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# Localizability in noncommutative spacetimes

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# Sommario

Nel presente lavoro si analizzano alcune proprità fondamentali della localizzabilità in due modelli di spazitempi noncommutativi. Nel Capitolo 1 si inizia motivando l'introduzione fisica di tali modelli di spazitempi e alcuni ragionamenti di carattere euristico che puntano verso la lunghezza di Planck come scala di noncommutatività. Viene presentata, poi, la teoria degli spazitempi noncommutativi e dei relativi gruppi di simmetrie, i cosiddetti Quantum Groups, definiti come deformazioni di particolari tipi di Hopf algebre. Infine sono definiti i concetti di stati, osservabili ed osservatori, dei quali faremo uso nel resto della trattazione. Nel Capitolo 2 la costruzione è applicata al ben noto spaziotempo di $\kappa$-Minkowski, ricavando i Quantum Groups di $\kappa$-Poincaré e studiando i relativi problemi di localizzabilità degli stati, seguendo ed estendendo la discussione effettuata in [43]. Emerge la proprietà che essendo il gruppo delle simmetrie deformato, differenti osservatori non concorderanno, in generale, sulla localizzabilità del medesimo stato. Nel Capitolo 3 una discussione analoga è portata avanti per la prima volta su uno spaziotempo noncommutativo di natura angolare meno conosciuto, il cosiddetto $\varrho$-Minkowski, introducendo nuove relazioni di indeterminazione per il Quantum Group e analizzando i risultanti vincoli sulle trasformazioni pure. Infine, nel Capitolo 4, si comparano i due modelli e si presentano prospetti per future indagini.

# Abstract

In this work we analyze the basic features of localizability in two models of noncommutative spacetimes. We start in Chapter 1 motivating the physical introduction of such kind of spacetimes and some heuristical reasoning pointing towards the Planck scale as the scale of noncommutativity. We, then, present the theory of noncommutative spacetimes and their symmetry groups, the so-called Quantum Groups, defined as deformations of particular types of Hopf algebras. Finally we define the notions of states, observables and observers we will make use of in the rest of the work. In Chapter 2 the construction is applied to the well-known $\kappa$-Minkowski spacetime, obtaining the $\kappa$-Poincaré Quantum Groups and studying the relative problems in localizability of states, following and extending the discussion made in [43]. It turns out that since the symmetry group is deformed, different observers will not agree, in general, on localizability properties of the same state. In Chapter 3 the analogous discussion is carried on for the first time on a lesser known noncommutative spacetime of angular nature, the so-called $\varrho$-Minkowski, introducing novel uncertainty relations for the Quantum Group and analyzing the resulting constraints on pure transformations. At last, in Chapter 4, we present a comparison between the two models and prospectives for future investigations.

# Contents


**1 Noncommutative spacetimes and Quantum Groups** **9**

1.1 Motivation . . . . . . . . . . 9

1.2 Classical commutative spacetimes . . . . . . . . . . 15

1.3 Noncommutative spaces and Quantum Groups . . . . . . . . . . 17

1.3.1 Connes construction . . . . . . . . . . 18

1.3.2 Quantum Group construction . . . . . . . . . . 21

1.4 Deformation of $\mathcal{C}(G)$ Hopf algebras . . . . . . . . . . 25

1.5 Deformation of $U(\mathfrak{g})$ Hopf algebras . . . . . . . . . . 27

1.6 Deformation of Poisson algebras . . . . . . . . . . 28

1.6.1 QM phase space and Moyal $\star$-product . . . . . . . . . . 29

1.6.2 Deformation quantization . . . . . . . . . . 31

1.6.3 Drinfel'd twist . . . . . . . . . . 33

1.7 Bicrossproduct Quantum Groups . . . . . . . . . . 37

1.8 States, observables and observers . . . . . . . . . . 39

**2 $\kappa$-deformation** **43**

2.1 Quantum Poincaré Group $\mathcal{C}_\kappa(P)$ . . . . . . . . . . 43

2.1.1 $\kappa$-Poincaré algebra sector . . . . . . . . . . 44

2.1.2 $\mathcal{C}_\kappa(P)$ algebra structure from $r$-matrix . . . . . . . . . . 45

2.1.3 $\mathcal{C}_\kappa(P)$ Hopf algebra structure . . . . . . . . . . 50

2.1.4 Bicrossproduct structure and $\mathcal{M}_\kappa$ derivation . . . . . . . . . . 52

2.2 Quantum Universal Enveloping Algebra $U_\kappa(\mathfrak{p})$ . . . . . . . . . . 53

2.2.1 $U_\kappa(\mathfrak{p})$ construction . . . . . . . . . . 53

2.2.2 Majid-Ruegg basis and $U_\kappa(\mathfrak{p})$ bicrossproduct structure . . . . 54

2.3 Observers, observables and the problem of localizability . . . . . . . . 56

2.3.1 $\mathcal{M}_\kappa$ coordinate realization . . . . . . . . . . 57

2.3.2 Localized states . . . . . . . . . . 60

2.3.3 $\kappa$-Poincaré realization . . . . . . . . . . 61

2.3.4 Observers and observables . . . . . . . . . . 65

**3 $\varrho$-deformation** **71**
3.1 Quantum Poincaré Group $\mathcal{C}_\varrho(P)$ . . . . . . . . . . 72
3.2 Quantum Universal Enveloping Algebra $U_\varrho(\mathfrak{p})$ . . . . . . . . . . 73
3.3 Localizability in $\varrho$-Minkowski . . . . . . . . . . 76
3.3.1 $\mathcal{M}_\varrho$ coordinate realization . . . . . . . . . . 77
3.3.2 Localized states . . . . . . . . . . 78
3.3.3 $\varrho$-Poincaré realization . . . . . . . . . . 79
3.3.4 Observers and observables . . . . . . . . . . 81

**4 Conclusions** **85**
4.1 Comparing the models . . . . . . . . . . 85
4.2 Future prospectives . . . . . . . . . . 87

**Appendices** **89**

**A Hopf algebras over rings and their deformation** **91**

**B General $r$-matrices and twists** **97**

# Chapter 1

# Noncommutative spacetimes and Quantum Groups

We begin our dissertation with some physical motivations to study noncommutative spacetimes in Section 1.1. After having briefly recalled the main classical commutative spacetimes' features in Section 1.2, we then turn our attention in Section 1.3 to define what a noncommutative space is and how it can be obtained via two different methods, the Connes' one and the Quantum Group's. The latter will be particularly relevant since Quantum Groups can be viewed as deformed symmetry groups of the relative noncommutative spacetime, so that in Sections 1.4-1.5 we analyze some possible ways to obtain physical relevant Quantum Groups as deformations of Hopf algebras. In Section 1.6 we present an explicit example of a well-known deformation of Poisson algebras and we introduce the fundamental notion of a *Drinfel'd twist*. Finally, Section 1.7 is devoted to the definition of a particular class of Quantum Groups called *bicrossproduct Quantum Groups*, that will turn out to be relevant in the case of deformations of the Poincaré group. The last Section of the Chapter, 1.8, introduces the ideas of states, observables and observers in both commutative as in noncommutative spacetimes.

## 1.1 Motivation

At present times the greatest majority of physical phenomena is successfully interpreted in the framework of one of two fundamental theories, General Relativity (GR), regarding the large scale gravitational physics, and Quantum Field Theory (QFT), concerning the laws of infinitesimal (with respect to human direct experience) quantum objects.

It is well known, however, that GR and QFT are in some way incompatible, in the sense that trying to combine the two theories leads to inconsistencies, and an hypothetical "Quantum Gravity" theory, unifying gravitation with the other fundamental interactions, is still at this time under development.

One of the main questions posed by researchers is at what scale we have to expect significant gravitational contributes in quantum effects. A series of purely heuristic reasonings, coming from a variety of different approaches (see for instance [28]) seem to point towards the so-called Planck scale.

Let us begin with the classical intensity comparison between gravitational and electric fields. Consider two classical massive charged particles in interaction. The Newtonian gravitational attraction force law is

$$F_G = G\frac{m_1 m_2}{r^2}, \tag{1.1}$$

where $m_1$, $m_2$ are the two particle masses and $r$ their relative distance; the electric interaction between them is given by Coulomb's law

$$F_C = k\frac{q_1 q_2}{r^2}, \tag{1.2}$$

with $q_1$, $q_2$ the two particle charges.

Consider now the case of two identical particles, such that $m_1 = m_2 = m$, $q_1 = q_2 = q$; we want to find at what scale $F_G$ and $F_C$ are of comparable intensity. Consider the ratio

$$\frac{F_G}{F_C} = \frac{Gm^2}{kq^2}; \tag{1.3}$$

at atomic scale (1.3) can be shown to be of the order of $10^{-40}$, letting us neglect gravitational interaction. On the contrary, for extremely high energies the running coupling constant of QED seems to point towards $\frac{ke^2}{\hbar c} \sim 1$ (with $e$ electron charge), so that, substituting in (1.3), one finds

$$\frac{F_G}{F_C} \approx \frac{Gm^2}{\hbar c}. \tag{1.4}$$

For (1.4) to be of order 1

$$m = \sqrt{\frac{\hbar c}{G}} = m_P, \tag{1.5}$$

the so-called *Planck mass.*

Therefore, for gravitational interactions to be significative at the small-scale regime, we must consider particles at the Planck scale[1]:

$$m_P = \sqrt{\frac{\hbar c}{G}} = 2.176434 \times 10^{-8} kg, \tag{1.6a}$$

$$E_P = \frac{\hbar c^5}{G} = 1.220890 \times 10^{28} eV. \tag{1.6b}$$

[1] All values for the constants cited in this chapter are taken from https://physics.nist.gov/cuu/Constants/index.html.

It is now clear why there is a heavy struggle to find experimental results of Quantum Gravity theories; the reached energy scale at LHC is just $2.36\times 10^{12} eV$, 16 magnitude orders less than the Planck energy scale.

Now that we have guestimated the scale at which gravity effects should be significative at quantum level, we are ready to show one of the fundamental inconsistencies of combining the two theories. Let us define the *Planck length*

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616255 \times 10^{-35} m, \tag{1.7}$$

and suppose that we are interested in measuring a spacetime event to the precision of $\ell_P$. Ordinary Quantum Mechanics tells us that we must employ a test particle with (reduced) Compton wavelength

$$\lambda = \frac{\hbar}{mc} \approx \ell_P. \tag{1.8}$$

Inverting the formula (1.8), such a particle needs to have mass

$$m \approx \frac{\hbar}{\ell_P c} = \sqrt{\frac{\hbar c}{G}} = m_P, \tag{1.9}$$

the Planck mass at which we expected gravity to be significative.

On the other side GR tells us that for a spherical solution of Einstein's field equations in the absence of angular momentum, electric charge and cosmological constant (a Schwarzschild solution), a mass distribution with radius less than a quantity called the *Schwarzschild Radius*

$$R_S = \frac{2Gm}{c^2}, \tag{1.10}$$

generates a black hole [54]. Substituting (1.9) in (1.10) one finds that for our test particle

$$R_S = 2\ell_P > \lambda = \ell_P, \tag{1.11}$$

and thus QM measures at Planck scale of a GR spacetime would result in a singularity of the theory.

It is now clear that a Quantum Gravity theory would be a great step towards a more deep understanding of the foundations of physics. There are, however, several problems to overcome in order to establish such a theory. A first issue in constructing a gravitational QM is the struggle with defining a self-adjoint gravitational field operator, which is strictly connected with the problem of misurability. As noted by Bronstein [13], following a reasoning similar to that of Bohr and Rosenfeld in QED's case [10], combining uncertainty QM relations and the equivalence principle in the case of the GR metric, a self-adjoint field operator which leads to sharp eigenvalues of the gravitational field cannot be defined due to upper bounds on mass densities of test bodies. Following the Bohr-Rosenfeld argument [28] suppose we want to measure the electromagnetic field average over a spacetime region, whose linear dimension and time duration are defined by $l$. This can be made by means of the analysis of initial and final momentum of a uniformly charged test body of linear dimension $l$, charge $q$ and mass $m$. We work here in natural units so that $c = \hbar = G = 1$. Requiring that the time interval for the momentum measurement is small compared to $l$, that any back-reaction can be neglected if the mass of the body is sufficiently high and that the borders of the body are separated by a spacelike interval, one obtains the following conditions

$$l \gtrsim \frac{1}{m}, \quad l \gtrsim \frac{q^2}{m}, \tag{1.12}$$

defining a classical test body both in quantum as in relativistic sense. Bohr and Rosenfeld noted that, in this regime, for the field strength $F$ one is led to an uncertainty relation

$$\Delta F l^3 \gtrsim \frac{q}{m}. \tag{1.13}$$

Eq.(1.13) tells us that, upon choosing an appropriate test body, infinite accuracy (i.e. sharp eigenvalues of the field operator) can be achieved.

Let us turn our attention on the gravitational case in the weak field approximation. Imposing the equivalence principle and carrying an analogous argument to that of the electromagnetic case, we obtain similar formulae where $F$ is substituted by $\frac{\Gamma}{\ell_P}$, with $\Gamma$ the metric connection, and $q$ by $\ell_P m$. Conditions (1.12) become now

$$l \gtrsim \frac{1}{m}, \quad l \gtrsim {\ell_P}^2 m, \tag{1.14}$$

from which follows $l \gtrsim \ell_P$. This last condition avoids the black-hole generation problem stated above. The uncertainty relation stated in terms of the connection, or equivalently the metric tensor $g$, is:

$$\Delta\Gamma l^3 \gtrsim {\ell_P}^2, \quad \Delta g l^2 \gtrsim {\ell_P}^2. \tag{1.15}$$

The problem with (1.15), compared to (1.13), is that apparently it is not possible to obtain infinite accuracy and sharp field eigenvalues since $\ell_P$ is a fixed constant.

Another aspect of incompatibility, this time between GR and QFT, is the fact that in GR metrics are determined as solutions of Einstein's field equations, and thus not given a priori, while QFT is constructed upon a fixed one. It is not clear, then, what field is necessary to quantize and how to do it in order to obtain a gravitational QFT.

A typical attempt is that of working in a weak field regime, where the GR metric can be expressed by

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.16}$$

with $\eta_{\mu\nu}$ a fixed Minkowski background metric and $h_{\mu\nu}$ the field perturbation representing gravitational effects we want to quantize.

Even in this apparently simple case problems arise. In analogy to other field theories, along with the quantization of $h_{\mu\nu}$ we must impose causality by means of commutation relations of the kind

$$[\hat{h}(x), \hat{h}(y)] = 0, \quad \forall (x, y) \text{ spacelike interval with respect to } \eta_{\mu\nu}. \tag{1.17}$$

The problem is that $\eta_{\mu\nu}$ is not the full spacetime metric $g_{\mu\nu}$, so that the condition imposed does not grant the full causality of the theory.

Another problem in quantizing the gravitational field is the (perturbative) non-renormalizability of the theory. Consider, at first, the GR Einstein-Hilbert action

$$\mathcal{S} = \int \left[ \frac{c^4}{16\pi G}(R - 2\Lambda) + \mathcal{L}_{\mathcal{M}} \right] \sqrt{-g} d^4x, \tag{1.18}$$

with $c$ the speed of light in vacuum, $R$ the Ricci scalar, $\Lambda$ the cosmological constant, $g$ a pseudo-Riemannian metric, $\mathcal{L}_{\mathcal{M}}$ a matter Lagrangian and $G \approx 6.674 \times 10^{-11} m^3 kg^{-1} s^{-2}$ the universal gravitational constant. If we try to define a gravitational interaction Lagrangian, as we do with the other types of interactions in QFT, a problem emerges. The coupling constant of such an interaction term is proportional to the inverse of the Einstein's constant $k = \frac{8\pi G}{c^4}$. We see that $G \approx \ell_P^2$ in natural units (i.e. taking $c = \hbar = 1$) so that $\frac{1}{k} \approx \ell_P^{-2}$.

Now, from QFT [56], we know that a necessary condition for an interaction to be (perturbatively) renormalizable is to have a divergence index $r \leqslant 0$, where $r$ is equal to minus the power of the mass dimension of the coupling constant. An interaction term with negative mass dimension is, therefore, non-renormalizable. In our case $[G] \simeq [\ell_P]^2 = [l]^2 = [m]^{-2}$ and we obtain the result that (at least in 4D) $r > 0$ and a quantum field GR theory would not be renormalizable.

The unrenormalizability of such a theory does not mean directly that GR cannot yield to a QFT, but that even if one succeeded in this task, physical results can not be obtained through the usual perturbation theory, which would imply the knowledge of infinite renormalization counterterms. Since exact solutions of a QFT are extremely difficult to find, at the moment a naive QFT of gravitation of this type does not lead to any experimental prediction, and is unverifiable. Some attempts to non-perturbative renormalization were proposed in time to overcome this problem, such as the *asymptotic safety* one (see [57] for reference).

Aside from these approaches, other possible paths to follow in order to obtain a Quantum Gravity theory were proposed, but until now none have been capable to formulate experimental predictions that can hold satisfying results with the actual technology, so that an entire new field of research called *phenomenological Quantum Gravity* arose (see for a brief outlook [33], and for further deepening [5]).

