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Trans-Planckian effects from Primordial Gravitational Waves

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To my mother
And to all the ones who gave me the love I needed to face my challenges

"The effort to understanding the Universe is one of the very few things that lifts human life a little above the level of a farce and gives it some of the grace of a tragedy"

Steven Weinberg

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Introduction

The objective of my master's thesis is to study the Primordial Gravitational Waves background generated through the Early Universe Cosmology. We know that the standard cosmological model needs to be improved considering an accelerated phase to solve some consistency problems. The study of such pre-Big Bang era, called Inflation, led to a very good understanding of the Cosmic Microwave Background thank the computation of the primordial scalar power spectrum. This result is in very good agreement with the experimental evidence obtained in the last years. The power spectrum from the CMB confirms that large scale structures we see today were originated from almost scale-invariant and almost adiabatic fluctuations which were present on the super-Hubble scale before the time of the recombination (i.e. when the CMB was produced). It is worth noting that this spectrum is dependent on both the Hubble parameter and the slow roll coefficient (which is model dependent). There is another observable from the inflation paradigm, the tensor power spectrum which, at the first order of the slow roll limit, it's dependent only on the Hubble parameter. Such detection could be crucial in the evaluation of the energy scale of such a primordial era. Nevertheless, if we were able to detect waves from the inflationary epoch, we could have the first spectacular probe of, at least, the possible existence of a quantum gravitational theory at the fundamental level. We may extract clues about modified gravity and particle physics beyond Standard Model or spot any violation of the various consistency conditions by testing the zoo of the inflationary models. On the other hand, we could have also a falsification of the Inflation theory and this may be even more exciting.

During my master's thesis, I'm analyzing the cosmological perturbation theory and the consequent quantization of the scalar field's perturbations away from its background classical value. From the original formulation of the inflationary paradigm by Guth and Starobinski in the 80s, one of the most puzzling features of Inflation is its duration. The duration of such an accelerated phase could be essential for the study of new emergent phenomena beyond our current understanding of fundamental laws. The inflation is supposed to last at least $N \approx 60$ e-folds. One of the simplest question that one may arise is what if that phase was longer. If so, it would mean that primordial fluctuations could have had wavelength smaller than the Planckian length l_P. In consequence, it's natural to think that this regime could be a massive chance to open a window to an unexplored sector of knowledge. In my dissertation, I took into consideration the possibility of a non-trivial vacuum choice, as pointed out by Danielsson in 2002. We know that when one studies quantum fields in a dynamical background (such as a deSitter one) a temporal Killing vector lacks. We have an intrinsic uncertainty in the choice vacuum. In literature, it's common to adopt the so-called "Bunch-Davies" vacuum which is equivalent to setting a hypothetical Minkowskian infinite past initial condition. Since the duration of Inflation cannot be

thought infinite, it is reasonable to set the initial condition at a precise time in the Universe's history. This lead consequently to a non-trivial choice of the vacuum status. We contemplate a one-parameter group of states which is commonly called α vacua. In this context, we show how the power spectrum of both tensorial and scalar perturbation can be modified depending on a cut-off energy scale. Moreover, we stress the concept of vacuum according to the particle physicist's point of view giving an overview of the basics of quantum fields in curved space-time. Nevertheless, one of the consequences of this general theory is the supplementary particles production due to the presence of non-vanishing Bogoliubov coefficients.
The vacuum in quantum field theory has always had a central role. Recent papers suggest the possibility to have a running vacuum in gravitational theory as well as other running quantities of Nature. It could be reasonable to take into account a modification of Einstein theory including a time-dependent "cosmological constant" term which can be a source of gravitational waves.

Another interesting development of the inflation theory is the enrollment of more fields. The simple addition of an auxiliary field could carry new ingredients in the theory. The presence of another field could be useful to contemplate both adiabatic and isocurvature initial conditions. Moreover, during the evolution, adiabatic and curvature modes can convert one into another through an oscillating phenomenon. From the GWs perspective, the presence of a second field could be of essential importance. The only field subjected by the slow roll condition is the Inflaton, while the dynamic of other (possible) fields is completely free. We can consider the production of primordial gravitational waves (second-order ones) if we promote as a source term the primordial scalar perturbations of the additional fields.
One of the hardest aspects of primordial gravitational waves is their weakness. It is extremely difficult to detect them. So, in [6] the authors took into consideration a mechanism of parametric resonance using the so-called "natural" inflation potential. By considering just an additional field in the early universe, the presence of an oscillating term in the potential led them to find a *Mathieau* equation which gives rise to resonant amplification of the gravitational waves spectrum detectable by future experiments.

The detection of the PGWs's spectrum will be the centre of the scientific investigation in the next years. From a theoretical point of view, we can examine alternatives to inflation that are on the market or adding modifications to the physics at fundamental level. In fact, we can have two kinds of GWs production. The first one is due to the vacuum oscillations. In that case, we can study how quantum fluctuations of the gravitational field are stretched by the accelerated expansion. Through this line of research, we can take into account both General Theory of Relativity or Modified Gravity (for example various Scalar-Tensor theories) or Effective Field Theories which can lead to violations of consistency relations. One of the most relevant extensions of the Standard Cosmological model is the modification of the dispersion relation at Planck scales. Several modern theories beyond the SM seem to conspire toward such modification and a consequent breaking of CPT invariance (and, of course, the Lorentz symmetry). For example, we can have this modification by the Loop Quantum Gravity theory, Non-Commutative Space-time or Strings. On the other hand, the GWs can be produced by classical objects. This is the case when we consider, for example, another scalar field that contributes to the stress-energy tensor giving rise to second-order tensorial perturbations. In this context, also particles produced during the inflationary stage could be a significant source term. Moreover, other models can be constructed if we ask ourselves what happened after the inflation ends. The main point is how the universe gets the temperature to start the Big Bang scenario. We can imagine that this condition is immediately obtained by an instantaneous reheating or we can also think that the Inflaton has a very slowly decaying. That could be another highly interesting window to a new sector of the Universe's history still not explored.

We are at the dawn of a new era of experiments that will give us, probably, the verdict about inflation. For example, the Laser Interferometer Space Antenna (LISA), promoted by ESA and NASA, will be the biggest experimental apparatus ever built and will give pieces of information about the gravitational waves with an unbelievable precision never reached so far. It consists of three spacecraft arranged in an equilateral triangle with million-kilometer arms and will be able to give us information about GWs with amplitude of order $\Omega_{GW} \lesssim 10^{-14}$. LISA will have the capabilities to give us constraints on both signals from cosmological sources, like the GWs production during first-order phase transition, from cosmic defects and probe the present expansion of the Universe. Moreover, LISA could give us essential information about the non-Gaussian effects incorporated in the three-point correlation functions. LISA is programmed to start, at least, in 10 years and by that date, we will have further hints or evidence of a model instead of another. LISA will be very useful in the case we look forward to a mechanism of resonance in the tensorial sector of primordial fluctuations. However, if we are looking for a net detection of a PGWs background produced during inflation, we have to refer to more futuristic experiments like BBO or DECIGO.

Chapter 1

Physics beyond Standard Cosmological Model

> There's no shame in not knowing. The only shame is when you pretend that you know everything
>
> R.P. Feynman

1.1 The need for Inflation

Our modern description of Nature is completely based on two cornerstones: General Relativity and the Standard Model of particle physics. Our Universe could be studied and conceived through those models bearing in mind some simplistic assumptions. Although these models agree with experimental evidence with an embarrassing precision, they seem to have some inconsistencies. These issues arise from the fact that initial conditions needed for the Hot Big Bang model seem to be unnatural.

The first conceptual problem is the *Flatness problem* which consist in the fact that the primordial energy density has to be nearly 1 according to recent observations. If we take into account the Friedmann equation:

$$\Omega - 1 = \frac{K}{a^2 H^2} \tag{1.1}$$

it is easy to see that such situation corresponds to a almost flat primordial Universe. If this wouldn't be the case, the Universe may collapse immediately ($\Omega > 1$) or cools in a too short time ($\Omega < 1$). The second problem which characterize the Big Bang model is the *Horizon problem*. In this case, it is not clear how the CMB (Cosmic Microwave Background) have essentially omogeneous and isotropic perturbations. It seems that, in a certain way, the Universe would have experience a period in which the perturbations would have been in causal connection one to another. Nevertheless, if the Hot Big Bang begins at very high temperature, there would be produced some exotic objects that we should see today, but observations seem to rule them out. These relics are for example: the gravitino (which is a supersymmetric version of the graviton), magnetic monopoles (that should emerge when a symmetry of a Grand Unified Theory breaks) and other topological defects. At this stage we can ask ourselves what are the main aspects which characterize a good inflation theory. The origin of the flatness problem is the growing of the term $|\Omega - 1|$. Is we take the time derivative of both sides of the Friedman equation, we convince ourselves that the problem arise from the fact that the Universe is decelerating, i.e. $\ddot{a} < 0$. So, the condition we desire in the model building of Inflation is that the Universe expansion has to be accelerated:

$$Inflation \Leftrightarrow \ddot{a} > 0 \tag{1.2}$$

Surprisingly, the second problem we are tackling (the horizon problem) can be linked to the same argument. In fact, from experiments we have that the ratio between the comoving horizon sizes at the time of decoupling and today is much smaller than unity.

$$\frac{d(t_{dec})}{d(t_0)} \sim \frac{a_0 H_0}{a_{dec} H_{dec}} \ll 1 \tag{1.3}$$

Again, the problem comes from the fact that the Universe is decelerating. Then, we can define the comoving Hubble parameter as follows: $\mathcal{H} = aH$. Consequently, if the Hubble parameter is constant, the comoving quantity as to grow up if we think the primordial stage as a deSitter expansion (i.e. $a \sim e^{Ht}$). If we define the comoving Hubble length as the distance over which we can have causal interaction we conclude that Inflation has to be a period during which:

$$Inflation \Leftrightarrow \frac{d}{dt}\mathcal{H}^{-1} < 0 \tag{1.4}$$

The first consequence of those conditions is that any inflationary period has to characterize the environment with a negative pressure. If we take the second Friedmann equation and require the positivity of both the rhs and the lhs:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.5}$$

we have:

$$Inflation \Leftrightarrow \omega < -\frac{1}{3} \tag{1.6}$$

where in the last step we have used the fact that the relation between the density and the pressure is: $p = \omega\rho$. This relation is called the Equation of State and ω is the EoS parameter. For the ordinary constituents of the Universe such parameter is constant, while the dark sector is usually parametrized like a fluid with time-dependent EoS parameter ω. Lastly, it is worth-mentioning that in the case we are dealing with a collection of fluids, the total EoS parameter is always a function of time.

1.2 The Inflaton as a solution

It easy to understand that these conditions don't describe a unique model. It is just a scenario that may be realized in many different, but equivalent, ways. If we are able to construct such scenario we will solve in one blow all the issue mentioned above. Inflation provides an elegant tool to restore consistency between theoretical analysis and experimental data. Moreover, if the inflationary conditions are satisfied we can also get rid of the unwanted relics. In fact, if such relics were produced before the start of Inflation, they would be diluted by the rapid expansion of the Universe. The only point that is matter of discussion still today is the theme of the primordial initial conditions from which the Inflaton starts. Those conditions rely on a stage of the Universe characterized by energy scales far away from the ones of the standard quantum field theory. We are closer to the typical energy scales of Grand Unification Theories than the scales of Standard Model. We will deal with the model proposed by Guth and Starobinsky. We will solve all the shortcomings the Standard Big Bang model has implementing the theory with an additional scalar field which is commonly called Inflaton ϕ. Adding a single scalar massless field driven by a potential $V(\phi)$ we can extract the energy momentum tensor:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right] \tag{1.7}$$

from which calculate the EoS parameter ω:

$$\omega = \frac{p_{infl}}{\rho_{infl}} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{1.8}$$

In order to achieve the desirable condition, we must impose what is called *the first slow-roll condition*:

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi) \tag{1.9}$$

So, the potential has to be much bigger than the kinetic term. Moreover, we want this period lasts over an extended period. This leads us to the *second slow-roll condition*:

$$|\ddot{\phi}| \ll |V(\phi)_\phi| \tag{1.10}$$

where we indicated with $V(\phi)_\phi$ the derivative of the potential with respect to the field ϕ.

The presence of this scalar field in the primordial stage of the Universe could figure out the inconsistencies we were talking about with one blow. So, a natural interest to find observables related to this new field was growing in recent years. In this sense, a future detection of a gravitational waves spectrum induced by the Inflaton, could represent a spectacular probe of the presence of new physics. Moreover, any test toward this purpose will be extremely useful to discriminate between different theories in competition. From now on, we will use the Inflaton field as a monitor of the Universe's history. We are allowed to do that because of we are working in an FLRW background and so, thanks to the consequent homogeneous evolution, we can use the same universal clock at each point. Hence, if different points in space has equal physical properties it imply that these points has synchronized clocks and, because the time coordinate is not a physical one, we are free to choose matter fields as a relational time. We can formulate all the objects in the theory as functions of the field ϕ (only if the field has non-singular Jacobian, i.e. $\dot{\phi}$ does not change sign). So, we will split the Inflaton field in two parts: $\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x})$. The background value $\bar{\phi}$ is the only part which is driven by the potential V, is dependent only on the cosmic time and plays the role of the clock. On the other hand, the perturbations $\delta\phi$ are dependent on both the time and the space and they play the role of the seeds in the Early Universe. Moreover, this field, driving the dynamics, magnifies tiny quantum fluctuations generated in the very early times into seeds that could generate the large scale structures we observe today. For that reason is quite important to pay attention on the role of the initial conditions. Facing this problem we are pushed forward to accommodate the Inflaton ϕ in the most appropriate theoretical context. The answer to that question is far away to be easily understood. The key point is that new physics may arise at the energy scale at which the inflation takes place and hence we are not be able to know if the laws we can handle are acceptable or not. In literature, it's common to conduct this study from a field theoretical perspective. Then, the natural theoretical environment is the quantum field theory in curved space-time. However, it is important to stress that this is just an effective approach to the problem because we don't know how physic's laws can change at the Planck scale. We can just ask ourselves if we can extract some observables that are strictly related to some modifications of the Planckian physics. Studying the evolution of the Inflaton field bring us to the conclusion that his power spectrum has to be quasi scale invariant. So, one of the main goal of theories which contemplate a departure from the standard analysis is to find a spectrum which may have some dependence from the an energy scale. Such a remarkable result may come from two different approaches. One can speculate on the ambiguity of the choice of the vacuum state or think about a modified commutation relation assuming a deformation of the well know CCRs (Canonical Commutation Relations) between the field and its conjugate momentum. The former case in something lying in the context of quantum field theory in curved space, while the latest is the typical approach of models beyond Standard Model such as Non-commutative field theories [3] or string theories.

1.3 On the primordial quantum fluctuation's faith

It's important to focus on the regimes in which cosmological perturbations evolve. In fact, from now on, we will distinguish two different regimes: Superhorizon and Subhorizon (also called in literature Super-Hubble and Sub-Hubble regimes). It is a crucial point to make this distinction because of we need to track the evolution of the perturbations and so we can focus our attention on the sector by which we can extract fruitful information.

Remember that $H \sim \sqrt{\rho}$ and so, being the energy density nearly constant during inflation, so it is the Hubble radius. Actually, the energy density decrease very little and so the Hubble radius H^{-1} grows, but at lowest order we will treat it as constant. Let's see the figure 1.3 and study the behaviour of the cosmological perturbations. There, we are plotting the comoving scale $(aH)^{-1}$ as function of $log(a)$. During inflation the physical perturbation's wavelength λ(which is proportional to the scale factor a) grows exponentially. So we can distinguish three different regimes. The first regime is the period when the cosmological perturbations wavelength is smaller that the Hubble radius ($\mathcal{H}^{-1}$) and this is called the Sub-Horizon regime (recall that $\mathcal{H} = aH$). We can also think about this classification in terms of the relation between the comoving physical momentum $k_{phys} = \frac{k}{a}$ and the Hubble parameter $\mathcal{H}$. In that sense the Sub-Horizon regime occurs when $k(\eta) \gg \mathcal{H}(\tau)$ During this period, the wavelength of the perturbations is so small that they don't really care about the fact that Universe is

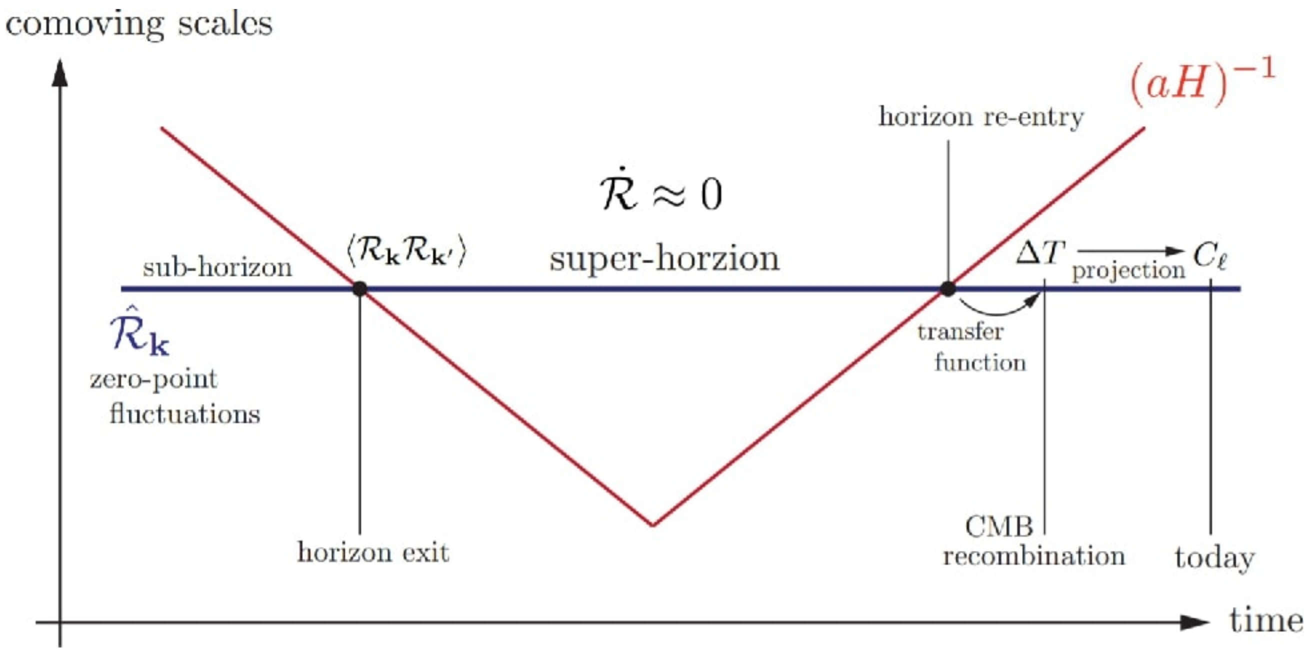

Figure 1.1: Comoving scales $(aH)^{-1}$ vs $log(a)$. The comoving horizon $(aH)^{-1}$ shrinks during inflation and after the reheating stage starts growing in the consequent FLRW period. The comoving scales k^{-1} exit at early times and re-enter in the cosmological scenario during the Hot Big Bang evolution. To study the dynamics of such perturbations we will use the gauge invariant quantity $\mathcal{R}$ which will be constant during the horizon exit. Credits: Daniel Baumann - Primordial Cosmology

expanding. In this scenario, perturbations are localized in a very tiny region in the Universe and basically, since $k \sim \frac{1}{\lambda} \sim a$, they behave like they are in a Minkowsky space-time. Then we can define the opposite situation which is called the Super Horizon regime when $k(\eta) \ll \mathcal{H}(\tau)$ or we may think that the correspondent wavelength is much bigger than the Hubble parameter. In this situation, the physical wavelength is longer than the Hubble radius ($\mathcal{H}^{-1}$). Within this regime the perturbations are seen by the Universe as constants. Here nothing really interesting happens. The very interesting moment in the history of the cosmological perturbations is the time when they crossed the Horizon. That means that at that time the Universe recognize the existence of the perturbations and we can think this time as the moment when perturbations are imprinted. In this context we will use the word horizon to refer to the Hubble length.

In summary, in the superhorizon regime we can think the variation of the metric, the energy density and the pressure over the horizon to be very small and we can safely think that they are constant in space in each causally connected region in the Early Universe. Remember that these regions evolve as independent homogeneous Universe and hence we cannot have propagations of any wave phenomenon.

Now, let's study the case in which are involved two perturbations with different wavelengths, say λ_1 e λ_2. If we take the graphic of the scales on the vertical axes and the time on the horizontal one, we can compare the behaviour of the wavelength of those perturbations with respect to the Hubble horizon. Suppose λ_1 is bigger that λ_2 . In this case, the former one will leave the horizon earlier. Hence perturbations with larger wavelengths are produced before perturbations with smaller wavelengths (and consequently the hierarchy is inverted for the momentum). That means that perturbations with greater wavelengths give me information about Inflation earlier in time.

Another fundamental tool to study inflationary cosmology is the number of *e-folds*N, which basically is an instrument to quantify the amount of Inflation.

$$N(t) \equiv \ln \frac{a(t_{end})}{a(t)} \tag{1.11}$$

and using the Inflaton field as the dynamical variable we can have these useful relations:

$$N(\phi) \equiv \ln \frac{a(t_{end})}{a(t)} = \int_t^{t_{end}} H(t)dt = \int_\phi^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi \approx \frac{1}{M_{Pl}^2} \int_{\phi_{end}}^{\phi} \frac{V}{V'} d\phi \tag{1.12}$$

where in the last equality we used the slow roll approximation. We can use this object to quantify the amount of inflation that was need to have the large scale structures we see today. Denoting the largest observable scales we see today as $k \approx \mathcal{H}_0 = H_0$ (where we used $a_0 = 1$) we can infer, from the thermal history of our Universe a complete expression for the number of e-folds the Inflation needed.

$$N(\phi_k) = -\ln \frac{k}{H_0} + 61 + \ln \frac{V_k^{1/4}}{\rho_{end}^{1/4}} - \frac{1}{3} \ln \frac{\rho_{end}^{1/4}}{\rho_{reh}^{1/4}} - \ln \frac{10^{16} GeV}{V_k^{1/4}} \tag{1.13}$$

This formula is dependent on the potential and the energy scale of the reheating stage other than the scales wave-number k and H_0. Since the potential is linked with the slow roll condition,

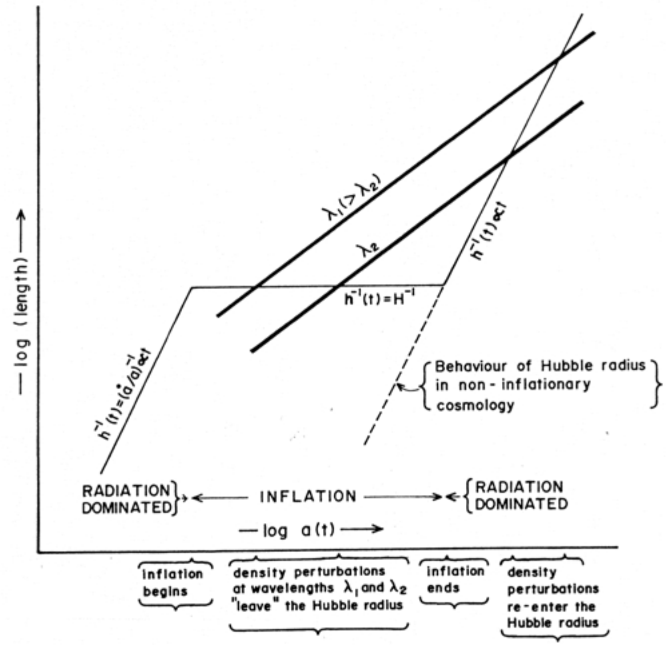

Figure 1.2: The figure illustrates the growths of the length scales of primordial fluctuations in relation to the Hubble radius during the inflationary and Freidman phases of expansion. Credits:Annual Review of Astronomy and Astrophysics

typically the third term is negligible while the fourth term is dependent on the model we use to describe the exit from the inflationary period and the entering to the Hot Big Bang scenario (for instance, in the case we think the reheating was instantaneous, the fourth term is negligible). It is common to think that it needs around 60 e-folds *after* the largest scales exit the horizon. On the other hand is not possible to give any information about the number of e-folds before that event; it will depend on how were the conditions before the inflationary stage and this is beyond our current understanding of the Universe's history.

To conclude this section is important to question what is the faith of quantum perturbations when they acquired super-Hubble scales. We will see that on super-Hubble scales, the quantum fields associated to the primordial perturbations, became constants. So, in this limit, the quantum ladder operators, which define a quantum field, commute. Consequently, on super-Hubble scales the quantum field operators must be replaced by classical Gaussian random variables, i.e.:

$$\langle e_v(\mathbf{k})\rangle = 0 \qquad \langle e_v(\mathbf{k})e_v^*(\mathbf{k}')\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{1.14}$$

So, replacing the quantum operators for the mode function with stochastic Gaussian field:

$$\hat{v}_{\mathbf{k}} \rightarrow v_{\mathbf{k}} = v_k(\tau)e_v(\mathbf{k}) \tag{1.15}$$

we can replace also the quantum vacuum average $\langle 0|...|0\rangle$ with a classical ensemble average $\langle ...\rangle$.

1.4 Quantization procedure in curved spacetime and the "natural" ambiguity of vacuum

At this point, it is relevant to point out some exotic features that may occur when one studies a field in a given metric background evolving in time.

The crucial point, in the following exposition, is the choice of vacuum. When studying quantum field theory in curved space, the choice of vacuum is not unique. There is a natural intrinsic uncertainty in the inflationary predictions, and we will see that they show to be generically small. One of the main issue we have to face is the definition of particles. In general we have to deal with such a definition problems when we move to quantize fields in presence of another background field. This is because, in curved space-time, we cannot associate temporal Killing vectors with the Hamiltonian describing the system. One could think to another recipe toward the vacuum's choice; one example is to take in consideration the stress-energy tensor which is covariant. However we still inherit the same ambiguity due to the boundary conditions and renormalization prescriptions. We will see that when one introduce a physical cut-off, it naturally induce a non conservation of the stress-energy tensor.

We are not able anymore to follow the standard Minkowskian procedure because of we cannot presume the existence of asymptotic states. To be more harmonious we can briefly outline the main results of the field's treatment in curved space (for further clarifications I suggest [4] and [5] and reference therein).

To fix the ideas, without loss of generality, we can consider the dynamics of a free massless scalar field evolving in a flat FLRW Universe which has the following infinitesimal line element:

$$ds^2 = dt^2 - a^2\delta_{ij}dx^i dx^j \tag{1.16}$$

It is immediate to see that this metric is conformally equivalent to one of a flat Minkowski type if we define the conformal time as follows:

$$\tau(t) = \int_{t_0}^{t} \frac{dt}{a(t)} \tag{1.17}$$

and so the line element takes the form:

$$ds^2 = a^2(\tau)[d\tau^2 - d\mathbf{x}^2] \tag{1.18}$$

The study of such scalar field will play a central role through the whole thesis because the fluctuations of the Inflaton around its classical background value will be, canonically, thought of as a free massless scalar field. The action is the usual one for a scalar field in Minkowski space-time but with the minimal substitution: $\eta_{Minkowski} \mapsto g^{\mu\nu}_{FLRW} \equiv g$

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi] \tag{1.19}$$

This theory lead to the well-known Klein-Gordon equation:

$$(\Box)\phi = \Big(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\Big)\phi = 0 \tag{1.20}$$

We recognize that this equation is solved by the usual plane wave solution of the form:

$$u_{\mathbf{k}}(x) = A_k e^{-ikx} \tag{1.21}$$

It is a well established fact that for a second order differential equation the Wronskian is a conserved quantity and we can safely use it to find the shape of the unknown factor A_k which may be a function of the momentum. In fact, we can see immediately that if we require the validity of the canonical commutation relations, we have a constraint between u_k and $\dot{u}_k$ that leads to: $[u_k, \dot{u}_k] = W = i$. Solving this simple relation we receive the final form of what we usually call the mode function. Remember that a mode function is the positive frequency solution of the classical equation of motion normalized by the CCRs is:

$$u_{\mathbf{k}}(x) = \frac{1}{\sqrt{2w_k}} e^{-ikx} \tag{1.22}$$

By means of those functions we usually expand a generic quantum field as a linear combination of the common ladder operators.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} [a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^{\dagger}_{\mathbf{k}} u^*_{\mathbf{k}}(x)] \tag{1.23}$$

In the last relation we have expressed the field ϕ as linear combination of the solutions u and its hermitian conjugate and we have also promoted the Fourier components a and a^* to operators that satisfy the usual canonical commutation relations: $[a_i(k), a^{\dagger}_j(k')] = \delta_{ij}\delta^3(\mathbf{k} \text{ - } \mathbf{k'})$ and $[a_i, a_j] = [a^{\dagger}_i, a^{\dagger}_j] = 0$. It is worth-noting that those operators are time independent and that ensure the validity of the commutation relations at all times. Those relations induce the respective equal time commutation relations between the field ϕ and its conjugate momentum π: $[\phi(k), \pi^{\dagger}(k')] = \delta^3(\mathbf{k} \text{ - } \mathbf{k'})$ and $[\phi, \phi] = [\phi^{\dagger}, \phi^{\dagger}] = 0$ and analogously for π and $\pi^{\dagger}$. Now it is time to spoil the features which make unique the theory in a time dependent background. Let's focus on the mode functions (solutions of the classical Klein-Gordon equation) and point out that they are not unique. In standard quantum theory, one associate the positive eigenvalues of the Killing vector $i\partial_t$ to the positive energy eigenfunctions. From a group-theoretical point of view, if we depart from the well known special relativistic symmetry group, we have not this useful correspondence anymore. When we study a quantum field evolving in a FLRW Universe we need to bear in mind what is the fundamental symmetry group of General Relativity: the reparametrization invariance. It means that we are not able anymore to have a definite choice of time and, consequently, of positive frequencies. Looking forward to a quantum mechanical treatment for this field, we define the rescaled field $f = a(\tau)\phi$ [1] .

