A single Dirac particle is found in the limiting case $A = B = 0$ [67] and can be regarded as composed of two Majorana fermion with the same mass.
Note that the Majorana mass terms violate the conservation of quantum numbers carried by ψ. Given the exact conservation of the electric charge, neutrinos are the only fermions of the SM particle content that can be described as Majorana fermions. Therefore, in order to extend the SM to take into account the oscillation there is an ambiguity in the construction of the neutrino mass terms because a priori they can be described both as Dirac and Majorana fermions.

C.2.2 The Seesaw mechanism

Having discussed the ambiguity in neutrino fermionic nature, it is now possible to tackle the problem of generating their mass consistently. The proposed mechanisms are many and can generally be distinguished on the basis of the type of extension used in the particle content of the SM. Neutrino mass terms can in fact be generated either by extending the lepton sector by adding fermions, or by extending the Higgs sector by adding scalar fields. Only the first type of models will be considered in this appendix.

The simplest way of massing neutrinos is to introduce right-handed neutrinos, N_R, so that Dirac masses can be built. In order to preserve the SM gauge symmetry these neutrinos must be gauge singlets and are then known as *sterile neutrinos*. They can couple to the lepton doublet L by the Yukawa term

$$\mathcal{L}_Y = Y^{\nu}_{ij}\bar{L}_{Lj}\tilde{H}N_{Rj} + h.c \tag{C.21}$$

where $\tilde{H} = i\sigma_2 H^*$, with H Higgs field doublet. The Higgs mechanism will then provide the neutrino Dirac masses

$$\mathcal{L}^{Dirac}_m = -\frac{vy_\nu}{\sqrt{2}}\bar{\nu}_L N_R + h.c. \, . \tag{C.22}$$

On the other hand, it is possible to consider non-renormalizable extensions of the SM by introducing the 5-dimensional operator

$$\mathcal{L}_5 = \frac{\lambda}{\Lambda_{SM}}(\bar{L}\tilde{H}^*)(\tilde{H}^\dagger L) \tag{C.23}$$

known as Weinberg operator. The interest of this operator for the purposes of neutrino physics lies in the fact that by means of the Higgs mechanism it produces a Majorana neutrino mass term

$$\mathcal{L}^{Majorana}_m = -\frac{\lambda v^2}{2\Lambda_{SM}}\bar{\nu}_L\nu^c_L + h.c. \, . \tag{C.24}$$

In fact the operator (C.23) produces a break in the conservation of the lepton number. The introduction of the Weinberg operator means that, in accordance with what was said in Chapter 2, the SM should be appropriately considered as an effective description of a more fundamental model that accounts for the mass of neutrinos in a renormalizable context, i.e. that reproduce the Weinberg operator in the low energy limit. A simple model that satisfies this perspective is provided by the *seesaw mechanism*.

Consider at least two sterile neutrinos as singlets of the SM gauge group, N_{Rj}

with $j > 2$ to account for the observations (C.12, C.13). The Lagrangian of the model is then given by

$$\mathcal{L}_{seesaw} = \mathcal{L}_{SM} - \sum_{j\alpha} \bar{L}_\alpha Y_{\alpha j} \tilde{H} N_{Rj} + \frac{1}{2} \sum_{jk} N_{Rj}^T C^\dagger M_{jk}^{(N)} N_{Rk} + h.c. \quad \text{(C.25)}$$

with $M^{(N)}$ Majorana mass matrix. The Higgs mechanism provides a mass Lagrangian of the type

$$\mathcal{L}_m^\nu = -m_D \bar{\nu}_L N_R - \frac{1}{2} m_M \bar{N}_R^c N_R + h.c. \quad \text{(C.26)}$$

in which there is the combination of Dirac and Majorana mass terms. Based on what has been said in Section C.2.1, such mass term can take the form

$$\mathcal{L}_m^\nu = \begin{pmatrix} \bar{\nu_L}, & \bar{N}_R \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & m_M \end{pmatrix} \begin{pmatrix} \nu_L \\ N_R \end{pmatrix} \quad \text{(C.27)}$$

where $m_D = y^\nu v/\sqrt{2}$ are Dirac masses and m_M Majorana ones. From the result (C.20) we see then that in the limit $m_D << m_M$, we obtain eigenstates corresponding to heavy neutrinos of masses $m_N \approx m_M$ and to light neutrinos with masses