As said before, the examples aformentioned and other heuristic reasonings seem to point toward a special scale (1.6a-b),(1.7), at which Quantum Gravity effects become significant. For example the emergence of the Planck length occurs in two of the most prominent theories pointing towards Quantum Gravity: string theory and loop quantum gravity [28]. String theory is a field theory in which point-like particles are substituted with 1D objects called "strings" living in a 2D space called "worldsheet" (a generalization of a worldline). From a series of thought experiments based upon high energy string collisions and renormalization group methods, an uncertainty relation of the kind $\Delta x \Delta p \gtrsim 1 + k\ell_P^2 \Delta_p$, with $k$ a constant, emerges. Furthermore, by string-duality, a length $r$ is indistinguishable from another $\ell_p^2/r$, so that at the same time a minimum and a maximum length arise. On the other side there is loop quantum gravity, an approach to the nonperturbative canonical quantization of the gravitational field, seen as a constraint theory, based on the use of Wilson loops (operators defined by the trace of the exponential of a gauge field transported along a closed loop) to resolve constraint equations. In this picture some operators used to define loop states have a discrete small scale structure; for example having the *area operator* a discrete spectrum, the area of a physical system is quantized in multiples of a *Planck area*.

At this point one is naturally lead to ask if there exists some theory that modifies the classical spacetime structure introducing Quantum Gravity effects at a length scale (1.7), and eventually overcoming the singularity problem defining $\ell_P$ as an asymptotic constant inferior limit in analogy with special relativity's introduction of the constant $c$.

Relevant to our discussion are a series of works by S. Doplicher et al. ([23, 24, 25]) that, starting from first principles and carrying on the Gedankenexperiment on the measure of spacetime events, introduce noncommutative spacetimes in quite a natural way with a noncommutativity scale given by $\ell_P$.

Another linked topic of great interest is to formulate a theory that tries to resolve the problem of the incompatibility between a constant fundamental length and the Lorentz-Fitzgerald special relativity's contractions, and many attempts have been made in this line of research such as the formulation of the so-called *Doubly Special Relativity* [5, 1].

The study of noncommutative spacetimes, born with Snyder and Yang to overcome QFT divergences in a covariant way [32], flourished in the spirit of Quantum Gravity prospectives, and in the next Sections we will discuss some possible approaches to rebuild the spacetime structures upon this new framework, introducing noncommutative structures and deformations of the classical symmetry groups, called Quantum Groups.

## 1.2 Classical commutative spacetimes

In general relativity, the notion of an ordinary commutative spacetime is given by means of differential geometry in terms of pseudo-Riemannian smooth manifolds.

Consider the *n-dimensional Minkowski spacetime* $\mathcal{M}$ a pseudo-Riemannian Lorentzian smooth manifold with constant metric. Being $\mathcal{M}$ pseudo-Riemannian implies that it is equipped with a tensor $g_{\mu\nu}$, where $\mu, \nu = 0, ..., n-1$, called the *metric tensor* such that:

(a) $g_{\mu\nu}$ is non degenerate,

(b) $g_{\mu\nu}$ is symmetric;

while being Lorentzian means that:

(c) the metric signature is $(1, n-1)$.

The *n-dimensional Poincaré group*[2] $P$ is the inhomogeneous group of isometries of $\mathcal{M}$, i.e. the group of transformations on $\mathcal{M}$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}, \tag{1.19}$$

that satisfy the pseudo-orthogonality relations

$$g_{\mu\nu} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} g_{\alpha\beta}. \tag{1.20}$$

Similarly we can define $\mathcal{M}$ starting from the associated Poincaré group. Given $(G, \cdot)$ a Lie group, we define $(G', \cdot)$ to be a *subgroup* of $(G, \cdot)$ if $G'$ is a submanifold of $G$ closed with respect to $\cdot$, i.e. if

$$h_1 \cdot h_2 \in G', \quad \forall h_1, h_2 \in G' \subset G. \tag{1.21}$$

$G'$ is said to be an *invariant* or *normal subgroup* of $G$ (denoted symbolically by $G' \lhd G$) if it is a subgroup and the following property holds:

$$g h g^{-1} \in G', \quad \forall h \in G', \forall g \in G. \tag{1.22}$$

Now, given $G$ and a subgroup $G'$ of $G$, we can consider the quotient set $G/G'$, in other words the set of equivalence classes

$$g \sim g' \leftrightarrow g' = gh, \quad g, g' \in G, \quad h \in G'. \tag{1.23}$$

Being $G, G'$ Lie groups and therefore manifolds, $G/G'$ is a manifold, and if $G' \lhd G$ this set can be shown to be a group.

We want to define, now, how a group acts on a manifold structure. Let $G$ be a Lie group and $\mathcal{M}$ a manifold, an action of $G$ on $\mathcal{M}$ is a map $\sigma : G \times \mathcal{M} \to \mathcal{M}$, $\sigma(g, p) = p' \in \mathcal{M}$, $\forall g \in G, \forall p \in \mathcal{M}$, such that:

[2] In the present work we will always consider the so-called *Poincaré special group*, i.e. the Poincaré group that has as subgroup the proper orthochronous Lorentz group SO(1,3) instead of the full Lorentz group O(1,3).

(i) $\sigma(e,p)=p$, $\forall p\in\mathcal{M}$, with $e$ neutral element of $G$,

(ii) $\sigma(g_1,\sigma(g_2,p))=\sigma(g_1g_2,p)$, $\forall g_1,g_2\in G$.

An *isotropy group* or *small group* or *stabilizer* of $G$ with respect to $p\in\mathcal{M}$ is a subgroup $H_p=\{g\in G:\sigma(g,p)=p\}$.

If $H_p$ is an isotropy group of $G$, $G/H_p$ is a manifold called *homogeneous G-space.*

Given now a group $G$, a subgroup $H$ and a normal subgroup $G'\triangleleft G$, if $G$ is the product of the two subgroups, and $G'\cap H=\{e\}$ (with $e$ the identity element of $G$), we say that $G$ is the *semidirect product* of $H$ acting on $G'$, denoted by $G=H\ltimes G'$ or $G=G'\rtimes H$.

The relevance of these structures can be made manifest by the following examples.

**Euclidean space construction** Consider the 3D (special) Euclidean group $E_3$, that is the group of direct isometries (translations and rotations but not reflections) of the Euclidean space $\mathbb{E}_3$, i.e. the set of transformations that preserve Euclidean distance $d(p,q)=\sqrt{\sum_{i=1}^{3}(q_i-p_i)^2}$, with $p,q\in\mathbb{E}_3$. $E_3$ has two relevant subgroups, the rotational $SO(3)$ and translational $T_3$ groups. It is easy to see that $T_3$ is a normal subgroup, since acting with a rotation or a translation on an element of $T_3$ gives again a translation; furthermore $SO(3)$ is the isotropy group of the origin of $E_3$. At this point we can write $E_3=SO(3)\ltimes T_3$ and note that the Euclidean space is nothing but the (dual of the) homogeneous space of $E_3$: $E_3/SO(3)=${elements of $E_3$ that differ only by a rotation}$=T_3\sim\mathbb{E}_3$, where $\sim$ is an isomorphism.

**Minkowski space construction** Repeating the reasoning made for the 4D Poincaré group $P$, we have $T_4$ the normal subgroup of translations, $SO(1,3)$ the isotropy group of the origin of $P$, and we can write $P=SO(1,3)\ltimes T_4$, and define the *Minkowski space* $\mathcal{M}$ by $P/SO(1,3)=T_4\sim\mathcal{M}$.

To complete the picture, it is important to recall that the *Poincaré algebra* $\mathfrak{p}$, obtained through the tangent space at the identity of the Poincaré group [55], is found to be:

$$[P_\mu,P_\nu]=0, \tag{1.24a}$$
$$[M_{\mu\nu},P_\lambda]=i(g_{\nu\lambda}P_\mu-g_{\mu\lambda}P_\nu), \tag{1.24b}$$
$$[M_{\mu\nu},M_{\lambda\sigma}]=i(g_{\mu\sigma}M_{\nu\lambda}-g_{\nu\sigma}M_{\mu\lambda}+g_{\nu\lambda}M_{\mu\sigma}-g_{\mu\lambda}M_{\nu\sigma}), \tag{1.24c}$$

with $P^\mu$ the translation generators and $M^{\mu\nu}$ the Lorentz generators. The usual 4D vectorial representation of the Lorentz sector is given by $(M_{\alpha\beta})^\mu_\nu=i(\delta^\mu{}_\alpha g_{\nu\beta}-\delta^\mu{}_\beta g_{\nu\alpha})$. On the contrary, given $\mathfrak{p}$ the (universal covering of the) Poincaré group can be found by means of the exponential map.

## 1.3 Noncommutative spaces and Quantum Groups

Provided we are working in a classical picture where we do not have nontrivial quantum phase space commutators, we can think to define some self-adjoint operators[3] $\hat{x}^\mu$ on a suitable Hilbert space such as $L^2(\mathbb{R}^n_x)$ associated with the classical spacetime events $x^\mu$, such that the algebraic classical condition

$$[x^\mu, x^\nu] = 0 \tag{1.25}$$

holds. This defines the notion of a *classical commutative spacetime* in algebraic terms. Note that $x^\mu$ are considered, in this case, observable operators obtained by classical observable coordinate functions through a quantization procedure. We will give a precise definition of observables in both the commutative and noncommutative cases in Section 1.8.

Although the definition of a quantum time operator is a complex subject (see for example the discussion carried on in [6]), in our case we will never analyze dynamical aspects, so that there is no quantum ambiguity.

Now, we want to "promote" our space to a noncommutative one so that a spacetime uncertainty scale arises, avoiding the measure problems mentioned in Chapter 1 (see again the works of Doplicher et al. [23, 24, 25]).

The straightforward way to do that would be to modify eq.(1.25) in such a way that the commutators would not be trivial anymore. At this point it is important to preliminarily note that an issue occurs in the definition of the self-adjoint operators. We still can define them on the positional Hilbert space, but since $x^\mu$ are unbounded operators the Gel'fand-Naimark construction we are about to show fails. A formal solution would be that of exponentiating the operators to make them bounded.

In general a noncommutative spacetime can be described by commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x), \tag{1.26}$$

with $x^\mu$ defined on a suitable Hilbert space and $\theta^{\mu\nu}(x)$ depending on coordinates. Note that in general the noncommutativity of coordinates involves also the time operator; this can be justified by reasoning on the synchronization of clocks as explained in [28]. A wide class of noncommutative spacetimes with physical interest is of the form $[x^\mu, x^\nu] = i\theta^{\mu\nu} + i{\zeta_\alpha}^{\mu\nu}x^\alpha$ with $\theta^{\mu\nu}$ a constant antisymmetric matrix. In this latter case if $\zeta = 0$, the spacetime is said to be a *canonical noncommutative spacetime*, while if $\theta = 0$ it is said to be a *Lie algebra-type spacetime*.

In this work we will concentrate on two Lie algebra-type noncommutative spacetimes; the first is one of the most studied ones, the so-called *$\kappa$-Minkowski spacetime*

[3] From here on, for simplicity of notation, we omit hats on self-adjoint operators when their nature is obvious from the context.

$\mathcal{M}_\kappa$, defined in $n$D by

$$[x^0, x^i] = i\lambda x^i, \qquad i = 1, ..., n-1, \tag{1.27a}$$
$$[x^i, x^j] = 0, \qquad i, j = 1, ..., n-1, \tag{1.27b}$$

with spacetime coordinate operators defined on some Hilbert space. A useful choice we will make in the discussion on localizability is that of taking $x^i$ as a complete set of observables defined over $L^2(\mathbb{R}^{n-1}_x)$, and $x^0$ as a self-adjoint operator defined on the same space.

The name $\kappa$-Minkowski is given by the fact that it is, as we will see, a deformation of the classical Minkowski space defined by a parameter $\frac{1}{\kappa} \doteq \lambda$. It is obvious that sending $\lambda \to 0$ we expect to obtain the classical Minkowski commutative spacetime, so we are postulating that noncommutative effects arise at an infinitesimal $\lambda$ scale. $\lambda$ has dimensions of a length and so it is generally identified with the Planck length due to a series of heuristical reasonings such the one on the localizability of spacetime events we mentioned before.

The second noncommutative Minkowski spacetime we will study in 4D is called *angular Minkowski* or *$\varrho$-Minkowski* (as it was named in [3]), and it is defined by

$$[x^0, x^i] = i\varrho\epsilon^i{}_{j3}x^j, \quad i, j = 1, 2, 3, \tag{1.28a}$$
$$[x^i, x^j] = 0. \tag{1.28b}$$

As in the previous case $\varrho$ has dimensions of a length, and the same discussion made about $\lambda$ and the Planck scale applies invariate.

At this point we would like to define a noncommutative version of the pseudo-Riemannian manifold $\mathcal{M}$, but such a noncommutative space does not fit the axioms of smooth manifolds and in general the very concept of points loses of significance, due to uncertainty relations induced by the commutators, and so cannot be described in terms of ordinary differential geometry. What we can do is to consider such a space in terms of an algebraic or groupal formal construction. There are two main approaches to do this, the Connes construction and the Quantum Groups one, that are deeply interconnected.

In this work we will mainly concentrate on the Quantum Group approach, but it is interesting to briefly summarize the main features of the Connes one to complete the picture and to give a formal definition of states and observables in the noncommutative case.

### 1.3.1 Connes construction

The Connes approach to noncommutative topology is founded on a theorem by Gel'fand and Naimark regarding $C^*$-algebras and topological spaces.

Recall that an (associative, unital) *algebra* $\mathcal{A}$ over a field $\mathbb{K}$ is a vector space equipped with two linear maps:

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad \textit{(product)}, \tag{1.29a}$$
$$\eta : \mathbb{K} \to \mathcal{A} \qquad \textit{(unit)}, \tag{1.29b}$$

satisfying the properties ($a, b, c \in \mathcal{A}$):

$$\mu(a \otimes (b + c)) = \mu(a \otimes b) + \mu(a \otimes c) \qquad \textit{(right-distributivity)}, \tag{1.30a}$$
$$\mu((a + b) \otimes c) = \mu(a \otimes c) + \mu(b \otimes c) \qquad \textit{(left-distributivity)}, \tag{1.30b}$$
$$\mu(\mu(a \otimes b) \otimes c) = \mu(a \otimes \mu(b \otimes c)) \qquad \textit{(associativity)}, \tag{1.30c}$$
$$\mu(\mathbf{1} \otimes a) = \mu(a \otimes \mathbf{1}) = a, \quad \mathbf{1} = \eta(1) \in \mathcal{A} \qquad \textit{(existence of unity)}. \tag{1.30d}$$

$\mathcal{A}$ is said to be *commutative* if, defined the *flip map* $\tau : \tau(a \otimes b) = b \otimes a$, the following condition holds:

$$\mu\tau(a \otimes b) = \mu(a \otimes b). \tag{1.31}$$

A *Banach algebra* $\mathcal{B}$ is an algebra over the complex field $\mathbb{C}$ that is also a Banach space, i.e. a space complete with respect to the metric induced by a norm $\| \cdot \|$: $\mathcal{B} \to \mathbb{C}$, with the additional property

$$\| ab \| \leq \| a \| \| b \|, \quad \forall a, b \in \mathcal{B}, \tag{1.32}$$

this ensures that the multiplication map is continuous.

A $C^*$-*algebra* $\mathcal{C}$ is a Banach algebra endowed with an involution $^*$ defining an adjoint map with the following properties:

(a) $a^{**} = a$,

(b) $(ab)^* = b^* a^*$,

(c) $(xa + yb)^* = \bar{x} a^* + \bar{y} b^*$,

(d) $\| a^* \| = \| a \|$,

(e) $\| a^* a \| = \| a \|^2$,

$\forall a, b \in \mathcal{C}$, $\forall x, y \in \mathbb{C}$, and with $\bar{\cdot}$ representing the usual complex conjugation on $\mathbb{C}$.

A relevant example of a commutative $C^*$-algebra is the algebra $\mathcal{C}(\mathcal{M})$ of the continuous complex-valued functions over a compact topological space $\mathcal{M}$, with the product given by

$$(f \cdot g)(x) = f(x) g(x), \quad f, g \in \mathcal{C}(\mathcal{M}), x \in \mathcal{M}, \tag{1.33}$$

unit map

$$\eta(x) = x\mathbf{1}, \quad \mathbf{1}(x) = 1, \quad \forall x \in \mathcal{M}, \tag{1.34}$$

and norm

$$\| f \|_\infty = \sup_{x \in \mathcal{M}} | f(x) | \, . \tag{1.35}$$

The core of the Connes costruction is a consequence of the *Gel'fand-Naimark theorem* [29] stating that a topological space $\mathcal{M}$ can be recovered from the $\mathcal{C}(\mathcal{M})$ algebra defined over it, because of the complete equivalence between compact Hausdorff spaces and commutative $C^*$-algebras and the fact that any commutative $C^*$-algebra can be realized as the $C^*$-algebra of continuous complex-valued functions on an Hausdorff space and viceversa (*Gel'fand representation*). The outline of this construction, that can be performed by means of the ideals of the algebra or equivalently the set of pure states or else irreducible representations of the algebra, is given, for example in [41, 1].