We will leverage on the fact that the dynamics of the field ϕ in a flat FLRW background is mathematically equivalent to the one of the scalar field f in a Minkowski metric. In terms of this auxiliary field the original theory can be expressed as:

$$S(\tau, \mathbf{x}) = \frac{1}{2} \int d^4x [f'^2 - (\nabla f)^2 + \frac{a''}{a} f^2] \tag{1.24}$$

[1] A word of caution about the notation needs here. We are calling ϕ the field under the quantization procedure, but it is important to stress that the we will quantize just the perturbations $\delta\phi$ around the classical background $\bar{\phi}$. Here we use just the symbol ϕ for notation's sake.

which lead to the following equation of motion:

$$f'' - \nabla^2 f - \frac{a''}{a} f = 0 \tag{1.25}$$

that is the same equation of a free scalar field in a Minkowskian space with an effective mass term:

$$m_{eff}^2 = -\frac{a''}{a} \tag{1.26}$$

In this term is incorporated all the information about the Universe expansion. Note that here we are dealing with a simple massless scalar field. If it would have been massive, the effective mass would get an extra term:

$$m_{eff}^2 = m^2 a^2 - \frac{a''}{a} \tag{1.27}$$

It is worth-noting that this action is time dependent and it implies a non conservation of the energy. We will come back later on this point. In order to quantize the theory we start calculating the conjugate momentum which is:

$$\pi = \frac{\partial L}{\partial f'} = f' - \mathcal{H} f \tag{1.28}$$

It will be useful to switch to the Fourier space, defining:

$$f(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} f_k(\tau) e^{i\mathbf{kx}} \tag{1.29}$$

So, when performing the Legendre transformation we have the following:

$$H = \frac{1}{2} \int d^3\mathbf{k} \Big[\pi(\mathbf{k})\pi^*(\mathbf{k}) + k^2 f(\mathbf{k}) v^*(\mathbf{k}) + \frac{a'}{a} \Big(f(\mathbf{k})\pi^*(\mathbf{k}) + \pi(\mathbf{k}) v^*(\mathbf{k}) \Big) \Big] \tag{1.30}$$

Recall that, due to the reality of the scalar field, we can also say that: $f(\mathbf{k}) = f^*(\mathbf{-k})$ and respectively holds for $f^\dagger(\mathbf{k})$. It's important to stress that in the classical phase space, a classical field configuration is completely specified by just an half of the whole Fourier space, but this doesn't hold anymore in quantum field theory (in general), but here we will not deal with such a complication. The theory we are dealing with is the usual one for an harmonic oscillator and so we can develop the usual treatment for such integrable system. The most natural environment to develop a quantum field theory is the Heisenberg picture, so we move to introduce the ladder operators which will be functions of time in this context. By means of those operators, we can recast our characters as follows:

$$\hat{f}_{\mathbf{k}} = \frac{\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}}{\sqrt{2k}} \qquad \hat{\pi}_{\mathbf{k}} = -i\sqrt{\frac{k}{2}} (\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}}) \tag{1.31}$$

One can easily reverse these two definitions and obtain the explicit form for the ladder operators:

$$\hat{a}_{\mathbf{k}} = \frac{1}{\sqrt{2}} \Big(\sqrt{k} \hat{f}_{\mathbf{k}} + i \frac{1}{\sqrt{k}} \hat{\pi}_{\mathbf{k}} \Big) \tag{1.32}$$

We can treat the field $\hat{f}$ in the standard way we do in ordinary quantum field theory and impose the standard equal time commutation relations:

$$[\hat{f}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = i\delta^{(3)}(\mathbf{x}\text{-}\mathbf{y}) \tag{1.33}$$

$$[\hat{f}(\tau, \mathbf{x}), \hat{f}(\tau, \mathbf{y})] = 0 \qquad [\hat{\pi}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = 0 \tag{1.34}$$

and, in Fourier space, using the reality of the field in our theory we have also the following:

$$[\hat{f}(\tau, \mathbf{k}), \hat{\pi}(\tau, \mathbf{k'})] = i\delta^{(3)}(\mathbf{k}\text{-}\mathbf{k'}) \tag{1.35}$$

These relations induce the fundamental relation between the creation and annichilation operators which is:

$$[\hat{a}(\tau, \mathbf{k}), \hat{a}^{\dagger}(\tau, \mathbf{k'})] = i\delta^{(3)}(\mathbf{k}\text{-}\mathbf{k'}) \tag{1.36}$$

In this formalism, the Hamiltonian takes the following simple form:

$$H = \frac{1}{2}\int d^3\mathbf{k}\Big[k(a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}} + a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}) + i\mathcal{H}(a^{\dagger}_{\mathbf{k}}a^{\dagger}_{\text{-}\mathbf{k}} - a_{\mathbf{k}}a_{\text{-}\mathbf{k}})\Big] \tag{1.37}$$

At this stage, we can use the Heisenberg equations for the operators a and $a^{\dagger}$ and arrive at the following system of operatorial equations of motion:

$$\begin{pmatrix} a'_{\mathbf{k}} \\ a'^{\dagger}_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} -ik & \mathcal{H} \\ \mathcal{H} & ik \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ a^{\dagger}_{-\mathbf{k}} \end{pmatrix} \tag{1.38}$$

Whose solutions could be cast as linear combinations of the operators evaluated at a given primordial time τ_0.

$$\hat{a_{\mathbf{k}}}(\tau) = u_k\hat{a_{\mathbf{k}}}(\tau_0) + v_k\hat{a}^{\dagger}_{-\mathbf{k}}(\tau_0) \tag{1.39}$$

$$\hat{a}^{\dagger}_{-\mathbf{k}}(\tau) = u^*_k\hat{a}^{\dagger}_{-\mathbf{k}}(\tau_0) + v^*_k\hat{a_{\mathbf{k}}}(\tau_0) \tag{1.40}$$

Where also these new quantity indicated by u_k and v_k are functions of time. These relations constitute what it's common to call in literature the "Bogoliubov transformation" which describe the mixing of the creation and annichilation operators due to the effect of time.

We can easily get another constraint between these Bogoliubov coefficients if we require the validity of the normalisation condition: $i(f^*_k f'_k - f'^*_k f_k) = 1$ which is substantially equivalent to the conservation of the Wronskian for the differential relative differential problem. (Note that the scalar product by with we defined the normalisation condition is the product that is defined in curved space-time taking into account a spatial Cauchy hypersurface Σ in a globally hyperbolic space-time). Nevertheless, we can think of the previous condition as the request of validity of the commutation relations between the ladder operators even after the time evolution. In formulas:

$$|u_k(\tau)|^2 - |v_k(\tau)|^2 = 1 \tag{1.41}$$

So, the quantized scaled fields and their conjugate momentum are:

$$\hat{f_{\mathbf{k}}}(\tau) = f_k(\tau)\hat{a}_{\mathbf{k}}(\tau_0) + f^*_k(\tau)\hat{a}^{\dagger}_{-\mathbf{k}}(\tau_0) \tag{1.42}$$

$$\hat{\pi}_{\mathbf{k}}(\tau) = -i\big(g_k(\tau)\hat{a}_{\mathbf{k}}(\tau_0) - g_k^*(\tau)\hat{a}^\dagger_{-\mathbf{k}}(\tau_0)\big) \tag{1.43}$$

where we defined the following two combinations of mode functions:

$$f_k(\tau) = \frac{1}{\sqrt{2k}}(u_k(\tau) + v_k^*(\tau)) \tag{1.44}$$

$$g_k(\tau) = \sqrt{\frac{k}{2}}(u_k(\tau) - v_k^*(\tau)) \tag{1.45}$$

Thank to the switch to the rescaled field f by the scale factor a we are in the position of writing the dynamical equation of the quantized scalar field in a Bessel-type form:

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0 \tag{1.46}$$

Moreover, we will settle down our description in a de Sitter space, which is the natural environment of Inflation. This means that the expansion parameter has the following form: $a = -\frac{1}{H\tau}$. By a direct use of algebra we find the following identity:

$$\frac{a''}{a} = \frac{2}{\tau^2} \tag{1.47}$$

The solution of such a Bessel equation is well known to be a linear combination of the Hankel function of first and second kind:

$$f_k(\tau) = \sqrt{-\tau}[C_1(k)H^{(1)}_{\frac{3}{2}}(-k\tau) + C_2(k)H^{(2)}_{\frac{3}{2}}(-k\tau)] \tag{1.48}$$

For later convenience it's useful to recall the asymptotic behaviour of the Hankel functions because they will be essential toward the evaluation of the power spectrum in the regimes of interest. The first important formula is the aspect of the Hankel function for the index ν we are concerning about: $\nu = \frac{3}{2}$.

$$H^{(1)}_{\frac{3}{2}}(z) = i\sqrt{\frac{2}{\pi z}}\frac{e^{iz}}{z}(1 - iz) \tag{1.49}$$

and the famous property: $(H^{(1)}_\nu(z))^* = H^{(2)}_\nu(z)$ At this stage we can easily compute, doing some straightforward simplifications, the general form of the mode functions of the quantized field f:

$$f_k = A_k\frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau}) + B_k\frac{e^{ik\tau}}{\sqrt{2k}}(1 + \frac{i}{k\tau}) \tag{1.50}$$

and its conjugate momentum g_k:

$$g_k(\tau) = \sqrt{\frac{k}{2}}\Big[A_k e^{-ik\tau} - B_k e^{ik\tau}\Big] \tag{1.51}$$

We can now exhibit the general form for the mode functions.

$$u_k = \frac{1}{2}\Big(A_k e^{-ik\tau}\Big(2 - \frac{i}{k\tau}\Big) + B_k e^{ik\tau}\frac{i}{k\tau}\Big) \tag{1.52}$$

$$v_k^* = \frac{1}{2}\Big(B_k e^{ik\tau}\Big(2 + \frac{i}{k\tau}\Big) - A_k e^{-ik\tau}\frac{i}{k\tau}\Big) \tag{1.53}$$

These quantities are subjected to the constraint (1.16) which can be put differently as:

$$|A_k|^2 - |B_k|^2 = 1 \tag{1.54}$$

It is clear that we have a complete one-parameter family of solutions all equivalent in principle. Now, we will show, at first, the common choice and, after, the general treatment for the initial conditions through which the observables get some modifications.

1.4.1 The Minkowskian assumption

Here we are in the position to set the initial conditions for the field we are interested in. The first, simplistic, choice we can made is the Euclidean vacuum, commonly known as the "Bunch-Davies vacuum". It corresponds to the assumption of thinking that in the infinite past the Universe was a Minkowsky type. This assumption lead to the following hypothesis:

$$\lim_{\tau \to -\infty} f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \tag{1.55}$$

with the previous limit we can infer the form of the elements we introduced before to be $A_k = 1$ and $B_k = 0$ and so:

$$f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau}) \tag{1.56}$$

In other words, the Minkowskian choice is the usual one occurring when one quantize fields in a flat background requiring the minimization of the energy density. Recall that the expression for the Hamiltonian operator for the quantum harmonic oscillator can be cast into te following form:

$$\hat{H} = \frac{1}{4}\int d^3k \Big[a_{\mathbf{k}} a_{-\mathbf{k}} F_k^* + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger F_k + (2a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta_D(0))E_k\Big] \tag{1.57}$$

where we have defined :

$$E_k(\tau) = |\dot{f}_k|^2 + \omega_k^2 |f_k|^2 \tag{1.58}$$

$$F_k(\tau) = \dot{f}_k^2 + \omega_k^2 f_k^2 \tag{1.59}$$

Requiring the flat space-time conditions on the mode functions f_k and minimizing the energy E_k bring us to the definition of the Bunch Davies vacuum.

Note that the only behaviour needed is for an infinite past time τ but no restrictions were made on k because the initial vacuum state ought to be imposed for all modes.

It is essential to stress again the crucial point of this procedure; that is the fact that we are only imposing an *instantaneous vacuum.* In fact, due to the time dependence of the

Hamiltonian, we cannot find any time-independent eigenvectors that can be used as a vacuum. The Euclidean vacuum we have just defined is somewhat emerging from the recipe we have outlined. Moreover, it is important to stress also that this minimization procedure holds only if $\omega_k^2(\tau_0) > 0$. Otherwise, the energy function has no minimum and we are not able to spot an instantaneous lowest-energy vacuum.[2] At this stage, we can compute the power spectrum of the field we are studying that is defined by the following:

$$\mathcal{P}_f = \frac{k^3}{2\pi^2}|f_k(\tau)|^2 \tag{1.62}$$

which represents the power spectrum and is the power per logarithmic interval k. We can point out that, by this definition, the power spectrum is a dimensionless quantity. It is easy to see that we arrive at a simple formula that is scale invariant in the super horizon regime. Now, recall that the field f is not the former one we were interested in, but it is what we called the rescaled field. Coming back to the original character we find finally the power spectrum for the scalar field ϕ evolving in a de Sitter space-time:

$$\mathcal{P}_\phi(k,\tau) = \frac{\mathcal{P}_f}{a^2(\tau)} = \left(\frac{H}{2\pi}\right)^2\left[1 + (k\tau)^2\right] \tag{1.63}$$

which, in super-horizon limit, namely $k\tau \to 0$ approaches to a constant value (with respect to the scale k):

$$\left(\frac{H}{2\pi}\right)^2 \tag{1.64}$$

This is an impressive result. Since H is slowly varying we can have the expression for the power spectrum of quantum scalar perturbations by just evaluating this expression at the horizon crossing $k = aH$.

1.4.2 The general treatment - alpha vacua

Using the same arguments that may be found in Danielsson [1], we can reasonably consider a vacuum state that is originated not in the infinite past, but in a specific finite moment τ_i.

$$a_k(\tau_i)|0,\tau_i\rangle = \underbrace{u_k(\tau_i)\hat{a}_k(\tau_i)|0,\tau_i\rangle}_{=0} + \underbrace{v_k(\tau_i)\hat{a}^\dagger_{-k}(\tau_i)|0,\tau_i\rangle}_{\neq 0} \tag{1.65}$$

[2] For completeness, it worth-mentioning that in literature are present other prescription to select the vacuum state along the ones we will talk about in this thesis. We can cite the *local or instantaneous vacuum* which is a generalization of the Bunch-Davies one. It transform the original problem into a Cauchy one at a finite initial time with initial conditions:

$$u_k(\tau_i) = \frac{1}{\sqrt{2k}} \qquad u_k'(\tau_i) = \pm iku_k(\tau_i) \tag{1.60}$$

This approach is equivalent to the infinite past assumption, this is the reason why sometimes in literature they are thought as a unique object even if they come out from two different ideas. Another prescription also used is the *minimal-energy vacuum*, so called because it minimizes the energy density:

$$u_k(\tau_i) = \frac{1}{\sqrt{2k^0}} \qquad u_k'(\tau_i) = \pm ik^0u_k(\tau_i) \tag{1.61}$$

According to the standard definition of the distruction operator a_k we have to impose the annichilation of the second term, which lead to the zero-setting of the mode function v at the time we set as the beginning of the Inflation. Such request leads to the following relation between the two coefficients A_k and B_k.

$$B_k = \frac{ie^{-2ik\tau_i}}{2k\tau_i + i} A_k \tag{1.66}$$

and bearing in mind the condition 1.54 we can make explicit the dependence of the Bogoliubov coefficient to the initial time τ_0 and so justify the definition of one parameter family.

$$|A_k|^2 = 1 + |B_k|^2 = 1 + \frac{ie^{-2ik\tau_0}}{2k\tau_0 + i}\frac{-ie^{-2ik\tau_0}}{2k\tau_0 - i}|A_k|^2 = 1 + \frac{1}{4k^2\tau_0^2 + 1}|A_k|^2 \tag{1.67}$$

So, we have:

$$|A_k|^2 = \frac{1}{1 - \frac{1}{4k^2\tau_0^2+1}} \equiv \frac{1}{1 - |\alpha_k|^2} \tag{1.68}$$

where we have defined in the last step: $\alpha_k = \frac{i}{2k\tau_0+i}$. Through these simple steps we have highlighted the nature of the α-vacua as a one parameter family of states. According to this easy relation we can perform the computation of the power spectrum which will carry completely new terms due to the crucial assumption we just made. By a straightforward algebra we find the following:

$$P_\phi = \frac{k^3}{2\pi^2 a^2}\langle 0|f_k^\dagger f_{k'}|0\rangle = \frac{k^3}{2\pi^2 a^2}\Big|A_k\frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau}) + B_k\frac{e^{ik\tau}}{\sqrt{2k}}(1 + \frac{i}{k\tau})\Big|^2 =$$
$$\left(\frac{H}{2\pi}\right)^2\Big[1 + |B_k|^2\Big(2 + \frac{2k\tau_i + i}{i}e^{2ik\tau_i} + \frac{2k\tau_i - i}{-i}e^{-2ik\tau_i}\Big)\Big] = \bar{P}_\phi + \delta P_\phi \tag{1.69}$$

So, it is self evident that we will receive an additional contribute to the power spectrum from the degeneracy of the initial quantum state. We will perform in detail this computation later.

1.5 Primordial quantum perturbations in slow-roll inflation regime

So far we have just highlighted basic properties peculiar of quantum field theory in curved space-time. Let's come back to the physics. Subhorizon scales are microscopic and therefore quantum effects are non negligible. Now, it's time to move on and tackle the original problem of studying the quantum fluctuations of the Inflaton field driven by a slow roll potential. This is the most important result of the inflation paradigm. We will derive in the slow roll approximation the power spectrum of the scalar perturbations in terms of the conserved gauge invariant quantity $\mathcal{R}$. It is important to stress again the crucial role the Hubble parameter H has for the faith of the perturbations. Remember that the value of the Hubble parameter give us crucial information about the instant of the history when fluctuations enter and exit the horizon. When we introduced the Inflaton field to fulfil the necessity to have an expansion era of the primordial

Universe, we convinced ourselves to use a slow roll potential to drive the scalar field ϕ. Soon after, it was natural to impose some conditions on the potential to guarantee the long-standing duration of the accelerated period. To keep track of those conditions it is common to use the so called "slow-roll parameters". It is necessary to introduce this objects which will play a very important role in the analysis. In the following we will follow the conventions used by Maggiore in [8] .

$$\epsilon \equiv -\frac{d\ln H}{d\ln a} = -\frac{\dot{H}}{H^2} \qquad \eta \equiv \frac{d\ln \epsilon}{d\ln a} = \frac{\dot{\epsilon}}{H\epsilon} \qquad \delta = \frac{\phi''}{H\phi'} \tag{1.70}$$

Using the slow roll conditions it is immediate to convince ourselves that the inflation can take place if and only if:

$$Inflation \Leftrightarrow \epsilon \ll 1 \tag{1.71}$$

Moreover, making use of the dynamical equation of that field we can write these parameters as functions of the potential:

$$\epsilon \equiv \frac{1}{2M_{Pl}^2}\frac{\dot{\phi}^2}{H^2} \approx \epsilon_V = \frac{M_{Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 \qquad \eta_V = M_{Pl}^2\frac{V_{\phi\phi}}{V} \tag{1.72}$$

These ones are related by the following:

$$\eta \approx 4\epsilon_V - 2\eta_V \qquad \delta = \epsilon_V - \eta_V \tag{1.73}$$

Finally, we are ready to tackle the problem of finding the power spectrum produced by scalar perturbation during inflation. If we take in consideration the scalar sector of the Einstein equation, it's easy to spot that we don't have any anisotropic contribution (at least, at first order of the perturbations). So, we have only to deal with two independent variable: one Bardeen potential Ψ and the quantum perturbation of the Inflaton field $\delta\phi(\tau, \mathbf{x})$. To solve the dynamics is useful to take into account two simple equation involving these two variables. The natural choice in this case is to take the (00) and the (oi) component of the perturbed Einstein equation, which are the following:

$$\nabla^2\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Psi) = -4\pi G\Big[(\phi_0')^2\Psi - \phi_0'\delta\phi' - a^2\frac{dV(\phi_0)}{d\phi_0}\delta\phi\Big] \tag{1.74}$$

$$\Psi' + \mathcal{H}\Psi = 4\pi G\phi_0'\delta\phi \tag{1.75}$$

At this stage is useful to introduce the famous variable u (sometimes is indicated with the letter Q in literature):

$$u \equiv -z\mathcal{R} \tag{1.76}$$

where $\mathcal{R}$ is the gauge invariant curvature perturbation on comoving curvature hyper-surface.

$$\mathcal{R} = -\Psi - \frac{\mathcal{H}\delta\phi}{\phi_0'} \tag{1.77}$$

and we also defined z as:

$$z = \frac{a\phi_0'}{\mathcal{H}} \tag{1.78}$$

The latter quantity has the role to encode the dependence of the model we are playing with. In fact it is easy to see that it depends only on the background evolution because the classical part of the Inflaton field ϕ_0 satisfies classical equation of motion which are the only ones influenced by the driven potential V subjected to the slow-roll conditions. The variable u is called in literature the Mukhanov-Sasaki variable. It is gauge invariant by construction and is related to the inflation perturbation $\delta\phi$ on a generic slicing and to a curvature perturbation Ψ in that gauge. It represents the inflation potential on spatially flat slices[3]:

$$u = \delta\phi|_{\Psi=0} \tag{1.80}$$

Performing the passage in the Fourier space we finally get the well known Mukhanov-Sasaki equation:

$$u''_{\mathbf{k}} + \left(k^2 - \frac{z''}{z}\right)u_{\mathbf{k}} = 0 \tag{1.81}$$

The dependence of the potential is encoded in the effective mass term in the brackets. Through the expression of z we can dive into the analysis of the primordial scalar power spectrum. In the previous chapter we already solved this kind of equation where the ratio $\frac{z''}{z}$ was replaced by $\frac{a''}{a}$. The previous analysis can be thought as the "zero-order" level of comprehension of the problem.

If we take into account the first slow roll parameter ϵ we can get immediately a modification of the Hubble parameter H (or its version through the conformal time $\mathcal{H}$). We know that the Hubble parameter is nearly constant during inflation. If we integrate the relation between ϵ and H we can obtain the following relation (in conformal time):

$$\mathcal{H}' = -\mathcal{H}^2(\epsilon_H - 1) \tag{1.82}$$

and so, at first order of the slow-roll perturbation theory we arrive at the following expression for the effective mass term in the Mukhanov-Sasaki equation

$$\frac{z''}{z} \sim \mathcal{H}^2[2 + 2\epsilon + 3\delta] = \mathcal{H}^2[2 + 5\epsilon + 3\eta] \tag{1.83}$$

and integrating we get a modification of the definition of conformal time given by the presence of the slow roll parameter:

$$\tau = -\frac{1}{(1-\epsilon)\mathcal{H}} \tag{1.84}$$

The effective mass term in the Mukhanov-Sasaki equation becomes:

$$\frac{z''}{z} = \frac{\nu^2 - 1/4}{\tau^2} \tag{1.85}$$

[3]Just for completeness, remember that the curvature perturbation on comoving hyper-surfaces $\mathcal{R}$ and the other gauge-invariant curvature perturbation on uniform energy density hyper-surfaces ζ are connected by the relation:

$$-\zeta = \mathcal{R} + \frac{2\rho}{9(\rho + P)}\left(\frac{k}{aH}\right)^2\Psi \tag{1.79}$$

and so, is evident that on large scales we have : $\mathcal{R} \approx -\zeta$

where $\nu^2 = \frac{9}{4} + 9\epsilon - 3\eta$ and consequently $\nu \approx \frac{3}{2} + 3\epsilon - \eta$ We have obtained, in some sense, a correction of the power spectrum of a scalar field evolving in a de Sitter Universe. This correction arise from the slow roll parameters (which are model dependent because their definition through the potential that drives the accelerated period).

1.5.1 Bunch-Davies initial condition

At this stage we have to face, again, the problem of the choice of the initial condition. Again, we can set the Bunch-Davies state as the vacuum and compute the power spectrum.Remember the general behaviour of the Hankel function for small argument:

$$H_\nu(x << 1) \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} x^{-\nu} \tag{1.86}$$

In the case of our interest, $x = -k\tau$. Within this assumption, the power spectrum of the field u is:

$$\mathcal{P}_u = \frac{k^2}{8\pi}(-k\tau)|H_\nu^{(1)}(-k\tau)|^2 \tag{1.87}$$

Here, we can switch to the curvature perturbation (which is a convenient choice in order to have a conserved quantity when the modes exit the horizon). The power spectrum of the curvature perturbation is obtained from the previous one by simply multiplying the factor $\frac{1}{z^2}$ and we obtain:

$$\mathcal{P}_\mathcal{R} = \frac{1}{z^2}\frac{k^2}{8\pi}(-k\tau)|H_\nu^{(1)}(-k\tau)|^2 \tag{1.88}$$

Restoring the dependence of the slow roll parameter and introducing a time τ_k which coincide with the instant when the modes crossed the horizon, we can finally get the general formula for the scalar power spectrum at first slow roll order:

$$\mathcal{P}_\mathcal{R} = (1 - 2\epsilon)\frac{1}{8\pi}\left(\frac{H^2}{\dot{\phi}_0}\right)_k^2 (-k\tau)^{3+6\epsilon-2\eta}|H^{(1)}_{\frac{3}{2}+3\epsilon-\eta}(-k\tau)|^2 \tag{1.89}$$

where the subscript k indicate that this quantity is evaluated at horizon exit. Nevertheless, from this formula and from the asymptotic behaviour for $(-k\tau) \to 0$, the dependence of conformal time cancels and we get the constant super-horizon scales power spectrum (which we already predicted). This fact set us free to choice the preferable time in which evaluate this expression. A very convenient time is the time when the modes crossed the horizon. That happens when:

$$\frac{k}{a(t_k)} = H(t_k) \qquad or \qquad k = \mathcal{H}(t_k) \tag{1.90}$$

and this lead to:

$$\tau(t_k) = -\frac{1}{(1 - \epsilon(t_k))} \approx -\frac{1}{k} \tag{1.91}$$

The last expression stress the concept of the dependence of the slow roll parameter from the scale k.

Due to this k-dependence, we can infer another useful relation between the Hubble parameter evaluated at the time of the horizon crossing of a given mode k and the time at which the last scale k_l leaves the horizon. It's easy to demonstrate that:

$$H(t_k) = H(t_l)\Big(\frac{k}{k_l}\Big)^{-\epsilon} \tag{1.92}$$

This equation says explicitly that fluctuations with smaller wavelengths leave the horizon later (as we already pointed out in the introduction of the Inflationary stage in chapter 1). In a similar way, we can also fin the k-dependence of the slow roll parameters.

$$\epsilon(t_k) = \epsilon(t_l)\Big(\frac{k}{k_l}\Big)^{2\epsilon+2\eta} \tag{1.93}$$

$$\eta(t_k) = \eta(t_l)\Big(\frac{k}{k_l}\Big)^{\epsilon-2\eta} \tag{1.94}$$

If we assume that the last scale leaves the horizon almost at the end of the inflation, then ϵ_l is something near to unity. After some manipulations we can write the power spectrum in the slow roll approximation:

$$\mathcal{P} = \frac{1}{2M_{Pl}^2\epsilon}\Big(\frac{H_k}{2\pi}\Big) \tag{1.95}$$

This is the formula we were looking for. In our general treatment of the power spectrum of the real scalar field we arrived at a scale independent formula. Now, we would like to quantify the dependence by the scale of the power spectrum introducing a convenient parametrization approximating the power spectrum with a power law of the scales.

$$\mathcal{P}_\phi(k) = A_\phi\Big(\frac{k}{k_*}\Big)^{n_s-1} \tag{1.96}$$

where we introduced the quantity $n_s - 1$ which is called the "spectral tilt" and is a fundamental object to parametrize the deviation from a flat spectrum. A power spectrum with a positive tilt is defined as a "blue spectrum", otherwise is called "blu spectrum". The limit case when $n_s = 1$ is the case of the Harrison-Zeldovich spectrum. We also introduced a pivot scale k_* which is something fixed by the energy scale of the experiment we project. It is worth-noting that both the amplitude A_ϕ and the spectral tilt are quantities that in general depend upon the model we are building. In order to compare the theoretical predictions with observational data one can introduce another parametrization which generalize the previous one.

$$\mathcal{P}_\phi(k) = A_\phi\Big(\frac{k}{k_*}\Big)^{n_s-1+\frac{1}{2}\frac{dn_s}{d\ln k}\ln(k/k_*)+\dots} \tag{1.97}$$

Here we have introduce the "running of the spectral index" $\frac{dn_s}{d\ln k}$. Our analysis was made up for a generic scalar field running in a curved background. From all these relations we earned, we can make the last step consisting in the computation of the tilt:

$$n_s(k) - 1 = \frac{d\log\mathcal{P}_{\mathcal{R}(k)}}{d\log k} \tag{1.98}$$

explicitly we find the dependence on the two slow roll parameters:

$$n_s - 1 = -6\epsilon + 2\eta \tag{1.99}$$

1.5.2 α vacua

Now, let's go back to the time when we choose the initial condition for the mode functions. Let's stress again that the inflation is just a limited period of the history of the Universe and so it is not reasonable to postulate and infinite past Minkowskian condition. Moreover, if the inflationary period lasts more than 60 e-folds the physical wavelengths of the primordial modes will be smaller than the Planck scale and so it is natural to think how introduce new effects in this scenario. A consistent quantum gravitational theory still lacks but we can introduce trans-Planckian (ultraviolet) new physics through the choice of the vacuum. Again, we will follow the Danielsson's prescription for that non trivial choice. Let's ho back to the eqn. 1.69 an recast the square of the modulus in the following way:

$$P_\phi = \frac{k^2}{4\pi a^2}\Big[|A_k|^2\Big(1+\frac{1}{k^2\tau^2}\Big)+A_k B_k^* e^{-2ik\tau}\Big(1-\frac{i}{k\tau}\Big)^2+B_k A_k^* e^{-2ik\tau}\Big(1+\frac{i}{k\tau}\Big)^2+|B_k|^2\Big(1+\frac{1}{k^2\tau^2}\Big)\Big] \tag{1.100}$$

From this expression we can extract the behaviour of the power spectrum in the two regimes we are mainly interested in. For first, if we take the limit: $|k\tau| \gg 1$ (i.e. modes well inside the horizon), the oscillating terms average to zero and we can also neglect the terms $\frac{1}{k^2\tau^2}$. So we have:

$$P_\phi \simeq \frac{k^2}{4\pi^2 a^2}[|A_k|^2 + |B_k|^2] = \frac{k^2 H^2 \tau^2}{4\pi}[1 + 2|B_k|^2] \tag{1.101}$$

where in the last step we made use of the deSitter scale factor $a = -\frac{1}{H\tau}$ and the condition between the Bogoliubov coefficient. Then, the other important case is when $k\tau \to 0$, i.e. super-horizon modes. In this case, all the relevant terms are those which are coupled to the inverse of $k\tau$:

$$\begin{aligned} P_\phi &\simeq \frac{k^2}{4\pi^2 a^2}\Big[\frac{|A_k|^2}{k^2\tau^2} + \frac{|B_k|^2}{k^2\tau^2} + A_k B_k^*\Big(-\frac{1}{k^2\tau^2}\Big) + A_k^* B_k\Big(-\frac{1}{k^2\tau^2}\Big)\Big] \\ &= \frac{H^2}{4\pi^2}(|A_k|^2 + |B_k|^2 - A_k B_k^* - A_k^* B_k) = \frac{H^2}{4\pi^2}|A_k - B_k|^2 \end{aligned} \tag{1.102}$$

At this point we can recall the explicit expression we have found for A_k and B_k, so that:

$$\begin{aligned} P_\phi &= \Big(\frac{H}{2\pi}\Big)^2\Big(\frac{1}{1-|\alpha_k|^2} + \frac{1}{4k^2\tau_0^2+1}\frac{1}{1-|\alpha_k|^2} - \frac{ie^{-2ik\tau_0}}{2k\tau_0+i}\frac{1}{1-|\alpha_k|^2} + \frac{ie^{2ik\tau_0}}{2k\tau_0}\frac{1}{1-|\alpha_k|^2}\Big)^2 \\ &= \Big(\frac{H}{2\pi}\Big)^2(1+|\alpha_k|^2 - \alpha_k e^{-2ik\tau_0} - \alpha_k^* e^{2ik\tau_0})\frac{1}{1-|\alpha_k|^2} \end{aligned} \tag{1.103}$$

Danielsson codified the inaccessibility of a remote Minkowskian vacuum through the use of a physical cut-off Λ. Remember that he assumed the physical modes began their evolution only after a certain time t_i (the initial time when the inflation starts). If we assume a de Sitter inflationary period (that means $a = -\frac{1}{\tau H}$) we can express the physical scale of a generic mode in the following way

$$k_{com} = a p_{phys} = -\frac{p_{phys}}{\tau H}. \tag{1.104}$$

Imposing the cut-off at a certain $p_{phys} = \Lambda$ at a precise value τ_i we have:

$$\tau(t_{in}) = -\frac{\Lambda}{k_{com} H} \tag{1.105}$$

The most important thing to note is that the value of the conformal time of the beginning is dependent by the value of the value of k. Basically, the main physical concept behind this discussion is that for modes that have largher wavelenghts today, we have to go further in time to impose initial condition.

$$\begin{aligned} P_\phi &= \left(\frac{H}{2\pi}\right)^2 \left(1 + \frac{1}{1+4\frac{\Lambda^2}{H^2}} - \frac{i}{i - 2\frac{\Lambda}{H}} e^{2i\frac{\Lambda}{H}} - \frac{i}{i + 2\frac{\Lambda}{H}} e^{-2i\frac{\Lambda}{H}}\right) \frac{1}{1 - \frac{1}{1+4\frac{\Lambda^2}{H^2}}} \\ &= \left(\frac{H}{2\pi}\right)^2 \left[\frac{2 + 4\frac{\Lambda^2}{H^2} + 2i\frac{\Lambda}{H}(e^{2i\frac{\Lambda}{H}} - e^{-2i\frac{\Lambda}{H}}) - (e^{2i\frac{\Lambda}{H}} - e^{-2i\frac{\Lambda}{H}})}{4\frac{\Lambda^2}{H^2}}\right] \end{aligned} \tag{1.106}$$

Taking the limit $\frac{\Lambda}{H} \gg 1$ we have:

$$P_\phi = \left(\frac{H}{2\pi}\right)^2 \left(1 - \frac{H}{\Lambda} \sin\left(\frac{2\Lambda}{H}\right)\right) \tag{1.107}$$

Doing some simple and straightforward calculations we arrive at the following expression for the scalar power spectrum including the slow roll approximation:

$$\mathcal{P}_s = \frac{1}{8\pi^2}\frac{H^2}{\epsilon}\left[1 + \frac{H}{\Lambda}\sin\left(\frac{2\Lambda}{H}\right) + ...\right] \tag{1.108}$$

In that way also the scalar spectral index get a modification:

$$n_s - 1 \approx 2\eta_V - 6\epsilon_V + 2\epsilon_V \cos\left(\frac{2\Lambda}{H}\right) \tag{1.109}$$

1.6 Gravitational waves background of Early Universe

One of the most robust and model-independent prediction of the Inflation is the production of gravitational waves. The evolution of such tensor perturbation is regulated by the transverse-traceless (TT) spatial part of the Einstein equations. We demonstrated that the power spectrum of the curvature perturbations were dependent of both the Hubble parameter during inflation and to the slow roll parameter ϵ which is characterized by the particular model we are dealing with for the Inflationary scenario. Now, the case of gravitational wave production is more subtle because it doesn't carry any information about the potential driving the accelerated primordial stage of the Universe, but its power spectrum will be only a function of the Hubble parameter during inflation.