$$m_\nu \approx m_D \frac{1}{m_M} m_D^T \, . \quad \text{(C.28)}$$

This expression shows how as m_M increases there is a decrease of m_ν, hence the name *seesaw*. The smallness of m_ν and the approximate expression of the corresponding eigenstate (see (refMassEigenstste))

$$\varphi_{L\nu} = \nu_L + \frac{m_D}{m_M} N_R^c \quad \text{(C.29)}$$

allow such states to be identified with observed ordinary neutrinos. In contexts of extensions of the SM using theories of grand unification it is natural to associate the masses m_D and m_M to the two different symmetry breaking scales. For example, for $SU(5) \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)$ we have $m_D \sim 10^2 GeV$, $m_M \sim 10^{14} GeV$ and then $m_\nu \sim 0.1 eV$. In this way we can reproduce the experimental observations and give a justification for why sterile neutrinos are not observed. Furthermore, the hierarchy problem of explaining why the neutrinos masses would be so much smaller than the other SM masses is thus solved.

Actually there are different versions of the seesaw mechanism: the one presented here is the so-called seesaw type I model based on the exchange of right-handed neutrinos. Alternatively, the mechanisms of seesaw type II and III have also been proposed, based respectively on the exchange of scalar and fermion triplets [68].

Appendix D

The Slow-roll Inflation

The *slow-roll inflationary models* constitute a particular class of inflationary models in which the accelerated expansion phase is limited in time and leads continuously to the standard cosmological evolution phase. Inflation is determined by the presence of a scalar field ϕ called inflaton with interaction potential $V(\phi)$. The classical theory of inflaton interaction with gravity is described by action

$$S = -\frac{M_P^2}{2}\int d^4x\sqrt{-g}\,R + \int d^4x\sqrt{-g}\,\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] . \qquad \text{(D.1)}$$

By means of the variational principle, for variations with respect to the metric, we find Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M_P^2}\left(g_{\mu\nu}\partial_\lambda\phi\,\partial^\lambda\phi - g_{\mu\nu}V(\phi)\right) \qquad \text{(D.2)}$$

while for variations with respect to the scalar field we have the equation of motion

$$\nabla_\mu\nabla^\mu\phi + \partial_\phi V = 0 . \qquad \text{(D.3)}$$

In order to study the evolution of the universe, we consider a Friedmann-Robertson-Walker metric of the type

$$ds^2 = dt^2 - a^2(t)|d\mathbf{x}|^2 . \qquad \text{(D.4)}$$

If we then assume that the scalar field is spatially homogeneous, $\phi = \phi(t)$, we find that the energy-momentum tensor associated with the inflaton is given by

$$T^0_0 = \frac{1}{2}\dot{\phi}^2 + V \qquad T^i_j = -\delta^i_j\left(\frac{1}{2}\dot{\phi}^2 - V\right) . \qquad \text{(D.5)}$$

We therefore deduce that the inflatonic field behaves like a perfect fluid with energy density ρ and pressure p such that the state equation $p = \omega\rho$ has

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \; . \tag{D.6}$$

Note that in the particular case of a constant scalar field, $\dot{\phi} = 0$, we find the state equation of the perfect fluid associated with the cosmological constant $\Lambda_{c.c.} > 0$ with $w = -1$ that is the case of an eternal de Sitter inflation.
The Friedmann equations governing cosmological evolution take the form

$$3H^2 = \frac{1}{M_P^2}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) \tag{D.7}$$

$$2\dot{H} = -\frac{1}{M_P^2}\dot{\phi}^2 \tag{D.8}$$

while the equation of motion for the scalar field is reduced to

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0 \; . \tag{D.9}$$

At this point we apply explicitly the *slow-roll approximation* for which we consider a slow evolution formalized by the following conditions

$$|\dot{H}| << H^2 \qquad |\ddot{\phi}| << |H\dot{\phi}| \qquad \dot{\phi}^2 << |V|. \tag{D.10}$$

By virtue of these approximations the equations of motion are simplified to

$$3H^2 = \frac{1}{M_P^2}V \tag{D.11}$$

$$3H\dot{\phi} = -\partial_\phi V \; . \tag{D.12}$$

If we derive the former with respect to ϕ and divide both by $3H^2$, they become

$$\frac{\partial_\phi H}{H} = \frac{1}{2}\frac{\partial_\phi V}{V} \qquad \frac{\dot{\phi}}{H} = -M_P^2\left(\frac{\partial_\phi V}{V}\right)^2 . \tag{D.13}$$