The idea is, at this point, to consider a noncommutative $C^*$-algebra instead of a commutative one and to define a *noncommutative space* as the topological space associated to it by means of the Gel'fand-Naimark construction. Unlike the commutative case, in a noncommutative space of this kind it is not possible, in general, to define the concept of a point. This is the main reason to introduce such spaces in this way, instead of the natural way of defining commutative spacetimes by means of geometrical points. But our goal was to define a noncommutative structure analogous to a pseudo-Riemannian manifold, not only a topological noncommutative space. The main result of Connes was the recognition that all the extra features of such a space can be encoded in a so-called *spectral triple* [16].

Before defining what a spectral triple is, we must first mention that the previously cited Gel'fand-Naimark theorem also states that any $C^*$-algebra can be faithfully represented as a subalgebra of the algebra of bounded operators on an infinite dimensional separable Hilbert space via the so-called *GNS* (Gel'fand-Naimark-Segal) *construction*.

Therefore, without loss of generality, we define a *spectral triple* $(\mathcal{B}(\mathcal{H}), \mathcal{H}, D)$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on a Hilbert space $\mathcal{H}$ and $D$ a (not necessarily bounded) operator on $\mathcal{H}$, called *Dirac operator*, with the following properties:

(a) $D$ is self-adjoint on $\mathcal{H}$,

(b) the commutator $[D, a]$ is bounded on a dense subalgebra of $\mathcal{C}$ $\forall a \in \mathcal{C}$,

(c) $D$ has compact resolvent.

At this point, considering the $C^*$-algebra of continuous complex-valued functions $\mathcal{C}(\mathcal{M})$ on a pseudo-Riemannian manifold $\mathcal{M}$, taking the smooth functions subalgebra $\mathcal{C}^\infty \subset \mathcal{C}$ (needed to obtain a differential structure) represented on an Hilbert space $\mathcal{H}$ via the GNS construction by an algebra of bounded operators $\mathcal{B}(\mathcal{H})$ and defining a spectral triple $(\mathcal{B}, \mathcal{H}, D)$, we can recover both the topology and the full geometrical structure of the starting manifold.

To clarify the importance of the Dirac operator, let us consider the commutative case of a pseudo-Riemannian manifold $\mathcal{M}$ with Lorentzian metric $g_{\mu\nu}$ and $\mathcal{A}$ the $C^*$-algebra of continuous complex-valued functions defined over it; the notion of distance between two points $x, y \in \mathcal{M}$ can be implemented via

$$d(x,y) = \sup_{a\in\mathcal{A}}\{\mid a(x) - a(y) \mid : \; \parallel [D,a] \parallel \leq 1\}, \tag{1.36}$$

and the Dirac operator assumes the standard QFT form $D = i\gamma_\mu\partial^\mu$, so that the metric tensor will be given by

$$[\gamma_\mu, \gamma_\nu] = 2g_{\mu\nu}. \tag{1.37}$$

From this example it is clear that the role of $D$ is to encode in an algebraic way the metric tensor and a differential calculus (see [40] for reference); furthermore, the notion of integration can be implemented via particular kind of traces.

In the noncommutative case the starting $C^*$-algebra is chosen to be noncommutative and the reasoning follows the same way as in the commutative case leading to the encoding of the required features of our noncommutative space in a spectral triple.

### 1.3.2 Quantum Group construction

Deeply connected with the Connes approach there is another construction based on deformations of Hopf algebras called *Quantum Groups*. The main idea in this case is to build the spacetime starting from the relative symmetry group as done in Section 1.2 with the classical Euclidean and Minkowski spaces. The problem in following straightforwardly the noncommutative construction is that eq.(1.26) breaks explicitly Poincaré invariance; one of the possible solutions is to consider a *deformed* theory, taking the classical Poincaré Lie group, promoting it to an Hopf algebra and deforming it to a so-called *Quantum Group* under which the commutation relations defining the noncommutative spacetime are covariant. This will be the approach we will mainly follow going forward.

At first, we want to define what an Hopf algebra is.

A (coassociative, counital) *coalgebra* $\mathcal{C}$ over a field $\mathbb{K}$ is a vector space endowed with two maps:

$$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \qquad \textit{(coproduct)}, \tag{1.38a}$$

$$\varepsilon : \mathcal{C} \to \mathbb{K} \qquad \textit{(counit)}, \tag{1.38b}$$

with the properties:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \qquad \textit{(coassociativity)}, \tag{1.39a}$$

$$(id \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes id) \circ \Delta = id \qquad \textit{(counity)}. \tag{1.39b}$$

$\mathcal{C}$ is said to be *cocommutative* if:

$$\tau\Delta(a) = \Delta(a), \tag{1.40}$$

with $a \in \mathcal{C}$ and $\tau$ the flip map.

The importance of the coproduct map is, for example, provided by the following reasoning. Given two representations of $\mathcal{C}$, $(\varrho_1, V_1)$, $(\varrho_2, V_2)$, where $V_1$, $V_2$ are the vectorial spaces on which $\mathcal{C}$ is represented, we ask if it is possible to construct a representation $(\varrho, V_1 \otimes V_2)$ on the tensor product of the two vectorial spaces. Without the coproduct structure it is possible to show [1] that a linear and homomorphic $\varrho$ that respects the associativity of the algebra cannot be constructed. Using $\Delta$, instead, one can define a representation $\varrho : \varrho(a)(v_1 \otimes v_2) = ((\varrho_1 \otimes \varrho_2) \cdot \Delta(a))(v_1 \otimes v_2)$, that is linear by definition of $\Delta$, homomorphic by (1.42a) and (co)associative by (1.39a). If (1.40) holds, $(\varrho, V_1 \otimes V_2)$ is isomorphic to $(\varrho, V_2 \otimes V_1)$.

Let us note that the notions of algebras and coalgebras are *categorical-dual*, in the sense that expressing coalgebra structures by means of commutative diagrams and reversing the arrows one obtains the commutative diagrams encoding the algebra structures. This construction is discussed in Appendix A, where all these algebraic structures are generalized over commutative rings, rather than defined on fields. More explicitly, given a coalgebra $\mathcal{C}$ we can construct the dual vector space of the linear functionals on $\mathcal{C}$, $\mathcal{C}^* = Lin(\mathcal{C})$, and define an inner product $\langle \cdot, \cdot \rangle : \mathcal{C}^* \otimes \mathcal{C} \to \mathbb{K}$ such that $\Delta$ and $\varepsilon$ define adjoint maps $\mu : \mathcal{C}^* \otimes \mathcal{C}^* \to \mathcal{C}^*$ and $\eta : \mathbb{K} \to \mathcal{C}^*$ by means of the relations:

$$\langle ab, c \rangle = \langle a \otimes b, \Delta(c) \rangle, \qquad a, b \in \mathcal{C}^*, c \in \mathcal{C}, \tag{1.41a}$$

$$\langle \mathbf{1}, c \rangle = \varepsilon(c), \qquad \mathbf{1} \in \mathcal{C}^*. \tag{1.41b}$$

A *bialgebra* $(\mathcal{B}, \mu, \eta, \Delta, \varepsilon)$ over a field $\mathbb{K}$ is then a vector space that is an algebra, a coalgebra and that satisfies the following *homomorphism* compatibility conditions $(a, b \in \mathcal{B})$:

$$\Delta(ab) = \Delta(a)\Delta(b), \tag{1.42a}$$

$$\Delta(1) = 1 \otimes 1, \tag{1.42b}$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \tag{1.42c}$$

$$\varepsilon(1) = 1. \tag{1.42d}$$

We can now define a *Hopf algebra* $(\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ over a field $\mathbb{K}$ as a bialgebra equipped with the following *antipode map* $(a, b \in \mathcal{H})$:

$$S : \mathcal{H} \to \mathcal{H}, \tag{1.43a}$$

$$\mu \circ (S \otimes id) \circ \Delta = \mu \circ (id \otimes S) \circ \Delta = \eta\varepsilon \quad \textit{(Hopf identity)}, \tag{1.43b}$$

$$S(ab) = S(b)S(a), \tag{1.43c}$$

$$S(1) = 1, \tag{1.43d}$$

$$(S \otimes S) \circ \Delta(a) = \tau \Delta S(a). \tag{1.43e}$$

Two relevant examples of Hopf algebras are now discussed.

Consider a finite group $G$, and the space of continuous functions on it $\mathcal{C}(G)$. Its algebra can be promoted to a commutative Hopf algebra if we define the following algebra, coalgebra and antipode:

$$(f \cdot g)(x) = f(x)g(x), \tag{1.44a}$$

$$\eta(\lambda) = \lambda 1, \quad 1(x) = 1, \quad \forall x \in G, \tag{1.44b}$$

$$\Delta(f)(x, y) = f(xy), \tag{1.44c}$$

$$\varepsilon(f) = f(e), \tag{1.44d}$$

$$S(f)(x) = f(x^{-1}), \tag{1.44e}$$

with $f, g \in \mathcal{C}(G)$, $x, y \in G$, $\lambda \in \mathbb{K}$ and $e$ the neutral element of $G$. The coproduct is defined by the isomorphism $\mathcal{C}(G) \otimes \mathcal{C}(G) \sim \mathcal{C}(G \times G)$. In this case the coproduct gives the group multiplication law, where the counit defines the identity element $e$ and the antipode the groupal inversion.

Let, now, $G$ be a compact topological group. The construction above still works with the caveat of taking $\mathcal{C}(G \times G)$ as an algebra completion of $\mathcal{C}(G) \otimes \mathcal{C}(G)$, as shown in [15]. In this case, starting with $G$, taking the $C^*$-algebra of continuous functions defined over it, promoting it to an Hopf algebra and deforming such an algebra (i.e. making it noncommutative, as we will formalize in the next Section and discuss more generally in Appendix A), we define what is called a *compact matrix Quantum Group*, whose name is due to the fact that such a structure admits a representation (or even a definition) in terms of matrices.

For the second example we start with a definition of what a universal enveloping algebra is (see [31] for reference). Given a Lie algebra $\mathfrak{g}$ it is possible to define a couple $(U(\mathfrak{g}), i)$ called the *universal enveloping algebra of* $\mathfrak{g}$, where $U(\mathfrak{g})$ is an associative unital algebra, and $i : \mathfrak{g} \to U(\mathfrak{g})$ a linear map satisfying the following properties:

(i) $\forall X, Y \in \mathfrak{g}$, $i([X, Y]) = i(X)i(Y) - i(Y)i(X)$,

(ii) $U(\mathfrak{g})$ is generated by elements $i(X)$,

(iii) if $\mathcal{A}$ is an associative unital algebra and $j : \mathfrak{g} \to \mathcal{A}$ a linear map satisfying (i), there exists a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to \mathcal{A}$ such that $\phi(1) = 1$ and $\phi(i(X)) = j(X)$, $\forall X \in \mathfrak{g}$.

The second noteworthy example of Hopf algebras is that of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ endowed with

$$\mu(x \otimes y) = [x, y], \tag{1.45a}$$
$$\Delta(x) = x \otimes 1 + 1 \otimes x, \tag{1.45b}$$
$$\eta(\lambda) = \lambda \mathbf{1}, \tag{1.45c}$$
$$\varepsilon(x) = 0 \quad \forall x \neq \mathbf{1}, \quad \varepsilon(\mathbf{1}) = 1, \tag{1.45d}$$
$$S(x) = -x. \tag{1.45e}$$

where $x, y \in U(\mathfrak{g})$, $\lambda \in \mathbb{K}$. In this case the coproduct defines the algebra actions on a tensor product of elements of the algebra, and since the coproduct is cocommutative, it gives the usual Leibniz rule, granting the algebra actions on the algebra a derivation structure.

If the Lie algebra $\mathfrak{g}$ is semisimple, deforming it via the introduction of a deformation parameter gives a so-called *Drinfel'd-Jimbo Quantum Group.*

It can be shown that if $\mathfrak{g}$ is the Lie algebra of $G$, $\mathcal{C}(G)$ and $U(\mathfrak{g})$ are dual Hopf algebras in the following sense [1].

Two Hopf algebras $\mathcal{H}, \mathcal{H}^*$ are dual if there exists a nondegenerate product $\langle \cdot, \cdot \rangle$ : $\mathcal{H}^* \otimes \mathcal{H} \to \mathbb{K}$ such that

$$\langle ab, x \rangle = \langle a \otimes b, \Delta(x) \rangle, \tag{1.46a}$$
$$\langle 1_{\mathcal{H}^*}, x \rangle = \varepsilon(x), \tag{1.46b}$$
$$\langle \Delta(a), x \otimes y \rangle = \langle a, xy \rangle, \tag{1.46c}$$
$$\varepsilon(a) = \langle a, 1_{\mathcal{H}} \rangle, \tag{1.46d}$$
$$\langle S(a), x \rangle = \langle a, S(x) \rangle, \tag{1.46e}$$

with $a, b \in \mathcal{H}^*$, $x, y \in \mathcal{H}$.

Eqs. (1.46a,c) show explicitly that the product in $\mathcal{H}^*$ is defined by the coproduct in $\mathcal{H}$ and the coproduct in $\mathcal{H}^*$ is defined by the product in $\mathcal{H}$. It is, therefore, possible to show that a commutative Hopf algebra defines a cocommutative dual Hopf algebra and viceversa.

In this picture, following the construction (1.45a-e), we can easily promote our classical Poincaré Lie algebra (1.24a-c) to an (undeformed) Hopf algebra endowing it with the additional structures:

$$\Delta P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu, \tag{1.47a}$$
$$\Delta M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}, \tag{1.47b}$$
$$S(P_\mu) = -P_\mu, \tag{1.47c}$$
$$S(M_{\mu\nu}) = -M_{\mu\nu}, \tag{1.47d}$$

and trivial counits.

A third type of Quantum Groups of relevance for our discussion, are the so-called *bicrossproduct Quantum Groups*. The main result here is the fact that the bicrossproduct structure of a Quantum Group gives a natural way of defining non-commutative spaces through a procedure similar to that of Section 1.2 employed in the recovery of the Euclidean and Minkowski spaces; on the contrary, the Quantum Group can be seen as the simmetry group of a noncommutative spacetime. We will discuss them in Section 1.7, after having defined some ways of deforming the Hopf algebras $\mathcal{C}(P)$ and $U(\mathfrak{p})$ in Sections 1.4 and 1.5 and an explicit way of deforming Poisson algebras in 1.6.

## 1.4 Deformation of $\mathcal{C}(G)$ Hopf algebras

We want, now, to find a general way to construct deformed $\mathcal{C}(G)$ Hopf algebras. We will start introducing undeformed structures that naturally lead to Hopf algebras and to deformation methods called "quantizations".

A *Lie bialgebra* is a bialgebra $\mathfrak{g}$ such that:

(i) $\mathfrak{g}$ is a Lie algebra ($\exists$ a Lie bracket $[,]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$),

(ii) the dual $\mathfrak{g}^*$ is a Lie algebra with Lie bracket $\delta^*:\mathfrak{g}^*\otimes\mathfrak{g}^*\to\mathfrak{g}^*$,

and defined the dual map $\delta$ to $\delta^*$, called the *cocommutator*, $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$, the following compatibility condition (*1-cocyclicity*) holds:

(iii) $\delta([X,Y])=(ad_X\otimes 1+1\otimes ad_X)\delta(Y)-(ad_Y\otimes 1+1\otimes ad_Y)\delta(X)$, with $X,Y\in\mathfrak{g}$ and $ad_XY=[X,Y]$ the adjoint action of the algebra on the algebra.

Note that, by symmetry of the definition, also $\mathfrak{g}^*$ has a Lie bialgebra structure.

Given a Lie algebra $\mathfrak{g}$ we define a *classical $r$-matrix* to be

(a) a tensor $r\in\bigwedge^2\mathfrak{g}$,

with the properties:

(b) the symmetric part of $r$ is $\mathfrak{g}$-invariant under $\mathfrak{g}\otimes\mathfrak{g}$,

(c) the following *Schouten bracket*

$$[[r,r]]=[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}] \tag{1.48}$$

is $\mathfrak{g}$-invariant in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$.

Eq.(1.48) is called *Modified Yang-Baxter Equation*, (MYBE).

The classical $r$-matrix has the important property of defining a Lie bialgebra structure on the Lie algebra $\mathfrak{g}$ through the cocommutator $\delta(X)=[X,r]$, $X\in\mathfrak{g}$.

Note that property (c) is automatically satisfied if $[[r, r]] = 0$ (*Classical Yang Baxter Equation*, CYBE). A Lie bialgebra determined by a CYBE solution is said to be *quasitriangular*, while one that arises from a skew CYBE solution ($r_{12} = r_{21}$) it is said to be *triangular*.

To clarify the two indices notation of $r$, we define $r_{\alpha\beta} \in \bigwedge^3 \mathfrak{g}$, $\alpha, \beta = 1, 2, 3$ as (summation implied over repeated indices):

$$r_{12} = c_{ij} a_i \otimes a_j \otimes 1, \tag{1.49a}$$

$$r_{23} = c_{ij} 1 \otimes a_i \otimes a_j, \tag{1.49b}$$

$$r_{13} = c_{ij} a_i \otimes 1 \otimes a_j, \tag{1.49c}$$

$$[r_{12}, r_{13}] = c_{ij} c_{mn} [a_i, a_m] \otimes a_j \otimes a_n, \tag{1.49d}$$

$$[r_{12}, r_{23}] = c_{ij} c_{mn} a_i \otimes [a_j, a_m] \otimes a_n, \tag{1.49e}$$

$$[r_{13}, r_{23}] = c_{ij} c_{mn} a_i \otimes a_m \otimes [a_j, a_n], \tag{1.49f}$$

with $a_i \in \mathfrak{g}$.