By the analysis of the Inflaton, we know that modes in our Universe at first were well inside the Hubble radius; then, they crossed it and, lastly, they re-entered. So, an hypothetical gravitational wave spectrum could carry useful information about the conditions of both the beginning and the end of the Inflation period.

Let's start considering the tensor sector of the metric perturbations. From a mathematical point of view, the treatment of tensor perturbation is easier with respect to the scalar sector because here we have not to deal with gauge ambiguities. From the well known cosmological perturbation theory we can cast the infinitesimal line element in the following manner through the well known Newtonian gauge: $ds^2 = a^2[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j]$ where we have used the conformal time τ and x^i as the comoving spatial coordinates. Through the SVT (sclar-vector-tensor) decomposition of the metric and the consequent study of gauge invariant quantities, we noted that h_{ij} is yet a gauge invariant quantity (unlike the other kind of perturbations). We recall that h_{ij} it's symmetric, traceless and transverse:

$$h_{ij} = h_{ji} \qquad h_{ii} = 0 \qquad \partial_i h_{ij} = 0 \tag{1.110}$$

These proprieties of the tensor perturbations make easier the computation of the Einstein tensor. The reason is that when we try to compute the Ricci scalar by the Ricci tensor $R = g^{\mu\nu} R_{\mu\nu}$, the former one will contain only terms of second order in the perturbations or terms like the ones in the previous equations. Nevertheless, these conditions tell us that the tensor perturbations h_{ij} has only two degree of freedoms.

From these considerations we can infer that the general form of the solutions is the following:

$$h_{ij}(\mathbf{x}, t) = \sum_{\lambda=(+,x)} h^{(\lambda)}(t)\epsilon_{ij}^{(\lambda)}(\mathbf{x}) \tag{1.111}$$

where the two tensor $\epsilon_{ij}^{(+,x)}(\mathbf{x})$ represent the polarization states which, obviusly, obey similar conditions h_{ij} do. For future convenience it is convenient to switch to the Fourier representation:

$$h_{ij} = \sum_{\lambda} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2} M_{Pl}} \epsilon_{ij}^{\lambda}(\mathbf{k}) h_{\mathbf{k}}^{\lambda}(\tau) e^{i\mathbf{k}\mathbf{x}} \tag{1.112}$$

Being $\mathbf{k}$ the direction of the propagation, we can write the polarization tensors in the following way:

$$\epsilon_{ij} \equiv \frac{1}{\sqrt{2}}[\epsilon_i(\mathbf{k})\epsilon_j(\mathbf{k}) - \bar{\epsilon}_i(\mathbf{k})\bar{\epsilon}_j(\mathbf{k})] \tag{1.113}$$

$$\epsilon_{ij} \equiv \frac{1}{\sqrt{2}}[\epsilon_i(\mathbf{k})\bar{\epsilon}_j(\mathbf{k}) - \bar{\epsilon}_i(\mathbf{k})\epsilon_j(\mathbf{k})] \tag{1.114}$$

To arrive at the equation of motion for the tensorial sector we could act as in the case of the scalar sector. One can start from the Einstein equations and perturb them to find the relations between the metric dependent l.h.s. and the matter perturbations induced by the Inflaton in the r.h.s.. Adopting this approach, it is immediate to see that the equation of motion is completely independent form the perturbations of the Inflaton because of we treat it as a perfect fluid and is well known that we do not have any anisotropic term. It also instructive to arrive at the dynamical equations by a parallel way. We can start considering the total action of the theory and varying it. We will see that the term coming from the matter action cancels automatically with one term of the variation of the metric one. From a quantum field theoretical point of view, h_{ij} can be thought as a couple of two scalar massless fields evolving in a dynamical FLRW background. Let's considerer the complete action as a combination of the two parts:

$$S = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{g}R + \int d^4x\sqrt{-g}\Big[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\Big] = S_{EH} + S_{inflaton} \tag{1.115}$$

where the Ricci tensor is the one calculated with respect to the FLRW metric. The complete computation is a little bit long, but we here outline the main aspects of the proof. We can use the fact that the FLRW metric is conformally equivalent to the Minkowskian one. In the case two metrics are related by: $\tilde{g}_{ab} = \Omega^2 g_{ab}$ one can demonstrate that the new Ricci scalar could be calculated from the former one by the use of the following formula:

$$\tilde{R} = \Omega^{-2}(R - 6\nabla^2\ln\Omega - 6\nabla_a\ln\Omega\nabla^a\ln\Omega) \tag{1.116}$$

Along with this remarkably result, we have to calculate the metric perturbation up to the second order. For this purpose it will be useful to remember the following relation:

$$\det(X+\epsilon A) = \det X\det(I+\epsilon B) = \det\Big(1 + \epsilon\operatorname{tr}B + \frac{\epsilon^2}{2}\Big[(\operatorname{tr}B)^2 - \operatorname{tr}(B^2)\Big]\Big) + O(\epsilon^3) \tag{1.117}$$

where we denoted: $B \equiv X^{-1}A$ Using this formula we can calculate the determinant of the metric that will take part in the computation of the action's variation:

$$^{(0)}g = -a^8 \tag{1.118}$$

$$^{(1)}g =^{(0)} g\operatorname{tr}(a^{-2}\eta^{\mu\nu}a^2h_{\nu\rho}) = 0 \tag{1.119}$$

$$^{(2)}g = \frac{a^8}{2}h_{\mu\nu}h^{\mu\nu} \tag{1.120}$$

Using these formulas we can calculate the Ricci scalar up to the second order and then use the conformal relations to find the Ricci scalar for a FLRW Universe. In summary, we have the perturbed Ricci scalar in Minkowki:

$$\bar{R} = -h^{ij}\ddot{h}_{ij} - \frac{1}{4}\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} - \frac{3}{4}\dot{h}_{ij}\dot{h}^{ij} \tag{1.121}$$

and then:

$$ {}^{(0)}R = \frac{6}{a^2}(\mathcal{H}' + \mathcal{H}) \tag{1.122} $$

$$ {}^{(1)}R = 0 \tag{1.123} $$

$$ {}^{(2)}R = -a^{-2}\Big(h^{ij}\ddot{h}_{ij} + \frac{1}{4}\bar{\partial}_i h_{jk}\bar{\partial}^{jk} + \frac{3}{4}\dot{h}_{ij}\dot{h}^{ij}\Big) - 3a^{-2}\mathcal{H}h^{ij}\dot{h}_{ij} \tag{1.124} $$

At this stage we have all the components to calculate the metric action up to the second order. That means that we will use only the following quantities:

$$ {}^{(2)}(\sqrt{-g}R) =^{(0)} \sqrt{-g}^{(2)}R +^{(2)} \sqrt{-g}^{(0)}R \tag{1.125} $$

Using the previous results we can write down the full Einstein-Hilbert action up to the second order:

$$ {}^{(2)}S_{EH} = \frac{1}{8}\int d^4xa^2\dot{h}_{ij}\dot{h}^{ij} - \frac{1}{8}\int d^4xa^2\bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk} - \frac{1}{4}\int d^4xa^2(\mathcal{H}^2 + 2\dot{\mathcal{H}})h_{ij}h^{ij} \tag{1.126} $$

(we have set $M_{Pl}^2 = 1$ for practical purpose, but we will restore it at the end of the computation). Now we can move on and evaluate the perturbed matter action of the scalar field in its tensorial sector up to the second order. We have:

$$ {}^{(2)}(\sqrt{-g}\mathcal{L}_\phi) =^{(2)} \sqrt{-g}^{(0)}\mathcal{L}_\phi +^{(0)} \sqrt{-g}^{(2)}\mathcal{L}_\phi \tag{1.127} $$

Doing some simplification we arrive at the following expression for the action:

$$ {}^{(2)}S_\phi = \frac{1}{4}\int d^4xa^2(\mathcal{H}^2 + 2\dot{\mathcal{H}})h_{ij}h^{ij} \tag{1.128} $$

From this analysis we have shown that the contribution of the Inflaton in the tensorial sector cancels exactly with the third element we calculated of the Einstein-Hilbert term. Consequently, the tensor sector is characterized by just the second order metric perturbations.

$$ {}^{(2)}S = \frac{M_{Pl}^2}{8}\int d^4xa^2(\dot{h}_{ij}\dot{h}^{ij} - \bar{\partial}_i h_{jk}\bar{\partial}^i h^{jk}) \tag{1.129} $$

In the Fourier space we can easily convert this action such that:

$$ {}^{(2)}S = \frac{M_{Pl}^2}{16}\sum_{\lambda=\pm 2}\int d\tau d^3ka^2\left[(h'^{\lambda})^2 + k^2(h^\lambda)^2\right] \tag{1.130} $$

From this expression it is easy to show that the Euler-Lagrange equations are the following:

$$ h_{ij}^{''} + 2\frac{a'(\tau)}{a(\tau)}h'_{ij} - \nabla^2 h_{ij} = 0 \tag{1.131} $$

The previous equation is a well known wave equation and consequently its solutions are gravitational waves. In general on the r.h.s. we could have another term which provides a source for the gravitational waves Π_{ij} (anisotropic stress tensor). In that case the equation of motion could get a modification:

$$h''_{ij} + 2\frac{a'(\tau)}{a(\tau)}h'_{ij} - \nabla^2 h_{ij} = 16\pi G a^2(\tau)\Pi_{ij}(\tau, \mathbf{x}) \tag{1.132}$$

Anyway, a tensorial power spectrum could raise even if this tensor is absent. In general, the source of the anisotropic component could be of different nature; for example, relativist particles (e.g. neutrinos) moving on geodesic of the metric that included the tensor perturbations, generate an isotropic stress that could affect the previous equation. Nevertheless, if we go further in perturbations, anistropic sources could arise from the first order perturbations as well. We will see that first order perturbations in the scalar sector could give rise to relevant second order gravitational waves background. For the moment we will stick with the discussion of gravitational waves emerging from a perturbed FLRW Universe. For the moment, let's consider a free theory which the anisotropic stress tensor vanish. We will come back to this point later to include sources for the tensorial sector. The previous homogeneous equation is a Bessel equation which is a common equation when one approach to cosmological perturbations. As usual, it is convenient to introduce an auxiliary field to cast the ordinary Bessel equation into a well understood shape: an oscillating harmonic oscillator with a time dependent frequency term. This is a very useful step toward the quantization project. So, we can define:

$$v_{ij} = \frac{aM_{Pl}}{\sqrt{2}}h_{ij} \tag{1.133}$$

By the use of this auxiliary field we can interpret the original theory such as a theory of two scalar massless fields in a FLRW background. In term of the new field we can write the Fourier expansion:

$$v_{ij} = \sum_{\lambda}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\epsilon^{\lambda}_{ij}(\mathbf{k})v^{\lambda}_{\mathbf{k}}(\tau)e^{i\mathbf{kx}} \tag{1.134}$$

So, we can conclude that the equations of motion, in the Fourier space, take the following form:

$$h''_k + 2\frac{a'(\tau)}{a(\tau)}h'_k + k^2 h_k = 0 \tag{1.135}$$

In term of the rescaled field:

$$v''^{\lambda}_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right)v^{\lambda}_{\mathbf{k}} = 0 \tag{1.136}$$

If we consider a generic cosmological epoch, i.e. $\frac{a''}{a} \sim \frac{1}{\tau^2}$ we can infer the behaviour of the perturbations. Inside the horizon ($k\tau >> 1$) we have the usual harmonic oscillator behaviour for all the well known epochs of the Universe (RD, MD, ΛD). Looking for the generic solution of that equation we have:

$$v_k(\tau) = \sqrt{-\tau}[C_1(k)H^{(1)}_{\frac{3}{2}}(-k\tau) + C_2(k)H^{(2)}_{\frac{3}{2}}(-k\tau)] \tag{1.137}$$

Those are the mode functions we are going to use to quantize the tensorial field. Coming back to our original object of interest, we can finally write down the canonical commutation relations which encode the quantum nature we are conferring to this tensorial field. So, defining the conjugate field $\pi_{\mathbf{k}}$ for each polarization mode ($r = 1, 2$)

$$\pi^r_{\mathbf{k}}(\tau) = a^2(\tau) h'^r_{\mathbf{k}}(\tau) \tag{1.138}$$

We impose the following equal time commutation relations:

$$\left[\hat{h}^r_{\mathbf{k}}(\tau), \hat{\pi}^s_{\mathbf{k'}}(\tau)\right] = i\delta^{rs}\delta^{(3)}(\mathbf{k} - \mathbf{k'}) \tag{1.139}$$

$$\left[\hat{h}^r_{\mathbf{k}}(\tau), \hat{h}^s_{\mathbf{k'}}(\tau)\right] = \left[\hat{\pi}^r_{\mathbf{k}}(\tau), \hat{\pi}^s_{\mathbf{k'}}(\tau)\right] = 0 \tag{1.140}$$

Finally, we express the tensor field as a linear combination of the mode function (obtained inverting the relation with the rescaled field v) and the usual creation and annichilation operators.

$$\hat{h}^r_{\mathbf{k}}(\tau) = h_k(\tau)\hat{a}^r_{\mathbf{k}} + h^*_k(\tau)\hat{a}^{r\dagger}_{\mathbf{-k}} \tag{1.141}$$

By virtue of the CCRs we just imposed on the field, the ladder operators satisfy the following:

$$\left[\hat{a}^r_{\mathbf{k}}, \hat{a}^s_{\mathbf{k'}}\right] = \delta^{rs}\delta^{(3)}(\mathbf{k} - \mathbf{k'}) \tag{1.142}$$

and all the others are null. Nevertheless, it's important to remember that if we require the consistency of the two sets of commutation relations we need to impose the following relation that may be thought as a Wronskian conservation.

$$h_k(\tau)h'^*_k(\tau) - h^*_k(\tau)h'_k(\tau) = \frac{i}{a^2(\tau)} \tag{1.143}$$

1.6.1 Bunch-Davies vacuum

Now it's time to make our choice of the initial conditions and compute the observables. Making use of the same formulas we used for the scalar field yet, we can compute the power spectrum of the tensor field, which, at first order of slow roll parameters and on super-horizon scales is given by the following:

$$\mathcal{P}_t = \frac{8}{M^2_{Pl}}\Big(\frac{H}{2\pi}\Big)\Big(\frac{k}{aH}\Big)^{-2\epsilon} \tag{1.144}$$

It is clear that evaluating it at the horizon crossing it will depends only on the value of H_k. Now we have found the expression for the tensor power spectrum, we can do the same parametrization we made for the scalar sector.

$$\mathcal{P}_t = A_t\Big(\frac{k}{k_*}\Big)^{n_t + \frac{1}{2}\frac{dn_t}{d\ln k}\ln k(k/k_*)+\ldots} \tag{1.145}$$

Here we finally introduce another very useful quantity which characterize the study of the various inflationary models: the tensor-to-scalar ratio r which is defined to be the ratio between the amplitude of the tensor and the scalar power spectrum

$$r \equiv \frac{A_t}{A_s} \tag{1.146}$$

In that case it is easy to see that:

$$r = 16\epsilon \tag{1.147}$$

Now, let's calculate the tensor spectral index:

$$n_t = \frac{d \ln \mathcal{P}_t}{d \ln a} \frac{d \ln a}{d \ln k} = -2\epsilon(1-\epsilon)^{-1} \approx -2\epsilon \tag{1.148}$$

Therefore, we can conclude showing what is commonly called in literature *the consistency relation*

$$r \approx -8 n_t \tag{1.149}$$

Any study of different models will provide modifications or extension of such relation.

1.6.2 α vacuum in deSitter space-time

In the same fashion we computed this corrected power spectrum of a massless scalar field in a de Sitter background (as can be thought the scalar perturbation of the Inflaton), we can also obtain the respective formula for the tensor power spectrum modified by the initial condition involving the cut-off scale Λ. As long as the two polarization states of the gravitational waves can be thought as a couple of massless scalar field rescaled by the scale factor a (as we will see later in this dissertation) we can safely use the same result we just obtained to write the following:

$$\mathcal{P}_t = 2\frac{H^2}{\pi^2}\Big[1 + \frac{H}{\Lambda}\sin\Big(\frac{2\Lambda}{H}\Big) + ...\Big] \tag{1.150}$$

Here we have all the information we may need to compute the two indexes that characterize the analysis. It is straightforward to see that if we introduce such modifications in the initial status at the beginning of the Inflation we will get consequently also modifications of the scalar and tensor index. Recalling the expression we found for the modified scalar spectral index:

$$n_s - 1 = \Big[2\eta - 4\epsilon + \epsilon\cos\Big(\frac{2\Lambda}{H}\Big)\Big](1+\epsilon) \approx 2\eta_V - 6\epsilon_V + 2\epsilon_V\cos\Big(\frac{2\Lambda}{H}\Big) \tag{1.151}$$

and making explicit the dependence of the number of e-folds, we obtain:

$$n_s - 1 = 2\eta_V - 6\epsilon_V + 2\epsilon_V\cos\Big(\frac{2\Lambda}{H_{infl}}\exp(\epsilon_V N_e)\Big) \tag{1.152}$$

In the same way we can also write down the modification that arise in the tensor sector:

$$n_t = -2\Big[\epsilon + \epsilon\cos\Big(\frac{2\Lambda}{H}\Big)\Big] = -2\Big[\epsilon_V + \epsilon_V\cos\Big(\frac{2\Lambda}{H_{infl}}\exp(\epsilon_V N_e)\Big)\Big] \tag{1.153}$$

We demonstrated that just taking the simple assumption of the presence of a physical cu-off we automatically introduced an important modification in both the scalar and tensor sector though the presence of an oscillatory term. The most important thing that arise from this analysis is the consequent violation of the inflationary consistency relation. In fact, computing the tensor-scalar ratio we find a consistent modification:

$$r = 8n_t\Big[-1 + \cos\Big(\frac{2\Lambda}{H_{infl}}\exp(\epsilon_V N_e)\Big)\Big]^{-1} \tag{1.154}$$

Thus, we can safely say that an accurate measure of the tensor-scalar ratio could provide a proof of the Inflation's duration. Such experimental evidence could be essential in order to understand if the Inflation has or not a predecessor era. Inflation is thought to be the first era when we can use the toolkit we built over the last century using Quantum Field Theory and General Relativity; if there would be another physical period before the start of such inflationary phase, the initial conditions of Inflation may be footprints of such unknown phase.

1.6.3 The energy density

In this section we want to face the problem of finding the energy of gravitational waves in the most general settings of a dynamical background. For references see [31], [7] and references therein. It is immediate to spot the difference between General Relativity and Electrodynamics. The latter one is described by linear dynamical equations, while the former one is described by the Einstein equations which are non linear and so any form of energy contributes to space-time curvature and vice-versa! The consequence is that the presence of gravitational waves constitutes a source of energy and momentum. Moreover, if we want to investigate about this topic, we are forced to go beyond the linearised (or Minkowskian) assumption for the background. In fact, assuming such strong condition, we automatically wash out from the beginning the possibility of generating any form of curvature in the background. To sum up, it is mandatory to take into account a general curved background. When we study the production of gravitational waves in a curved background is customary to split the metric in two contributions:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \tag{1.155}$$

In the case we are interested in, i.e. the GWs background generate in a FLRW Universe, both the background and the metric are functions of time. Here we have to face a new problem that didn't arise when one studies the GWs background in the weak-field limit (i.e. when the background part $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ is constant). It is not so obvious which one can be considered as the background and which one the "perturbation". If every component of such splitting is dynamic, we need a new recipe to study the background and the perturbations separately. In general, there could be two possibility to make such distinction. The first one is to identify the background as the one which has a clear symmetry (e.g. the ones which characterize the FLRW Universe), but the simplest way is to make a distinction by mean of the scales and frequencies. In fact, we can discriminate the two components simply taking into account the different typical wavelengths. We will call L_B the typical scale of the background metric and λ the characteristic one of the tensor perturbations. So we can spot the two contributions in a clear way if we notice that the relation between these two has to be:

$$\lambda \ll L_B \tag{1.156}$$

or, alternatively, we can think such relation as a condition on the frequencies:

$$\nu_B \gg \nu \tag{1.157}$$

So, the "ripples" of a generic metric will be identified by an high frequency and, on the contrary, the background will be studied by the low energy ones. To study their evolution let's consider the trace-reversed form of the Einstein equation:

$$R_{\mu\nu} = 8\pi G\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big) \tag{1.158}$$

Then, we can expand the Ricci tensor, with respect to the perturbations $h_{\mu\nu}$, up to the second order in the following way:

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ... \tag{1.159}$$

where just the first term in the expansion contains only the background metric, from the second term on there will be the presence of perturbations with powers specified by the upper label. Thank to this expansion we can split the dynamics in two parts: the high and the low frequency sectors. In fact, it it easy to see that the $\bar{R}_{\mu\nu}$ term contains only the low frequency modes, while $R^{(1)}_{\mu\nu}$, being linear in $h_{\mu\nu}$, contains only the high frequency modes and, finally, the $R^{(2)}_{\mu\nu}$ contains both of them.

$$\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{low} + 8\pi G\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{low} \tag{1.160}$$

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{high} + 8\pi G\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{high} \tag{1.161}$$

The first equation will tell us the dynamics of the background, while the second one will tell us the evolution of the perturbations $h_{\mu\nu}$ as waves propagating on a curved background. Now, we want to define the energy-momentum tensor for these gravitational waves. In order to do that, let's consider the average of both the previous equations on a spatial length l such as:

$$\lambda \ll l \ll L_B \tag{1.162}$$

In this way the typical size of the background will be recognized as constant over this spatial region, while the perturbations will be understand as oscillation over such region. Doing this average, it is clear that the equation for the high frequency modes is trivially null because of the average wash out the short wavelength (high frequency) contribution.

$$\underbrace{\bar{R}_{\mu\nu}}_{\mathcal{O}\left(\frac{1}{L_B}\right)^2} = -\langle R^{(2)}_{\mu\nu}\rangle + 8\pi G\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle \equiv -\underbrace{\langle R^{(2)}_{\mu\nu}\rangle}_{\mathcal{O}\left(\frac{h}{\lambda}\right)^2} + 8\pi G\Big(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\Big) \tag{1.163}$$

where in the last step we defined the the average of the second term on the right hand side as the respective quantity calculated with respect to the background metric (also $\bar{T}$ it's the trace calculate by mean of $\bar{g}$). From this equation it is explicit that not only the matter content of the Universe affect the background, but also the contribute from $\langle R^{(2)}_{\mu\nu}\rangle$ plays an active role on the curvature of the background. Moreover, we can infer an important relation between the magnitude of the perturbations h and the wavelength of the signal. In fact, if the matter term would not appear we would have approximately an equivalence of scales:

$$\Big(\frac{1}{L_B}\Big)^2 \simeq \Big(\frac{h}{\lambda}\Big)^2 \tag{1.164}$$

In the case the matter contribute was present, to balance the three contributes we obtain the following relations which clarifies the condition of validity of perturbation theory:

$$\frac{h}{\lambda} \lesssim \frac{1}{L_B} \rightarrow h \lesssim \frac{\lambda}{L_B} \tag{1.165}$$

At this stage we define the following object in the TT gauge:

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \tag{1.166}$$

and, taking the trace of this tensor, it is immediate to write this definition in the following way:

$$-\langle R^{(2)}_{\mu\nu}\rangle = 8\pi G\Big(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\Big) \tag{1.167}$$

and consequently the equation 1.163 becomes:

$$\bar{R}_{\mu\nu} = 8\pi G\Big(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\Big) + 8\pi G\Big(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\Big) \tag{1.168}$$

This equation can be recast in a more useful way:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = 8\pi G(\bar{T}_{\mu\nu} + t_{\mu\nu}) \tag{1.169}$$

This relation is particularly interesting because of it shows that the gravitational waves act on the background geometry as well as a matter field described by $t_{\mu\nu}$. By a straightforward computation we have:

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle \qquad \langle R^{(2)}\rangle = 0 \tag{1.170}$$

and so:

$$t_{\mu\nu} = \frac{1}{32\pi G}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle \tag{1.171}$$

Finally we can calculate the energy density:

$$t_{00} = \frac{1}{32\pi G}\langle\dot{h}^{TT}_{ij}\dot{h}^{TTij}\rangle \tag{1.172}$$

So, explicitly:

$$\rho_{GW} = -T_0^0 = \frac{1}{64\pi G}\frac{(h'_{ij})^2 + (\nabla h_{ij})^2}{a^2} \tag{1.173}$$

and computing the vacuum expectation value:

$$\langle 0|\hat{\rho}_{GW}|0\rangle = \int_0^\infty \frac{k^3}{2\pi^2}\frac{|h'_k|^2 + k^2|h_k|^2}{a^2}\frac{dk}{k} \tag{1.174}$$

From now on, will be very useful to introduce the notion of energy density for the gravitational waves spectrum.

$$\Omega_{GW} \equiv \frac{1}{\rho_{crit}(\tau)}\frac{d\langle 0|\hat{\rho}_{GW}|0\rangle}{dlnk} \tag{1.175}$$

where we defined the critical density:

$$\rho_{GW}(\tau) = \frac{3H^2(\tau)}{8\pi G} \tag{1.176}$$

and consequently the energy density has the following expression:

$$\Omega_{GW}(k,\tau) = \frac{8\pi G}{3H^2(\tau)}\frac{k^3}{2\pi^2}\frac{|h'_k|^2 + k^2|h_k|^2}{a^2} \tag{1.177}$$

Note again that, from our definition, the energy density is dimensionless.