To control the inflationary expansion, the slow-roll parameter is introduced

$$\epsilon \equiv -\frac{\dot{H}}{H^2} << 1 \tag{D.14}$$

such that $\dot{\epsilon} \sim 0$, and by double integration we find the following trend of the scale factor

$$a(t) \sim t^{1/\epsilon} \tag{D.15}$$

from which we can qualitatively see the reproduction of the inflationary phase. By virtue of the equations of motion, the parameter ϵ takes on the explicit expression in terms of the inflatonic field. Slow-roll inflation can then be fully specified by parameters

$$\epsilon = \frac{M_P^2}{2}\left(\frac{\partial_\phi V}{V}\right)^2 \tag{D.16}$$

$$\eta = M_P^2\left(\frac{\partial_\phi^2 V}{V}\right) \tag{D.17}$$

for which it is requested

$$\epsilon, \eta \ll 1 \ . \tag{D.18}$$

Inflationary models in slow-roll approximation constitute a wide class of models that differ for the expression of the scalar potential associated with the inflatonic field. Typically we consider simple power models of the type (Figure D.1)

$$V(\phi) \sim \phi^n \qquad\qquad n > 0 \tag{D.19}$$

for which it results

$$\frac{\partial_\phi V}{V} = \frac{n}{\phi} \qquad\qquad \frac{\partial_\phi^2 V}{V} = \frac{n(n-1)}{\phi^2} \ . \tag{D.20}$$

We see then that in these cases the approximation is satisfied for scalar field $\phi^2 \gg M_P^2$.

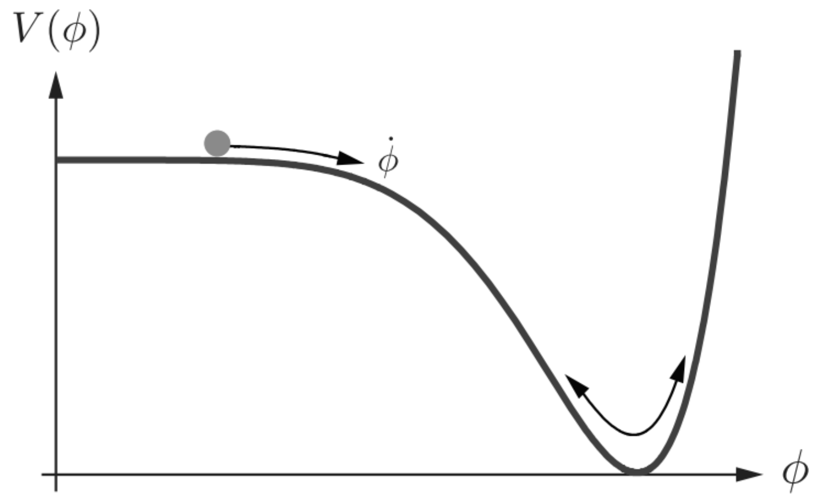

Figure D.1: Typical trend of the inflationary potential

In the cosmological context, the duration of the inflationary phase is often measured through the *e-folding parameter* N. If a_i and a_f denote respectively the scale factor at at initial and final instant of inflation, N is defined by

$$N \equiv \ln \frac{a_f}{a_i} = \int_{t_i}^{t_f} dt\, H(t)\, . \tag{D.21}$$

In order for the inflationary phase to have a sufficient duration to adapt to the current observed standard phase, e-folding parameters are required such that

$$N \gtrsim 60\, . \tag{D.22}$$

In slow-roll regime the (D.21) can be expressed in terms of field variation by means of the following approximation

$$H\, dt = \frac{H}{\dot{\phi}} d\phi \approx -\frac{3H}{\partial_\phi V} H d\phi \approx \frac{1}{\sqrt{2\epsilon}} \frac{d\phi}{M_P} \tag{D.23}$$

for which it is located

$$N = \int_{\phi_i}^{\phi_f} \frac{d\phi}{M_P} \frac{1}{\sqrt{2\epsilon}}\, . \tag{D.24}$$

In the case of inflationary models with power scalar potential, it is found [52]

$$N = \frac{1}{2\pi M_P^2} (\phi_i^2 - \phi_f^2) \tag{D.25}$$

which, being $\phi_i^2 \gg M_P^2$, for small n can be compatible with the condition (D.22). Moreover, if the potential $V(\phi)$ has a minimum point, inflation parameters tend to grow near this point until they break the slow-roll requirement. In this way, the phase of slow inflation continuously adapts to the standard expansion phase. We conclude that slow-roll inflation models, however simplified, allow us to solve the evolution problems of the universe and reproduce a type of phenomenology that agrees with current observations.

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