Now that we have defined the idea of a Lie bialgebra and the fundamental properties of the classical $r$-matrix, we want to define a group structure that is for Lie bialgebras what a Lie group was for a Lie algebra. This structure will be called a *Poisson-Lie group.*

A *Poisson Manifold* $\mathcal{M}$ is a smooth manifold of finite dimension equipped with a bivector $\Lambda \in \bigwedge^2 \mathcal{M}$ that defines a Poisson bracket structure:

$$\Lambda(df, dg) = \{f, g\}, \quad f, g \in \mathcal{F}(\mathcal{M}). \tag{1.50}$$

Given $\mathcal{M}, \mathcal{N}$ Poisson manifolds, a smooth map $F : \mathcal{N} \to \mathcal{M}$ is said to be a *Poisson Map* if it preserves the Poisson brackets on $\mathcal{M}$ and $\mathcal{N}$, i.e. if

$$\{f, g\}_{\mathcal{M}} \circ F = \{f \circ F, g \circ F\}_{\mathcal{N}}, \quad \forall f, g \in C^\infty(\mathcal{M}). \tag{1.51}$$

We are now ready to define a Poisson-Lie group $G$ as a Poisson manifold and a Lie group with the two structures being compatible in the following sense:

(i) the multiplication map $\mu : G \times G \to G$ is a Poisson map,

(ii) given $H$ another Poisson-Lie group, every homomorphism $\Phi : G \to H$ is an homomorphism of Lie groups and a Poisson map.

The reason we are interested in Poisson-Lie groups, is due to a theorem (see [15] for details) that states that given a Poisson-Lie group, then its Lie algebra has a natural Lie bialgebra structure (the *tangent Lie bialgebra*), and conversely given a Lie bialgebra it is uniquely defined a Poisson structure that makes the bialgebra the tangent bialgebra of a Poisson-Lie group. A *Poisson algebra* is a commutative algebra endowed with a Poisson bracket, satisfying the usual Jacobi identity and Leibniz rule. If $G$ is a Poisson-Lie group, then the associated Poisson algebra can be constructed in terms of the set $\mathcal{C}^\infty(G)$ of $C^\infty$ functions defined over $G$. This Poisson algebra $\mathcal{C}^\infty(G)$ is also an Hopf algebra and it is called a *Poisson-Hopf algebra*, with the Poisson and Hopf structures automatically satisfying some compatibility relations. Summarizing, we have defined a Poisson-Hopf algebra $\mathcal{C}^\infty(G)$, corresponding to a Poisson-Lie group $G$ related to a Lie bialgebra structure on the Lie algebra of $G$.

Returning to our explicit construction, it can be shown that the classical $r$-matrix defines a Poisson-Lie structure through the following *Sklyanin bracket*:

$$\{f,g\} = r^{\alpha\beta}(X^R_\alpha f X^R_\beta g - X^L_\alpha f X^L_\beta g), \qquad f,g \in \mathcal{C}^\infty(G), \tag{1.52}$$

where $X^L$, $X^R$ are the left and right invariant vector fields.

Now that we have defined an Hopf algebra of functions on a group, the last step to analyze is the question of the deformation. In the following we will consider a particular class of deformations called "quantization". Let $\mathcal{A}$ be a commutative Poisson-Hopf algebra with Poisson bracket $\{,\}$; a *quantization* of $\mathcal{A}$ is an *Hopf algebra deformation* $\mathcal{A}_q$ of $\mathcal{A}$ (essentially a new Hopf algebra isomorphic to the starting one with multiplication and coproduct equivalent to the undeformed ones modulo $q$, as we will discuss more formally in Appendix A) such that

$$\{x,y\} \equiv \frac{ab-ba}{q} \pmod{q}, \tag{1.53}$$

if $a,b \in \mathcal{A}_q$ reduce to $x,y \in \mathcal{A}$ modulo $q$, where the meaning of "modulo $q$" is intended in the general sense of elements in an algebra congruent modulo an ideal of the algebra (see Appendix B); naively in our case two elements of an algebra are congruent modulo $q$ if they differ by terms proportional to a power of $q$. Conversely is said that $\mathcal{A}$ is the *quasiclassical limit* of $\mathcal{A}_q$.

A quantization of a Poisson-Lie group $G$ is a quantization $\mathcal{C}^\infty_q(G)$ of the algebra $\mathcal{C}^\infty(G)$ regarded as a Poisson algebra.

The approach we will use to quantize the Poisson-Hopf algebra will be the usual canonical quantization $\{,\} \to \frac{1}{i}[,]$. Note that the deformation parameter $q$ does not enter the quantization rule because we will define the classical $r$-matrix already including it. An alternative way to carry on the discussion would be to define the $r$-matrix without the deformation parameter and to introduce it in the quantization rule.

## 1.5 Deformation of $U(\mathfrak{g})$ Hopf algebras

Starting with the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, our goal would now be to define a Quantum Group via the deformation induced by a parameter $q$.

Since, as said in Subsection 1.3.2, the Hopf algebras $\mathcal{C}(G)$ and $U(\mathfrak{g})$ are dual, we have two starting points to obtain $U(\mathfrak{g})_q$. The first is to consider the Quantum Group $\mathcal{C}(G)_q$ and dualize it through (1.46a-e); this method, however, despite its conceptual simplicity, is not easy to work out due to the complexity of calculations involved. The second one is to follow a dual corresponding procedure to that outlined in the previous Section based on the definition of *co-Poisson-Hopf algebras* $U(\mathfrak{g})$ related to Lie bialgebra structures on $\mathfrak{g}$.

A *co-Poisson algebra* $\mathcal{A}$ is a cocommutative algebra endowed with a skew map $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ called the *Poisson co-bracket* satisfying a *co-Jacobi identity* and a *co-Leibniz rule* (see [15] for the details). A *co-Poisson-Hopf algebra* is a co-Poisson algebra and a Hopf algebra, the two structures satisfying a compatibility condition.

Now, in analogy with the dual case, if the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ has a co-Poisson structure given by a Poisson co-bracket $\delta$ making it a co-Poisson-Hopf algebra, then there is a natural Lie bialgebra structure on $\mathfrak{g}$ given evaluating the co-bracket in the elements of the algebra; on the contrary, given a a Lie bialgebra structure $\delta \to \mathfrak{g} \otimes \mathfrak{g}$, there is a unique extension of $\delta$ to a Poisson co-bracket on $U(\mathfrak{g})$ that makes $U(\mathfrak{g})$ a co-Poisson-Hopf algebra.

In this case the quantization is performed by taking a cocommutative co-Poisson-Hopf algebra and deforming its dual Poisson bracket (called *Poisson cobracket*) $\delta$ such that

$$\delta(x) \equiv \frac{\Delta_q(a) - \tau \circ \Delta_q(a)}{q} \pmod{q}, \tag{1.54}$$

with $a \in \mathcal{A}_q$ such that it reduces to $x \in \mathcal{A}$ modulo $q$.

A quantization of a Lie bialgebra $(\mathfrak{g}, \delta)$ is, then, a quantization $U(\mathfrak{g})_q$ of the universal enveloping algebra $U(\mathfrak{g})$ equipped with a co-Poisson-Hopf structure.

The procedure is subject to an existence theorem by Drinfel'd that states the following. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $r$ a skew-symmetric classical $r$-matrix satisfying the CYBE; then there exists a deformation $U(\mathfrak{g})_q$ of $U(\mathfrak{g})$, such that its quasiclassical limit is $\mathfrak{g}$ with a Lie bialgebra structure given by $r$. The proof of this theorem entails an explicit construction of the quantum group via a so-called *twist operator*.

To discuss these constructions we outline in the following Section an explicit procedure to deform by quantization Poisson algebras.

## 1.6 Deformation of Poisson algebras

We present, now, an explicit example of quantization seen as deformation. In Subsection 1.6.1 we start with a well-known noncommutative space, the QM phase space, showing a procedure to obtain it starting from the classical phase space and deforming its Poisson algebra introducing a noncommutative product called *Moyal*

*$\star$-product.* In Subsection 1.6.2 we generalize the results obtained to a generic case of Poisson algebra deformation, employing a procedure called *quantization deformation.* Finally, in Subsection 1.6.3 we introduce the notion of a Drinfel'd twist operator and see how it can be used to define $\star$-products and deformation of universal enveloping Hopf algebras.

### 1.6.1 QM phase space and Moyal $\star$-product

To begin the discussion let us start with a well-known example of a noncommutative space, the QM phase space. This, in the 3D case, is the 6D space generated by the operators $(q^i, p_i)$, $i = 1, 2, 3$, endowed with the commutation relations

$$[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i{}_j, \quad i, j = 1, 2, 3. \tag{1.55}$$

The canonical quantization, leading from the classical phase space to the QM phase space, is realized through a map that sends classical commutative functions of commutative coordinates to quantum operatorial functions of operatorial coordinates, called the *Weyl map* or *quantizer* $\Omega : f(q^i, p_j) \to \hat{F}(\hat{q}^i, \hat{p}_j)$. The explicit action of the map on monomials $q_j^m p_l^n$ is

$$q_j^m p_l^n \to \Omega(q_j^m p_l^n) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\hat{p}_l^{n-k}\hat{q}_j^m\hat{p}_l^k, \tag{1.56}$$

and the resulting operators are ordered as in the *symmetrical ordering* prescription. (1.56) can be restated in differential form (see [1]) employing the *normal ordering* prescription (consisting in putting every $q$ to the left and $p$ to the right, and denoted by $\mathcal{N}$) by means of

$$\Omega(q_j^m p_l^n) = \mathcal{N}e^{-\frac{i}{2}\hbar\frac{\partial^2}{\partial q_j \partial p_l}} q_j^m p_l^n|_{q\to\hat{q}, p\to\hat{p}}, \tag{1.57}$$

and reordering the resulting terms as in the symmetrical ordering (for a review on the connections between different ordering prescriptions and Weyl maps see [44]). Generalizing to a generic function of the phase space and applying the Fourier transform one obtains the following differential and integral expressions:

$$\Omega(f(q,p)) = \mathcal{N}e^{-\frac{i}{2}\hbar\frac{\partial^2}{\partial q \partial p}} f(q,p)|_{q\to\hat{q}, p\to\hat{p}}, \tag{1.58a}$$

$$\Omega(f(q,p)) = \frac{1}{2\pi}\int d^3\alpha d^3\beta \tilde{f}(\alpha,\beta)e^{\alpha\cdot\hat{q}+\beta\cdot\hat{p}}, \tag{1.58b}$$

where $\tilde{f}(\alpha,\beta) = \int d^3q d^3p f(q,p)e^{-i(\alpha q+\beta p)}$ is the Fourier transform of $f$ and $\alpha$, $\beta$ are the Fourier-conjugate variables to $q$ and $p$. The scalar product notation stands for $\alpha \cdot \hat{q} = \sum_i \alpha_i \hat{q}_i$.

It is now possible to define the inverse map of $\Omega$, called the *Wigner map* or *dequantizer* $\Omega^{-1}$ [40]:

$$\Omega^{-1}(\hat{F}(\hat{q},\hat{p})) = \int \frac{d^3\alpha d^3\beta}{(2\pi)^2\hbar} e^{-i(\alpha q+\beta p)} Tr\{\hat{F} e^{-i(\alpha\cdot\hat{q}+\beta\cdot\hat{p})}\}. \tag{1.59}$$

This map associates real functions to operatorial functions, provided their Fourier transform is well-defined, and a noncommutative product in the algebra of functions to the noncommutative operatorial product. It is possible, in fact, to define a new product upon functions on the phase space in terms of these maps, the so-called *Moyal $\star$-product*:

$$f \star g \doteq \Omega^{-1}(\Omega(f)\Omega(g)). \tag{1.60}$$

It is straightforward to see that this new product is associative but noncommutative.

There are several integral expressions for the $\star$-product; here we will use the following, obtained by means of the Baker-Campbell-Hausdorff formula and inverse Fourier transform (for the explicit derivation see [1]):

$$(f \star g)(x) = \frac{1}{2\pi} \int d^3s d^3t e^{isx} e^{-\frac{i\hbar}{2}sJt} \tilde{f}(s-t)\tilde{g}(t), \tag{1.61}$$

where $s = (\alpha_i, \beta_i)$, $t = (\alpha'_i, \beta'_i)$ (primed quantities refers to $g$), $x = (q^i, p_i)$ and $sJt = s_i J_{ij} t_j$ with $J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$.

At this point eq.(1.61) can be rewritten in differential form as

$$(f \star g)(q,p) = f(q,p) e^{\frac{i\hbar}{2}\overleftarrow{\partial_i}\Lambda^{ij}\overrightarrow{\partial_j}} g(q,p), \tag{1.62}$$

where the arrow notation indicates on which side the differential operator acts and $\Lambda^{ij}$ is the Poisson tensor defining the Poisson brackets:

$$f\overleftarrow{\partial_i}\Lambda^{ij}\overrightarrow{\partial_j} g = \{f,g\} = \partial_q f \partial_p g - \partial_p f \partial_q g \tag{1.63}$$

in the Darboux basis.

At this point applying (1.62) to $q$ and $p$ it is easy to see that the $\star$-product reproduces the commutator (1.55):

$$q^i \star p_j - p_j \star q^i = i\hbar\delta^i{}_j. \tag{1.64}$$

In this picture we have shifted the perspective from the operator's to the product's noncommutativity, obtaining a new equivalent description of phase space relations.

A naive adaptation of the Moyal product to the canonical 4D noncommutative spacetime case $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ would lead to

$$(f \star g)(x) = \frac{1}{(2\pi)^4} \int d^4s d^4t e^{-is_\mu x^\mu} e^{-\frac{i}{2}s_\mu\theta^{\mu\nu}t_\nu} \tilde{f}(s-t)\tilde{g}(t). \tag{1.65}$$

The differential expansion

$$(f \star g)(x) = e^{\frac{i}{2}\theta_{\mu\nu}\partial_y^\mu \partial_z^\nu} f(y)g(z)|_{y=z=x}, \tag{1.66}$$

give rise to the correct commutators

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}. \tag{1.67}$$

The introduction via the Weyl map of deformed Fourier transforms and noncommutative exponentials allows one to define noncommutative plane waves, in analogy with the classical case. This is the basis of the development of field theories in noncommutative spacetimes (for a review in the $\kappa$-Minkowski case see [37] and [1]).

### 1.6.2 Deformation quantization

We are now ready to introduce a more general approach to obtain noncommutative products as the Moyal $\star$-product through deformations of classical commutative products, the so-called *deformation quantization* or *$\star$-quantization.*

Let us start with the example of eq.(1.64). Defining the *$\star$-commutator* or *Moyal bracket* $[f,g]_\star \doteq f \star g - g \star f$ and recalling (1.62):

$$\begin{aligned}
[f,g]_\star &= f(q,p)e^{\frac{i\hbar}{2}\overleftarrow{\partial_i}\Lambda^{ij}\overrightarrow{\partial_j}}g(q,p) - g(q,p)e^{\frac{i\hbar}{2}\overleftarrow{\partial_i}\Lambda^{ij}\overrightarrow{\partial_j}}f(q,p) \approx \\
&\approx i\frac{\hbar}{2}(f\overleftarrow{\partial_i}\Lambda^{ij}\overrightarrow{\partial_j}g - g\overleftarrow{\partial_i}\Lambda^{ij}\overrightarrow{\partial_j}f) + O(\hbar^2) = \\
&= i\hbar\{f,g\} + O(\hbar^2),
\end{aligned} \tag{1.68}$$

one notices that the $\star$-commutator reduces to the ordinary Poisson bracket at first order in the deformation parameter $\hbar$. This can analogously be seen by expanding directly (1.62):

$$f \star g = fg + i\hbar\{f,g\} + O(\hbar^2), \tag{1.69}$$

that shows explicitly that the Moyal $\star$-product is a deformation of the classical commutative product.

To reconnect ourselves to the discussion carried on in the previous Sections, we note that this $\star$-product is a *Poisson algebra deformation*

$$\frac{f \star g - g \star f}{q} \equiv \{f,g\} \pmod{q}, \tag{1.70}$$

with deformation parameter $q = i\hbar$. Comparing (1.70) with (1.53), it is easy to see that this is a particular case of Hopf algebra deformations.