After the analysis of the energy density, it is instructive to compute carefully the contribute which comes from the first order term. If we insert the linear term in the Einstein equation, we obtain the equation 1.161. In this case, the first term on the right hand side is clearly negligible with respect to the left hand side. It represent the non linear interaction of the perturbations with itself.

$$\underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}\left(\frac{h}{\lambda^2}\right)} = \underbrace{-[R^{(2)}_{\mu\nu}]^{high}}_{\mathcal{O}\left(\frac{h}{\lambda}\right)^2} + 8\pi G\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{high} \tag{1.178}$$

So, this relation simplifies:

$$R^{(1)}_{\mu\nu} \cong +8\pi G\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)^{high} \tag{1.179}$$

In this case we are left with two possibilities. If matter contributes only to the background curvature, its contribute to the perturbations dynamics vanish and consequently:

$$[T^{\mu\nu}]^{high} = 0 \rightarrow R^{(1)}_{\mu\nu} = 0 \tag{1.180}$$

The last expression reduces to the following equation for the perturbations:

$$\bar{\nabla}_\sigma\bar{\nabla}^\sigma + 2\bar{R}_{\mu\alpha\nu\rho}h^{\alpha\rho} - \bar{R}_{\alpha\mu}h^{\alpha\nu} - \bar{R}_{\alpha\nu}h^{\alpha\mu} \approx \bar{\nabla}_\sigma\bar{\nabla}^\sigma + 2\bar{R}_{\mu\alpha\nu\rho}h^{\alpha\rho} = 0 \tag{1.181}$$

Where we neglected the last two terms. Now, in the case the background is flat we obtain again the well known wave equation in the case of weak field:

$$\Box h_{\mu\nu} = 0 \tag{1.182}$$

On the contrary, if the background is curved, the 1.181 tell us about the effects due to the wave propagation such as the redshift or the gravitational lensing. What if the matter contribute is not negligible? In that situation we will have a high frequency matter contribute which leads us to the following:

$$R^{(1)}_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu}R^{(1)} + h_{\mu\nu}\bar{R}) \approx 8\pi G[T_{\mu\nu}]^{high} \tag{1.183}$$

If we evaluate these objects using the FLRW metric we will obtain, again, the following equation:

$$\ddot{h}_{ij}(\mathbf{x},t) + 3H\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}h_{ij}(\mathbf{x},t) = 16\pi G\Pi_{ij}(\mathbf{x},t) \tag{1.184}$$

In conclusion, this analysis concerning just the splitting of the dynamical equations by means of the energy contributions, leads us, once again, to the wave equation of the tensor modes in curved space-time.

1.6.4 Energy density of PGWs

It is instructive to compute the explicit form of the energy density of the perturbations we have calculated in the last section. For first, let us recall the explicit form of the quantum field induced by the quantum fluctuation of the Inflaton.

$$\hat{\phi}(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}[u_k(\tau)\hat{a}_k(\tau_0)e^{i\mathbf{k}\cdot\mathbf{x}} + u^*_k(\tau)\hat{a}^\dagger_k(\tau_0)e^{-i\mathbf{k}\cdot\mathbf{x}}] \tag{1.185}$$

where the mode functions are:

$$\begin{aligned} u_k(\tau) &= \frac{A_k}{a\sqrt{2k}}\Big(1 - \frac{i}{k\tau}\Big)e^{-ik\tau} + \frac{B_k}{a\sqrt{2k}}\Big(1 + \frac{i}{k\tau}\Big)e^{ik\tau} \\ &\equiv A_k\gamma(\tau) + B_k\gamma^*(\tau) \end{aligned} \tag{1.186}$$

The stress-energy tensor:

$$T^\mu_\nu = g^{\mu\lambda}(\partial_\lambda\phi\partial_\nu\phi^\dagger - \frac{1}{2}g_{\lambda\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi^\dagger) \tag{1.187}$$

so, the $0-0$ component is:

$$T^0_0 = \frac{1}{2a^2}[\dot{\phi}\dot{\phi}^\dagger + k^2\phi\phi^\dagger] \tag{1.188}$$

Now, we can take the expectation value with respect to the quantum vacuum:

$$\langle\rho\rangle = \langle 0|T^0_0|0\rangle = \frac{1}{2a^2}\int\frac{d^3k}{(2\pi)^3}\langle 0|(u'_k\hat{a} + u'^*_k\hat{a}^\dagger)(u'^*_k\hat{a}^\dagger + u'_k\hat{a}) + k^2(u_k\hat{a} + u^*_k\hat{a}^\dagger)(u^*_k\hat{a}^\dagger + u_k\hat{a})|0\rangle \tag{1.189}$$

and taking advantage of the normalization condition $\langle 0|0\rangle = 1$, this expression reduces to:

$$\langle\rho\rangle = \frac{1}{2a^2}\int \frac{d^3k}{(2\pi)^3}[u_k' u_k'^* + k^2 u_k u_k^*] \tag{1.190}$$

So, we have just obtained the general form of the expectation value for the energy density for a scalar field in a FLRW space-time. This result, of course, depend only on the general form of the mode functions and, hence, on the parameters which are functions of the initial conditions. It is worth noting that this expression will be in general divergent, so we will need to give a precise prescription to regularize it. One of the most common way to regularize such integral is to subtract the value of the "zero-point energy" which, in this case, is equivalent to subtract the value of this expression calculated with respect to the Bunch-Davies vacuum. Let's start compute this simple expression. The fundamental state is described by the simplistic assumption: $B_k = 0$. In this case:

$$\langle\rho\rangle_{vac} = \frac{1}{2a^2}\int \frac{d^3k}{(2\pi)^3}|A_k|^2(\gamma'(\tau)\gamma'^*(\tau) + k^2\gamma(\tau)\gamma(\tau)^*) \tag{1.191}$$

where:

$$\gamma'(\tau) = -\frac{e^{-ik\tau}}{a\sqrt{2k}}(ik) \approx^{dS} \frac{H\tau e^{-ik\tau}}{a\sqrt{2k}}(ik) \tag{1.192}$$

Putting all together in the 1.191 we obtain:

$$\begin{aligned}\langle\rho\rangle_{vac} &= \frac{1}{2a^2}\int \frac{d^3k}{(2\pi)^3}|A_k|^2\Big[\frac{(H\tau)^2}{2k}k^2 + k^2\frac{(H\tau)^2}{2k}\Big(1+\frac{1}{(k\tau)^2}\Big)\Big] \\ &= \frac{1}{2a^4}\int \frac{d^3k}{(2\pi)^3}|A_k|^2 k\Big[1+\frac{1}{2(k\tau)^2}\Big]\end{aligned} \tag{1.193}$$

Now, we can compute the complete version of the energy density in presence of a non vanishing B_k. In this more general case we have:

$$u'(\tau) = \frac{H\tau}{\sqrt{2k}}[A_k e^{-ik\tau} - B_k e^{ik\tau}](ik) \tag{1.194}$$

The energy density:

$$\begin{aligned}\langle\rho\rangle &= \frac{1}{2a^2}\int \frac{d^3k}{(2\pi)^3}[u_k' u_k'^* + k^2 u_k u_k^*] \\ &= \frac{1}{2a^4}\int \frac{d^3k}{(2\pi)^3}\Big\{\frac{k^2}{2k}\Big[|B_k|^2 - B_k A_k^* e^{2ik\tau} - B_k^* A_k e^{-2ik\tau} + |A_k|^2\Big] \\ &\quad + \frac{k^2}{2k}\Big[|A_k|^2\Big(1+\frac{1}{(k\tau)^2}\Big) + |B_k|^2\Big(1+\frac{1}{(k\tau)^2}\Big) + B_k A_k^* e^{2ik\tau}\Big(1+\frac{i}{k\tau}\Big)^2 + B_k^* A_k e^{-2ik\tau}\Big(1-\frac{i}{k\tau}\Big)^2\Big]\Big\}\end{aligned} \tag{1.195}$$

At this stage, we can take another simplification and take also a time average on the previous expression. In that way we can get rid of all the oscillating terms.

$$\langle\rho\rangle = \frac{1}{4a^4}\int \frac{d^3k}{(2\pi)^3}k\Big\{(|A_k|^2 + |B_k|^2)\Big[1+\Big(1+\frac{1}{(k\tau)^2}\Big)\Big]\Big\} \tag{1.196}$$

Here, the normalized energy density is:

$$\langle\rho\rangle_{ren} = \langle\rho\rangle - \langle\rho\rangle_{vac} = \frac{1}{2a^4}\int\frac{d^3k}{(2\pi)^3}k\Big\{|B_k|^2\Big[\Big(1+\frac{1}{2(k\tau)^2}\Big)\Big]\Big\} \tag{1.197}$$

1.7 The smoking gun

There are several good reasons to invest time to the study of the primordial gravitational waves background. The first crucial point is that GW could give the first glimpse of a quantum gravitational theory. The main reason why we are concerning about the detection of inflationary gravitational waves is that we are looking for a robust evidence of the Inflationary theory. In fact, several models which provide the same accelerated expansion without introducing the Inflaton field are on the market.

Gravitational waves from inflationary stage of the Universe could carry many useful information about the history of the Universe between the end of the inflation and the start of the electroweak phase transition. Two are the things that characterize a good model for the inflationary era; the first one is, obviously, the potential which drives the Inflaton and the second is the way in which the inflation ends and the Standard Big Bang model starts. Different models have different forecasts about the tensor to scalar ratio r which is the discriminant we may use to select the model which has the best fit with the experimental evidence.

It is important to stress that gravitational waves from the very early Universe cosmology offer unsullied proves of its primordial status. Gravitons are thermally decoupled, so the imprinting of the gravitational waves are not washed out by the thermal history of the universe. That is both the power and the weakness of the gravitational waves. They carry direct information of the very early universe by their weak coupling with other constituents of the universe, but they are extremely difficult to detect because their tiny amplitude.

Nevertheless, the presence of the gravitational waves background could be crucial because it could be a clear evidence of a quantum nature of the metric perturbations. In fact, our discussion of the metric perturbations was based on the assumption that we can think them as a couple of quantum fields (one for each polarization state) when we described it as a combination of ladders quantum operators.

From the observation of the CMB we was able to get information about the power spectrum of the scalar perturbations generated in the Early Universe. Recall that, according to the previous computations, we demonstrated that:

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2 M_{Pl}^2\epsilon} \tag{1.198}$$

or, introducing the k-dependence, the scalar power spectrum is given by:

$$\mathcal{P}_\zeta = \frac{H_*^2}{8\pi^2 M_{Pl}^2\epsilon}\Big(\frac{k}{aH_*}\Big)^{n_s-1} \tag{1.199}$$

This result is in very good agreement with the observational data obtained by PLANCK mission. However, this formula is dependent on both the Hubble parameter and the slow roll term, so we have only an information which mix the two characters of this game. If we want to have some clearer information about the history of our Universe we must look at the tensor power spectrum which is dependent only by the Hubble parameter and consequently can give us the information we may need for a clear evidence of the energy scale of the inflation. A measure of r (and consequently the tensor tilt) will be essential toward the study of the energy scale of inflation. In fact, making use of the Friedmann equation in slow roll regime $H^2 = \frac{V}{3M_{Pl}^2}$ we can link the potential which drives the inflation with the tensor-to-scalar ratio r in the following manner: $V = (3\pi^2 A_S/2)M_{Pl}^4 r$ where we used the amplitude A_S of scalar perturbations. Using the value estimated by Planck Collaboration we arrive at the following relation between the energy scale of Inflation and r:

$$V = (1.88 \times 10^{16} GeV)^4 \frac{r}{0.10} \tag{1.200}$$

It is clear that this is much closer to the typical scale of a GUT than the scale of human-made particle colliders. Up to date, the only information we could get from the Standard Model about the physics at those energy scales is the fact that at this energy we may expect an unification of all the gauge couplings.

Another fundamental forecast of the tensor-to-scalar ratio could be the excursion of the Inflaton field. We can easily relate them in the following way:

$$\frac{\Delta\phi}{M_{Pl}} \geq \sqrt{\frac{r_*}{8}} N_* \approx \frac{N_*}{60}\sqrt{\frac{r_*}{0.002}} \tag{1.201}$$

where we indicated with the subscript * the quantities evaluated at a pivot scale commonly taken $k_* = 0.05 Mpc^{-1}$ and so $r_* \approx 0.01$ This formula is useful to discriminate also various models of Inflation proposed in literature. Depending on the ratio between the field excursion and the reduced Planck mass we can have large or small field model of Inflation.

1.7.1 Cosmological parameters for PGWs from various models

As we have stressed, the energy scale of the inflation is directly linked with the tensor-to-scalar ratio r which, basically, depend upon the model which drives the inflationary stage. So, it is interesting at this stage to point out the main aspects which characterize the world of inflationary models. Generally those models are discriminated by the values of the fields compared with the Planck mass $M_{Pl} = \frac{\hbar c}{G}$ (or the reduced one m_{Pl} which is defined such that: $m_{Pl}^2 = \frac{M_{Pl}^2}{8\pi}$). To make this distinction between different models it's fundamental to take into consideration the number of the e-folds N. We used this concept to have some hints about the lasting of the inflationary stage. In the introduction we linked the number N with the physical parameters which comes from the observations. Here, we would like to link this quantity with the typical cosmological parameter of the inflation. We will define the number of e-folds in the following manner:

$$N = \int_{t_1}^{t_2} dt H(t) = \int_{\phi_1}^{\phi_2} \frac{H(\phi)}{\dot{\phi}} = -8\pi G \int_{\phi_1}^{\phi_2} \frac{V}{V_\phi} d\phi = 8\pi G \int_{\phi_2}^{\phi_1} \frac{V}{V_\phi} d\phi \tag{1.202}$$

So, once the potential is given ,the number of e-folds required and fixed and the value of the field at the end of inflation setted to ϕ_e we will compute easily the value of the inflation at the beginning of the process ϕ_i.

$$\Delta N = 8\pi G \int_{\phi_e}^{\phi_i} \frac{V}{V_\phi} d\phi \tag{1.203}$$

We will discriminate two kinds of models:

- Large field inflation: $\phi \gtrsim m_{Pl}$ or, in terms of the potential, $V_\phi > 0$ and $\phi_i > \phi_e$. A typical example of this kind of models is given by the power-law coupling, such as: $V(\phi) = \frac{1}{2}m^2\phi^2$ or, more generally, $V(\phi) = g_n\phi^n$. In this case the slow roll coefficients will be proportional to $\sim \frac{m_{Pl}^2}{\phi^2}$ and so the field ϕ has to have a very large field excursion to preserve the validity of the slow-roll conditions. Then, if we put this class of potential in the 1.203 and require to have an inflationary stage that leasts around 60 e-folds, we will find that $\phi_i \gtrsim 15m_{Pl}$. One of the consequence of such consistency conditions is to have a potential (evaluated at the beginning of the inflation) such that: $V(\phi_i) \ll m_{Pl}^4$. However, this condition is not sufficient to guarantee the consistency of the dynamics involving super-Planckian field values. In fact, even if we known that the potential $V(\phi)$ is responsible just on the dynamics of the classical part of the inflaton field, at fundamental level the theory is driven by the effective potential $V_{eff}(\phi)$ which takes into account the contributes from the quantum loop corrections. So, in order to construct a consistent field-theoretical model of inflation one must be careful in the consideration of the potential which drives this stage. The only way to forbid the generation of the quantum corrections in the effective potential is to invoke a precise symmetry which will help us to get rid of such complications. One of the most famous example is the so called "*natural inflation*" which basically is a theory which benefits of an exact shift symmetry. The potential of such case is given by:

$$V(\phi) = \Lambda^4\Big[1 + \cos\Big(\frac{\phi}{f}\Big)\Big] \tag{1.204}$$

  where Λ and f are mass scales.

- Small field inflation: $\phi \ll m_{Pl}$ or, in terms of the potential, $V_\phi < 0$ and $\phi_i < \phi_e$. So, in this case, the models are characterized by sub-Panckian excursions of the field. One of the most famous example (and the ordinary one we think about when studying the inflation) is the so called *hil-top* potential where is assumed that near the origin the potential has the form: $V(\phi) = V_0 - \frac{\lambda}{4}\phi^4$ and so the slow roll potential are satisfied if and only if the the values of the field is sufficiently small.

Whatever is the theoretical model which drives the Inflaton field, or the theory which supply the inflation, the production of Gravitational Waves in the Early Universe leave in the cosmic history a clear signature of new physics. So, we can think of detecting it through the new space-based interferometer. It is customary to relate the energy fraction $\Omega_{GW}(f)$ (that is the experimental quantity we are interested in) with the primordial power spectrum $P_{T,in}$ in the

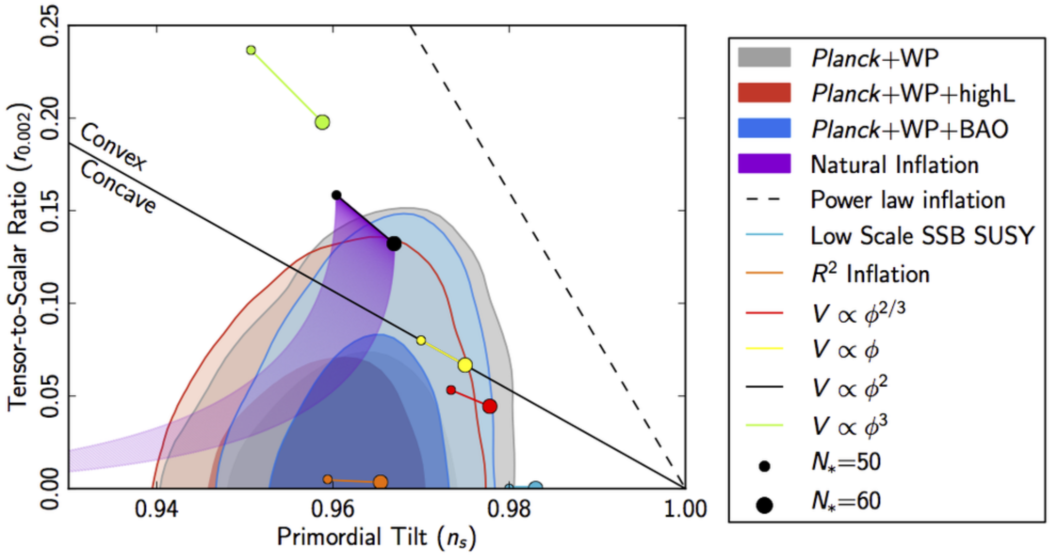

Figure 1.3: Tensor-to-scalar ratio vs the primordial tilt n_s for a sample of inflationary models. From [Planck Collaboration], Ade et al. (2016bc)

following way:

$$\Omega_{GW}(f) = \frac{\pi^2}{3H_0} f^2 |T_{GW}(f)|^2 P_{T,in}(f) \tag{1.205}$$

We have parametrized the primordial tensor power spectrum through a power law of k: $P_{T,in} = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T}$ where the tensor amplitude $A_T(k_*)$ is proportional to the amplitude of the curvature perturbations $\mathcal{R}$ on comoving surfaces through the tensor to scalar ratio evaluated at a fiducial pivot scale[4] $A_T(k_*) = r(k_*)A_{\mathcal{R}}(k_*)$. Then, remembering that $f = \frac{k}{2\pi}$ we can express the energy fraction in the following way:

$$\Omega_{GW} = \frac{\pi^2}{3H_0^2} f^2 |T_{GW}(f)|^2 r(k_*) A_{\mathcal{R}}(k_*) \left(\frac{f}{f_*}\right)^{n_T} \tag{1.206}$$

Therefore, is clear that the information about the model we are playing with is encoded into the tensor-to-scalar ratio. Different models will give different predictions of r and hence of $A_{\mathcal{R}}$. A very powerful way to visualize the behaviour of these different models is to plot each predictions of the tensor-to-scalar ratio with respect to the scalar tilt n_s. The figure 1.3 shows the marginalized 68% and 95% c.l. regions for n_s and $r_{0.002}$ from the Planck2015 data for different predictions which came from different models. From this plot is clear that the concave potentials (i.e. $V \sim \phi^n$ with $n < 1$) are compatible with data, while convex ones with $n > 2$ are excluded. The current best bound on r gives $r < 0.07$ at 98% c.l. for $k_* = 0.05 Mpc^{-1}$ which leads to a present time spectral energy-density $\Omega_{GW}(f) \simeq 10^{-15}$ for $f \simeq 10^{-17}$ Hz. Before ending this subsection, it is necessary to introduce the only quantity not yet specified in the 1.206. The quantity T_{GW} is called the transfer function and plays a fundamental role toward the

[4]for example, for $k_* = 0.002 Mpc^{-1}$ we have a correspondent frequency $f_* \sim 3.09 \times 10^{-18} Hz$, or for $k_* = 0.05 Mpc^{-1}$, we get $f_* = 7.73 \times 10^{-17} Hz$

goal of measure the power spectrum of perturbations produced in the Early Universe today. In fact, its definition can be obtained as follows. The primordial tensor power spectrum $\mathcal{P}_h(k,\tau_i)$ (where τ_i is a conformal time soon after the end of Inflation) can be related to the tensor power spectrum at a time τ by a multiplicative transfer function:

$$\mathcal{P}_h(k,\tau) = T_h(k,\tau)\mathcal{P}_h(k,\tau_i) \tag{1.207}$$

where this function can be obtained by the ratio between the mode functions evaluated at different times:

$$T_h(k,\tau) = \Big|\frac{h_k(\tau)}{h_k(\tau_i)}\Big|^2 \tag{1.208}$$

1.7.2 The smooth transition into classicality

The last point that makes so interesting the tests we can made about the primordial Universe, and the signals we can get from it, is about the faith of the quantum primordial perturbations. Through the study of the primordial fluctuations in the early universe, we assumed to treat them as a quantum objects. So, one of the most natural question that can arise in this kind of context is about the faith of the quantum behaviour. In other words, the wide studied phenomenon of the quantum decoherence. Along the Copenaghen interpretation, we know that, when a quantum system interacts with the environment, the system experience a spontaneous transition between a quantum superposition to a statistical mixture. Due to this transition, the original coherent state vanish by a time dependent of the number n of components of the quantum state. Our goal is to contextualize this concept toward a cosmological environment. The problems of decoherence is a problem that arise very frequently when studying Quantum Cosmology at different stages of comprehension and different theoretical settings.

In order to better understand the faith of the quantum perturbations we need to go back to the general theory of fields evolving in an expanding Universe. Formerly, all the modes induced by the quantum fluctuations of the inflaton are quantum mechanical excitation of the vacuum state. The process of creation of these modes in purely quantum mechanical and, in fact, all the observable effects we can extract from this theory go to zero from the moment that we neglect the Planck constant h. The Hamiltonian 1.37 is the one of a collection of harmonic oscillators with a coupling term that is due to the Universe's expansion. This term will be responsible of the squeezing of the states. In fact, by virtue of the 1.41 the two mode functions f_k and g_k can be parametrized in term of a squeezing parameter $r_k(\tau)$ and a squeezing angle $\phi_k(\tau)$:

$$\begin{aligned} f_k &= e^{-i\theta_k(\tau)}\cosh(r_k) \\ g_k &= e^{i\theta_k(\tau)+2\phi_k}\sinh(r_k) \end{aligned} \tag{1.209}$$

where we introduced a global phase θ. The squeezing parameter $r_k(\tau)$ grows with time and, at the horizon crossing τ_* is such that: $|r_k(\tau_*)| \gg 1$. Then when the modes cross the horizon we have all squeezed states. The squeezed states are the ones which exhibit explicitly the significance of the Heisenberg uncertainty principle. If a variable gets its minimum uncertain, his counterpart gets maximum uncertain. It is worth-noting that the squeezed states are "classical states" with respect to the WKB point of view; so, basically, thinking the classical mechanics as a limit of quantum mechanics. The main point is our observation of structures we guess to have quantum nature. We are dealing with the perturbations $\delta\phi$ (i.e. the fluctuations of the Inflaton) and we are playing with them as a quantum perturbations. However, the key point is that we record them in a classical way. So, if the nature of those perturbations is quantum, a phenomenon of decoherence had to be happened. The interactions we play with when studying cosmology between quantum fluctuations and the Universe are classical. Nevertheless, a classical transition had to happen because of the squeezing of the quantum states independently from the horizon crossing. At some point there should be something that enshrine the division between the quantum era and the classical one. This entity replace the role of the laboratory measurement apparatus in the usual quantum mechanical setting of the problem. It is natural to think that this role can be played by the horizon. This is still a matter of discussion at the present stage of Quantum Cosmology. In conclusion, whatever was the faith of the cosmological perturbations, after their journey through the cosmic history, when they come back inside the horizon they will constitute a stochastic background. The only way to track their quantum origin are the initial conditions from which they are drawn. Any new hints of new fundamental physics can be extracted by the indirect observation of such primordial nature of those perturbations.

1.8 A particle physicist's perspective

1.8.1 The Trans-Planckian backreaction and the relation between two vacuum state

Another interesting point of view of the same problem is the particle physicist's perspective. We claimed that, for a dynamical background, we cannot associate a unique formulation of the vacuum state. We use to define the vacuum as a state without the presence of particles. So, accepting this definition, the next step is defining the concept of particle. The typical response at the question: "What is a particle?" is the following "A particle is what a particle detector detects". [5] The key point is the motion of a detector that can influence its capacity of register particles. The Bogoliubov coefficient thought us that starting from a a definite set of mode functions we can always make a transformation that bring us to the different one. In that way we can switch from a definition of the generic field we are quantizing:

$$\chi(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} [v_k \hat{a}_{\mathbf{k}} e^{i\mathbf{kx}} + v_k^* \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}}] \tag{1.210}$$

We can always define a new mode function in that way:

$$u_k = \alpha_k v_k(\tau) + \beta_k v_k^*(\tau) \tag{1.211}$$

And write the field χ in an equivalent way

$$\chi(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} [u_k \hat{b}_{\mathbf{k}} e^{i\mathbf{kx}} + u_k^* \hat{b}_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}}] \tag{1.212}$$

where we defined a new set of ladder operators $(\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}}^\dagger)$. If we require the consistence of these two definitions for the quantum field χ we find the following relation:

$$v_k \hat{a}_{\mathbf{k}} + v_k^* \hat{a}_{-\mathbf{k}}^\dagger = u_k \hat{b}_{\mathbf{k}} + u_k^* \hat{b}_{-\mathbf{k}}^\dagger \tag{1.213}$$

And substituting 1.211 in this equation we can find the following rules of transformation of the ladder operators:

$$\hat{a}_{\mathbf{k}} = \alpha_k^* \hat{b}_{\mathbf{k}} + \beta_k \hat{b}_{-\mathbf{k}}^\dagger \tag{1.214}$$

$$\hat{a}_{\mathbf{k}}^\dagger = \alpha_k \hat{b}_{\mathbf{k}}^\dagger + \beta_k^* \hat{b}_{-\mathbf{k}} \tag{1.215}$$

which can be easily inverted.

$$\hat{b}_{\mathbf{k}} = \alpha_k \hat{a}_{\mathbf{k}} - \beta_k \hat{a}_{-\mathbf{k}}^\dagger \tag{1.216}$$

$$\hat{b}_{\mathbf{k}}^\dagger = \alpha_k^* \hat{a}_{\mathbf{k}}^\dagger - \beta_k^* \hat{a}_{-\mathbf{k}} \tag{1.217}$$

[5] For completeness it's important to cite another very good answer to the previous question. A particle, from a group theoretical point of view, can be defined as an irriducible representation of the Poincarè group.

At this stage we can ask ourselves how to build the Fock space. We can use both the two sets of annichilation and creation operators. However, these two sets are physically inequivalent. In fact, we can define vacuum states in two different ways at the same time.

$$\hat{a}_{\mathbf{k}}|0\rangle_a = 0 \qquad\qquad \hat{b}_{\mathbf{k}}|0\rangle_b = 0 \tag{1.218}$$

It is immediate to convince ourselves that the two definitions are not equivalent if the coefficient β_k are non vanishing. In fact, if we compute the expectation value of the number operator $N^{(a)}$ (defined by the first set of ladder operators) with respect to the vacuum states defined by the second one, we have:

$${}_b\langle 0|\hat{N}^{(a)}_{\mathbf{k}}|0\rangle_b = {}_b\langle 0|a^\dagger_{\mathbf{k}} a_{\mathbf{k}}|0\rangle_b = {}_b\langle 0|(\alpha\hat{b}^\dagger + \beta^*\hat{b}_{-\mathbf{k}})(\alpha^*\hat{b} + \beta\hat{b}^\dagger_{-\mathbf{k}})|0\rangle_b = {}_b\langle 0|\beta^*\hat{b}_{-\mathbf{k}}\beta\hat{b}^\dagger_{-\mathbf{k}}|0\rangle_b = |\beta|^2\delta^{(3)}(0) \tag{1.219}$$

So, $|\beta_k|^2$ is the number density in the phase space. This means that a certain vacuum state for, e.g., the first set $(\hat{a}, \hat{a}^\dagger)$ corresponds to a multiparticle state from the point of view of the second one $(\hat{b}, \hat{b}^\dagger)$ and vice-versa. In some sense, bacause of this ambiguity is originated from the impossibility of finding mode functions with a definite energy's sign, we can think evocatively that these particles are coming from the geometry. This remarkably result may be obtained also in a different way. One can calculate the vacuum expectation with respect to a given state of the general stress energy tensor for the quantum scalar field we are playing with:

$$\langle 0_b|(-T^0_0)|0_b\rangle = \frac{H^2\tau^2}{2}\int\frac{d^3k}{(2\pi)^2}\Big[(\alpha_{\mathbf{k}}u'_{\mathbf{k}} + \beta_{\mathbf{k}}u'^*_{\mathbf{k}})(\alpha^*_{\mathbf{k}}u'^*_{\mathbf{k}} + \beta^*_{\mathbf{k}}u'_{\mathbf{k}}) + \mathbf{k}^2(\alpha_{\mathbf{k}}u_{\mathbf{k}} - \beta_{\mathbf{k}}u^*_{\mathbf{k}})(\alpha^*_{\mathbf{k}}u^*_{\mathbf{k}} - \beta^*_{\mathbf{k}}u_{\mathbf{k}})\Big] \tag{1.220}$$

Of course, this expectation value depends from the Bogoliubov coefficients and, moreover, it will be divergent for any choice of them. After regularizing it by simply subtracting the expectation value of the stress-energy tensor with respect to the Bunch-Davies vacuum, we arrive at the same result of the 1.219. It is clear that this is critical point of this theory. One of the consequence of this fact is that, if we want to take into account a non trivial initial state, we will have to deal with a time dependent particle density. If one want to work out forecasts in this context, he will need to build some mechanism that will get rid of such back-reactions or to study the faith of these collateral products. Now, one may ask what is the relation between these two vacuum states. For definition, if we apply the 1.216 to the b-vacuum state $|0\rangle_b$ the result is zero. So, if we think the b-vacuum state to be in some functional relation with the other vacuum $|0\rangle_a$ we have the following relation:

$$(\alpha_k\hat{a}_{\mathbf{k}} - \beta_k\hat{a}^\dagger_{-\mathbf{k}})\hat{f}|0\rangle_a = 0 \tag{1.221}$$

Because of the function $\hat{f}$ is a combination of just creation operators we have the following differential equation:

$$\alpha_k\frac{\partial\hat{f}}{\partial\hat{a}^\dagger_{\mathbf{k}}} - \beta_k\hat{a}^\dagger_{-\mathbf{k}}\hat{f} = 0 \tag{1.222}$$

which admit the general solution:

$$\hat{f}(\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{k}}) = C(\hat{a^\dagger}_{-\mathbf{k}})e^{\frac{\beta_k}{\alpha_k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}} \tag{1.223}$$

Normalizing the vacuum states we finally arrive at the following relation between the two definition of vacuum:

$$|0\rangle_b = \prod_k \frac{1}{\alpha^{1/2}} \exp\left(\frac{\beta_k}{2\alpha_k}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}\right)|0\rangle_a \tag{1.224}$$

A state of this form is called in literature a *squeezed state* and so we will say that the b-vacuum is a squeezed state with respect to the a-vacuum and vice-versa. To conclude, it is important to stress that the convergence of this quantity is assured by the fact that $|\beta_k|^2$ goes to zero faster than k^{-3} for large k. This condition is essential to get also the finiteness of the total number density.