In this sense it is possible to show that quantum mechanics is a $\star$-deformation of classical mechanics, induced by the deformation parameter $\hbar$. Let us recall the Poisson time-evolution equation for an observable $f$, with $H$ the Hamiltonian function:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f,H\}. \tag{1.71}$$

The quantum analog for the operators $\hat{f}$, $\hat{H}$, obtained through the Weyl quantization procedure is the Heisenberg evolution equation:

$$\frac{d\hat{f}}{dt} = \frac{\partial \hat{f}}{\partial t} + i\frac{[\hat{f}, \hat{H}]}{\hbar}. \tag{1.72}$$

In analogy with the discussion made for QM phase space relations, eq.(1.72) can be restated in terms of commutative functions equipped with a $\star$-product:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{i}{\hbar}[f, H]_\star = \frac{\partial f}{\partial t} + \frac{i}{\hbar}(f \star H - H \star f) \approx \frac{\partial f}{\partial t} + \{f, H\} + O(\hbar^2), \tag{1.73}$$

and therefore, taking the classical limit $\hbar \to 0$, classical mechanics is recovered.

It has been proved by Kontsevich [35] that given a Poisson manifold (i.e. a smooth manifold equipped with a Poisson bracket) it is always possible to introduce a $\star$-product that quantizes the Poisson structure. This, in particular, means that all classical systems defined by a Poisson bracket can be "quantized" in the sense of the example shown above.

In the canonical noncommutative case $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ we can take a Poisson structure of the kind

$$\{x^\mu, x^\nu\} = \theta^{\mu\nu}, \tag{1.74}$$

leading to a $\star$-deformation

$$[f(x), g(x)]_\star = i\{f, g\} + O(\theta^3), \tag{1.75}$$

that reproduces the wanted result

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}. \tag{1.76}$$

In particular for the $\kappa$-Minkowski case there is a Poisson manifold structure given by

$$\{f, g\} = \sum_i x^i \left( \frac{\partial}{\partial x^0} f \frac{\partial}{\partial x^i} g - \frac{\partial}{\partial x^i} f \frac{\partial}{\partial x^0} g \right), \tag{1.77}$$

that allows $\star$-products of the type

$$[f, g]_\star = i\lambda\{f, g\} + O(\lambda^2), \tag{1.78}$$

leading to commutators

$$[x^0, x^i]_\star = i\lambda x^i, \tag{1.79a}$$

$$[x^i, x^j]_\star = 0, \tag{1.79b}$$

that retains the same form of (1.27a-b) (remember that in this case $x^\mu$ are coordinate functions, while in (1.27a-b) they were self-adjoint operators). Note that in this case (1.78) gives rise to a multitude of different $\star$-products that are equal up to the first order in $\lambda$ but in general differ at highest orders. A deep analysis in this regard is conducted for example in [1], where the notion of *Weyl systems* is discussed, leading to the construction of various $\star$-products. Two relevant observations in this context are that it is possible to relate different $\star$-products with different ordering prescriptions and to different representations of $\kappa$-Poincaré translation generators.

### 1.6.3 Drinfel'd twist

In the previous Subsections we have seen how a possible way to "quantize" a space (in the general sense of giving it a noncommutative structure) is through the introduction of a noncommutative $\star$-product depending on a deformation parameter. The main requirements are that this product must reproduce the commutation relations of the spacetime and must reduce to the standard commutative product in the limit of the deformation parameter going to 0.

Consider now the Moyal $\star$-product (1.66); we want to implement such a structure starting from the usual classic commutative product and turn our attention to the universal enveloping algebra deformation. A way to do so is to define the *Drinfel'd twist* [8]. Given a Lie algebra $\mathfrak{g}$ defined on $\mathcal{M}$ and $U(\mathfrak{g})$ its universal enveloping algebra, promoted to a Hopf algebra via the procedure (1.45a-e), the *Drinfel'd twist* $\mathcal{F}$ is an invertible map $\mathcal{F} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ with action

$$\mathcal{F} : \mathcal{C}(\mathcal{M}) \otimes \mathcal{C}(\mathcal{M}) \to \mathcal{C}(\mathcal{M}) \otimes \mathcal{C}(\mathcal{M}), \tag{1.80}$$

that satisfies the following cocycle condition [7]

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}, \tag{1.81}$$

and the following normalization condition [50]

$$(\varepsilon \otimes 1)\mathcal{F} = (1 \otimes \varepsilon)\mathcal{F} = 1. \tag{1.82}$$

In terms of (1.80) it is possible to write

$$f \star g = \mu_\star(f \otimes g) \doteq \mu \circ \mathcal{F}^{-1}(f \otimes g), \qquad f, g \in \mathcal{C}(\mathcal{M}); \tag{1.83}$$

in this way $\mathcal{F}$ realizes the notion of the deformation $\mu_\star$ of the classical commutative product $\mu : \mathcal{C}(\mathcal{M}) \otimes \mathcal{C}(\mathcal{M}) \to \mathcal{C}(\mathcal{M})$. The cocycle and the normalization conditions imply that the $\star$-product is associative and the existence of the neutral element 1: $f \star 1 = 1 \star f = f$.

Being $\mathcal{F} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$, we can pose

$$\mathcal{F} = \mathbf{f}^\alpha \otimes \mathbf{f}_\alpha, \qquad \mathcal{F}^{-1} = \bar{\mathbf{f}}^\alpha \otimes \bar{\mathbf{f}}_\alpha, \tag{1.84}$$

with $\alpha$ a multi-index; in this notation

$$f \star g = \bar{\mathbf{f}}^\alpha(f)\bar{\mathbf{f}}_\alpha(g). \tag{1.85}$$

In our case (1.66) such a twist is of the form

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}}, \qquad \mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}}, \tag{1.86}$$

and explicitly in notation (1.84)

$$\mathcal{F}^{-1} = \sum \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1} \cdots \theta^{\mu_n\nu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} \otimes \partial_{\nu_1} \cdots \partial_{\nu_n}. \tag{1.87}$$

In the example above we have employed the notion of the twist to make the pointwise product of the algebra of functions noncommutative, i.e. a $\star$-product. The Drinfel'd twist, however, allows one to deform more general bilinear maps. Some examples (all discussed in [7] and [8]) are the product between vector fields, 1-form fields, tensor fields, exterior forms and Lie-derivatives (that allow for the definition of a deformed differential calculus), and so on. Other important cases in which this procedure can be employed are, for instance, the $\star$-deformation of classical mechanics and field theories, as discussed in [8], and the $\star$-deformation of differential geometry and General Relativity, as shown in [7].

Another important feature of the Drinfel'd twist is its connection with the classical $r$-matrix if the Hopf algebra is cocommutative. To show this let us start with the canonical spacetime case. Eq.(1.68) can be rewritten in terms of the Sklyanin bracket (1.52) as

$$[f, g]_\star \approx \mu \circ [r, f \otimes g] + O(\theta^3). \tag{1.88}$$

The connection between the Moyal bracket and the twist is given by means of (1.83) by

$$[f, g]_\star = \mu \circ \mathcal{F}^{-1}(f \otimes g) - \mu \circ \mathcal{F}^{-1}(g \otimes f). \tag{1.89}$$

Approximating (1.89) to the first order in $\theta$ and equating (1.88),(1.89):

$$\mu \circ [r, f \otimes g] = \mu \circ \mathcal{F}^{-1}_{(I)}(f \otimes g) - \mu \circ \mathcal{F}^{-1}_{(I)}(g \otimes f) \tag{1.90}$$

where the index $(I)$ means the first order. Following Drinfel'd [26], defining $\mathcal{F}_{21} = \mathbf{f}_\alpha \otimes \mathbf{f}^\alpha$, one obtains that $ir = \mathcal{F}^{-1}_{(I)} - \mathcal{F}^{-1}_{21(I)}$. In our canonical $\theta^{\mu\nu}$-noncommutative case since by the commutator antisymmetry $\theta^{\mu\nu}$ must be antisymmetric, $\mathcal{F}_{21} = \mathcal{F}^{-1}$; this will be true also in the $\varrho$-deformation case, since as we will see the twist is antisymmetrical also in that case, containing a wedge product. Furthermore, since in our cases the twists are exponentials (this is true in general for every Lie algebra-type Minkowski noncommutative spacetime as seen in Appendix B): $\mathcal{F}^{-1}_{21(I)} = \mathcal{F}_{(I)} = -\mathcal{F}^{-1}_{(I)}$. Twists of the form we are studying are called *abelian twists*, and since $r = \mathcal{F}^{-1}_{(I)} - \mathcal{F}^{-1}_{21(I)} = 2\mathcal{F}^{-1}_{(I)}$, the classical $r$-matrix can be obtained by multiplying by 2 the first order of the twist expansion:

$$\mathcal{F}^{-1} \approx 1 \otimes 1 + \frac{1}{2} r + \ldots \tag{1.91}$$

Another way to see the connection is by introducing the *universal $\mathcal{R}$-matrix*

$$\mathcal{R} \doteq \mathcal{F}_{21}\mathcal{F}^{-1}. \tag{1.92}$$

The meaning of $\mathcal{R}$ is to give a natural representation of the permutation group, as shown in [8]. In our cases (1.92) reduces to

$$\mathcal{R} = \mathcal{F}^{-2}; \tag{1.93}$$

Furthermore, it is possible to demonstrate that expanding $\mathcal{R}$ in the deformation parameter, the classical $r$-matrix is exactly its linear term:

$$\mathcal{R} \approx 1 \otimes 1 + r + O(\theta^2). \tag{1.94}$$

Note that in the expansion (1.94) we have written $r$ instead of $r\theta$ following our convention to incorporate the deformation parameter in the classical $r$-matrix. From (1.93) and (1.94) one recovers easily the result (1.91).

Relevant for our discussion is at this point the question of the deformation of the enveloping algebra $U(\mathfrak{g})$ for a Lie-algebra-type noncommutative spacetime. Starting from the Drinfel'd twist there are two ways to achieve our desired result [8]. The first is to take the Lie algebra $\mathfrak{g}$, $\star$-deforming its commutators obtaining a $\star$-Lie algebra $\mathfrak{g}_\star$ (with a $\star$-deformed Leibniz rule and so a deformed coproduct) and considering its universal enveloping algebra $U(\mathfrak{g})_\star$. A second possible way is to take $\mathfrak{g}$, considering its universal enveloping algebra $U(\mathfrak{g})$ and deforming it keeping its algebra sector undeformed and twisting the coproduct (see Appendix B for further details) as follows:

$$\Delta_{\mathcal{F}} = \mathcal{F} \Delta \mathcal{F}^{-1}. \tag{1.95}$$

In this way one obtains a twisted Hopf algebra $U(\mathfrak{g})_{\mathcal{F}}$ isomorphic to $U(\mathfrak{g})_\star$, and therefore the choice between the two methods is a matter of convenience. One of the possible reasons to choose the "$\mathcal{F}$-case" is that, being undeformed the Lie sector, the Casimir operators are undeformed as well and so the Wigner classification of unitary irreducible representations (and so the particle classification) still holds [49]. This construction was applied to the Poincaré case for the first time in [14]. In the "$\star$-case", in turn, vector fields still have the geometric meaning of infinitesimal generators and the $\star$-Lie derivative is linear [8].

We can, now, reconnect to the discussion carried on in Section 1.5. We said, there, that a quantization $U(\mathfrak{g})_q$ of $U(\mathfrak{g})$ can be given in terms of twist operators. It is, in fact, possible to prove that the Hopf algebra $U(\mathfrak{g})_{\mathcal{F}}$ obtained deforming the coproduct as in (1.95) (and in general the antipode as in (B.7), although it is irrelevant for our cases) is a quantization of the Lie bialgebra $(\mathfrak{g}, \delta)$ if the twist satisfies the following properties:

$$\mathcal{F} \equiv 1 \otimes 1 \qquad \text{(mod q)}, \tag{1.96a}$$

$$\frac{\mathcal{F} - 1 \otimes 1}{q} \equiv -\frac{1}{2} r \qquad \text{(mod q)}; \tag{1.96b}$$

this condition is obtained from (1.54) writing the Poisson co-bracket in terms of the classical $r$-matrix and manipulating the left hand side [15]. From (1.96b) it is possible to see that the connection between the twist and the classical $r$-matrix previously demonstrated explicitly for our particular cases is general to the twist method of quantization.

At this point we anticipate that unlike the case of $\varrho$-Poincaré, we will not use this method to derive the Quantum Universal Enveloping Algebra of $\kappa$-Poincaré. To show why, we introduce at first the notion of a *quasitriangular Hopf algebra*. In (1.92) we have defined, via the twist operator, an element called universal $\mathcal{R}$-matrix; this tensor can actually be defined more rigorously for a subclass of Hopf algebras.

An Hopf algebra $\mathcal{H}$ is said to be *quasitriangular* if there exists an element $\mathcal{R} = s_\alpha \otimes s^\alpha \in \mathcal{H} \times \mathcal{H}$ (the *universal $\mathcal{R}$-matrix*) such that

$$(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{1.97a}$$
$$(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{1.97b}$$
$$(\tau \circ \Delta)(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \qquad a \in \mathcal{H}, \tag{1.97c}$$

with

$$\mathcal{R}_{12} = s_\alpha \otimes s^\alpha \otimes 1, \tag{1.98a}$$
$$\mathcal{R}_{23} = 1 \otimes s_\alpha \otimes s^\alpha, \tag{1.98b}$$
$$\mathcal{R}_{13} = s_\alpha \otimes 1 \otimes s^\alpha. \tag{1.98c}$$

If the condition $\mathcal{R}_{12}\mathcal{R}_{21} = 1$ is verified, the Hopf algebra is called *triangular*.

This construction resembles that of the classical $r$-matrix introduced in Section 1.4. We now show that there is an analog to the CYBE also for the universal $\mathcal{R}$-matrix. Let us start with a mindless computation, using the definitions (1.98a-c):

$$(1 \otimes \tau \circ \Delta)(\mathcal{R}) = (id \otimes \tau)(id \otimes \mathcal{R}) = (id \otimes \tau)\mathcal{R}_{13}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{R}_{13}; \tag{1.99}$$

recalling the properties (1.99):

$$\begin{aligned}(1 \otimes \tau \circ \Delta)(\mathcal{R}) &= s_\alpha \otimes \tau \circ \Delta(s^\alpha) = s_\alpha \otimes \mathcal{R}\Delta(s^\alpha)\mathcal{R}^{-1} = \\ &= (1 \otimes \mathcal{R})(s_\alpha \otimes \Delta(s^\alpha))(1 \otimes \mathcal{R}^{-1}) = \\ &= \mathcal{R}_{23}(id \otimes \Delta)\mathcal{R}\mathcal{R}_{23}^{-1} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23}^{-1}.\end{aligned} \tag{1.100}$$

Equating (1.99),(1.100) and multiplying by right by $\mathcal{R}_{23}$ one obtains the *Yang-Baxter equation* (YBE):

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1.101}$$

Another thing to be noted is that for a quasitriangular Hopf algebra, given two representations on tensor products of vectorial spaces $V_1 \times V_2$ and $V_2 \times V_1$, these are isomorphic with one another; in fact the coproduct $\Delta$ of $V_1 \times V_2$ is related to that

$\tau \otimes \Delta$ of $V_2 \times V_1$ by $\mathcal{R}$ through the isomorphism (1.97c). In this sense $\mathcal{R}$ provides a representation of the permutation group.

Given, now, a quasitriangular Hopf algebra with universal $\mathcal{R}$-matrix $\mathcal{R}$ and an admissible twist $\mathcal{F}$ (a twist that satisfies the cocycle condition), it is possible to twist the Hopf algebra into a new one with an $\mathcal{R}$-matrix given by $\tilde{\mathcal{R}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$ that satisfies again the YBE (for the proof see, for example, [22]). Note that in our previous discussion we introduced an $R$-matrix on the twisted Hopf algebra in the form (1.92), i.e. the starting universal $R$-matrix was $\mathcal{R} = 1 \otimes 1$. Since our starting Hopf algebra (1.47a-d) is a trivial extension of the Poincaré Lie algebra, we can always consider condition (1.92) to be valid. While for $\varrho$-Poincaré an admissible twist does exists, in the $\kappa$-case it does not, and we cannot apply our construction.

Taking, now, the quasiclassical limit (first order approximation of (1.94)) one can show that the classical analog of YBE is the CYBE [26]; conversely Etingof and Kazhdan [27] have proven that every classical $r$-matrix satisfying the CYBE can be quantized to a universal $R$-matrix satisfying the YBE, and we have completed the correspondence between "classical" Lie bialgebras and "quantum" Hopf algebras. For "classical" Poisson-Lie groups a YBE is satisfied by an element called *classical R-matrix* $\mathcal{R} : G \times G \to G \times G$, as shown in [15].

Now, following the Drinfel'd existence theorem stated in Section 1.5, the twisted Hopf algebras $U(\mathfrak{p})_{\mathcal{F}}$ (in the cocommutative case) are shown to be quasitriangular Hopf algebras [15, 11]; therefore the classical $r$-matrix of the associated "classical" Lie bialgebra must satisfy the CYBE. While this is the case of $\varrho$-Minkowski (as we will see in eqs.(3.3a-c)), for $\kappa$-Minkowski the $r$-matrix satisfies the MYBE (2.13), so we have recovered from another perspective the stated result that in this latter case the twisting procedure cannot be applied. Some solutions were proposed to overcome this problem, as to enlarge the algebra [11] or to consider a Hopf algebroid instead of the $\kappa$-Poincaré Hopf algebra [34], but they are beyond the scope of this present work.

## 1.7 Bicrossproduct Quantum Groups

Before turning our attention to explicit cases of Minkowski deformations, one last thing to analyze are the aforementioned bicrossproduct Quantum Groups.

In the noncommutative case the analogous of the decomposition of $P$ into a semidirect product of the Lorentz and the translational subgroups is realized through the notion of the *bicrossproduct structure.*

Let us start introducing a *right action* of an Hopf algebra $\mathcal{H}$ on an algebra $\mathcal{A}$ as a linear map $\triangleleft : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A}$ with the properties:

$$a \triangleleft (hg) = (a \triangleleft h) \triangleleft g, \tag{1.102a}$$

$$1 \triangleleft h = \varepsilon(h)1, \tag{1.102b}$$

$h, g \in \mathcal{H}$, $a \in \mathcal{A}$.

The right action is said to be covariant (i.e. preserves the structure of the algebra $\mathcal{A}$) if

$$(a \cdot b) \triangleleft h = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)}), \tag{1.103}$$

where $h \in \mathcal{H}$, $a, b \in \mathcal{A}$ and $h_{(1)}, h_{(2)}$ are defined by the coproduct $\Delta(h)$ in the (sumless) *Sweedler's notation*: $\Delta(h) = \sum_i h_{(1)_i} \otimes h_{(2)_i} = h_{(1)} \otimes h_{(2)}$.

For comparison recall that for a right action of a group $G$ on an algebra $\mathcal{A}$, properties (1.55b) and (1.56) are stated as follows:

$$1 \triangleleft g = 1, \tag{1.104a}$$
$$(a \cdot b) \triangleleft g = (a \triangleleft g)(b \triangleleft g), \tag{1.104b}$$

due to the trivial coalgebra structure.

The introduction of a covariant right action allows one to construct (in our case Poincaré-) invariant products of elements of $\mathcal{A}$.

Dual to the notion of an action there is that of a *coaction*. Given an algebra $\mathcal{A}$ and a coalgebra $\mathcal{C}$, a left coaction $\beta_L : \mathcal{A} \to \mathcal{C} \otimes \mathcal{A}$ is a linear mapping satisfying

$$(id \otimes \beta_L) \circ \beta_L = (\Delta \otimes id) \circ \beta_L \qquad \text{(coassociativity)}, \tag{1.105a}$$
$$(\varepsilon \otimes id) \circ \beta_L = id \qquad \text{(counitality)}. \tag{1.105b}$$

The coaction is said to be covariant if it is an homomorphism:

$$\beta_L(ab) = \beta_L(a)\beta_L(b), \qquad a, b \in \mathcal{A}, \tag{1.106a}$$
$$\beta_L(1) = 1 \otimes 1; \tag{1.106b}$$

in this case it preserves the algebra structure on which it coacts.

Given, now, two Hopf algebras $\mathcal{X}, \mathcal{A}$ a *bicrossproduct algebra* $\mathcal{X} \bowtie \mathcal{A}$ is the tensor product $\mathcal{X} \otimes \mathcal{A}$ endowed with a right action and a left coaction

$$\triangleleft : \mathcal{A} \otimes \mathcal{X} \to \mathcal{A}, \tag{1.107a}$$
$$\beta_L : \mathcal{X} \to \mathcal{A} \otimes \mathcal{X}, \tag{1.107b}$$

and a Hopf algebra structure given by:

$$\mu((x \otimes a), (y \otimes b)) = (x \otimes a) \cdot (y \otimes b) = xy_{(1)} \otimes (a \triangleleft y_{(2)})b, \tag{1.108a}$$
$$1_{\mathcal{X} \bowtie \mathcal{A}} = 1_{\mathcal{X}} \otimes 1_{\mathcal{A}}, \tag{1.108b}$$
$$\Delta(x \otimes a) = (x_{(1)} \otimes x_{(2)}{}^{(\bar{1})} a_{(1)}) \otimes (x_{(2)}{}^{(\bar{2})} \otimes a_{(2)}), \tag{1.108c}$$
$$\varepsilon(x \otimes a) = \varepsilon(x)\varepsilon(a), \tag{1.108d}$$
$$S(x \otimes a) = (1_{\mathcal{X}} \otimes S(x^{(\bar{1})} a)) \cdot (S(x^{(\bar{2})}) \otimes 1_{\mathcal{A}}), \tag{1.108e}$$

where $x, y \in \mathcal{X}$, $a, b \in \mathcal{A}$ and we have defined $\beta(x) = x^{(\bar{1})} \otimes x^{(\bar{2})}$, with $x^{(\bar{1})} \in \mathcal{A}$ and $x^{(\bar{2})} \in \mathcal{X}$. The action and coaction must also satisfy the following compatibility requests:

$$\varepsilon(a \triangleleft x) = \varepsilon(a)\varepsilon(x), \tag{1.109a}$$

$$\Delta(a \triangleleft x) = (a_{(1)} \triangleleft x_{(1)}){x_{(2)}}^{(\bar{1})} \otimes (a_{(2)} \triangleleft {x_{(2)}}^{(\bar{2})}), \tag{1.109b}$$

$$\beta_L(xy) = (x^{(\bar{1})} \triangleleft y_{(1)}){y_{(2)}}^{(\bar{1})} \otimes x^{(\bar{2})}{y_{(2)}}^{(\bar{2})}, \tag{1.109c}$$

$${x_{(1)}}^{(\bar{1})}(a \triangleleft x_{(2)}) \otimes {x_{(1)}}^{(\bar{2})} = (a \triangleleft x_{(1)}){x_{(2)}}^{(\bar{1})} \otimes {x_{(2)}}^{(\bar{2})}. \tag{1.109d}$$

The meaning of the symbol $\cdot \bowtie \cdot$ is that the first factor acts on the second, while the second coacts back on the first.

A bicrossproduct algebra $\mathcal{X} \bowtie \mathcal{A}$ can be viewed as the universal enveloping algebra generated by elements of the type $X = x \otimes 1$, $A = 1 \otimes a$, modulo the commutation relations

$$[X, A] = x \otimes a - x_{(1)} \otimes (a \triangleleft x_{(2)}); \tag{1.110}$$

in fact, by eq.(1.108a)

$$XA = (x \otimes 1) \cdot (1 \otimes a) = x \otimes (1 \triangleleft 1)a = x \otimes a, \tag{1.111a}$$

$$AX = (1 \otimes a) \cdot (x \otimes 1) = x_{(1)} \otimes (a \triangleleft x_{(2)}). \tag{1.111b}$$

In Chapter 2 we will see explicitly that $\kappa$-Poincaré is (in a particular basis) an example of a bicrossproduct Quantum Group and in what sense this structure is the deformed version of a semidirect product. Furthermore, in Subsection 2.3.3 we will make the construction more explicit working in 1+1D.

## 1.8 States, observables and observers

Since the main results presented in this work rely on the notions of states, observables and observers, we state, now, a series of definitions generalizing these concepts to both the commutative and the noncommutative cases that will be useful in the following discussions.

**States.** We start from the definition of functional states, generalizing both the classical and the quantum cases. To do this we start from the Connes picture of Subsection 1.3.1. A *state* $\phi$ is a linear functional from a $C^*$-algebra $\mathcal{C}$ to the complex field [41]:

$$\phi : \mathcal{C} \to \mathbb{C}, \tag{1.112}$$

positive defined

$$\phi(a^*a) \geq 0 \quad \forall a \in \mathcal{C}, \tag{1.113}$$

and normalized

$$\| \phi \| = \sup_{\|a\| \leq 1} \{\phi(a)\} = 1. \tag{1.114}$$

The space of states can be shown to be convex: if $\phi_1$ and $\phi_2$ are two states, then $\forall \lambda \in \mathbb{C}$, $\psi = \cos^2 \lambda \phi_1 + \sin^2 \lambda \phi_2$ is another state. Any state that can be expressed as a convex combination is said to be a *mixed state*, while states that cannot are called *pure states*.

To give a feel of this definition, consider the case of the $C^*$-algebra of $n \times n$ complex-valued matrices $M_n$. A state is given by a matrix $\phi : \phi(a) = Tr\{\phi a\}$, $\forall a \in M_n$. From (1.113) follows that $\phi$ must be self-adjoint and from (1.114) that it must have unit trace. Being self-adjoint $\phi$ can be diagonalized; if more than one eigenvalue is different from zero it can be written as a convex sum of diagonal matrices of trace 1, if instead only one eigenvalue is different from zero (and it is 1) it is not possible to write it in terms of such a sum since (1.113) requires diagonal elements to be positive numbers less than 1. Pure states are, then, projectors and are in a 1 to 1 correspondence with the rays of the space (i.e. normalized n-dimensional vectors). The construction can be extended to the infinite dimensional case considering bounded operators on a separable Hilbert space, and mixed states are now represented by density matrices. This construction works both in the commutative and in the noncommutative case.

As mentioned in Subsection 1.3.1 from a commutative algebra and its set of pure states it is possible to define a topology and thus obtain the associated topological space, expliciting the Connes construction (see [41]). Furthermore, through the GNS construction we can associate the notion of (functional) states to that of *vector states* on a Hilbert space. Given, in fact, an algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, any normalized vector $|\xi\rangle$ defines a state with expectation value $\phi_\xi(a) = \langle \xi | \hat{a} | \xi \rangle$, $\hat{a} \in \mathcal{B}(\mathcal{H})$. On the contrary, to any state $\phi$ it corresponds a vector state $\xi_\phi \in \mathcal{H}$ such that $\langle \xi_\phi | \hat{a} | \xi_\phi \rangle = \phi(a)$. If the variance $\Delta(a) = \sqrt{\phi(a^2) - \phi(a)^2} = \sqrt{\langle \xi_\phi | \hat{a}^2 | \xi_\phi \rangle - (\langle \xi_\phi | \hat{a} | \xi_\phi \rangle)^2} = 0$, the state is said to be *localized*.

**Observables.** An *observable* $\mathcal{A}$ is, heuristically, a physical quantity that can be measured. Formally, in classical mechanics it is defined as a real-valued function on the phase space, while in quantum mechanics as a self-adjoint operator defined on a Hilbert space.

Generalizing the discussion in the spirit of the definition of states we have given, an observable $\mathcal{A}$ is a self-adjoint element of the $C^*$-algebra $\mathcal{C}$. In this way we can say that a state is a mapping from physical observables to their measured value.

**Observers.** The notion of observers is a more subtle one. Heuristically an observer in classical mechanics is something that performs a measure on a physical system and associates a real numerical value to the corresponding observable function; in QM, instead, it is a filter procedure that sends, after having performed a measure on a quantum object, a quantum state to a classical one associating numerical eigenvalues to observable operators with discrete spectra, or continuous density eigenvalues to operators with continuous spectra. Although these definitions are far from being rigorous, in this work we will avoid the problem considering an observer in relation to its reference frame.

An *observer* $O$ is a reference frame with respect to which the ordinary theory of measurement (i.e. the possibility of finding mean values, variances and other higher moments of one or more observables in a state) can be applied. Note that we are making an abuse of terminology, since in general the concepts of observers and reference frames are not identified and it is possible to consider observers located in a different position than the origin of their reference frame, but since for our purposes we can always consider observers located in the origin of their reference frame without loss of generality, the correspondence [observer]$\leftrightarrow$[origin of a reference frame]$\leftrightarrow$[reference frame] can be made without issues. To underline this feature in the following we will often refer to "observers located in the origin of their reference frame" even if the statement is somewhat redundant. One last thing to be noted is that since we are dealing only with special-relativistic theories, not taking into account GR features, when we consider an observer we mean an inertial one.

# Chapter 2

# $\kappa$-deformation

After having introduced the notion of Quantum Group deformations of the ordinary Poincaré group and the structure of noncommutative Minkowski spacetimes, in this chapter we study the archetypical $\kappa$-deformation. In Section 2.1 we start from the algebra of continuous functions over the Poincaré group trying to define the relative Quantum Group, while in Section 2.2 we examine the dual case of the deformed universal enveloping algebra. Finally, in Section 2.3, following and extending the discussion made in [43], the problem of localizability in relation to transformations of observers and observables is analyzed.

## 2.1 Quantum Poincaré Group $\mathcal{C}_\kappa(P)$

We study, now, the Quantum Group $\mathcal{C}_\kappa(P)$[1]. In 2.1.1 we apply a naive reasoning to obtain the algebra structure of this Quantum Group based on the covariance of $\kappa$-Minkowski commutation relations, but it turns out that although this method leads to the correct translational and Lorentz commutation rules, the cross-relations between the two sectors are not fully specified. At this point, in Subsection 2.1.2, we apply the discussion outlined in Section 1.4, finding the entire algebra. Subsection 2.1.3 is devoted to the discussion of the cosector and antipodes of the Quantum Group, while in 2.1.4 the bicrossproduct structure of $\mathcal{C}_\kappa(P)$ is analyzed.

[1] We remark we are using the label $\mathcal{C}_\kappa(P)$ to distinguish between this type of $\kappa$-Poincaré construction and its dual, obtained via the universal enveloping algebra procedure, that will lead to a Quantum Group we will call $U_\kappa(\mathfrak{p})$. Another definition used in literature is that of "Quantum Group" and "Quantum Algebra", that shows the different nature of the two approaches. Although we will use, sometimes, this naming in the following, we remark that also the Quantum Algebra is, ultimately, a Quantum Group.

### 2.1.1 $\kappa$-Poincaré algebra sector

We will, now, apply the previous results to the case study of $\kappa$-Minkowski, whose commutation relations are

$$[x^0, x^i] = i\lambda x^i, \qquad i = 1, ..., n-1, \tag{2.1a}$$
$$[x^i, x^j] = 0, \qquad i, j = 1, ..., n-1. \tag{2.1b}$$

In analogy with the classical case, we would like to define the *$\kappa$-Minkowski space* $\mathcal{M}_\kappa$ as the quotient space, with respect to the Lorentz subgroup, of a Hopf algebra deformation of the Poincaré group called *$\kappa$-Poincaré*. Before doing so, however, we begin our analysis in this Section considering the opposite path, trying to define *$\kappa$-Poincaré* as the simmetry group of *$\kappa$-Minkowski*.

We could think to define, therefore, the *$\kappa$-Poincaré Quantum Group $\mathcal{C}_\kappa(P)$* as the noncommutative Hopf algebra of continuous functions on the Poincaré group that preserve the $\kappa$-Minkowski commutation relations, i.e. the algebra generated by $\{\Lambda^\mu_\nu, a^\mu\}$ that leaves (2.1a-b) invariate under the transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1, \tag{2.2}$$

from $\mathcal{M}_\kappa$ to $\mathcal{C}_\kappa(P) \otimes \mathcal{M}_\kappa$. Note that we are representing $\Lambda^\mu{}_\nu$ and $a^\mu$ as self-adjoint operators on a suitable Hilbert space (for example in Subsection 2.3.3 we will choose $L^2(SO(1,3) \times \mathbb{R}^3)$). Note also that (2.2) has the form of a left-coaction of $\mathcal{C}_\kappa(P)$ on $\mathcal{M}_\kappa$ (we will verify properties (1.105a-b) in Section 2.1.3 after the introduction of a suitable coproduct and counity); to ensure the invariance of relations (2.1a-b), we require, then, (2.2) to be a covariant left-coaction. In other words, recalling (2.1a-b), and since from (1.106a) $\beta_L(ab - ba) = \beta_L(a)\beta_L(b) - \beta_L(b)\beta_L(a)$, we ask that

$$[x'^\mu, x'^\nu] = i\lambda(\delta^\mu{}_0 x'^\nu - \delta^\nu{}_0 x'^\mu). \tag{2.3}$$

At this point, one is lead to think that on imposing eq.(2.3) the full algebra structure of $\mathcal{C}_\kappa(P)$ is recovered. As we are about to show, this is not the case.

$$\begin{aligned}[x'^\mu, x'^\nu] =&[\Lambda^\mu{}_\alpha \otimes x^\alpha + a^\mu \otimes 1, \Lambda^\nu{}_\beta \otimes x^\beta + a^\nu \otimes 1] = \\ =&\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \otimes x^\alpha x^\beta - \Lambda^\nu{}_\beta \Lambda^\mu{}_\alpha \otimes x^\beta x^\alpha + \Lambda^\mu{}_\alpha a^\nu \otimes x^\alpha - a^\nu \Lambda^\mu{}_\alpha \otimes x^\alpha + \\ &+ a^\mu \Lambda^\nu{}_\beta \otimes x^\beta - \Lambda^\nu{}_\beta a^\mu \otimes x^\beta + [a^\mu, a^\nu] \otimes 1.\end{aligned} \tag{2.4}$$

The right-hand side of (2.3) assumes the form

$$i\lambda(\delta^\mu{}_0 x'^\nu - \delta^\nu{}_0 x'^\mu) = i\lambda(\delta^\mu{}_0(\Lambda^\nu{}_\sigma \otimes x^\sigma + a^\nu \otimes 1) - \delta^\nu{}_0(\Lambda^\mu{}_\rho \otimes x^\rho + a^\mu \otimes 1)), \tag{2.5}$$

so that, equating terms at order 0 in $x$, it follows straightforwardly

$$[a^\mu, a^\nu] = i\lambda(\delta^\mu{}_0 a^\nu - \delta^\nu{}_0 a^\mu), \tag{2.6}$$

and the translational sector, unlike the classical Poincaré group case, does not commute. This will pose, as we will see in discussing observables and reference frames transformations, problems in localizability of $\kappa$-Poincaré transformed observables. Note, also, that this algebra relation is isomorphic (if not the same) to the $\kappa$-Minkowski algebra, a feature we will discuss in the following construction of $\mathcal{M}_\kappa$ from $\mathcal{C}_\kappa(P)$. Taking the limit $\lambda \to 0$ we recover the classical commutativity we expected.