1.8.2 adiabatic vacuum

Before moving to the adiabatic prescription for the vacuum, it is important to stress the concept of particle in standard quantum field theory. Recall that in QFT in Minkowsky space-time we associate to a "particle" a wave packet picked in a central value k which is the wave number that characterize the particle we are talking about. This wave packet has a width Δk and we usually assume that $\Delta k << k$ and this is equivalent to say that $\lambda >> \frac{1}{k}$ This procedure is not valid anymore if the background structure is not trivial (i.e. flat). Indeed, if the geometry is a function of time in a region of the same size of the wavelength λ, we cannot have the usual plan wave solution necessary for the wave packet. So, we are forced to move on and go after this issue using another path to define the same concept.

In order to define the adiabatic vacuum, is important to introduce the standard notion of particle number operator which has to satisfy some conditions postulated by Parker that a scalar field must satisfy in a FLRW Universe. These conditions are the following:

1. N_k has to be Hermitian

2. When we think the expansion frozen at one time, the operator n_k ha to become the usual one for the Minkowski space-time

$$N_k(\tau_1) = a_k^{(\tau_1)} a_k^{\dagger(\tau_1)} \tag{1.225}$$

3. The "minimization postulate": If the expansion rate $\frac{\dot{a}}{a}$ of the Universe become slow, so there will be the number operator. This condition force the number of particles to be constant as long as possible.

The adiabatic vacuum prescription relies on the WKB method for the solution of the harmonic oscillator.

$$v_k'' + \omega_k^2(\tau) v_k = 0 \tag{1.226}$$

with

$$\omega_k(\tau) = \sqrt{k^2 + m^2 a^2 - \frac{a''}{a}} = \sqrt{k^2 + m_{eff}^2} \tag{1.227}$$

We will look for an asymptotic solution. The first step toward this project is to make a simple ansatz on the solution's shape (this method is just a simple example of a more comprehensive theory called *the phase integral method*):

$$v_k(\tau) = \frac{1}{\sqrt{W_k(\tau)}} \exp\Big[i \int_{\tau_0}^{\tau} W_k(\tau')d\tau' \Big] \tag{1.228}$$

Inserting this expression into 1.226 we obtain the following relation:

$$W_k^2 = \omega^2 - \frac{1}{2}\Big[\frac{W_k''}{W_k} - \frac{3}{2}\Big(\frac{W_k'}{W_k}\Big)^2\Big] \tag{1.229}$$

If the frequency is a slowly varying function of time we can use the previous relation as a recurrence formula which allow us to find an asymptotic solution as a power series of the parameter $(\omega T)^{-1}$. So, at leading order we have:

$$^{(0)}W_k = \omega_k \tag{1.230}$$

and at second order:

$$^{(2)}W_k = \omega_k \Big(1 - \frac{1}{4}\frac{\omega_k''}{\omega_k^3} - \frac{3}{8}\frac{\omega_k'^2}{\omega_k^4}\Big) \tag{1.231}$$

This iterative prescription leads us to find step by step a new solution until a best value N after which the accuracy get worse. So, this best value N can be found if we set the exact solution at a given time τ_i to be:

$$v_k(\tau_i) = v_k^{(N)}(\tau_i), \qquad v_k'(\tau_i) = v_k'^{(N)}(\tau_i) \tag{1.232}$$

In the case the frequency is time dependent the vacuum definition is not so clear anymore. Here the vacuum fluctuations are deformed by external field giving rise to the effect of the *vacuum polarization*. However, it is worth-mentioning that the forecasts on physical observables are not affected by these additional ambiguities.

It is therefore clear that we cannot have a unique definition for the vacuum because once again a dependence by the initial time appeared. Moreover there emerge another uncertainty around the choice of the order at which stop the series. To make contact to the previous analysis, we can make explicit the relation between the zero-th order of the adiabatic prescription and the Bunch-Davies vacuum. In the zeroth order approximation, the equation we have to solve is the following:

$$v_k'' + (k^2 - \frac{a''}{a})v_k = 0 \tag{1.233}$$

is of the following form :

$$v_k = \frac{1}{\sqrt{\omega}} e^{\pm i\omega\tau} \tag{1.234}$$

The necessary condition of validity has to be: $\frac{d}{d\tau} \ln \frac{a''}{a} \ll \omega$. Then, this is equivalent (in the deSitter case) to:

$$k\tau \gg 1 \tag{1.235}$$

which is exactly the super-Hubble condition. So we infer that this is the only regime where this approximation is acceptable. By this analysis it is also straightforward to see that the conjugate momentum is:

$$\pi_k = ikv_k \tag{1.236}$$

and we get back the exact condition we have taken in the case of the Bunch-Davies vacuum. At this point we can make a little stint and generalize the condition we have just found. If we define the following:

$$Q_{WKB} = \frac{3}{4}\Big(\frac{W_k'}{W_k}\Big)^2 - \frac{1}{2}\frac{W_k''}{W_k} \tag{1.237}$$

the WKB approximation is valid only if the ratio between the WKB charge is negligible if compared with the square of the frequency :

$$\left|\frac{Q_{WKB}}{\omega^2}\right| << 1 \tag{1.238}$$

Chapter 2

Extension to two fields Inflation

The road to hell is paved with good intentions

British dictum

After this close look at one field inflation, one can think about some extensions of this setting and look forward to a cosmological model involving more fields. We convinced ourselves the Inflaton plays the role of a clock for the universe's primordial stage. The presence of a single scalar field in the dynamics was able to figure out the main issues that the standard cosmological paradigm have. Through this theory, we extracted some predictions and constraints on observables which, unfortunately, led to a challenging, but hard, experimental situation.

A natural extension of the Inflation paradigm is to allow the presence of, at least, another scalar field in the early universe. It is of growing theoretical interest include more fields in the inflationary paradigm in light of modern theories beyond Standard Model such as supersymmetric and supergravitational ones. The scenario with more fields is quite thrilling due to several additional features that characterize it with respect to the single-field slow roll inflation. Multi-field inflation gives us the possibility to split the labour which was covered by the Inflaton ϕ which was responsible for both the expansion of the universe and the generation of the cosmological perturbations. More fields carry additional ingredients to the theory. In this more general environment, one can contemplate features experimentally desirable: an extension of the consistency relation, a modification of the spectral index and the possibility to generate non-Gaussianities. Nevertheless, the presence of different fields could give rise to the production of the so-called classical production of gravitational waves because of perturbations of one field can be viewed as a source term by the other and vice-versa. In this section we will follow the formalism presented in [41] and [36] and reference therein.

2.1 A new basis: $(\sigma$, s)

Up to now we didn't focus our attention on the nature of the primordial perturbations. The reason is that we just studied the one field model for the inflationary stage of the Universe. There was natural to consider only the adiabatic perturbations. Here we will define the two kind of perturbations we may have. They are distinguished by their orientation with respect to the trajectory in the phase space of the evolution of the background solution. We define the *Adiabatic or curvature perturbations* the perturbations which are along the same trajectory in phase-space of the background solution. On the other hand, we define *Isocurvature perturbations (or entropic perturbations)* the ones which perturb the solutions orthogonally to the backgorund solution.

Physically, the isocurvature perturbations arise as baryon modes or cold dark matter modes or neutrino ones. It is important to do a little stint and being more consistent in our definitions.

$$S = \int d^4x\sqrt{-g}\left[\frac{m_{Pl}^2}{2}R - \frac{1}{2}G_{ab}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - W(\phi)\right] \tag{2.1}$$

Let's call these two fields φ_1 and φ_2 and cast them in a vector field of two components, say $\vec{\varphi} = (\varphi_1, \varphi_2) \equiv (\phi, \chi)$. In the field space, the direction of the background is given by the direction of the velocity vector field given by the derivative of the two fields: $\dot{\vec{\varphi}} = (\dot{\varphi}_1, \dot{\varphi}_2)$ and consequently, the angle of that direction is given by:

$$\tan\theta \equiv \frac{\dot{\varphi}_2}{\dot{\varphi}_2} \tag{2.2}$$

From now on we will switch to a new coordinate system which is defined locally at each point of the trajectory of the fields in the field configuration space.

$$(\varphi_1, \varphi_2) \Rightarrow (\sigma, s) \tag{2.3}$$

The coordinate σ will be called the *adiabitc field coordinate* while the second one s is called the *entropy field coordinate* (we will call it also the *isocurvature coordinate* for reasons that will be clearer soon). These two new coordinates are defined as the integrated path length along the trajectory and the ortogonal distance from it respectively. So, by that definition follows:

$$s = \dot{s} = \ddot{s} = 0 \tag{2.4}$$

To switch from the former one to the new coordinate system we will take advantage of a new parameter θ which will be the angle defined by the 2.2. We can do this passage thank to a rotation in the field space encoded in the following relation:

$$\begin{pmatrix}\dot{\sigma}\\ \dot{s}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix} = S^T\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix} \tag{2.5}$$

where we defined $S = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ the rotation matrix which belongs to the group $SU(2)$ (see Fig).

Consequently the inverse rotation may be expressed as follows:

$$\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix} = S\begin{pmatrix}\dot{\sigma}\\ \dot{s}\end{pmatrix} \tag{2.6}$$

Moreover, we can also rotate the components of the potential's gradient $\nabla V = (\frac{\partial V}{\partial\varphi_1}, \frac{\partial V}{\partial\varphi_2}) \equiv (V_1, V_2)$ in the new basis we have just defined. That is possible by the same rotation matrix:

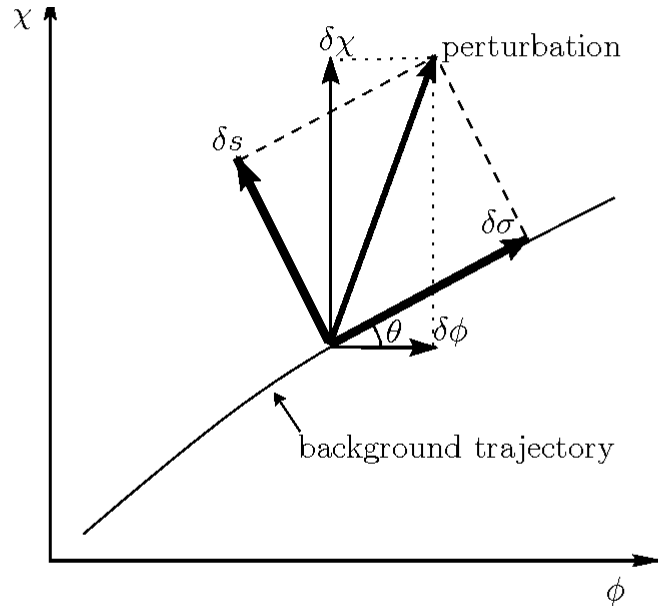

Figure 2.1: The decomposition of fields in the field space with respect to the background trajectory into adiabatic and entropic components. Credits: Gordon, Wands, Bassett and Maartens 41

$$\begin{pmatrix} V_\sigma \\ V_s \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = S^T \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{2.7}$$

Lastly, we can do the same work for the second derivatives which can be obtained by two rotations. So we arrive at the following:

$$V_{\sigma\sigma} \equiv \cos^2\theta V_{11} + 2\cos\theta\sin\theta V_{12} + \sin^2\theta V_{22} \tag{2.8}$$

$$V_{ss} \equiv \sin^2\theta V_{11} - 2\cos\theta\sin\theta V_{12} + \cos^2\theta V_{22} \tag{2.9}$$

$$V_{\sigma s} \equiv -\sin\theta\cos\theta V_{11} + (\cos^2\theta - \sin^2\theta)V_{12} + \cos\theta\sin\theta V_{22} \tag{2.10}$$

To conclude this section on the study of the background quantities. The Klein-Gordon equations for the two fields are the usual ones:

$$\ddot{\varphi}_1 + 3H\dot{\varphi}_1 + V_1 = 0 \tag{2.11}$$

$$\ddot{\varphi}_2 + 3H\dot{\varphi}_2 + V_2 = 0 \tag{2.12}$$

And, making use of the 2.2, 2.5, 2.4 we can find the dynamical equations in the new coordinate system.

$$\ddot{\sigma} + 3H\dot{\sigma} + V_\sigma = 0 \tag{2.13}$$

$$\ddot{\theta} - 3H\dot{\theta} + V_{\sigma s} - 2\frac{V_\sigma}{\dot{\sigma}}\dot{\theta} = 0 \tag{2.14}$$

2.2 Adiabatic and Isocurvature perturbations

If we want to tackle the problem of finding the power spectra of the scalar and tensor perturbations up to the first order in the perturbation theory, it will be necessary to do a little stint and generalize the previous definitions of the slow-roll parameters when more fields are involved in the theory.

$$\epsilon_{IJ} = \frac{M_{Pl}^2}{2}\frac{V_I V_J}{V^2} \qquad \eta_{IJ} = M_{Pl}^2 \frac{V_{IJ}}{V} \tag{2.15}$$

We also define the followings:

$$\epsilon = \frac{M_{Pl}^2}{2}\frac{(\nabla V)^2}{V^2} = \frac{M_{Pl}^2}{2}\frac{V_\sigma^2}{V^2} \tag{2.16}$$

and:

$$\eta_{\sigma\sigma} = M_{Pl}^2\frac{V_{\sigma\sigma}}{V} \qquad \eta_{\sigma s} = M_{Pl}^2\frac{V_{\sigma s}}{V} \qquad \eta_{ss} = M_{Pl}^2\frac{V_{ss}}{V} \tag{2.17}$$

We can also obtain the three slow-roll parameter we have just defined as combinations of the original ones by using the same rotation matrix argument we used from the beginning. Now, it's time to move on and study the dynamics of the perturbations. We will use the main results of the cosmological perturbation theory. In the spatially flat gauge, the dynamical equations for the Sasaki varibale $Q = \delta\varphi_I + \frac{\bar{\varphi}_I'}{\mathcal{H}}\psi$ (where ψ is a Bardeen potentials) is given by:

$$\ddot{Q}_I + 3H\dot{Q}_I - \frac{1}{a^2}\nabla^2 Q_I + \sum_J \Big[V_{IJ} - \frac{8\pi G}{a^3}\frac{d}{dt}\Big(\frac{a^3}{H}\dot{\varphi}_I\dot{\varphi}_J\Big)\Big] = 0 \tag{2.18}$$

The last term in the square brackets in general can be ignored because it can be demonstrated that bring a negligible contribute with respect to the others. Using this formula, we can write down the two equation of motion for the two perturbations we are studying.

$$\ddot{\delta\sigma} + 3H\dot{\delta\sigma} + \Big[-\frac{1}{a^2}\nabla^2 + V_{\sigma\sigma} - \dot{\theta}^2 - \frac{8\pi G}{a^3}\frac{d}{dt}\Big(\frac{a^3}{H}\dot{\sigma}^2\Big)\Big]\delta\sigma = 2\frac{d}{dt}(\dot{\theta}\delta s) - 2\Big(\frac{V_\sigma}{\dot{\sigma}} + \frac{\dot{H}}{H}\Big)\dot{\theta}\delta s \tag{2.19}$$

$$\ddot{\delta s} + 3H\dot{\delta s} + \Big(-\frac{1}{a^2}\nabla^2 + V_{ss} + 3\dot{\theta}^2\Big)\delta s = -\frac{\dot{\theta}}{\dot{\sigma}}\frac{1}{2\pi G a^2}\nabla^2\Phi \tag{2.20}$$

The main goal of this section is to compute the power spectrum of the curvature perturbation that, for a system of two fields, is defined in this way:

$$\mathcal{R} = -H\frac{\dot{\varphi}_1\delta\varphi_1 + \dot{\varphi}_2\delta\varphi_2}{\dot{\varphi}_1^2 + \dot{\varphi}_2^2} = -H\frac{\delta\sigma}{\dot{\sigma}} \tag{2.21}$$

From this formula it is immediate to see that σ in this scenario plays the role of the Inflaton field because of its perturbations are the source of the curvature perturbation $\mathcal{R}$. In this formula there is no presence of s. In that sense, the entropic perturbations don't affect the curvature perturbations and this is the reason why we usually call them *isocurvature perturbations*. Now we can also get the dynamical equation for the gauge invariant curvature perturbation which is the following:

$$\dot{\mathcal{R}} = \frac{H}{\dot{H}}\frac{1}{a^2}\nabla^2\Phi - 2H\frac{\dot{\theta}}{\dot{\sigma}}\delta s \tag{2.22}$$

From that equation we can make an immediate conclusion: if $\dot{\theta} = 0$ the two modes of perturbations are decoupled and δs is just another field which evolve independently from the adiabatic one. In this case, metric perturbations are influenced just by the field σ. On the other hand, if $\dot{\theta} \neq 0$, adiabatic and isocurvature (entropic) modes mix and the curvature perturbation $\mathcal{R}$ is no more constant on super-Hubble scales. Along with the curvature perturbation is also important to define the entropy perturbation analogously:

$$\mathcal{S} \equiv H\frac{\delta s}{\dot{\sigma}} \tag{2.23}$$

Here we are mainly interested in the computation of the scalar power spectrum of such perturbations. It is important to stress that $\delta\sigma$ and δs are coupled by the equations 2.19 2.20 and so their evolution is not independent. Moreover, in 2001 Bartolo et al. [**??**] demonstrated that a phenomenon of oscillation between the two fields occurs in the same fashion of the phenomenon of neutrino oscillations. If we define the

$$u_I = aQ_I \tag{2.24}$$

the dynamical equation transforms into:

$$u_I'' + \left(k^2 - \frac{2}{\tau^2}\right)u_I = \frac{3}{\tau^2}M_{IJ}u_J \tag{2.25}$$

where M_{IJ} is a matrix describing the mixing between the two perturbations. However, we can always make a rotation in the field space diagonalizing the matrix M_{IJ} with an ortogonal transformation $(U : U^T U = 1)$ and write down the equations for independent fields: v_I. So, defining:

$$\vec{u} = \begin{pmatrix} a\delta\varphi_1 \\ a\delta\varphi_2 \end{pmatrix} = \begin{pmatrix} aQ_1 \\ aQ_2 \end{pmatrix} \tag{2.26}$$

we can use another rotation matrix U which will be characterized by a new angle that for simplicity we will call Θ and by which we will diagonalize the mixing matrix M.

$$U = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \tag{2.27}$$

In that way:

$$U^T M U = diag(\lambda_1, \lambda_2) \tag{2.28}$$

So, though this rotation we are able to go from the original field perturbations into a couple of independent two scalar fields v_1 and v_2 but we are looking for studying the evolution of the adiabatic and entropic perturbations. Hence, we can use the first rotation matrix we introduced to rotate this independent perturbations into the ones we are interested in.

$$\begin{pmatrix} a\delta\sigma \\ a\delta s \end{pmatrix} = S^T U \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos(\Theta-\theta) & -\sin(\Theta-\theta) \\ \sin(\Theta-\theta) & \cos(\Theta-\theta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.29}$$

These relations give us the possibility to reduce the computation of the correlation functions of the two new variables ($\delta\sigma$ and δs) using the well known formulas for a real scalar field v_i. Skipping a lot of algebra we can find the power spectrum of the adiabatic and the isocurvature perturbations and, then, link them with the variables $\mathcal{R}$ and $\mathcal{S}$. Finally we can get the form of these spectra evaluating them at the horizon crossing k_*

$$\mathcal{P}_{\mathcal{R}*}(k) = \left(\frac{H*}{\dot{\sigma}*}\right)^2 \frac{k^3}{2\pi^2}\langle|\sigma_{\vec{k}}|^2\rangle = \left(\frac{H^2*}{2\pi\dot{\sigma}*}\right)^2 [1+(-2+6C)\epsilon - 2C\eta_{\sigma\sigma}] \tag{2.30}$$

$$\mathcal{C}_{\mathcal{RS}*}(k) = -\left(\frac{H*}{\dot{\sigma}*}\right)^2 \frac{k^3}{2\pi^2}\langle|\delta\sigma_{\vec{k}}\delta s^*_{\vec{k}}|^2\rangle = 2C\eta_{\sigma s}\left(\frac{H^2*}{2\pi\dot{\sigma}*}\right)^2 \tag{2.31}$$

$$\mathcal{P}_{\mathcal{S}*}(k) = \left(\frac{H*}{\dot{\sigma}*}\right)^2 \frac{k^3}{2\pi^2}\langle|\delta s_{\vec{k}}|^2\rangle = \left(\frac{H^2*}{2\pi\dot{\sigma}*}\right)^2 [1-2(1-C)\epsilon - 2C\eta_{ss}] \tag{2.32}$$

where we defined $C \equiv 2 - \ln 2 - \gamma \approx 0.729637$. We can see that we have not just the two power spectra but also a new function $\mathcal{C}_{\mathcal{RS}}$ that encode the correlation between the two perturbations. Moreover, it is easy to point out that, unlike the case of one field Inflation, the curvature perturbation here is not constant outside the horizon anymore. Now, if we move on the Fourier space we can write the two perturbations evaluated in a time $t > t_*$ as a linear combination of their initial values through the use of transition functions $T(t, t_*)$.

$$\begin{pmatrix} \mathcal{R}_{\vec{k}}(t) \\ \mathcal{S}_{\vec{k}}(t) \end{pmatrix} = T_k(t, t_*) \begin{pmatrix} \mathcal{R}_{\vec{k}_*} \\ \mathcal{S}_{\vec{k}_*} \end{pmatrix} \tag{2.33}$$

An immediate simplification can be made if we notice that in the case of pure abiabatic modes the curvature perturbations are constants and then $T_{\mathcal{RR}} = 1$. Moreover, adiabatic perturbations cannot generate any isocurvature perturbations (the contrary is not true) and hence $T_{\mathcal{SR}} = 0$.

$$\begin{pmatrix} \mathcal{R}_{\vec{k}}(t) \\ \mathcal{R}_{\vec{k}}(t) \end{pmatrix} = \begin{pmatrix} 1 & T_{\mathcal{RS}}(t, t*) \\ 0 & T_{\mathcal{SS}}(t, t*) \end{pmatrix} \begin{pmatrix} \mathcal{R}_{\vec{k}_*} \\ \mathcal{R}_{\vec{k}_*} \end{pmatrix} \tag{2.34}$$

These transfer functions play a fundamental role in the evaluation of the power spectra too. The power spectra will be linear combinations of their value at the horizon crossing with products of these transfer functions as coefficients. To calculate the spectral indices to the first order of the slow-roll parameters we just need to consider the zeroth order of the spectra. If our goal is to evaluate the spectral tilt of all these quantities, we will compute the logaritmic derivative of the power spectra at a generic time after t_*. Hence, we have to calculate the logaritmic derivative of the following quantities:

$$\mathcal{P}^{(0)}_* \equiv \mathcal{P}_{\mathcal{R}_*} \equiv \mathcal{P}_{\mathcal{S}_*} \equiv \left(\frac{H^2*}{2\pi\dot{\sigma}*}\right)^2 \tag{2.35}$$

and

$$\mathcal{C}_{\mathcal{RS}_*} = 0 \tag{2.36}$$

So, using the transfer functions:

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_*^{(0)} + T_{\mathcal{RS}}^2 \mathcal{P}_*^{(0)} \tag{2.37}$$

$$\mathcal{C}_{\mathcal{RS}}(k) = T\mathcal{RS}T_{\mathcal{SS}}\mathcal{P}_*^{(0)} \tag{2.38}$$

$$\mathcal{P}_{\mathcal{S}}(k) = T_{\mathcal{SS}}^2 \mathcal{P}_*^{(0)} \tag{2.39}$$

At this stage it is important to define another fundamental parameter which characterize the correlation between the two kind of perturbations. In literature is common to adopt a parametrization that is consequence of the fact that the quantity$T_{\mathcal{RS}}$ has a limited range; it varies between 0 and 1 and so it is natural to adopt the following:

$$\cos\Delta \equiv \frac{T_{RS}}{\sqrt{1+T_{\mathcal{RS}}^2}} \tag{2.40}$$

Doing some careful computations we finally arrive at the following physical quantities.

$$n_{\mathcal{R}} = -(6-4\cos^2\Delta)\epsilon + 2\eta_{\sigma\sigma}\sin^2\Delta - 4\eta_{\sigma s}\cos\Delta\sin\Delta + 2\eta_{ss}\cos^2\Delta \tag{2.41}$$

$$n_{\mathcal{S}} = -2\epsilon + 2\eta_{ss} \tag{2.42}$$

$$n_{\mathcal{C}} = -2\epsilon - 2\eta_{\sigma s}\tan\Delta + 2\eta_{ss} \tag{2.43}$$

Here, in order to compute the tensor-to-scalar ratio, it is useful to recall the form of the tensor power spectrum which is not affected by this new formalism and has the same shape as in the case of the single field Inflation.

$$\mathcal{P}_t = \frac{k^3}{2\pi^2}\langle h_{\mu\nu}(\vec{k})h^{*\mu\nu}(\vec{k})\rangle = \frac{k^3}{2\pi^2}\langle 2|h_+(\vec{k})|^2 + 2|h_x(\vec{k})|^2\rangle = \frac{8}{M_{Pl}^2}\Big(\frac{H}{2\pi}\Big)_{k=aH} \tag{2.44}$$

or:

$$\mathcal{P}_t = \frac{2}{3\pi^2 M_{Pl}^4}V \tag{2.45}$$

Again, through a straightforward calculation we have the spectral index for the tensorial sector to be:

$$n_t = -2\epsilon \tag{2.46}$$

Finally, we can compute the new tensor-to-scalar ratio that will be dependent on both the slow-roll parameter and also on the new parameter Δ which parametrize the correlation between adiabatic and isocurvature perturbations.

$$r = 16\epsilon\sin^2\Delta \tag{2.47}$$

In this section we convinced ourselves that from a general environment in which more fields are taken into account, we have two main departures form the original forecasts we made in single-field slow-roll inflation. Whenever we add just another one field in the Early Universe we will deal with perturbations no longer purely adiabatic. Nevertheless we don't have anymore a scalar power spectrum which is scale invariant and that leads us to a modification of the consistency relation.

2.3 Classical production of gravitational waves in two field Inflation

2.3.1 Parametric resonance in two field Inflation

A case of interest about a two field scenario was investigated by Zhou et al. in [13]. They considered a model in which two fields are involved in the driving of the primordial stage of the Universe. As we just pointed out, the presence of a second field give us the possibility to include entropic modes. Moreover, a very impressive property is that the entropy perturbations can be converted into curvature ones. Therefore, we will have a resonant peak in the curvature perturbation power spectrum. Nevertheless, this contribute from the second order perturbations will dominate the GWs background and so the resulting power spectrum will be in the range detectable by future planned experiments like LISA. In this paper the authors propose a driven mechanism for the inflation divided in two phases. The first phase is dominated by a first field ϕ and then by χ. When the first period stops, the perturbations of the first field start oscillating and then the perturbations of χ get a remarkable enhancement. In order to amplify this kind of phenomena we will take into account a typical potential used in the axion monodromy inflation such as:

$$V(\phi,\chi) = g\Lambda_0^3\phi + \Lambda^4(\phi)\cos\left(\frac{\phi}{f_a}\right) + \xi\Lambda_0^3\chi + V_0 \tag{2.48}$$

which basically is given by a sum of a power-law potential and a "natural" part (which has a shift symmetry). In detail, the mass scale Λ is given by:

$$\Lambda(\phi) = \Lambda_0\Big(1 + \alpha\frac{\phi}{M_{Pl}}\Big) \tag{2.49}$$

The characteristic energy scale Λ_0 sets the typical energy scale of the process under investigation and the scale f_a sets the period of the oscillating phenomenon. In order to achieve the desirable behaviour, we need to specialize the analysis with some conditions on the parameters and the fields excursions.

- The evolution of ϕ has to dominate in the first phase, then: $|g| \gg |\xi|$
- Defining $b_*(\phi) = \frac{\Lambda^4(\phi)}{|g|\Lambda_0^3}f_a$, in order to have a flat potential in the early stage we need to have: $b_*(\phi \ll \phi_0) \ll 1$
- in order to stop the rolling of the ϕ field is to require that: $b_*(\phi_e) \gtrsim 1$

- To have a domination of the classical contribute with respect to the quantum one: $\frac{\dot{\phi}}{H} > \frac{H}{2\pi}$
- The effective mass for $\delta\phi$ on flat slicing is dominated by $V''(\phi)$ when the parametric resonance happens: $V_0 \gg 2|g|\Lambda_0^3 f_a$

where in the last steps we have defined the start and the end of the period of ϕ domination with ϕ_0 and ϕ_e respectively. So, the complete theory looks like:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}(\partial_\mu\chi)^2 - V(\phi,\chi) \tag{2.50}$$

Then, we can move on and look forward to the dynamical equations in the standard fashion. We will arrive at the usual Klein-Gordon equations for a scalar field on a curved background guided by a potential V

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \\ \ddot{\chi} + 3H\dot{\chi} + V_{,\chi} = 0 \end{aligned} \tag{2.51}$$

Then, as usual, it is convenient to introduce suitable slow roll coefficients in the standard fashion:

$$\begin{aligned} \epsilon_{\phi\phi} &= \frac{\dot{\phi}^2}{2H^2M_{Pl}^2} \qquad & \epsilon_{\chi\chi} &= \frac{\dot{\chi}^2}{2H^2M_{Pl}^2} \\ \eta_{\phi\phi} &= \frac{\dot{\epsilon}_{\phi\phi}}{H\epsilon} \qquad & \eta_{\chi\chi} &= \frac{\dot{\epsilon}_{\chi\chi}}{H\epsilon} \end{aligned} \tag{2.52}$$

where in the last step we defined with ϵ the sum of the two first slow-roll parameters $\epsilon = \epsilon_{\phi\phi} + \epsilon_{\chi\chi} = -\frac{\dot{H}}{H^2}$. So, by leveraging on this assumption, we can safely compute the expression of $\ddot{\phi}$, $\dot{\phi}$, $\dot{\chi}$ in the following way:

$$\begin{aligned} \ddot{\phi} &= 3Hb_*(\phi)\dot{\phi}\sin\left(\frac{\phi}{f_a}\right) \\ \dot{\phi} &= \dot{\phi}_0 - 3Hf_a b_*(\phi)\cos\left(\frac{\phi}{f_a}\right) \\ \dot{\chi} &= \dot{\chi}_0 \equiv -\frac{\xi\Lambda_0^3}{3H} \\ \dot{\phi}_0 &= -\frac{g\Lambda_0^3}{3H} \end{aligned} \tag{2.53}$$

At this stage, we can perform the analysis we made in the general case of multi-field inflation and write down the dynamical equations for each components using the mass matrix. We are looking at the dynamical equations for the perturbations of the two fields. Then, we arrive at the following set of equations:

$$\ddot{\delta\chi_k} + 3H\dot{\delta\chi} + \frac{k^2}{a^2}\delta\chi +^2_{\chi\chi} + m^2_{\chi\phi}\delta\phi_k = 0 \tag{2.54}$$

$$\ddot{\delta\phi}_k + 3H\dot{\delta\phi} + \frac{k^2}{a^2}\delta\phi + m^2_{\phi\phi} + m^2_{\chi\phi}\delta\chi_k = 0 \tag{2.55}$$

$$m^2_{\chi\chi} = \frac{\partial^2 V}{\partial\chi^2} - \frac{1}{M^2_{Pl}}\Big(3\dot{\chi}^2 + \frac{2\dot{\chi}\ddot{\chi}}{H} - \frac{\dot{H}\dot{\chi}^2}{H^2}\Big) \tag{2.56}$$

$$m^2_{\phi\phi} = \frac{\partial^2 V}{\partial\phi^2} - \frac{1}{M^2_{Pl}}\Big(3\dot{\phi}^2 + \frac{2\dot{\phi}\ddot{\phi}}{H} - \frac{\dot{H}\dot{\phi}^2}{H^2}\Big) \tag{2.57}$$

$$m^2_{\chi\phi} = \frac{\partial^2 V}{\partial\chi\partial\phi} - \frac{1}{M^2_{Pl}}\Big(3\dot{\phi}\dot{\chi} + \frac{\dot{\chi}\ddot{\phi} + \dot{\phi}\ddot{\chi}}{H} - \frac{\dot{H}\dot{\phi}\dot{\chi}}{H^2}\Big) \tag{2.58}$$

If we take advantage of the conditions under which the model is constrained, we can spot a hierarchy in the mass matrix: $m^2_{\phi\phi} \gg m^2_{\chi\phi} \gg m^2_{\chi\chi}$. Consequently, the two equations for the perturbations look like:

$$\ddot{\delta\chi}_k + 3H\dot{\delta\chi}_k + \frac{k^2}{a^2}\delta\chi_k \simeq \frac{\dot{\chi}\ddot{\phi}}{M^2_{Pl}H}\delta\phi_k \tag{2.59}$$

$$\ddot{\delta\phi}_k + 3H\dot{\delta\phi}_k + \Big(\frac{k^2}{a^2} - \frac{\Lambda^4(\phi)}{f_a^2}\cos\Big(\frac{\phi}{f_a}\Big)\Big)\delta\phi_k = 0 \tag{2.60}$$

Defining the rescaled field $\delta\Phi_k = a^{3/2}(t)\delta\phi_k$ we can get an equation o an harmonic oscillator with time dependent frequency:

$$\ddot{\delta\Phi}_k + \omega_k^2(t)\delta\Phi_k = 0 \tag{2.61}$$

where we condensed in the frequency term the following sum:

$$\omega_k^2 = \frac{k^2}{a^2} - \frac{\Lambda^4(\phi)}{f_a^2}\cos\Big(\frac{\phi}{f_a}\Big) - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} \tag{2.62}$$

We have found what is commonly called in literature as a Mathieu equation. This kind of equation is famous for having two behaviour depending on the sign of the exponential which follows the two solutions. As a result, due to the growing exponential behaviour, we get a resonant amplification of the modes such that:

$$|\delta\phi_k| \propto e^{\lambda_k H t} \tag{2.63}$$

where λ_k are the eigenvalues of the differential problem and they are expressed in terms of the so-called Floquet number μ_k in the following way:

$$\lambda_k = \mu_k \frac{|g|\Lambda_0^3}{6H^2 f_a} - \frac{3}{2} \tag{2.64}$$

As we may see, the perturbation $\delta\phi_k$ acts as a source term for the perturbation $\delta\chi$ in 2.59. Then, a resonance of the first field induces automatically a remarkable enhancement in the second field involved in this model. Then, in the case the coefficient on the right hand side of

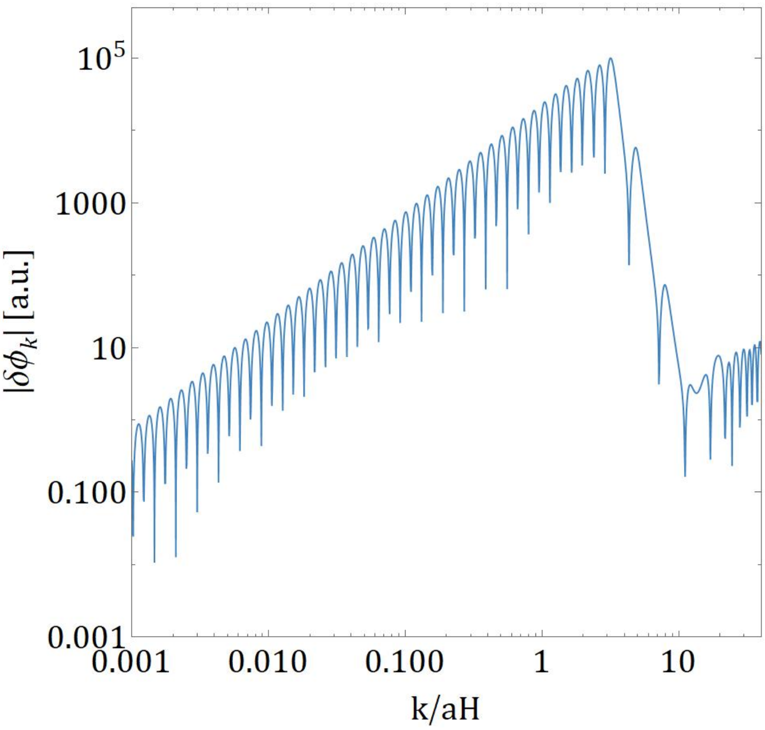

Figure 2.2: Induced resonance of k_* mode with the following set of parameters: $g = -0.05$, $\xi = -0.1g$, $\alpha = 0.55$, $\Lambda_0 = 4 \times 10^{-4} M_P$, $f_a = 5 \times 10^{-3} M_P$, $V_0 = 3 \times 10^{-11} M_P^4$, $k_* = 10^{12} Mpc^{-1}$

the 2.59 does not vanish with respect to the coefficient of $\delta\chi$ (alias $\frac{k^2}{a^2}$), the perturbations $\delta\chi$ assumes the same behaviour of $\delta\phi$. So:

$$|\delta\chi| \propto e^{\lambda_k H t} \tag{2.65}$$

$$H\delta t_{\phi_{k*}} \approx \ln\left(\sqrt{\frac{4(1+Q)}{9P_0}}\right) \tag{2.66}$$

where we defined:

$$P_0 = \left(\frac{2 f_a H}{\dot{\phi}_0}\right)^2 \qquad Q = 2\frac{\Lambda^4}{\dot{\phi}_0} \tag{2.67}$$

In this case, for sake of simplicity, we take the initial conditions of the field's perturbations in the Bunch-Davies vacuum:

$$\lim_{\tau \to -\infty} a\delta\phi_k = \lim_{\tau \to -\infty} a\delta\chi_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{2.68}$$

In Fig. A it is clear the behaviour of the perturbation versus k/aH for a given set of parameters. Then, if we use the formulas introduced in Appendix B, we can solve the non-homogeneous differential equation in the case there is a presence of a source term like this:

$$S_k^\lambda(\tau) = \frac{2}{M_{Pl}^2} = \int \frac{d^3p}{(2\pi)^3} \epsilon^\lambda(k,p)\delta\phi_p(\tau)\delta\phi_{k-p}(\tau) + (\phi \leftrightarrow \chi) \tag{2.69}$$

Then, using the deSitter approximation we can compute the Green function for the problem which is the one calculated in 34:

$$G_k(\tau,\tau') = \frac{1}{k^3\tau'}[-k(\tau-\tau')\cos k(\tau-\tau') + (1+k^2\tau\tau')\sin k(\tau-\tau')]\Theta(\tau-\tau') \tag{2.70}$$

So, the power spectrum for tensor modes looks like:

$$\begin{aligned} P_h(k,\tau_{end}) =& \frac{4}{\pi^4 M_{Pl}^4} k^3 \int_0^\infty dp p^6 \int_{-1}^1 d\cos\theta \sin^4\theta \times \Big| \int_{\tau_0}^{\tau_{end}} d\tau_1 G_k(\tau_{end},\tau_1)(\delta\phi_p(\tau_1)\delta\phi_{|k-p|}(\tau_1)+ \\ & \delta\chi_p(\tau_1)\delta\chi_{|k-p|}(\tau_1))\Big|^2 \end{aligned} \tag{2.71}$$

where we just defined $\tau_{end} \approx 0$ the time when the inflation stops.

Chapter 3

Dynamical vacuum: footprints in Primordial Gravitational Waves

3.1 The Cosmological Constant problem

"I would rather have questions that can't be answered than answers that can't be questioned"

Richard P. Feynman

One of the most thrilling puzzle of the century is the Cosmological Constant problem. Here we will briefly recall the main aspects of this issue and, after that, we will examine a mechanism by which the quantum vacuum energy could give imprints in the Universe. Before starting, it's important to stress that a satisfactory solution of this shortcoming still lacks. Probably, the cosmological constant problem (CC from now on) is the key toward the project of having a unified theory of quantum fields and gravity. Firstly, the presence of this term in Einstein equations was proposed by Albert Einstein himself but the same Einstein was persuaded and this idea was left. He was looking for a modification of his equations in order to describe a finite, static and closed Universe. Recall the Einstein equation with such new term:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.1}$$

The addition of that simple additional CC term perfectly fit with the principle of General Covariance, i.e. the covariant derivative of both sides ha to be equal to zero. Assuming the Newton gravitational coupling G as a constant and requiring the usual conservation of the common stress energy tensor for the ordinary constituents of our Universe, we can safely fulfil the Bianchi identity even in the presence of that term. Requiring such condition, inasmuch as the Bianchi identity is fulfilled by the Einstein tensor $\nabla^\mu G_{\mu\nu} = \nabla^\mu(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0$, it's immediate to conclude that the Λ term has to be constant.

$$\nabla^\mu \Lambda = \partial^\mu \Lambda = 0 \Leftrightarrow \Lambda = 0 \tag{3.2}$$

Several years later, when the cosmological observation became more and more sophisticated, this concept received a complete restoration and still today plays a central role in modern theories. The role of the CC is to address the unnatural accelerated behaviour of our Universe driven by a kind of energy which seems to have strange physical properties. For that reason, it was historically called Dark Energy (DE).

In the last decades it is undeniable that the quantum vacuum got a growing interest from the scientific community, especially after triumph of the Higgs-Englert-Brout-Guralnik-Hagen-Kibble theory. This was one of the most spectacular probe of a tangible effect of the quantum vacuum in modern fundamental physics. So, one of the main attempt to introduce a quantum vacuum contribution in the Einstein theory of General Relativity was to propose the Higgs vacuum as a source. The quantum nature of the CC term could be the keystone to spot any quantum contributions in geometry. In that way, the CC Λ was associated to the energy of quantum vacuum, but this association it easy to see that bring us to a profound problem of fine tuning. Let's start considering the tree-level Higgs potential:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \tag{3.3}$$

We will conduct a simple calculation using the methods of quantum field theory in curved space-time; so we will treat as quantum fields just the matter ones and the gravitational field as an external one. The complete theory looks like:

$$S = S_{EH} + S[\phi] = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}R + \int d^4x\sqrt{|g|}\Big[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \rho_{vac} - V(\phi)\Big] \tag{3.4}$$

where we denoted:

$$\rho_{vac} = \frac{\Lambda}{8\pi G} \tag{3.5}$$

the vacuum energy density. Note that here we have absorbed the vacuum energy term in the matter field action. Such theory, is responsible, through a spontaneous symmetry breaking mechanism of the acquisition of mass by the particles of the Standard Model. Then, a natural question is to ask ourselves what is the impact of the Higgs particle itself to the vacuum energy balance. In some sense, we can think the cosmological constant as the sum of the various quantum contributes to the vacuum energy density. The global net effect is to produce a "classical" cosmological constant. To the first approximation, the electroweak theory gives a contribute to the total energy density as follows:

$$\rho^{(0)}_{vac} \approx \Big(\frac{\Lambda_0}{\kappa^2} + V_0\Big) - \frac{m_h^2 v^2}{8} \tag{3.6}$$

where κ is the curvature of the space-time and v is the value of the Higgs vacuum (which is approximately $v^2 \approx 6.06 \times 10^4 GeV^2$) and the Higgs mass $m_h \approx 125 GeV$. So, putting all together, we arrive at the following result:

$$|\rho^{(0)}_{vac}| \simeq 1.2 \times 10^8 GeV^4 \simeq 10^{-68} m^4_{Pl} \tag{3.7}$$

which is such a dramatic result! If we compare that value with the one which comes from the observation we have a significant discrepancy of about 56 order of magnitude! From the observations, the hypothetical energy contribution should be:

$$\rho_\Lambda \simeq 10^{-48} GeV^4 \simeq 10^{-124} m^4_{Pl} \tag{3.8}$$

So, a cosmological constant term is not easily associated with a a vacuum from a particle field theory perspective. Then, we should move toward other directions. Quantum field theory teaches us that constants are not really constants. So, inspired by the concept of dynamical vacuum we may think to replace the ordinary cosmological constant to a time-dependent cosmological "constant": $\Lambda \to \Lambda(t)$. This term could constitute a fifth contribution to the energy density of the Universe along the others made by baryons, neutrinos, radiation and dark matter. So, this is the reason why such new ingredient has been called *quintessence*. The presence of such term, along the Inflaton field, would supply the energy balance the Universe needs to describe its actual dynamics.

Along the quintessential term, that in principle could modify the evolution of the Hubble parameter, we could ask ourselves if there exist another method to obtain the same final result of an accelerating period. It is therefore tempting to associate the inflaton field to the quintessence one. Unfortunately this is not so easy because of the fine tuning of the features which have to characterize the dynamics of them. Recently, was proposed another suggestive mechanism which could supply the role of the Quintessence in [30]. In this paper the author point out how a ultra-fast quantum fluctuation in the primordial Universe can be the cause of the late time accelerated expansion of our Universe. In this sense the time evolution of the scale factor a can be described in perfect analogy of a Kapitza pendulum which basically consist in a pendulum anchored to a fast oscillating point. This quantum inverted pendulum is a beautiful and simple tool to recover the dynamics of our Universe. The Kapitza prescription consist in decomposing the scale factor in the following way:

$$a(t) = a_s(t) + a_f(t) = a_s(t) + a_f(t)\sin(\omega t) \tag{3.9}$$

but nothing prevents us to consider a dependence of the space coordinate $\mathbf{x}$ in a quantum chaotic primordial scenario. By the uncertain principle, we can say that the amplitude of such fluctuations has to depend on a_s. Consequently, we may expect to find some modification of the dynamical equations for tensor modes through the damping term.

$$h_k'' + 2\tilde{\mathcal{H}}h_k' + k^2 h_k = 0 \tag{3.10}$$

or, in term of the cosmic time:

$$\ddot{h}_k + +3\tilde{H}\dot{h}_k + \frac{k^2}{a^2}h_k = 0 \tag{3.11}$$

Here, we could considerer the Hubble parameter as a sum of two contributes:

$$\tilde{H} = H_s + H_f \tag{3.12}$$

Being the Hubble parameter the inverse of a typical time scale $H \sim \frac{1}{t}$, we can infer an immediate conditions between the typical period of oscillations which characterize the fast-oscillating term $\mathcal{H}_f$ and the time scale τ which characterize the period we are looking at. Then, the condition which guarantees the dominance of the new term over the ordinary one is when the time scale we are exploring is bigger than the period of such oscillations. Explicitly:

$$H_f > H_s \Longleftrightarrow t > T \tag{3.13}$$

where we indicated T the period of such oscillations at a certain fundamental scale $T = \frac{2\pi}{\omega}$. Then, in order to compute the new form of the equation 3.11, we move on evaluating the new Hubble term.

$$H = H_s + H_f = H_s + \frac{a_f'}{a_s}\sin(\omega t) + \frac{a_f}{a_s}\omega\cos(\omega t) \tag{3.14}$$

where we defined $H_s = \frac{\dot{a}_s}{a_s}$. We may wonder whether or not the third term on the r.h.s. could be neglected taking advantage of the condition that the amplitudes of the fast oscillating term has to be negligible compared to the ordinary one, i.e. $\frac{a_f}{a_s} \ll 1$. The presence of the ultra-fast frequency ω could overcompensate this condition and give rise to an additional term. Anyway, if we insert this expression in the wave equation, we obtain:

$$\ddot{h}_k + 3(H_s + \frac{a_f'}{a_s}\sin(\omega t) + \frac{a_f}{a_s}\omega\cos(\omega t))\dot{h}_k + \frac{k^2}{a_s^2}h_k = 0 \tag{3.15}$$

We are left with a non linear second order differential equation with an oscillating viscous term. At this stage, thinking the "slow" part as the usual exponential law $a_s = e^{Ht}$, the only ingredient still missing is the functional form of the fast amplitude a_f. This term will probably be a function of its slow counterpart: $a_f = a_f(a_S)$. A complete analysis of such term would led us to find a sort of power law for a_f modified by a fundamental scale which enter the game due to the minimal uncertain length. To conclude this section it is worth mentioning that such a contribute could match other theories beyond the Standard Cosmological Model such as theories which contemplate some modified dispersion relations.

Chapter 4

Gravitational waves from a quantum stochastic background

> "What we observe is not Nature in itself, but Nature exposed to our method of questioning"
>
> Werner Karl Heisenberg

As every theoretical physicist know, the nature of a quantum vacuum is not actually an empty space. In Quantum Field Theory, all the space-time is filled by superposition of fields. We could model what happens in the whole space-time as ultra-fast oscillations at every point point. Consequently, we can assume that an anisotropic stress tensor emerges from such stochastic behaviour. In this way, the wave-equation acquire a source term as follows:

$$h_k'' + 2\mathcal{H}h_k' + k^2 h_k = 16\pi G a^2 \tilde{\Pi}_k(\tau, \mathbf{k}) \tag{4.1}$$

where we are thinking $\tilde{\Pi}$ as a stochastic function, so that:

$$\begin{aligned} \langle \tilde{\Pi}_k(\tau, \mathbf{k}) \rangle &= 0 \\ \langle \tilde{\Pi}^*_{k_1}(\tau_1) \tilde{\Pi}_{k_2}(\tau_2) \rangle &= N \delta^3(k_1 - k_2) \delta(\tau_1 - \tau_2) \end{aligned} \tag{4.2}$$

We have indicated with $\langle ... \rangle$ the expectation value over the vacuum states and over the typical time of the anisotropic source. In order to fulfil the dimensional coherence, we may conclude that the constant N has to have the following behaviour: $[N] \approx \left(\frac{k}{\tau}\right)^3$. Now, we want to use a stochastic approach to extract shadows of such stochastic background in the power spectrum of the primordial tensor perturbations induced by the Inflation. We will treat this computation a la Langevin. The general solution of the inhomogeneous problem will be the superposition of the following two functions:

$$h_k(\tau, \mathbf{k}) = h_k^{homogeneous}(\tau, \mathbf{k}) + h_k^{particular}(\tau, \mathbf{k}) \tag{4.3}$$

We have just found the solution to the homogeneous problem that, in its general form, looks like:

$$
\begin{aligned}
h_k &= \frac{A_k}{a}\frac{e^{-ik\tau}}{\sqrt{2k}}(1-\frac{i}{k\tau}) + \frac{B_k}{a}\frac{e^{ik\tau}}{\sqrt{2k}}(1+\frac{i}{k\tau}) \\
&\equiv A_k\gamma_k^{(1)} + B_k\gamma_k^{(2)}
\end{aligned}
\tag{4.4}
$$

Note that, according to the assumption we just made on the nature of the source term, if we take the ensamble average of the complete solution we will get easily back the ordinary homogeneous solution. The effect of that source will be reflected in the computation of the two-point function, i.e. the power spectrum. Now, we can look forward finding the solutions of the complete equation. We will make use of the common Lagrange variation of constants method. Let' assume the particular solution could be thought as a combination of the two independent solution of the homogeneous one weighted by two functions of conformal time.

$$
\gamma_k^p(\tau,\mathbf{k}) = C_1(\tau)\gamma_k^{(1)} + C_2(\tau)\gamma_k^{(2)}
\tag{4.5}
$$

So, our next task is to compute the form of these two unknown functions. To be consistent with our previous analysis, we will get a vanishing contribute at linear order. We will spot the imprints of such presence when compute the two point function. In order to find those unknown Lagrange functions we need to solve the following differential system:

$$
\begin{aligned}
C_1'(\tau)\gamma_k^{(1)} + C_2'\gamma_k^{(2)} &= 0 \\
C_1'(\tau)\gamma_k'^{(1)} + C_2'\gamma_k'^{(2)} &= 16\pi G a^2(\tau)\tilde{\Pi}_k(\tau)
\end{aligned}
\tag{4.6}
$$

To solution can be easily obtained by the determinant method in terms of the original mode functions associated to the homogeneous problem. So, we have:

$$
\begin{aligned}
C_1'(\tau) &= -16\pi G\frac{\gamma_k^{(2)}\tilde{\Pi}_k(\tau)}{\gamma_k'^{(2)}\gamma_k^{(1)} - \gamma_k'^{(1)}\gamma_k^{(2)}} \\
C_2'(\tau) &= 16\pi G\frac{\gamma_k^{(1)}\tilde{\Pi}_k(\tau)}{\gamma_k'^{(2)}\gamma_k^{(1)} - \gamma_k'^{(1)}\gamma_k^{(2)}}
\end{aligned}
\tag{4.7}
$$

This expressions integrated will help us toward the goal of expressing the global solutions. Of course, such functions have ensamble average equal to zero. Having the complete solution we can move on to the computation of the power spectrum. We will have

$$
\begin{aligned}
\langle\gamma_k^*(\tau_1)\gamma_k(\tau_2)\rangle &= \langle[\gamma_k^{*hom}(\tau) + \gamma_k^{*p}(\tau)][\gamma_k^{hom}(\tau) + \gamma_k^p(\tau)]\rangle \\
&= \gamma_k^{*hom}(\tau_1)\gamma_k^{hom}(\tau_2) + \langle\gamma_k^{*p}(\tau_1)\gamma_k^p(\tau_2)\rangle
\end{aligned}
\tag{4.8}
$$

Where we have neglected all the mixed terms because of they are washed out by the average due to the first one of the 4.2. We just calculated the first term in the previous expression, so our next task is to elaborate the two point function of the particular solutions. This result will contribute to the total power spectrum of the primordial gravitational waves background which brings information of the very early Universe. Here the detailed passages:

$$\begin{aligned}
&\langle \gamma^{*p}(\tau_1)\gamma^p(\tau_2)\rangle = \\
&= (16\pi G)^2 \Big\langle \underbrace{\Big[\frac{-ie^{ik\tau_1}}{a(\tau_1)\sqrt{2k}}\Big(1+\frac{i}{k\tau_1}\Big)\int^{\tau_1} d\tau' \frac{e^{-ik\tau'}}{\sqrt{2k}}\Big(1-\frac{i}{k\tau'}\Big)a^3(\tau')\Pi_k^*(\tau') +}_{1} \\
&\underbrace{+\frac{ie^{-ik\tau_1}}{a(\tau_1)\sqrt{2k}}\Big(1-\frac{i}{k\tau_1}\Big)\int^{\tau_1} d\tau' \frac{e^{ik\tau'}}{\sqrt{2k}}\Big(1+\frac{i}{k\tau'}\Big)a^3(\tau')\Pi_k^*(\tau')\Big]}_{2} \times \\
&\underbrace{\times\Big[\frac{ie^{-ik\tau_2}}{a(\tau_2)\sqrt{2k}}\Big(1-\frac{i}{k\tau_2}\Big)\int^{\tau_2} d\tau'' \frac{e^{ik\tau''}}{\sqrt{2k}}\Big(1+\frac{i}{k\tau''}\Big)a^3(\tau'')\Pi_k(\tau'') +}_{3} \\
&\underbrace{-\frac{ie^{ik\tau_2}}{a(\tau_2)\sqrt{2k}}\Big(1+\frac{i}{k\tau_2}\Big)\int^{\tau_2} d\tau'' \frac{e^{-ik\tau''}}{\sqrt{2k}}\Big(1-\frac{i}{k\tau''}\Big)a^3(\tau'')\Pi_k(\tau'')\Big]}_{4}\Big\rangle
\end{aligned} \tag{4.9}$$

Now, let's evaluate carefully all these terms. We have to compute four products. The first multiplied the third gives:

$$\begin{aligned}
&\frac{e^{ik(\tau_1-\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1+\frac{i}{k\tau_1}\Big)\Big(1-\frac{i}{k\tau_2}\Big)\int^{\tau_1} d\tau' \int^{\tau_2} d\tau'' e^{-ik\tau'}e^{ik\tau''}\Big(1-\frac{i}{k\tau'}\Big)\Big(1+\frac{i}{k\tau''}\Big)\times \\
&\times a^3(\tau')a^3(\tau'')\underbrace{\langle\tilde{\Pi}_k^*(\tau')\tilde{\Pi}_k(\tau'')\rangle}_{N\delta(\tau'-\tau'')} = \\
&= \frac{e^{ik(\tau_1-\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1+\frac{i}{k\tau_1}\Big)\Big(1-\frac{i}{k\tau_2}\Big)\int^{\tau_1} d\tau' \Big(1+\frac{1}{(k\tau')^2}\Big)\Big(\frac{1}{H\tau'}\Big)^6 N \\
&= -\frac{e^{ik(\tau_1-\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1+\frac{i}{k\tau_1}\Big)\Big(1-\frac{i}{k\tau_2}\Big)\frac{N}{H^6}\Big[\frac{1}{5\tau'^5}+\frac{1}{7k^2\tau'^7}\Big]_{\tau_0}^{\tau_1}
\end{aligned} \tag{4.10}$$

Now the first and fourth:

$$\begin{aligned}
&\frac{-e^{ik(\tau_1+\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1+\frac{i}{k\tau_1}\Big)\Big(1+\frac{i}{k\tau_2}\Big)\int^{\tau_1} d\tau' \int^{\tau_2} d\tau'' e^{-ik\tau'}e^{-ik\tau''}\Big(1-\frac{i}{k\tau'}\Big)\Big(1-\frac{i}{k\tau''}\Big)\times \\
&\times a^3(\tau')a^3(\tau'')\underbrace{\langle\tilde{\Pi}_k^*(\tau')\tilde{\Pi}_k(\tau'')\rangle}_{N\delta(\tau'-\tau'')} = \\
&= -\frac{e^{ik(\tau_1+\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1+\frac{i}{k\tau_1}\Big)\Big(1+\frac{i}{k\tau_2}\Big)\int^{\tau_1} d\tau' N e^{-2ik\tau'}\Big(1-\frac{i}{k\tau'}\Big)^2\Big(\frac{1}{H\tau'}\Big)^6 \\
&= -\frac{e^{ik(\tau_1+\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1+\frac{i}{k\tau_1}\Big)\Big(1+\frac{i}{k\tau_2}\Big)\frac{N}{H^6}\int^{\tau_1} d\tau'\Big[\frac{e^{-2ik\tau'}}{\tau'^6}-\frac{e^{-2ik\tau'}}{k^2\tau'^8}-2i\frac{e^{-2ik\tau'}}{k\tau'^7}\Big]
\end{aligned} \tag{4.11}$$

The second with the fourth:

$$
\begin{aligned}
&\frac{-e^{-ik(\tau_1+\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1-\frac{i}{k\tau_1}\Big)\Big(1-\frac{i}{k\tau_2}\Big)\int^{\tau_1}d\tau'\int^{\tau_2}d\tau''e^{ik\tau'}e^{ik\tau''}\Big(1+\frac{i}{k\tau'}\Big)\Big(1+\frac{i}{k\tau''}\Big)\times\\
&\times a^3(\tau')a^3(\tau'')\underbrace{\langle\tilde{\Pi}^*_k(\tau')\tilde{\Pi}_k(\tau'')\rangle}_{N\delta(\tau'-\tau'')} =\\
&=\frac{-e^{-ik(\tau_1+\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1-\frac{i}{k\tau_1}\Big)\Big(1-\frac{i}{k\tau_2}\Big)\int^{\tau_1}d\tau'e^{21k\tau'}\Big(1+\frac{i}{(k\tau')}\Big)^2\Big(\frac{1}{H\tau'}\Big)^6 N\\
&=-\frac{e^{-ik(\tau_1+\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1-\frac{i}{k\tau_1}\Big)\Big(1-\frac{i}{k\tau_2}\Big)\frac{N}{H^6}\int^{\tau_1}d\tau'e^{2ik\tau'}\Big[\frac{1}{\tau'^6}-\frac{1}{k^2\tau'^8}+\frac{2i}{k\tau'^7}\Big]
\end{aligned}
\tag{4.12}
$$

And, finally, the second with the fourth gives:

$$
\begin{aligned}
&\frac{e^{-ik(\tau_1-\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1-\frac{i}{k\tau_1}\Big)\Big(1+\frac{i}{k\tau_2}\Big)\int^{\tau_1}d\tau'\int^{\tau_2}d\tau''e^{ik\tau'}e^{-ik\tau''}\Big(1+\frac{i}{k\tau'}\Big)\Big(1-\frac{i}{k\tau''}\Big)\times\\
&\times a^3(\tau')a^3(\tau'')\underbrace{\langle\tilde{\Pi}^*_k(\tau')\tilde{\Pi}_k(\tau'')\rangle}_{N\delta(\tau'-\tau'')} =\\
&=\frac{e^{-ik(\tau_1-\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1-\frac{i}{k\tau_1}\Big)\Big(1+\frac{i}{k\tau_2}\Big)\int^{\tau_1}d\tau'\frac{N}{H^6}\Big(1+\frac{i}{(k\tau')^2}\Big)\frac{1}{\tau'^6}\\
&=-\frac{e^{-ik(\tau_1-\tau_2)}}{a(\tau_1)a(\tau_2)(2k)^2}\Big(1-\frac{i}{k\tau_1}\Big)\Big(1+\frac{i}{k\tau_2}\Big)\frac{N}{H^6}\Big[\frac{1}{5\tau'^5}+\frac{1}{7k^2\tau'^7}\Big]^{\tau_1}_{\tau_0}
\end{aligned}
\tag{4.13}
$$

From this analysis, we are left with four new contributes. Two of them scales as power-law in time, while the other two contributes are expressed by various Exponential Integrals $Ei(z)$. It is reasonable to think that those rapidly oscillatory terms provide a vanishing contribute to the total budget, but here we will perform the computation without discarding any term. Then, we can take $\tau_1 = \tau_2$ and sum up all these contributes. The initial time τ_0 can be set the time at which the inflationary stage starts. So, according to the previous argument, we can take this expression from 1.105. Then, one more time, the value of such energy scale Λ will enter the game. Moreover, for sake of simplicity we will make use of the easy relation $a = -\frac{1}{H\tau}$ thinking the Hubble parameter as constant. Finally, after extracting the real part of such complex expression we find the additional contribute to the total power spectrum to be:

$$
\begin{aligned}
\delta\tilde{P} = \frac{1}{140H^4k^4\Lambda^7k\tau^2\tau^5}N\Big[\Lambda^7\Big(8k^7(k\tau^2-1)\tau^7\cosh(\tau i2k)Im\Big(Ei\Big(\frac{2i\Lambda}{H}\Big)-Ei(-2ik\tau)\Big)+\\
+4k\tau\sinh(\tau i2k)\Big((2k^6\tau^6-k^4\tau^4+3k^2\tau^2-5)\sin(2k\tau)+4k^7\tau^7Re\Big(Ei\Big(\frac{2i\Lambda}{H}\Big)-Ei(-2ik\tau)\Big)\Big)+\\
+k\tau(2k^4\tau^4-3k^2\tau^2-20)((k\tau^2-1)\sin(2k\tau)\cosh(\tau i2k)-2k\tau\cos(2k\tau)\sinh(\tau i2k))\Big)-\\
-2Hk^7\tau^7\cos\left(\frac{2\Lambda}{H}\right)[(k\tau^2-1)(-3H^4\Lambda^3+H^2\Lambda^4+5H6-2\Lambda^6)\cosh(2ik\tau)+\\
+H\Lambda k\tau(3H^2\Lambda^2+20H^4-2\Lambda^4)\sinh(2ik\tau)]+\\
+Hk^7\tau^7\sin\left(\frac{2\Lambda}{H}\right)[H\Lambda(k\tau^2-1)(-3H^2\Lambda^2-20H^4+2\Lambda^4)\cosh(2ik\tau)+\\
+4k\tau(-3H^4\Lambda^2+H^2\Lambda^4+5H^6-2\Lambda^6)\sinh(2ik\tau)]-\\
-2(k\tau^2+1)[H^5k^7\tau^7(5H^2+7\Lambda^2)+7k^2\Lambda^7\tau^2+5\Lambda^7])+\\
+2\Lambda^7(k\tau^2-1)(2k^6\tau^6-k^4\tau^4+3k^2\tau^2-5)\cos(2k\tau)\cosh(\tau i2k)\Big]
\end{aligned}
\tag{4.14}
$$

Again, is clear that in a very next future, the observational data will be essential toward a complete characterization of such primordial era. In this computation we encoded inside the constant N and in the energy scale Λ fruitful informations about the energy scales sensible to a modification of the laws of physics we know up to now.