Consider, now, terms in $\Lambda\Lambda$:

$$\begin{aligned}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta \otimes x^\alpha x^\beta - &\Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha \otimes x^\beta x^\alpha = \\ &= \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta \otimes x^\alpha x^\beta - \Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha \otimes x^\beta x^\alpha + \\ &\quad + \Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha \otimes x^\alpha x^\beta - \Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha \otimes x^\alpha x^\beta = \\ &= [\Lambda^\mu{}_\alpha, \Lambda^\nu{}_\beta] \otimes x^\alpha x^\beta + \Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha \otimes [x^\alpha, x^\beta] = \\ &= [\Lambda^\mu{}_\alpha, \Lambda^\nu{}_\beta] \otimes x^\alpha x^\beta + \Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha \otimes i\lambda(\delta^\alpha{}_0 x^\beta - \delta^\beta{}_0 x^\alpha),\end{aligned} \tag{2.7}$$

from which follows, since (2.3) has no second order terms in $x$ in the right side,

$$[\Lambda^\mu{}_\alpha, \Lambda^\nu{}_\beta] = 0. \tag{2.8}$$

It is important to note that in this construction of the Hopf algebra the Lorentz sector remains undeformed, still having trivial commutators. This, as we will show later, gives the result that upon performing only deformed Lorentz transformations, the uncertainty of an observable does not increase under the (co)action of the group.

Let us equate the remaining terms on the left and right hand side:

$$\begin{aligned}\Lambda^\nu{}_\beta\Lambda^\mu{}_\alpha \otimes i\lambda(\delta^\alpha{}_0 x^\beta - \delta^\beta{}_0 x^\alpha) + \Lambda^\mu{}_\alpha a^\nu \otimes x^\alpha - a^\nu\Lambda^\mu{}_\alpha \otimes x^\alpha + \\ + a^\mu\Lambda^\nu{}_\beta \otimes x^\beta - \Lambda^\nu{}_\beta a^\mu \otimes x^\beta = i\lambda(\delta^\mu{}_0\Lambda^\nu{}_\sigma \otimes x^\sigma - \delta^\nu{}_0\Lambda^\mu{}_\rho \otimes x^\rho).\end{aligned} \tag{2.9}$$

It is easy to see that this last condition does not completely fix the remaining commutators $[\Lambda^\mu_\nu, a^\rho]$:

$$[\Lambda^\mu{}_\alpha, a^\nu] + [a^\mu, \Lambda^\nu{}_\alpha] = i\lambda(\Lambda^\nu{}_0\Lambda^\mu{}_\alpha - \Lambda^\nu{}_\alpha\Lambda^\mu{}_0 + \delta^\mu{}_0\Lambda^\nu{}_\alpha - \delta^\nu{}_0\Lambda^\mu{}_\alpha). \tag{2.10}$$

The reason this construction fails is that relations (2.1a-b) admit more than one single covariance group (see for example the $\kappa$-Galilei group [30]).

### 2.1.2 $\mathcal{C}_\kappa(P)$ algebra structure from $r$-matrix

To fully compute the commutators between coordinate functions of $\mathcal{C}_\kappa(P)$ we now follow the method described in Section 1.4 based upon the introduction of the classical $r$-matrix.

A classical $r$-matrix for $\mathcal{C}_\kappa(P)$ is found to be [38]:

$$r = \frac{i}{\kappa} M_{0\nu} \wedge P^\nu, \tag{2.11}$$

where we have defined $\frac{1}{\kappa} \doteq \lambda$. We remind that according to a common convention we are incorporating the deformation parameter in the definition of the classical $r$-matrix, even if in more formal context (e.g. [15]) this is not always done.

Let us evaluate the Schouten bracket recalling eqs.(1.24a-c):

$$\begin{aligned}
[r_{12}, r_{13}] &= -\frac{1}{\kappa^2}[M_{0\nu}, M_{0\mu}] \wedge P^\nu \wedge P^\mu = \\
&= -\frac{i}{\kappa^2}(g_{0\mu}M_{\nu 0} - g_{\nu\mu}M_{00} + g_{\nu 0}M_{0\mu} - g_{00}M_{\nu\mu}) \wedge P^\nu \wedge P^\mu = \\
&= -\frac{i}{\kappa^2}(M_{\nu 0} \wedge P^\nu \wedge P_0 + M_{0\mu} \wedge P_0 \wedge P^\mu - g_{00}M_{\nu\mu} \wedge P^\nu \wedge P^\mu), \qquad (2.12a)\\
[r_{12}, r_{23}] &= -\frac{1}{\kappa^2}M_{0\nu} \wedge [P^\nu, M_{0\mu}] \wedge P^\mu = \frac{i}{\kappa^2}M_{0\nu} \wedge (\delta^\nu{}_\mu P_0 - \delta^\nu{}_0 P_\mu) \wedge P^\mu = \\
&= \frac{i}{\kappa^2}(M_{0\mu} \wedge P_0 \wedge P^\mu - M_{00} \wedge P_\mu \wedge P^\mu) = \frac{i}{\kappa^2}M_{0\mu} \wedge P_0 \wedge P^\mu, \qquad (2.12b)\\
[r_{13}, r_{23}] &= -\frac{1}{\kappa^2}M_{0\nu} \wedge M_{0\mu} \wedge [P^\nu, P^\mu] = 0. \qquad (2.12c)
\end{aligned}$$

Summing the three terms:

$$[[r, r]] = \frac{i}{\kappa^2}g_{00}M_{\nu\mu} \wedge P^\nu \wedge P^\mu - \frac{i}{\kappa^2}M_{\nu 0} \wedge P^\nu \wedge P_0, \tag{2.13}$$

which one can explicitly show to be invariant under the action of $\kappa$-Poincaré algebra generators, satisfying condition (c) of the $r$-matrices definition (eq.(1.48)). Note that this $r$-matrix satisfies a MYBE instead of the CYBE; this feature will be particularly relevant in the discussion of the construction of $U(\mathfrak{p})$, since it tells us that we cannot apply the Drinfel'd twist approach outlined in Subsection 1.6.3.

At this point our goal is to compute explicitly the Sklyanin brackets (1.52) of the elements of the algebra, and to do this we must obtain the Poincaré left- and right-invariant vector fields. We will employ the following standard procedure [55]. Consider for simplicity the D=4 case for Poincaré. At first we consider an ISO(1,3) matrix representation of $P$, given by elements in the following form:

$$g = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}, \tag{2.14}$$

where $\Lambda$ are 4x4 matrices and $a$ 4x1 column vectors; then we want to compute left and right invariant Maurer-Cartan 1-forms, which for a matrix group are given by:

$$\Theta^L = g^{-1}dg, \qquad (2.15a)$$

$$\Theta^R = dgg^{-1}. \qquad (2.15b)$$

Inverting the matrix gives:

$$g^{-1} = \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1}a \\ 0 & 1 \end{pmatrix}, \tag{2.16}$$

and differentiating:

$$dg = \begin{pmatrix} d\Lambda & da \\ 0 & 0 \end{pmatrix}; \tag{2.17}$$

so that the Maurer-Cartan forms are:

$$\Theta^L = \begin{pmatrix} \Lambda^{-1}d\Lambda & \Lambda^{-1}da \\ 0 & 0 \end{pmatrix}, \tag{2.18a}$$

$$\Theta^R = \begin{pmatrix} \Lambda^{-1}d\Lambda & -\Lambda^{-1}ad\Lambda + da \\ 0 & 0 \end{pmatrix}. \tag{2.18b}$$

$\Theta^L$ and $\Theta^R$ can be written in terms of a matricial representation of Lie algebra generators in the following way:

$$\Theta^L = (\Lambda^{-1})_\alpha{}^\lambda d\Lambda_{\lambda\beta} J^{\alpha\beta} + (\Lambda^{-1})_\beta{}^\alpha da_\alpha \mathcal{E}^\beta = \Theta^L_{\alpha\beta} J^{\alpha\beta} + \Theta^L_\beta \mathcal{E}^\beta \tag{2.19a}$$

$$\Theta^R = d\Lambda_{\alpha\lambda}(\Lambda^{-1})^\lambda{}_\beta J^{\alpha\beta} - d\Lambda_\beta{}^\gamma(\Lambda^{-1})_\gamma{}^\alpha a_\alpha \mathcal{E}^\beta + da_\beta \mathcal{E}^\beta = \Theta^R_{\alpha\beta} J^{\alpha\beta} + \Theta^R_\beta \mathcal{E}^\beta; \tag{2.19b}$$

here greek letters are 4-indices running from 0 to 3, and contractions of the type $\Theta_\beta \mathcal{E}^\beta$ stand for $\Theta_{\beta 4}\mathcal{E}^{\beta 4}$, while the generators are

$$J^{12} = -J^{21} = i\begin{pmatrix} 0&0&0&0&0 \\ 0&0&-1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \quad J^{31} = -J^{13} = i\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&-1&0&0&0 \\ 0&0&0&0&0 \end{pmatrix},$$

$$J^{23} = -J^{32} = i\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&-1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \quad J^{01} = -J^{10} = i\begin{pmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix},$$

$$J^{02} = -J^{20} = i\begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \quad J^{03} = -J^{30} = i\begin{pmatrix} 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \tag{2.20}$$

$$\mathcal{E}^0 = \begin{pmatrix} 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \quad \mathcal{E}^1 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \quad \mathcal{E}^2 = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix},$$

$$\mathcal{E}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

We can now write down the most general form of invariant vector fields in the $\{\Lambda^\alpha{}_\beta, a^\rho\}$ basis:

$$X^\mu = f^{\mu\nu}\frac{\partial}{\partial a^\nu} + f'_\nu \frac{\partial}{\partial \Lambda_{\nu\mu}}, \tag{2.21a}$$

$$X^{\mu\nu} = l^\mu{}_\rho \frac{\partial}{\partial \Lambda_{\rho\nu}} + l'^\nu{}_\rho \frac{\partial}{\partial \Lambda_{\rho\mu}} + h^\mu \frac{\partial}{\partial a_\nu} + h'^\nu \frac{\partial}{\partial a_\mu}, \tag{2.21b}$$

where $f$, $f'$, $l$, $l'$, $h$, $h'$ can be obtained imposing the duality relations:

$$\langle \Theta_\alpha, X^\mu \rangle = \delta^\mu{}_\alpha, \tag{2.22a}$$

$$\langle \Theta_{\alpha\beta}, X^\mu \rangle = 0, \tag{2.22b}$$

$$\langle \Theta_\alpha, X^{\mu\nu} \rangle = 0, \tag{2.22c}$$

$$\langle \Theta_{\alpha\beta}, X^{\mu\nu} \rangle = \delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\nu{}_\alpha \delta^\mu{}_\beta. \tag{2.22d}$$

For the left invariant ones we have:

$$\langle \Theta_\beta, X^\mu \rangle = (\Lambda^{-1})_\beta{}^\alpha f^\mu{}_\alpha = \delta^\mu{}_\beta \to f^\mu{}_\alpha = \Lambda_\alpha{}^\mu, \tag{2.23a}$$

$$\langle \Theta_{\alpha\beta}, X^\mu \rangle = f'_\lambda (\Lambda^{-1})_\alpha{}^\lambda \delta^\mu{}_\beta = 0 \to f'_\lambda = 0, \tag{2.23b}$$

$$\langle \Theta_\beta, X^{\mu\nu} \rangle = (\Lambda^{-1})_\beta{}^\nu h^\mu + (\Lambda^{-1})_\beta{}^\mu h'^\nu = 0 \to h^\mu = h'^\mu = 0 \tag{2.23c}$$

$$\begin{aligned} \langle \Theta_{\alpha\beta}, X^{\mu\nu} \rangle &= (\Lambda^{-1})_\alpha{}^\rho l^\mu{}_\rho \delta^\nu{}_\beta + (\Lambda^{-1})_\alpha{}^\rho l'^\nu{}_\rho \delta^\mu{}_\beta = \delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\nu{}_\alpha \delta^\mu{}_\beta \to \\ &\to -l'^\nu{}_\rho = l^\nu{}_\rho = \Lambda_\rho{}^\nu, \end{aligned} \tag{2.23d}$$

while for the right invariant ones:

$$\langle \Theta_{\alpha\beta}, X^\mu \rangle = (\Lambda^{-1})^\mu{}_\beta f'_\alpha = 0 \to f'_\alpha = 0, \tag{2.24a}$$

$$\langle \Theta_\beta, X^\mu \rangle = -(\Lambda^{-1})^{\mu\alpha} a_\alpha f'_\nu \delta^\mu{}_\beta + f^\mu{}_\beta = \delta^\mu{}_\beta \to f^\mu{}_\beta = \delta^\mu{}_\beta, \tag{2.24b}$$

$$\begin{aligned} \langle \Theta_{\alpha\beta}, X^{\mu\nu} \rangle &= (\Lambda^{-1})^\nu{}_\beta l^\mu{}_\alpha + (\Lambda^{-1})^\mu{}_\beta l'^\nu{}_\alpha = \delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\nu{}_\alpha \delta^\mu{}_\beta \to \\ &\to l'^\nu{}_\alpha = -l^\nu{}_\alpha = \Lambda_\alpha{}^\nu \end{aligned} \tag{2.24c}$$

$$\begin{aligned} \langle \Theta_\beta, X^{\mu\nu} \rangle &= h^\mu \delta^\nu{}_\beta + h'^\nu \delta^\mu{}_\beta - (\Lambda^{-1})^{\nu\alpha} a_\alpha l^\mu{}_\beta - (\Lambda^{-1})^{\mu\alpha} a_\alpha l'^\nu{}_\beta = \\ &= (h^\mu + a^\mu)\delta^\nu{}_\beta + (h'^\nu - a^\nu)\delta^\mu{}_\beta = 0 \to h'^\nu = -h^\nu = a^\nu; \end{aligned} \tag{2.24d}$$

substituting into (2.21a-b) gives:

$$X_L^{\alpha\beta} = \Lambda^{\mu\alpha} \frac{\partial}{\partial \Lambda^\mu{}_\beta} - \Lambda^{\mu\beta} \frac{\partial}{\partial \Lambda^\mu{}_\alpha}, \tag{2.25a}$$

$$X_L^\alpha = \Lambda^{\mu\alpha} \frac{\partial}{\partial a^\mu}, \tag{2.25b}$$

$$X_R^{\alpha\beta} = \Lambda^\beta{}_\nu \frac{\partial}{\partial \Lambda_{\alpha\nu}} - \Lambda^\alpha{}_\nu \frac{\partial}{\partial \Lambda_{\beta\nu}} + a^\beta \frac{\partial}{\partial a_\alpha} - a^\alpha \frac{\partial}{\partial a_\beta}, \tag{2.25c}$$

$$X_R^\alpha = \frac{\partial}{\partial a_\alpha}. \tag{2.25d}$$

Since the Lie algebra of the left- or right- invariant vector fields on a Lie group is isomorphic to the tangent space at identity of the group, the Lie algebra of the group can be identified with the Lie algebra of the invariant vector fields [39]; we can set, therefore, the following relations with the Poincaré Lie algebra generators:

$$M^{\alpha\beta} = iX^{\alpha\beta}, \tag{2.26a}$$

$$P^\alpha = X^\alpha, \tag{2.26b}$$

which enable us to rewrite (1.52) as

$$\begin{aligned}\{f, g\} &= \frac{i}{\kappa}(M^R_{0\nu} \wedge P^{R\nu} - M^L_{0\nu} \wedge P^{L\nu})(df, dg) = \\ &= -\frac{1}{\kappa}(X^R_{0\nu} \wedge X^{R\nu} - X^L_{0\nu} \wedge X^{L\nu})(df, dg).\end{aligned} \tag{2.27}$$

Performing the calculation for $a^\rho$ and $a^\sigma$, omitting terms in $\frac{\partial}{\partial\Lambda}$ which acting on $a$ give zero,

$$\begin{aligned}\{a^\rho, a^\sigma\} &= -\frac{1}{\kappa}\left(a_\nu \frac{\partial}{\partial a^0} - a_0 \frac{\partial}{\partial a^\nu}\right) \wedge \frac{\partial}{\partial a_\nu}(a^\rho, a^\sigma) = \\ &= -\frac{1}{\kappa}[(a_\nu \delta^\rho{}_0 - a_0 \delta^\rho{}_\nu) g^{\sigma\nu} - (a_\nu \delta^\sigma{}_0 - a_0 \delta^\sigma{}_\nu) g^{\rho\nu}] = \\ &= -\frac{1}{\kappa}(a^\sigma \delta^\rho{}_0 - a^\rho \delta^\sigma{}_0);\end{aligned} \tag{2.28}$$

the commutators are obtained via the canonical prescription $\{,\} \to \frac{1}{i}[,]$, and we find the previously stated result (2.6), quantizing the Poisson-Hopf algebra to a deformed one.