Chapter 5

Conclusions and Outlook

> "Culture is that which remains with an individual when he has forgotten all he learned"
>
> Edouard Herriot

In summary, in this thesis we have studied the very early universe extending the well established quantum field theory for a generic dynamical background. Such description of Nature at fundamental scale can be thought as a rough model, because a graceful and comprehensive quantum gravitational theory still lacks. When using such extensions some problems come up such as the natural ambiguity of the vacuum state. In this context the power spectrum of the primordial gravitational waves acquires a non trivial modification that can be responsible of violations of the consistency relations and, most importantly, can give us a new window to the typical energy scale of such new phenomena. In that sense, primordial gravitational waves represent an uncorrupted messenger of the very early Universe. An hypothetical signal registered in the newt decades would be a powerful way to see a tangible effect of the trans-Planckian transition and have a look on the primordial stage of the Cosmos long before the start of the reheating period and the consequent decoupling of the species from the primordial plasma. Nevertheless, any other fundamental phenomenon in such primitive scenario can leave a characteristic mark in the primordial power spectrum. In fact, if we think the onset of the Inflation as a chaotic scenario, it is reasonable to have many ways to produce anisotropies. Such presence is guaranteed by the non linearity of the Einstein equations themselves and, consequently, a deviation of the homogeneity of the scale factor is expected. Moreover, a quantum noise background can affect the final computation of the power spectrum. The presence of an anisotropic stochastic stress energy source can induce a new correction and can also carry more informations about the sensibility of the Inflation's period start to the initial and the environmental conditions.

Appendix A

Cosmological perturbation theory

Throughout this thesis we made use of the so called cosmological perturbation theory. We assumed that the main character in the primordial stage of the Universe was the Inflaton field. So, any perturbations of this scalar field will automatically generate a perturbed stress-energy tensor $\delta T_{\mu\nu}$. Then, through the Einstein equation, this perturbation $\delta T_{\mu\nu}$ generate immediately a perturbation of the metric $\delta g_{\mu\nu}$. On the other hand, if we start from a perturbation of the metric, we will easily convince ourselves that, through the Klein-Gordon equation for a scalar quantum field evolving on a generic background, we get back a perturbation of the scalar field $\delta\phi$. So, it is clear that metric perturbations and field perturbations are tightly coupled. Consequently, it seems to be evident the interest in developing a perturbation theory setted in a dynamical background. Now, when one wants to study the perturbation theory in General Relativity, it is customary to split the metric in two contributions: the background and the perturbations.

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}) \tag{A.1}$$

In that way, any perturbed space-time can be thought as a sum of a given background plus a "little" perturbation. Here we will concern about the first order cosmological perturbation theory and that means that in the calculations we will discard all the terms which will be of the second order in the perturbations and also in their derivatives. Our interests are mainly focused on a particular cosmological background which is the one compatible with the Cosmological Principle, i.e. the FLRW. In particular, we will make use of the simplest form of such background: the flat FLRW universe. As well known, in this case the time slices have a Minkowskian geometry and, consequently, we will be free to use the usual Fourier analysis. For generalizations to more general settings and the complete theory of harmonic analysis see [29]. Moreover, as yet pointed out, this condition guarantees us the possibility to switch, thanks to the definition of the conformal time, to a metric which is conformally equivalent to the Minkowskian one. Let's start describing how the theory works. The first step toward a cosmological perturbation theory is to considerer the issue of mapping. In order to examine the real world (which will be called the "perturbed Universe") we usually start considering the unperturbed one (the background). When one wants to map a point $\bar{P}$ from the background space-time to a respective point $\hat{P}$ living in the perturbed Universe has just to take into account a coordinate system $\hat{x}$. Now, the key point is that we have several ways to associate a point in the real Universe from the one of the background. For example, we can consider another

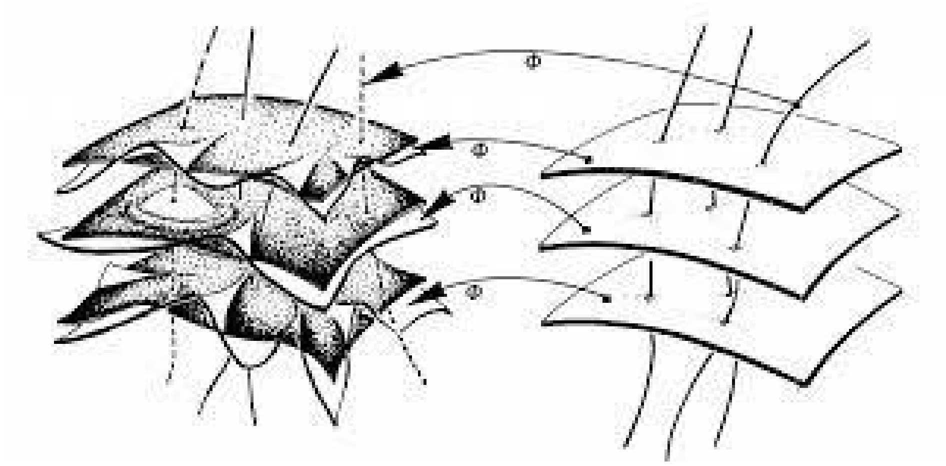

Figure A.1: A gauge transformation between the background slicing of the space-time and the perturbed one

coordinate system $\tilde{x}$ which associates another point $\tilde{P}$ in the perturbed space-time. The only constraint between these different description in the "real manifold" is the following:

$$\tilde{x}^{\alpha}(\tilde{P}) = \hat{x}^{\alpha}(\hat{P}) \equiv x^{\alpha}(\bar{P}) \tag{A.2}$$

Specifying the particular choice of the map used is called a "gauge choice".

Then, we can investigate about the relation between the two coordinate systems in the perturbed space-time. It is important to stress again that we will deal with only the infinitesimal transformation. Then, we can link them through the use of a vector field ξ:

$$\tilde{x}^{\alpha} = \hat{x}^{\alpha} + \xi^{\alpha} \tag{A.3}$$

Whenever computing the perturbations of some object of interest we are facing the problem of making the difference between the quantity which lives in the real space-time (perturbed) and the background one. These elements live in two different manifolds and, so, it's not immediate to compare them. Our goal is to translate the perturbations as functions of the background space-time, as they would live there.

Any "real" physical quantity lives in the perturbed Universe and so it may be expressed by:

$$Q = \bar{Q} + \delta Q \tag{A.4}$$

According to what we have just discussed, we are not able to define a unique point in the perturbed space-time in which this quantity lives and this is due to the ambiguity we previously pointed out A.2. Different points in the real universe correspond to the same point in the background. So, the perturbations δQ cannot be defined unequivocally, but they are clearly gauge dependent. We can think of them at least in two different ways (depending on the choice of the coordinates):

$$\begin{aligned} \delta\hat{Q}(x^{\alpha}) &= Q(\hat{P}) - \bar{Q}(\bar{P}) \\ \delta\tilde{Q}(x^{\alpha}) &= Q(\tilde{P}) - \bar{Q}(\bar{P}) \end{aligned} \tag{A.5}$$

From the well known differential geometry, these two quantities can be related by the Lie derivative along the line of the vector field ξ^α.

$$\tilde{\delta Q} = \hat{\delta Q} + \mathcal{L}_\xi \mathcal{Q} \tag{A.6}$$

This change of the perturbations depending by the vector field which links two different coordinate choice is called *gauge transformation.* A *gauge choice* is a precise correspondence between the background and the perturbed space-time. Note that all the objects in the previous relation are function of the coordinate x^α of the background space-time. Here we are mainly interested in the gauge transformation of the following quantities: scalar s, vector field $w^\mu = (w^0, w^i)$ a (1,1) tensor field A^μ_ν and finally a (0,2) tensor field $B_{\mu\nu}$. For sake of brevity, we will recall here the rules of transformation of those quantities under a gauge redefinition.

$$\begin{aligned}
\tilde{\delta s} &= \delta s - \bar{s}'\xi^0 \\
\tilde{\delta w}^0 &= \delta w^0 + \xi^0{}_{,0}\,\bar{w}^0 - \bar{w}^0{}_{,0}\,\xi^0 \\
\tilde{\delta w}^i &= \delta w^i + \xi^i{}_{,0}\,\bar{w}^0 \\
\tilde{\delta A}^0_0 &= \delta A^0_0 - \bar{A}^0_{0,0}\xi^0 \\
\tilde{\delta A}^0_i &= \delta A^0_i + \frac{1}{3}\xi^0{}_{,i}\,\bar{A}^k_k - \xi^0{}_{,i}\,\bar{A}^0_0 \\
\tilde{\delta A}^i_0 &= \delta A^i_0 + \xi^i{}_{,0}\,\bar{A}^0_0 - \frac{1}{3}\xi^i{}_{,0}\,\bar{A}^k_k \\
\tilde{\delta A}^i_j &= \delta A^i_j - \frac{1}{3}\delta^i_j\bar{A}^k_{k,0}\xi^0 \\
\tilde{\delta B}_{\mu\nu} &= \delta B_{\mu\nu} - \xi^\rho_{,\mu}\bar{B}_{\rho\nu} - \xi^\sigma_{,\nu}\bar{B}_{\mu\sigma} - \bar{B}_{\mu\nu,\alpha}\xi^\alpha
\end{aligned} \tag{A.7}$$

Then, it is easy to see that if we consider the trace of the tensor A, taking $i = j = k$, we have:

$$\tilde{A}^k_k = \delta A^k_k - A^k_{k,0}\xi^0 \tag{A.8}$$

an thus, we can note a very important property of the perturbations which is the following:

$$\tilde{\delta A}^i_j = \delta A^i_j - \frac{1}{3}\delta^i_j\tilde{\delta A}^k_k = \delta A^i_j - \frac{1}{3}\delta A^k_k \tag{A.9}$$

We have found the useful peculiarity of the traceless part of the tensor perturbations of being gauge invariant. Note that in all of these relations, a central role was played by the temporal component of the Killing vector field ξ.

After this quick introduction to the concept of gauge transformation, we can specialize the previous arguments for the case of the metric.

A.1 Helicity decomposition

Here we will discuss the separation of a generic tensor in its spin components. To start with, we can recall a general result from differential geometry which states that a generic field of p-forms on a Riemannian space Σ can be decomposed according to the following rule:

$$\bigwedge^{p}(\Sigma) = d\bigwedge^{p-1}(\Sigma)\bigoplus ker\delta \tag{A.10}$$

where we have called with $ker\delta$ the kernel of the co-differential of those forms. This is a consequence of the Hodge decomposition theorem. In the same way, a generic tensor field on Σ can be split out in two orthogonal sectors:

$$\chi_{(\Sigma)} = \chi^{S}\bigoplus\chi^{V} \tag{A.11}$$

The first one consists of the gradient of scalar fields while the second one is made up by vector fields with vanishing divergence. Analogously, we can decompose a symmetric tensor t in three sectors:

$$t_{ij} = t^{S}_{ij} + t^{V}_{ij} + t^{T}_{ij} \tag{A.12}$$

where:

$$\begin{aligned} t^{S}_{ij} &= Tr(t)\gamma_{ij} + (\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\Delta)f \\ t^{V}_{ij} &= \nabla_i \upsilon_j + \nabla_j \upsilon_i \\ t^{T}_{ij} &: Tr(t^T) = 0; \nabla\cdot t^T = 0 \end{aligned} \tag{A.13}$$

where we introduced a scalar function f and a vector field υ_i with vanishing divergence [1].

We may apply this theory to the metric perturbations. In fact, we can safely spit the metric perturbations according to its spin decomposition. Consequently, we will have spin 0 modes (scalars), spin 1 modes (vectors) and finally spin 2 modes (tensor perturbations or *gravitational waves*). We know that the metric tensor in a generic spacetime of dimension $D = n+1$ has $\frac{1}{2}n(n+1)$ real degrees of freedom. Let's do a simple calculation in the case of our interest, i.e. when $D = 4$ and so $n = 3$. In that case, we expect the degrees of freedom to be 6. In this 6 degrees are encoded the three sectors: scalar, vectors and tensors. Thanks to the Helmholtz's theorem any vector field U_i can be decomposed in the following way: $U_i = \partial_i\upsilon + \upsilon_i$, where υ is a scalar (the potential flow) curl free $\upsilon_{[i,j]} = 0$ and υ_i is a vector (vorticity) which is characterized of having vanishing divergence: $\nabla\cdot\upsilon = 0$. The last condition reduces the number of degrees of freedom of the vector sector from 3 to 2. Then, according to the third equation of A.13 we can see that the degrees of freedoms of the tensorial sector reduces to 2. So, at this stage, is immediate to see that also the scalar sector gets 2 degrees of freedom. The key point of this splitting is that, at linear order in perturbation theory, this three sectors are completely decoupled. That means that the dynamical equations which govern the evolutions of such perturbations are independent and we can treat them as independent fields. This will not hold anymore when we go further in perturbations. We will recall at the end of this section a modern picture to treat such complications.

[1]There is also another important decomposition through spherical harmonics but we will not give all the details here

Now, let's move on to the central object of our treatment: the metric tensor. We can adopt a simple parametrization for the perturbed metric:

$$ds^2 = a^2(\eta)\{-(1+2\Phi)d\eta^2 - 2\partial_i B d\eta dx_i + [(1-2\Psi)\delta_{ij} + 2D_{ij}]dx^i dx^j\} \tag{A.14}$$

Then the complete metric looks like:

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1+2\Phi) & \partial_i B \\ \partial_i B & (1-2\Psi)\delta_{ij} + D_{ij}E \end{pmatrix} \tag{A.15}$$

where: $D_{ij} = (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)$ These quantities under a gauge transformation will change. It is always possible to perform such change of reference such that perturbations we may register in one coordinate system disappear in another one and vice-versa. This is clearly a very important issue and matter of concerning for a good development of the theory. According to the previous rules we have found in the last section, if we consider a vector field ξ through which perform the transformation of the perturbations, such that: $\xi^\mu = (\xi^0, \xi^i = \partial^i\beta + v^i)$ (where $\partial_i v^i = 0$). We can see that the transformation rules for those perturbations are the following:

$$\begin{aligned} \tilde{\Phi} &= \Phi - \xi'^0 - \mathcal{H}\xi^0 \\ \tilde{B} &= B + \xi^0 + \beta' \\ \tilde{\Psi} &= \Psi - \frac{1}{3}\nabla^2\beta + \mathcal{H}\xi^0 \\ \tilde{E} &= E + 2\beta \end{aligned} \tag{A.16}$$

So, once again, it is clear that the components depend upon the gauge transformation and this makes clear that when one wants to deal with cosmological perturbations has to pay great attention. In fact, in some coordinate systems one can see some perturbations which can be set to zero by simply imposing another gauge choice. In this case those phantom perturbations are called *pure gauge* and are not physical.

At this stage we can move on our analysis and calculate the other fundamental objects in order to develop the complete cosmological perturbation theory based on the perturbed Einstein equations. The first quantities we want to show in terms of the perturbations are the Christoffel symbols, which are given by:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}(g_{\rho\gamma,\beta} + g_{\rho\beta,\gamma} - g_{\beta\gamma,\rho}) \tag{A.17}$$

Because we are developing the theory at linear order in the perturbations we can split the computation in the following easier way:

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\delta g^{\alpha\rho}(g_{\rho\gamma,\beta} + g_{\rho\beta,\gamma} - g_{\beta\gamma,\rho}) + \frac{1}{2}g^{\alpha\rho}(\delta g_{\rho\gamma,\beta} + \delta g_{\rho\beta,\gamma} - \delta g_{\beta\gamma,\rho}) \tag{A.18}$$

Then, the components are the following:

$$
\begin{aligned}
\delta\Gamma^0_{00} &= \Phi' \\
\delta\Gamma^0_{0i} &= \partial_i\Phi + \mathcal{H}\partial_i B \\
\delta\Gamma^i_{00} &= \mathcal{H}\partial^i B + \partial^i B' + \partial^i\Phi \\
\delta\Gamma^0_{ij} &= -2\mathcal{H}\Phi\delta_{ij} - \partial_i\partial_j B - 2\mathcal{H}\Psi\delta_{ij} - \Psi'\delta_{ij} - \mathcal{H}D_{ij}E + \frac{1}{2}D_{ij}E' \\
\delta\Gamma^i_{0j} &= -\Psi'\delta_{ij} + \frac{1}{2}D_{ij}E' \\
\delta\Gamma^i_{jk} &= \partial_j\Psi\delta^i_k - \partial_k\Psi\delta^i_j + \partial^i\Psi\delta_{jk} - \mathcal{H}\partial^i B\delta_{jk} + \frac{1}{2}\partial_j D^i_k E + \frac{1}{2}\partial_k D^i_j E - \frac{1}{2}\partial^i D_{jk}E
\end{aligned}
\tag{A.19}
$$

Then, the consequent quantity we are interested in is the Ricci tensor, which, again, has to be calculated bearing in mind to take into account just the linear terms. Then it can be expressed as follows:

$$
\delta R_{\mu\nu} = \partial_\alpha\delta\Gamma^\alpha_{\mu\nu} - \partial_\mu\delta\Gamma^\alpha_{\nu\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} + \Gamma^\alpha_{\sigma\alpha}\delta\Gamma^\sigma_{\mu\nu} - \delta\Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu}\delta\Gamma^\sigma_{\mu\alpha} \tag{A.20}
$$

And, making explicit the components:

$$
\begin{aligned}
\delta R_{00} &= \mathcal{H}\partial_i\partial^i B + \partial_i\partial^i B' + \partial_i\partial^i\Phi + 3\Psi'' + 3\mathcal{H}\Psi' + 3\mathcal{H}\Phi' \\
\delta R_{0i} &= \frac{a''}{a}\partial_i B + \mathcal{H}^2\partial_i B + 2\partial_i\Psi' + 2\mathcal{H}\partial_i\Phi + \frac{1}{2}\partial_k D^k_i E' \\
\delta R_{ij} &= \Big(- \mathcal{H}\Phi' - 5\mathcal{H}\Psi' - 2\frac{a''}{a}\Phi - 2\mathcal{H}^2\Phi - 2\frac{a''}{a}\Psi - 2\mathcal{H}^2\Psi - \Psi'' + \partial_k\partial^k\Psi - \mathcal{H}\partial_k\partial^k B\Big)\delta_{ij} \\
&\quad - \partial_i\partial_j B' + \mathcal{H}D_{ij}E' + \frac{a''}{a}D_{ij}E + \mathcal{H}^2 D_{ij}E + \frac{1}{2}D_{ij}E'' + \partial_i\partial_j\Psi - \partial_i\partial_j\Phi - 2\mathcal{H}\partial_i\partial_j B \\
&\quad + \frac{1}{2}\partial_k\partial_i D^k_j E + \frac{1}{2}\partial_k\partial_j D^k_i E - \frac{1}{2}\partial_k\partial^k D_{ij}E
\end{aligned}
\tag{A.21}
$$

Finally, the last object we need for the writing of the Einstein equations is the Ricci scalar which can be obtained by the Ricci tensor:

$$
R = g^{\mu\alpha}R_{\alpha\mu} \tag{A.22}
$$

So:

$$
\delta R = \delta g^{\mu\alpha}R_{\alpha\mu} + g^{\mu\alpha}\delta R_{\alpha\mu} \tag{A.23}
$$

$$
\begin{aligned}
\delta R = \frac{1}{a^2}\Big(&- 6\mathcal{H}\partial_i\partial^i B - 2\partial_i\partial^i B' - 2\partial_i\partial^i\Phi - 6\Psi'' \\
&- 6\mathcal{H}\Phi' - 18\mathcal{H}\Psi' - 12\frac{a''}{a}\Phi + 4\partial_i\partial^i\Psi + \partial_k\partial^i D^k_i E\Big)
\end{aligned}
\tag{A.24}
$$

Now, we have all the ingredients we need to compute the Eisntein tensor $G_{\mu\nu}$. From now on we will use the notation with indices mixed, i.e. G^{μ}_{ν}. The reason of that choice is due to the convenience in calculations. We will take advantage of gauge choice to get rid of some components of the metric tensor we are working with, but if we would like to raise (or lower) indexes of some tensors, we have to bear in mind that this kind of operations can be performed through the complete metric $g_{\mu\nu}$ which encode other perturbation variables. In this way new variables will proliferate. The only way to avoid this kind of troubles is to perform all the calculation using the mixed notation. Then, here we will give the components of the mixed Einstein tensor:

$$\delta G^{\mu}_{\nu} = \delta(g^{\mu\alpha}G_{\alpha\nu}) = \delta g^{\mu\alpha}G_{\alpha\nu} + +g^{\mu\alpha}\delta G_{\alpha\nu} \tag{A.25}$$

So:

$$\begin{aligned}
\delta G^0_0 &= R^0_0 - \frac{1}{2}R = \frac{1}{a^2}\Big[6\mathcal{H}^2\Phi + 6\mathcal{H}\Psi' + 2\mathcal{H}\partial_i\partial^i B - \partial_i\partial^i\Psi - \frac{1}{2}\partial_k\partial^i D^k_i E\Big] \\
\delta G^0_i &= R^0_i = \frac{1}{a^2}[-2\mathcal{H}\partial_i\Phi - 2\partial_i\Psi' - \frac{1}{2}\partial_k D^k_i E'] \\
\delta G^i_j &= \frac{1}{a^2}[(2\mathcal{H}\Phi' + 4\frac{a''}{a}\Phi - 2\mathcal{H}^2\Phi + \partial_i\partial^i\Phi + \mathcal{H}\Psi' + 2\Psi'' - \partial_i\partial^i\Psi + 2\mathcal{H}\partial_i\partial^i B + \partial_i\partial^i B' + \frac{1}{2}\partial_k\partial^m D^k_m E)\delta^i_j \\
&\quad - \partial^i\partial_j\Phi + \partial^i\partial_j\Psi - 2\mathcal{H}\partial^i\partial_j B - \partial^i\partial_j B' + \mathcal{H}D^i_j E' + \frac{1}{2}D^i_j E'' + \frac{1}{2}\partial_k\partial^i D^k_j E + \frac{1}{2}\partial_k\partial_j D^{ik}E - \frac{1}{2}\partial_k\partial^k D^i_j E]
\end{aligned} \tag{A.26}$$

Now we have shown how the lhs of the Einstein equation look like. Then, the other character which plays a fundamental role in the theory is the stress-energy tensor T^{μ}_{ν}. The helicity decomposition is the same we made for the metric tensor. Explicitly:

$$\begin{aligned}
T^0_0 &= -\rho \\
T^i_0 &= S^i + \partial^i S \\
T^i_j &= p\delta^i_j + \underbrace{\underbrace{(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma}_{\Sigma^{scalar}_{ij}} + \underbrace{\frac{1}{2}(\partial_i\sigma_j + \partial_j\sigma_i)}_{\Sigma^{vect}_{ij}} + \underbrace{\sigma^{TT}_{ij}}_{\Sigma^{tensor}_{ij}}}_{\Sigma_{ij}}
\end{aligned} \tag{A.27}$$

It is worth-mentioning the important propriety of a perfect fluid of non producing any anisotropic stress-energy contribute. So that, for a perfect fluid, the Σ^{tensor}_{ij} always vanish. The FLRW, which is the background of our perturbed Universe, can be thought as a perfect fluid. So, the first order perturbations of the stress-energy tensor can be obtained simply replacing the energy density ρ and the pressure p in the A.27 with $\rho \to \bar{\rho} + \delta\rho$ and $p \to \bar{p} + \delta p$.

Let's specialize the computation of the perturbations of a stress-energy tensor for the case of a scalar field ϕ. Such scalar field, like the Inflaton, can thought as a perfect fluid in an expanding background. Then, the stress-energy tensor for a scalar field, is:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\Big(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\Big) \tag{A.28}$$

then, computing carefully the single components, we arrive at:

$$\begin{aligned}
\delta T_0^0 &= \Phi\phi'^2 - \delta\phi'\phi' - \delta\phi\frac{\partial V}{\partial\phi}a^2 \\
\delta T_0^i &= \partial^i B\phi'^2 + \partial^i\delta\phi\phi' \\
\delta T_i^0 &= -\partial^i\delta\phi\phi' \\
\delta T_j^i &= \Big(- \Phi\phi'^2 + \delta\phi'\phi' - \delta\phi\frac{\partial V}{\partial\phi}a^2\Big)\delta_j^i
\end{aligned} \tag{A.29}$$

From this expression is once again explicit the tight relation between the matter fields and the metric perturbations.

Gauge choice and gauge invariant quantities

In order to develop a consistent gauge invariant theory, we need to find new variables to describe the physics of the problems we will face which are gauge invariant. We can define different gauge invariant quantities; here, we will analyse some of them. The first step toward the definition of gauge invariant variables is the setting of the slicing. Let's stress again the concept of the gauge choice. We have said that doing a gauge choice means to fix the correspondence between two coordinate system. So, fixing a gauge means to define a precise "threading" of the entire space-time (for fixed spatial coordinates) and a "slicing" into hypersurfaces (at fixed time). So, firstly, we need for first to set a certain slicing. After that we will make a transformation which will bring us from a generic slice of the space-time to the one we choice. The first fundamental variable we have used in this thesis is the *Comoving curvature perturbation*: $\mathcal{R}$. This variable is defined such that the perturbations of the inflaton in such slicing are null. $\delta\phi_{com} = 0$. Now, if we recall that a generic gauge transformation for $\delta\phi$ is performed by:

$$\tilde{\delta\phi} = \delta\phi - \phi'\xi^0 \tag{A.30}$$

Then, in this case, the vector field through which we can switch to the new slice has to have:

$$\xi^0 = \frac{\delta\phi}{\phi'} \tag{A.31}$$

Now, recalling that the intrinsic curvature for an hypersurface is given by the laplacian of the curvature perturbation Ψ, ${}^{(3)}R = \frac{4}{a^2}\nabla^2\Psi$, and that the curvature perturbation transforms in the following:

$$\Psi' = \Psi + \mathcal{H}\xi^0 \tag{A.32}$$

it is straightforward to define the following:

$$\mathcal{R} \equiv \Psi + \mathcal{H}\frac{\delta\phi}{\phi'} \tag{A.33}$$

which is clearly gauge invariant for construction. In this case, $\mathcal{R}$ [2] represents the gravitational potential on the comovent slices. The second variables of interest in the one we can define starting from the uniform energy density slicing. In this case, we will characterize the vector ξ requiring: $\delta\rho = 0$. Bearing in mind that the density perturbation transforms like $\tilde{\delta\rho} = \delta\rho - \rho'\xi^0$, then if we switch from a generic slicing to the one in which holds $\tilde{\delta\rho}_{com} = 0$, we have:

$$\xi^0 = \frac{\delta\rho}{\rho'} \tag{A.34}$$

So, recalling again the A.32, we can define the *uniform density curvature perturbation*ζ:

$$\zeta \equiv \Psi + \mathcal{H}\frac{\delta\rho}{\rho'} \tag{A.35}$$

We may wonder what is the relation between these two variables. It can be shown that, during the inflationary stage and for super Hubble scales, they coincide. The most simple proof of this statement is based on noticing that during inflation: $\rho + P = \dot{\phi}^2$ and, thanks to the energy-conservation equation: $\rho' + 3\mathcal{H}(\rho + P) = 0 \rightarrow \rho' + 3\mathcal{H}\dot{\phi}^2$, then:

$$\zeta = \Psi + H\frac{\delta\phi}{\dot{\phi}} \equiv \mathcal{R} \tag{A.36}$$

where, in the last equality we made use of the fact that the perturbations of the energy density during inflation are apprximated in the following manner: $\delta\rho \simeq -3H\dot{\phi}\delta\phi$. The last property that is important to stress is that the comoving curvature perturbation $\mathcal{R}$ is conserved for large scales and for just adiabatic modes. Lastly, the fundamental object we used in the computations in this thesis is the Mukhanov-Sasaki variable which basically is defined on a *spatially flat slicing* such that $\Psi_{flat} = 0$. In this case we have: $\xi^0 = -\frac{\Psi}{\mathcal{H}}$ and the fluctuations of the inflaton field is $\tilde{\delta\phi} = \delta\phi + \frac{\phi'}{\mathcal{H}}\Psi$. Then the Mukhanov-Sasaki variable is:

$$\mathcal{Q} = \delta\phi + \frac{\dot{\phi}}{H}\Psi \equiv \frac{\dot{\phi}}{H}\mathcal{R} \tag{A.37}$$

After having defined these gauge invariant quantities, one can face the problem of solving the dynamical equations in cosmological perturbation theory. Thanks to these variables, we are able to find the explicit expression for the gauge invariant (so, that have a non ambiguous physical meaning) variables. The price we pay taking this way is the intricacy of the calculations. A gauge invariant computation is affected by the proliferation of a multitude of terms but the results will be given in terms of just the gauge invariant quantities. On the other hand, making an explicit gauge fixing by hand from the start will simplify the computations.

[2]The comoving curvature perturbation is also the main character of a very famous theorem by Steven Weinberg. The so called adiabatic theorem states that there always exist two adiabatic scalar modes in which the comoving curvature perturbation is conserved on super-horizon scales. In the special case the perturbations are generated by a single source, then the both of the two allowed solutions are adiabatic and conserved on super-horizon scales.

Appendix B

Induced gravitational waves from scalar perturbations

It is instructive at this stage to introduce the formalism we can use through the study of the gravitational wave background. It is well understood from the SVT decomposition in the Fourier space of the metric perturbations that the three sectors (scalar-vector-tensor) are uncoupled at the first order in the perturbations. That means that the dynamical equation in those three sectors don't mix with each other. This is not the case when we go further in the perturbation theory and consider higher orders. There we can see a mixing between modes of different nature. This conduct us to a new phenomenon that may be observable. Fortunately, at second order, we don't have mixing between second order quantity of the same nature. To be specific, we won't see any equations which involve second order objects on the right and left side of Einstein equations, but only a dynamical relation between a second order object on the l.h.s. and a quadratic first order element at the r.h.s. . So, according to that, we can see the first order dynamical perturbations as a new sources for the second order modes. Let's stress one more time the deep difference between this production of the gravitational waves and the production we examined in the first chapter. During inflation or during the reheating stage a production of a gravitational waves background may occur. The one we looked at in the first chapter was based on the vacuum fluctuations. We have setted our treatment of cosmological perturbations in the General Relativity and we dealt with vacuum oscillations for our main purpose. In this perspective, we can look at other metric theories beyond the Einstein's theory for the treatment of vacuum oscillations and spot new physics from the vacuum. On the other hand, we can treat a classical mechanism where the source of the tensorial modes are produced by the first order perturbations. The topic of second order Gravitational Waves spectrum induced by the first order scalar perturbations.

$$\hat{\Pi}_{ij}^{lm} G_{lm}^{(2)} = k^2 \hat{\Pi}_{ij}^{lm} T_{lm}^{(2)} \tag{B.1}$$

where we have used the projection operator $\hat{\Pi}_{ij}^{lm} = \Pi_l^i \Pi_m^j - \frac{1}{2}\Pi_{ij}\Pi^{lm}$ where $\Pi_{ij} = \delta_{ij} - \partial_i \partial_j / \Delta$

$$h_{ij}'' + 2\mathcal{H} h_{ij}' - \nabla^2 h_{ij} = -4\hat{\Pi}_{ij}^{lm} \mathcal{S}_{lm} \tag{B.2}$$

where the source term has the following expression

$$\mathcal{S}_{lm} = 2\Psi\partial^l\partial_m\Psi - 2\Phi\partial^l\partial_m\Psi + 4\Phi\partial^l\partial_m\Phi + 4\Psi\partial^l\partial_m\Psi + \partial^l\Psi\partial_m\Psi - \partial^l\Psi\partial_m\Phi - \partial^l\Phi\partial_m\Psi + 3\partial^l\Phi\partial_m\Phi$$
$$- \frac{4}{3(1+\omega)\mathcal{H}^2}\partial_l(\Phi' + 3\mathcal{H}\Psi)\partial_m(\Phi' + 3\mathcal{H}\Psi) - \frac{2c_s^2}{3\omega\mathcal{H}^2}[3\mathcal{H}(\mathcal{H}\Psi - \Phi') + \nabla^2\Phi]\partial_l\partial_m(\Psi - \Phi) \tag{B.3}$$

and the projection operator in the TT gauge is the combination of the projections operators in the Fourier space looks like:

$$\Pi_{ij}^{lm}(\hat{\mathbf{k}}) = P_i^l(\hat{\mathbf{k}})P_j^m(\hat{\mathbf{k}}) - \frac{1}{2}P^{lm}(\hat{\mathbf{k}})P_{ij}(\hat{\mathbf{k}}) \tag{B.4}$$

where $P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$. Through the Fourier transformation:

$$\Pi_{ij}^{lm}(\mathbf{x}) = \sum_{\lambda=+\times}\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}e^{i\mathbf{k}\ \mathbf{x}}\epsilon_{ij}^{\lambda}(\mathbf{k})\epsilon^{\lambda,lm}(\mathbf{k}) \tag{B.5}$$

Then, the wave equation in the Fourier space take the form:

$$\left[\frac{d^2}{d\tau^2} + 2\mathcal{H}\frac{d}{d\tau} + k^2\right]h_k^{\lambda}(\tau) = S_k^{\lambda}(\tau) \tag{B.6}$$

and the solution can be recast expressed by the use of the Green function in the following way:

$$h_k^{\lambda}(\tau) = \int_{-\infty}^{+\infty}d\tau_1 g_k(\tau,\tau_1)S_k^{\lambda}(\tau_1) \tag{B.7}$$

where $g_k(\tau,\tau')$ is the solution of the differential equation where on the rhs there is just a Dirac delta function of the form: $\delta(\tau-\tau')$. Then, one can compute the usual power spectrum which will be the sum of the single power spectrum corresponding to the two polarizations.

Appendix C

Statistics

In this this appendix I would like to stress some concept that are the back bone of every cosmological analysis. The mathematical structures we used to characterize the spectrum of the primordial gravitational waves inherited peculiarities from the Cosmological model we are dealing with.

First of all, let us describe what means the word perturbations. Even if we consider our Universe as isotropic and homogeneous, if we look closer some portion of it, it seems clearly inhomogeneous. The galaxy structures we observe today are simply the evolution of some seeds originated in the Early Universe. We explained that those seed came from the quantum fluctuations of the inflaton field. Here, we can look closer the concept of perturbations. Every inhomogeneity can be thought as a sum of a homogeneous term plus a "little" additional contribute which are non-homogeneous that we will call perturbations. So, we can take as a classical example the energy density ρ, which is in general a function of the comoving coordinates $\mathbf{x}$ and time.

$$\rho(t, \mathbf{x}) = \bar{\rho}(t) + \delta\rho(t, \mathbf{x}) \tag{C.1}$$

In literature is common to define the variable "relative density of perturbation" as the following object:

$$\delta(t, \mathbf{x}) \equiv \frac{\delta\rho(t, \mathbf{x})}{\bar{\rho}(t)} \tag{C.2}$$

How about the background value $\bar{\rho}$? It can be thought as the average over volumes of the real (or perturbed) quantity $\rho(t, \mathbf{x})$, such that: $\bar{\rho}(t) \equiv \langle \rho(t, \mathbf{x}) \rangle_{vol.average}$. We can imagine $\bar{\rho}$ is a solution for an homogeneous and isotropic Universe, as like the perturbations don't exist.

Those variables are random generated by some process we may don't know. However, our goal is to point out the statistical proprieties of physical quantities which are functions of those stochastic variables. In order to accomplish this goal, we need to define ways to compute the expectation values of such physical quantities. We will define two kinds of average.

The first concept, more physical, is the concept of "volume average" and it is nothing else than the integral media over a volume V:

$$\bar{f} \equiv \frac{1}{V} \int_V f(\mathbf{x}) d^3\mathbf{x} \tag{C.3}$$

The second concept we can introduce is the "ensamble average". This is somewhat more familiar with mathematicians and is based on the concept of the random variables. Every observables in the Universe can be cast as a function of such random variables that we called "inhomogeneity" $\delta(t, \mathbf{x})$ and we will represent them this way: $f = f(\delta)$. We can think this ensamble average as the average over many equivalent realizations of the system under investigation. These equivalent realizations can be extract by many samples of the random variables. So, the best would be having the exact espression of the probability distribution of δ. In that way, we can consequently define the ensamble average as the expectation value of a given function $f(\delta)$ over the infinite number of equivalent configurations of samples:

$$\langle f \rangle \equiv \int d\delta Prob(\delta) f(\delta) \tag{C.4}$$

From a theoretical point of view, this expression can tell us the distribution probability of some proprirties of one of the possible Universe emergent by the various samples of stochastic variables. Even though this concept could seems fair enough for a mathematician, it seems unphysical due to the fact that we live in single Universe and we cannot perform any measurement over an ensable of equivalent Universes. At this stage, we can save the theory by adding the "ergodic" assumption. We will think the plethora of distant section of our Universe as if they were different realizations of different Universes. We will define the "ergodic fields" the ones which satisfy the following proprierty:

$$\bar{f} \equiv \langle f \rangle \tag{C.5}$$

Of course, this equality must hold when considering the entire Universe, not just limited portions. In the case we are just limiting the analysis to a finite volume V, the discrepancy between these two averages is called the *Cosmic Variance* $= \bar{f}_V - \langle f \rangle$. This is a very common and useful tool in cosmology because it provides a good test to compare the theoretical prediction (by $\langle f \rangle$) and the observations $\bar{f}_V$. At this stage, we would like to see how our assumptions about the Universe could reflect on the statistical properties of the physical observables. The FLRW Universe is the one which match the requirements of homogeneity and isotropy. These work assumptions are well justified by the current cosmological observations and the Inflation theory is the strongest candidate to make our Universe so homogeneous and isotropic. Consequently, every physical quantity we can use to describe the Cosmos will inherited such properties. In particular, the homogeneity of the space means that the expectation value of a certain function is completely independent by the position in the real space $\mathbf{x}$, so: $\langle f(\mathbf{x}) \rangle = \langle f \rangle$; while the isotropy means that the statistical properties don't depend on the direction. That means that any vectorial quantity, say $\vec{v}$, we can define in our Universe, it will have : $\langle \vec{v} \rangle = 0$

The perturbation density we introduced at the beginning cannot be used to measure the homogeneous content of the Universe. So, it is natural to look forward to some other quantities. The natural further step is the square of the perturbations δ^2 and its expectation value, the *variance*:

$$\langle \delta^2 \rangle = \langle \frac{\delta\rho^2}{\bar{\rho}^2} \rangle = \frac{\langle \delta\rho^2 \rangle}{\bar{\rho}^2} \tag{C.6}$$

This is a measure of the inhomogeneity's amplitude, but it doesn't tell nothing about the shape of the inhomogeneity. We need to go further and define other elements. Let's introduce the *correlation function* ξ:

$$\xi(x_1, x_2) \equiv \langle \delta(x_1)\delta(x_2) \rangle \tag{C.7}$$

Due to the statistical homogeneity, the correlation function is just a function of the separation between the two points x_1 and x_2 and due to the isotropy, the only thing that really matters is the modulo $|\vec{r}| = r$, so:

$$\xi = \xi(|\vec{r}|) = \xi(|\vec{x}_1 - \vec{x}_2|) = \langle \delta(x_1)\delta(x_1 + r) \rangle \tag{C.8}$$

Through the concept of the correlation function we have information about how much the perturbations in different point of space-time are correlated. Of course, ξ is much bigger when the displacement is smaller and vice-versa. In the limit when r reaches the zero we obtain again the variance: $\xi(0) = \langle \delta(x)\delta(x) \rangle$ While, for $r \to \infty$, ξ starts oscillating around zero. The next step is to switch to the Fourier space. As we just have pointed out, switching the analysis in the Fourier space is a powerful way to make clear the physics at different scales. Thank to the Fourier analysis, any function in the real space can be decomposed in its Fourier modes through the Fourier transform:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3} \int f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k \tag{C.9}$$

and vice-versa:

$$f(\mathbf{k}) = \int f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x \tag{C.10}$$

Analogously to the correlation function, we can define the respective quantity in the Fourier space.

$$\langle \delta(x_1)\delta(x_2) \rangle \longrightarrow \langle \delta^*_{k_1} \delta_{k_2} \rangle \tag{C.11}$$

This dual quantity is called the *Power Spectrum*, but we will define it better soon. The presence of the complex conjugate is justified by the fact that the variance has to be defined positive. In fact, in the case $k_1 = k_2 = k$, one has:

$$\langle \delta^*_k \delta_k \rangle = \langle |\delta_k|^2 \rangle \geq 0 \tag{C.12}$$

To make the definition clearer, we will write down explicitly the two-point function in the Fourier space:

$$\begin{aligned}
\langle \delta^*_k \delta_{k'} \rangle &= \frac{1}{V^2} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \int d^3x' e^{-i\mathbf{k}'\cdot\mathbf{x}'} \langle \delta(x)\delta(x') \rangle \\
&= \frac{1}{V^2} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \int d^3r e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})} \langle \delta(x)\delta(x+r) \rangle \\
&= \frac{1}{V^2} \int d^3r e^{-i\mathbf{k}'\cdot\mathbf{r}} \xi(r) \int d^3x e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} \\
&= \frac{1}{V^2} \int d^3r e^{-i\mathbf{k}'\cdot\mathbf{r}} \xi(r) \delta_{k'k} \cdot V \longrightarrow^{V\longrightarrow\infty} (2\pi)^3 \delta^3_D(\mathbf{k}'-\mathbf{k}) P(\mathbf{k})
\end{aligned} \tag{C.13}$$

where we have defined the *Power Spectrum P*, as it is called in literature:

$$P(\mathbf{k}) \equiv \int d^3r e^{-i\mathbf{k}'\cdot\mathbf{r}} \xi(r) \tag{C.14}$$

It is worth-noting that, unlikely the correlation function, the power spectrum is always positive defined. The power spectrum is a very useful tool to describe the perturbations over large scales, while the correlation function suits better in the analysis on smaller scales. From the previous formula, it may seems that the power spectrum could be dependent on the direction of the momentum $\mathbf{k}$ but, due to the statistical isotropy, it is dependent just by the modulus: $P(\mathbf{k}) = P(k)$. Thanks to all these assumptions we can make this relation explicit by performing the integral in polar coordinates.

$$\begin{aligned} P(k) &= \int d^3 r e^{-i\mathbf{k}'\cdot\mathbf{r}} \xi(r) = \int_0^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi e^{-ikr\cos\theta} \xi(r) \\ &= 2\pi \int_0^\infty r^2 \xi(r) dr \frac{1}{ikr} \left[e^{-ikr\cos\theta} \right]_0^\pi = 4\pi \int_0^\infty r^2 \xi(r) \frac{\sin(kr)}{kr} dr \end{aligned} \tag{C.15}$$

Vice-versa:

$$\xi(r) = \frac{1}{(2\pi)^3} \int d^3 k e^{ikx} P(k) = \frac{4\pi}{(2\pi)^3} \int_0^\infty P(k) \frac{\sin(kr)}{kr} k^2 dk \tag{C.16}$$

According to this formula, it is immediate to evaluate the variance $\xi(0)$:

$$\xi(0) \cong \frac{1}{2\pi^2} \int_{-\infty}^\infty P(k) \frac{kr}{kr} k^2 k \frac{dk}{k} = \frac{1}{2\pi^2} \int_{-\infty}^\infty P(k) k^3 d\ln k \equiv \int_{-\infty}^\infty \mathcal{P}(k) d\ln k \tag{C.17}$$

where in the last step we have defined:

$$\mathcal{P}(k) \equiv \frac{k^3}{2\pi^2} P(k) \tag{C.18}$$

It is common, in literature, to call also $\mathcal{P}(k)$ the Power Spectrum. It has the privilege of being adimensional and it gives the contribute on logaritmic scales to the variance.

Power Spectrum of the SGWB

At this stage we would like to apply the previous definitions to the gravitational waves world. In this section we will recall the main properties of the SGWB induced by classical sources. At the end we will take a little stint to clarify the meaning of the expectation value when studying primordial gravitational waves. The first property which characterize the power spectrum is the homogeneity and the isotropy of the space-time. Making this simple request means to have a two point functions between tensor modes of the following form:

$$\langle h_{ij}(\mathbf{x}, \eta_1) h_{lm}(\mathbf{y}, \eta_2) \rangle = F_{ijlm}(|\mathbf{x} - \mathbf{u}|, \eta_1, \eta_2) \tag{C.19}$$

where $\langle ... \rangle$ is the ensamble average. The other fundamental assumption we usually made for the cosmological gravitational waves background is that it is unpolarized and this is a consequence of an apparent absence of a source of parity violation in the Universe. In terms of the polarized states, that means that:

$$\langle h_+(\mathbf{k}, \eta) h_\times(\mathbf{k}, \eta) \rangle = \langle h_{+2}(\mathbf{k}, \eta) h_{+2}(\mathbf{k}, \eta) - h_{-2}(\mathbf{k}, \eta) h_{-2}(\mathbf{k}, \eta) \rangle = 0 \tag{C.20}$$

From the previous considerations, we can finally define the power spectrum as follows:

$$\langle h_r(\mathbf{k},\eta)h_p^*(\mathbf{q},\eta)\rangle = \frac{8\pi^5}{k^3}\delta^{(3)}(\mathbf{k}-\mathbf{q})\delta_{rp}h_c^2(k,\eta) \tag{C.21}$$

Here the Dirac delta function remark the principle of statistical homogeneity and isotropy and the Kronecker delta reflects the unpolarized property. For any gaussian field the two point function is the only essential which will give the entire statistical information about the system. The strange factor $8\pi^5$ in the previous definition has been chosen so that:

$$\langle h_{ij}(\mathbf{x},\eta)h_{ij}(\mathbf{x},\eta)\rangle = 2\int_0^\infty \frac{dk}{k}h_c^2(k,\eta) \tag{C.22}$$

Bibliography

[1] U. H. Danielsson, "A Note on inflation and transPlanckian physics," Phys. Rev. D **66** (2002), 023511 [arXiv:hep-th/0203198 [hep-th]].

[2] T. Tanaka, "A Comment on transPlanckian physics in inflationary universe," [arXiv:astro-ph/0012431 [astro-ph]].

[3] F. Lizzi, G. Mangano, G. Miele and M. Peloso, "Cosmological perturbations and short distance physics from noncommutative geometry" JHEP **06** (2002), 049 [arXiv:hep-th/0203099 [hep-th]].

[4] L. E. Parker and D. Toms, "Quantum Field Theory in Curved Spacetime: Quantized Field and Gravity"

[5] N. D. Birrell and P. C. W. Davies, "Quantum Fields in Curved Space"

[6] N. Bartolo, S. Matarrese and A. Riotto, "Oscillations during inflation and cosmological density perturbations" Phys. Rev. D **64** (2001), 083514 [arXiv:astro-ph/0106022 [astro-ph]].

[7] M. Maggiore, "Gravitational Waves. Vol. 1: Theory and Experiments"

[8] M. Maggiore, "Gravitational Waves. Vol. 2: Astrophysics and Cosmology"

[9] P. Peter and J. P. Uzan, "Primordial Cosmology"

[10] S. Dodelson, "Modern Cosmology"

[11] M. C. Guzzetti, N. Bartolo, M. Liguori and S. Matarrese, "Gravitational waves from inflation" Riv. Nuovo Cim. **39** (2016) no.9, 399-495 [arXiv:1605.01615 [astro-ph.CO]].

[12] Y. F. Cai, C. Lin, B. Wang and S. F. Yan, "Sound speed resonance of the stochastic gravitational wave background" Phys. Rev. Lett. **126** (2021) no.7, 071303 [arXiv:2009.09833 [gr-qc]].

[13] Z. Zhou, J. Jiang, Y. F. Cai, M. Sasaki and S. Pi, "Primordial black holes and gravitational waves from resonant amplification during inflation" Phys. Rev. D **102** (2020) no.10, 103527 [arXiv:2010.03537 [astro-ph.CO]].

[14] D. S. Gorbunov and V. A. Rubakov, "Introduction to the theory of the early universe: Cosmological perturbations and inflationary theory"

[15] L. A. Boyle and P. J. Steinhardt, "Probing the early universe with inflationary gravitational waves" Phys. Rev. D **77** (2008), 063504 [arXiv:astro-ph/0512014 [astro-ph]].

[16] H. Bouzari Nezhad and F. Shojai, "The Effect of α-Vacua on the Scalar and Tensor Spectral Indices: Slow-Roll Approximation" Phys. Rev. D **98** (2018) no.6, 063512 [arXiv:1802.05537 [gr-qc]].

[17] D. J. H. Chung, A. Notari and A. Riotto, "Minimal theoretical uncertainties in inflationary predictions" JCAP **10** (2003), 012 [arXiv:hep-ph/0305074 [hep-ph]].

[18] D. Polarski and A. A. Starobinsky, "Semiclassicality and decoherence of cosmological perturbations" Class. Quant. Grav. **13** (1996), 377-392 [arXiv:gr-qc/9504030 [gr-qc]].

[19] B. J. Broy, "Corrections to n_s and n_t from high scale physics" Phys. Rev. D **94** (2016) no.10, 103508 [arXiv:1609.03570 [hep-th]].

[20] Brandenberger, R.H. Is the spectrum of gravitational waves the "Holy Grail" of inflation?. Eur. Phys. J. C 79, 387 (2019)

[21] V. Mukhanov and S. Winitzki "Introduction to quantum effects in gravity"

[22] L. Amendola and S. Tsujikawa, "Dark Energy: Theory and Observations"

[23] G. Calcagni, "Classical and Quantum Cosmology"

[24] E. Barausse, E. Berti, T. Hertog, S. A. Hughes, P. Jetzer, P. Pani, T. P. Sotiriou, N. Tamanini, H. Witek and K. Yagi, *et al.* "Prospects for Fundamental Physics with LISA" Gen. Rel. Grav. **52** (2020) no.8, 81 [arXiv:2001.09793 [gr-qc]].

[25] M. Maggiore, C. Van Den Broeck, N. Bartolo, E. Belgacem, D. Bertacca, M. A. Bizouard, M. Branchesi, S. Clesse, S. Foffa and J. García-Bellido, *et al.* "Science Case for the Einstein Telescope" JCAP **03** (2020), 050 [arXiv:1912.02622 [astro-ph.CO]].

[26] A. Ricciardone, "Primordial Gravitational Waves with LISA" J. Phys. Conf. Ser. **840** (2017) no.1, 012030 [arXiv:1612.06799 [astro-ph.CO]].

[27] G. Amelino-Camelia, "Introduction to quantum-gravity phenomenology" Lect. Notes Phys. **669** (2005), 59-100 [arXiv:gr-qc/0412136 [gr-qc]].

[28] G. Esposito, "An Introduction to quantum gravity" [arXiv:1108.3269 [hep-th]].

[29] H. Kodama, M. Sasaki "Cosmological Perturbation Theory", Progress of Theoretical Physics Supplement, Volume 78, January 1984, Pages 1–166

[30] I. I. Smolyaninov, “Effect of Fast Scale Factor Fluctuations on Cosmological Evolution,” Universe **7** (2021) no.6, 164 [arXiv:2105.12567 [gr-qc]].

[31] C. Caprini and D. G. Figueroa, “Cosmological Backgrounds of Gravitational Waves,” Class. Quant. Grav. **35** (2018) no.16, 163001 [arXiv:1801.04268 [astro-ph.CO]].

[32] L. J. Garay, “Quantum gravity and minimum length,” Int. J. Mod. Phys. A **10** (1995), 145-166 [arXiv:gr-qc/9403008 [gr-qc]].

[33] G. Mangano, “Shadows of trans-planckian physics on cosmology and the role of the zero-point energy density,” Phys. Rev. D **82** (2010), 043519 [arXiv:1005.2758 [astro-ph.CO]].

[34] M. Biagetti, M. Fasiello and A. Riotto, “Enhancing Inflationary Tensor Modes through Spectator Fields,” Phys. Rev. D **88** (2013), 103518 [arXiv:1305.7241 [astro-ph.CO]].

[35] Y. Wang, “Inflation, Cosmic Perturbations and Non-Gaussianities,” Commun. Theor. Phys. **62** (2014), 109-166 [arXiv:1303.1523 [hep-th]].

[36] Hannu Kurki-Suonio, Lecture notes - Cosmological perturbation theory. Part 1 - Part 2, Helsinki University.

[37] A. de la Macorra, G. Piccinelli, General scalar fields as quintessence. Phys. Rev. D 61, 123503 (2000). [arXiv:hep-ph/9909459]

[38] J. O. Gong, “Multi-field inflation and cosmological perturbation” Int. J. Mod. Phys. D **26** (2016) no.01, 1740003 [arXiv:1606.06971 [gr-qc]].

[39] G. Domènech, “Scalar induced gravitational waves review” [arXiv:2109.01398 [gr-qc]].

[40] D. Baumann, “Primordial Cosmology,” PoS **TASI2017** (2018), 009 [arXiv:1807.03098 [hep-th]].

[41] C. Gordon, D. Wands, B. A. Bassett and R. Maartens, “Adiabatic and entropy perturbations from inflation,” Phys. Rev. D **63** (2000), 023506 [arXiv:astro-ph/0009131 [astro-ph]].

[42] N. Bartolo, S. Matarrese and A. Riotto, “Nongaussianity from inflation,” Phys. Rev. D **65** (2002), 103505 doi:10.1103/PhysRevD.65.103505 [arXiv:hep-ph/0112261 [hep-ph]].

[43] J. M. Bardeen, “Gauge Invariant Cosmological Perturbations,” Phys. Rev. D **22** (1980), 1882-1905 doi:10.1103/PhysRevD.22.1882

[44] J. Sola, “Cosmological constant and vacuum energy: old and new ideas,” J. Phys. Conf. Ser. **453** (2013), 012015 [arXiv:1306.1527 [gr-qc]].

[45] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, “Universe reheating after inflation,” Phys. Rev. D **51** (1995), 5438-5455 [arXiv:hep-ph/9407247 [hep-ph]].

[46] A. Shukla, S. P. Trivedi and V. Vishal, “Symmetry constraints in inflation, α-vacua, and the three point function,” JHEP **12** (2016), 102 [arXiv:1607.08636 [hep-th]].

[47] S. Kundu, “Inflation with General Initial Conditions for Scalar Perturbations,” JCAP **02** (2012), 005 [arXiv:1110.4688 [astro-ph.CO]].

[48] M. Sasaki, T. Suyama, T. Tanaka and S. Yokoyama, “Primordial black holes—perspectives in gravitational wave astronomy,” Class. Quant. Grav. **35** (2018) no.6, 063001 [arXiv:1801.05235 [astro-ph.CO]].

[49] Wang, Mike S. “Primordial Gravitational Waves from Cosmic Inflation.” (2017).

[50] K. N. Ananda, C. Clarkson and D. Wands, “The Cosmological gravitational wave background from primordial density perturbations,” Phys. Rev. D **75** (2007), 123518 [arXiv:gr-qc/0612013 [gr-qc]].

[51] D. Baumann, P. J. Steinhardt, K. Takahashi and K. Ichiki, “Gravitational Wave Spectrum Induced by Primordial Scalar Perturbations,” Phys. Rev. D **76** (2007), 084019 [arXiv:hep-th/0703290 [hep-th]].

[52] K. Turzyński and M. Wieczorek, “Floquet analysis of self-resonance in single-field models of inflation,” Phys. Lett. B **790** (2019), 294-302 [arXiv:1808.00835 [astro-ph.CO]].

[53] Y. F. Cai, C. Lin, B. Wang and S. F. Yan, “Sound speed resonance of the stochastic gravitational wave background,” Phys. Rev. Lett. **126** (2021) no.7, 071303 [arXiv:2009.09833 [gr-qc]].

[54] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP **05** (2003), 013 [arXiv:astro-ph/0210603 [astro-ph]].

[55] S. Weinberg, “Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity,”

[56] M. Nakahara, “Geometry, topology and physics,”

[57] L. Hollenstein, M. Jaccard, M. Maggiore and E. Mitsou, “Zero-point quantum fluctuations in cosmology,” Phys. Rev. D **85** (2012), 124031 [arXiv:1111.5575 [astro-ph.CO]].

[58] Q. Wang, Z. Zhu and W. G. Unruh, “How the huge energy of quantum vacuum gravitates to drive the slow accelerating expansion of the Universe,” Phys. Rev. D **95** (2017) no.10, 103504 [arXiv:1703.00543 [gr-qc]].

[59] S. S. Cree, T. M. Davis, T. C. Ralph, Q. Wang, Z. Zhu and W. G. Unruh, “Can the fluctuations of the quantum vacuum solve the cosmological constant problem?,” Phys. Rev. D **98** (2018) no.6, 063506 [arXiv:1805.12293 [gr-qc]].

[60] J. Martin and R. H. Brandenberger, "The TransPlanckian problem of inflationary cosmology," Phys. Rev. D **63** (2001), 123501 [arXiv:hep-th/0005209 [hep-th]].

[61] C. P. Burgess, J. M. Cline, F. Lemieux and R. Holman, "Are inflationary predictions sensitive to very high-energy physics?," JHEP **02** (2003), 048 [arXiv:hep-th/0210233 [hep-th]].

[62] U. H. Danielsson, "Inflation as a probe of new physics," JCAP **03** (2006), 014 [arXiv:hep-th/0511273 [hep-th]].

[63] J. Lesgourgues, D. Polarski and A. A. Starobinsky, "Quantum to classical transition of cosmological perturbations for nonvacuum initial states," Nucl. Phys. B **497** (1997), 479-510 [arXiv:gr-qc/9611019 [gr-qc]].

[64] C. Kiefer, "Quantum Gravity,"

[65] E. Poisson, "A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics,"

[66] M. D. Schwartz, "Quantum Field Theory and the Standard Model,"

[67] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory,"