A calculation of $\{\Lambda^\alpha{}_\beta, \Lambda^\mu{}_\nu\}$ gives identically 0, since $P^\mu$ does not contain derivatives in $\Lambda$ neither in left nor in right bases; so the result (2.8) comes straightforwardly.

Differently from what we found employing the covariance method, we can now fix the mixed brackets:

$$
\begin{aligned}
\{\Lambda^{\alpha}{}_{\beta}, a^{\rho}\} =& -\frac{1}{\kappa}\left(\Lambda_{\nu\mu}\frac{\partial}{\partial\Lambda^{0}{}_{\mu}} - \Lambda_{0\mu}\frac{\partial}{\partial\Lambda^{\nu}{}_{\mu}} + a_{\nu}\frac{\partial}{\partial a^{0}} - a_{0}\frac{\partial}{\partial a^{\nu}}\right) \wedge \frac{\partial}{\partial a_{\nu}}(\Lambda^{\alpha}{}_{\beta}, a^{\rho}) \\
&+ \frac{1}{\kappa}\left(\Lambda^{\mu}{}_{0}\frac{\partial}{\partial\Lambda^{\mu\nu}} - \Lambda^{\mu}{}_{\nu}\frac{\partial}{\partial\Lambda^{\mu 0}}\right) \wedge \Lambda^{\kappa\nu}\frac{\partial}{\partial a^{\kappa}}(\Lambda^{\alpha}{}_{\beta}, a^{\rho}) = \\
=& -\frac{1}{\kappa}\left(\Lambda_{\nu\mu}\delta^{\alpha}{}_{0}\delta^{\mu}{}_{\beta} - \Lambda_{0\mu}\delta^{\alpha}{}_{\nu}\delta^{\mu}{}_{\beta}\right) g^{\rho\nu} + \\
&+ \frac{1}{\kappa}\left(\Lambda^{\mu}{}_{0}\delta^{\alpha}{}_{\mu}g_{\beta\nu} - \Lambda^{\mu}{}_{\nu}\delta^{\alpha}{}_{\mu}g_{\beta 0}\right) \Lambda^{\kappa\nu}\delta^{\rho}{}_{\kappa} = \\
=& -\frac{1}{\kappa}(\Lambda^{\rho}{}_{\beta}\delta^{\alpha}{}_{0} - \Lambda_{0\beta}g^{\rho\alpha} - \Lambda^{\alpha}{}_{0}\Lambda^{\rho}{}_{\beta} + \underbrace{\Lambda^{\alpha}{}_{\nu}\Lambda^{\rho\nu}}_{\Lambda^{\alpha\mu}\Lambda^{\rho\nu}g_{\mu\nu}=g^{\alpha\rho}} g_{\beta 0}) = \\
=& \frac{1}{\kappa}((\Lambda^{\alpha}{}_{0} - \delta^{\alpha}{}_{0})\Lambda^{\rho}{}_{\beta} + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho}).
\end{aligned}
\tag{2.29}
$$

Considering the commutators[2] we obtain

$$
[\Lambda^{\alpha}{}_{\beta}, a^{\rho}] = -\frac{i}{\kappa}((\Lambda^{\alpha}{}_{0} - \delta^{\alpha}{}_{0})\Lambda^{\rho}{}_{\beta} + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho}). \tag{2.30}
$$

Having completed the algebra structure of $\mathcal{C}_{\kappa}(P)$, we note that in this formulation the Lorentz sector is undeformed, while the translational one and the cross-relations are noncommutative, giving intuitively an increase in uncertainty of transformed observables.

Again, taking $\kappa \to \infty$, the classical Poincaré limit is obtained.

### 2.1.3 $\mathcal{C}_{\kappa}(P)$ Hopf algebra structure

So far we have dealt only with the algebra sector of the Poisson-Hopf algebra $\mathcal{C}_{\kappa}(P)$; we now turn our attention to the coalgebra sector and the definition of the antipode, which complete the picture giving rise to the group laws of $\mathcal{C}_{\kappa}(P)$.

Since we are dealing with a Poisson-Lie structure, the group laws must be compatible with the usual Poisson brackets, which satisfy the undeformed Leibniz rule. Following the reasoning made for example (1.44a-e), the groupal laws are encoded in the cosector and the antipode, so we expect these structures to be undeformed as well.

At first we define the group composition law through a *coproduct* $\Delta : \mathcal{C}_{\kappa}(P) \to \mathcal{C}_{\kappa}(P) \otimes \mathcal{C}_{\kappa}(P)$:

$$
\Delta(a^{\mu}) = \Lambda^{\mu}{}_{\nu} \otimes a^{\nu} + a^{\mu} \otimes 1, \tag{2.31a}
$$

$$
\Delta(\Lambda^{\mu}{}_{\nu}) = \Lambda^{\mu}{}_{\alpha} \otimes \Lambda^{\alpha}{}_{\nu}, \tag{2.31b}
$$

[2]We note that the canonical substitution prescription is ordering-unambiguous due to the commutativity of Λs.

that is the categorical-dual structure of the algebra *product* $\mu : \mathcal{C}_\kappa(P) \otimes \mathcal{C}_\kappa(P) \to \mathcal{C}_\kappa(P)$, and satisfies eq.(1.39a); then we define a *counit* $\varepsilon : \mathcal{C}_\kappa(P) \to \mathbb{C}$:

$$\varepsilon(a^\mu) = 0, \tag{2.32a}$$

$$\varepsilon({\Lambda^\mu}_\nu) = {\delta^\mu}_\nu, \tag{2.32b}$$

the categorical-dual structure to the *unit* $\eta : \mathbb{C} \to \mathcal{C}_\kappa(P)$, that satisfies the *counity* property (1.39b). These two structures define the coalgebra sector. Requiring the homomorphism conditions (1.42a-d) hold gives $(\mathcal{C}_\kappa(P), \mu, \eta, \Delta, \varepsilon)$ the structure of a *bialgebra*.

Consider, now, the $\kappa$-Minkowski algebra (2.1a-b). Applying (2.31a) to $x^\mu$ one obtains $\Delta(x^\mu) = {\Lambda^\mu}_\nu \otimes x^\nu + a^\mu \otimes 1$, the transformation rule (2.2). This is the required left coaction of $\mathcal{C}_\kappa(P)$ on the $\kappa$-Minkowski spacetime $\mathcal{M}_\kappa$. We are, now, ready to demonstrate that (2.2) satisfies the left-coaction properties (1.105a-b). Let us begin with the left hand side of (1.105a):

$$\begin{aligned}(id \otimes \beta_L) \circ \beta_L(x^\mu) &= (id \otimes \beta_L)({\Lambda^\mu}_\nu \otimes x^\nu + a^\mu \otimes 1) = \\ &= {\Lambda^\mu}_\nu \otimes {\Lambda^\nu}_\alpha \otimes x^\alpha + {\Lambda^\mu}_\nu \otimes a^\nu \otimes 1 + a^\mu \otimes 1 \otimes 1,\end{aligned} \tag{2.33}$$

since $\beta_L(1) = 1 \otimes 1$. For the right hand side:

$$\begin{aligned}(\Delta \otimes id) \circ \beta_L(x^\mu) &= (\Delta \otimes id)({\Lambda^\mu}_\nu \otimes x^\nu + a^\mu \otimes 1) = \\ &= {\Lambda^\mu}_\alpha \otimes {\Lambda^\alpha}_\nu \otimes x^\nu + {\Lambda^\mu}_\nu \otimes a^\nu \otimes 1 + a^\mu \otimes 1 \otimes 1,\end{aligned} \tag{2.34}$$

therefore (1.105a) is satisfied. For (1.105b):

$$(\varepsilon \otimes id) \circ \beta_L(x^\mu) = (\varepsilon \otimes id)({\Lambda^\mu}_\nu \otimes x^\nu + a^\mu \otimes 1) = {\delta^\mu}_\nu x^\nu = x^\mu = id(x^\mu), \tag{2.35}$$

and thus (2.2) is a left-coaction. It is also covariant since the commutators between $\Lambda$'s and $a$'s are compatible with those obtained in Section 2.1.1 imposing the covariance of $\kappa$-Minkowski relations.

To complete the picture we must introduce an object in terms of which we can define the group inversion law; such an object is the *antipode map* $S : \mathcal{C}_\kappa(P) \to \mathcal{C}_\kappa(P)$:

$$S(a^\mu) = -a^\nu {(\Lambda^{-1})^\mu}_\nu, \tag{2.36a}$$

$$S({\Lambda^\mu}_\nu) = {(\Lambda^{-1})^\mu}_\nu, \tag{2.36b}$$

with the properties (1.43a-e).

Comparing (2.36a-b) with (2.16) shows explicitly that the antipode provides the group inversion.

The structure $(\mathcal{C}_\kappa(P), \mu, \eta, \Delta, \varepsilon, S)$ with the properties here defined is a (deformed) *Hopf algebra*, and in particular the *$\kappa$-Poincaré Quantum Group*.

### 2.1.4 Bicrossproduct structure and $\mathcal{M}_\kappa$ derivation

Now that we have defined the $\kappa$-Poincaré Quantum Group as a covariance group of $\kappa$-Minkowski, we are interested in seeing if it is possible to reverse the reasoning, defining $\kappa$-Minkowski from $\kappa$-Poincaré, to complete a picture analogous to that of the classical case, where the Minkowski spacetime could be viewed as the homogeneous $P$-space with respect to the Lorentz subgroup.

It has been shown (see [48, 51]) that the Quantum Universal Enveloping Algebra of the Poincaré group assumes different forms based upon the chosen basis of the generators of the Hopf algebra, related one to another by taking nonlinear combinations of generators, as we will see in the next Section. This reflects in the dual Quantum Group $\mathcal{C}_\kappa(P)$ as can be seen noting that commutation relations (2.1a-b) are nonlinear, and so nonlinear changes of coordinates are allowed.

It has been shown that in a particular basis (called the "Majid-Ruegg basis", while the results so far obtained are in the so-called "standard basis") $\kappa$-Poincaré assumes a bicrossproduct structure, namely

$$\mathcal{C}_\kappa(P) = T^* \triangleright\!\blacktriangleleft \mathcal{C}(SO(1,3)), \tag{2.37}$$

where $C(SO(1,3))$ is the classical commutative algebra of continuous functions on the Lorentz group and $T^*$ the algebra of functions on the dual of the translational sector, defined in this basis by

$$[a^\mu, a^\nu] = i\lambda(\delta^\mu{}_0 a^\nu - \delta^\nu{}_0 a^\mu), \tag{2.38a}$$

$$\Delta(a^\mu) = a^\mu \otimes 1 + 1 \otimes a^\mu, \tag{2.38b}$$

$$S(a^\mu) = -a^\mu, \tag{2.38c}$$

$$\varepsilon(a^\mu) = 0. \tag{2.38d}$$

The left coaction and the right action are given by

$$\beta_L(x^\mu) = \Lambda^\mu{}_\nu \otimes x^\nu, \tag{2.39a}$$

$$\Lambda^\alpha{}_\beta \triangleleft x^\varrho = -\frac{i}{\kappa}((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho}), \tag{2.39b}$$

where the right action is given by commutators, as can be seen comparing (2.39a-b) with (2.30), and the left coaction is connected with the coproduct, as discussed in [51].

Recalling now the classical Minkowski space construction of Section 1.2, it is now clear why the commutation relations (2.6) have the same form of the $\kappa$-Minkowski commutators (2.1a-b)[3]: $T^* \sim \mathcal{M}_\kappa$, and we finally have recovered the noncommutative space starting from the deformation of the group. We will make this construction more explicit in Subsection 2.3.3 where we will introduce a convenient representation of the group algebra.

[3] In the Majid-Ruegg basis the classical $r$ matrix is $r = \frac{i}{\kappa} M_{0\nu} \otimes P^\nu$ [48] and commutators retain the same form as in the standard basis case.

## 2.2 Quantum Universal Enveloping Algebra $U_\kappa(\mathfrak{p})$

We are now interested in analyzing the dual (in the sense of (1.46a-e)) of $\mathcal{C}_\kappa(P)$ obtained in Section 2.1, namely the Quantum Group $U_\kappa(\mathfrak{p})$ obtained through the deformation of the universal enveloping Hopf algebra of Poincaré. In Subsection 2.2.1 we outline the procedure employed to find its structure, and in 2.2.2 we show how it is possible to give it a bicrossproduct structure working in a particular "generator basis".

### 2.2.1 $U_\kappa(\mathfrak{p})$ construction

To deform the universal enveloping algebra the first attempt we could make is that of working with a Drinfel'd twist. However, since in our case the classical $r$-matrix satisfies the MYBE (2.13) instead of the CYBE, we must follow another way to quantize the algebra.

We briefly outline the discussion carried on in [47] that leads to a common definition of the $\kappa$-Poincaré Quantum Universal Enveloping Algebra. At first we consider the *anti-De Sitter algebra* $\mathfrak{o}(3,2)$ instead of the Poincaré one, in order to overcome the problem of the latter not being semisimple[4]. Then we consider the universal enveloping algebra $U(\mathfrak{o}(3,2))$ and we deform it via the introduction of a parameter $q$, leading to the algebra $U_q(\mathfrak{o}(3,2))$. At this point we perform the limits

$$R \to \infty, \quad iR\log(q) \to \kappa^{-1}, \quad 0 < \kappa < \infty, \tag{2.40}$$

where $R$ is a real parameter of the $\mathfrak{o}(3,2)$ algebra interpreted in the classical spacetime framework as the De Sitter Radius[5]. We notice in the limit $q \to 1$ the appearance of the mass parameter $\kappa = \frac{1}{\lambda}$.

In this way one has obtained the Quantum Universal Enveloping Algebra $U_\kappa(\mathfrak{p})$ in the so-called "standard basis" [48], with algebra and coalgebra sectors and antipodes given by:

$$[P_\mu, P_\nu] = 0, \tag{2.41a}$$
$$[M_j, P_0] = 0, \tag{2.41b}$$
$$[M_j, P_k] = i\epsilon_{jkl}P_l, \tag{2.41c}$$
$$[N_j, P_0] = iP_j, \tag{2.41d}$$

[4]Note that our approach starts again with a spacetime and its group of simmetries, while later we will be dealing with the opposite reasoning, obtaining the spacetime from the bicrossproduct structure of its quantum group.

[5]This procedure is made in analogy with the classical case, where the Poincaré algebra can be obtained via an İnönü-Wigner contraction from the anti-De Sitter one in the limit $R \to \infty$ [9], and the Minkowski spacetime is seen as a "flat limit" of the anti-De Sitter one, sending to infinity the curvature radius $R$.

$$
\begin{aligned}
[N_j, P_k] =& i\delta_{jk}\frac{\sinh(\lambda P_0)}{\lambda}, && \text{(2.41e)}\\
[M_j, M_k] =& i\epsilon_{jkl}M_l, && \text{(2.41f)}\\
[M_j, N_k] =& i\epsilon_{jkl}N_l, && \text{(2.41g)}\\
[N_j, N_k] =& -i\epsilon_{jkl}(M_l\cosh(\lambda P_0) - \frac{\lambda^2}{4}P_lP_iM^i), && \text{(2.41h)}\\
\Delta P_0 =& P_0\otimes 1 + 1\otimes P_0, && \text{(2.41i)}\\
\Delta P_j =& P_j\otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}}\otimes P_j, && \text{(2.41j)}\\
\Delta M_j =& M_j\otimes 1 + 1\otimes M_j, && \text{(2.41k)}\\
\Delta N_j =& N_j\otimes e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}}\otimes N_j +\\
&+\frac{\lambda}{2}\epsilon_{jkl}(P_k\otimes M_l e^{\frac{\lambda P_0}{2}} + e^{-\frac{\lambda P_0}{2}}M_k\otimes P_l), && \text{(2.41l)}\\
S(P_\mu) =& -P_\mu, && \text{(2.41m)}\\
S(M_j) =& -M_j, && \text{(2.41n)}\\
S(N_j) =& -N_j + \frac{3}{2}i\lambda P_j, && \text{(2.41o)}
\end{aligned}
$$

where $P_\mu$ are the momenta generators, $M_j \doteq {\epsilon_j}^{kl}M_{kl}$ the rotations generators and $N_j \doteq M_{0j}$ the boosts generators. All the counits are 0.

A few things are to be noted. Firstly, in the limit $\kappa \to \infty$ ($\lambda \to 0$) the usual Poincaré algebra (1.24a-c) is recovered as we expected to be; secondly, from eqs.(2.41a,f) we note that at the algebra level the translational and rotational sectors are undeformed, while the deformation appears in the boost sector (2.41h) and in the boost-momentum cross-relations (2.41e). So, apparently, the Lorentz sector does not form a subalgebra, and an eventual bicrossproduct construction fails.

It is also relevant to note that (2.41a-o) are dual, with respect to relations (1.46a-e), to the Quantum Group structure given by (2.6), (2.8), (2.30)-(2.32a-b), (2.36a-b).

Given, now, the pair of dual Hopf algebras $(\mathcal{C}_\kappa(P), U_\kappa(\mathfrak{p}))$, it is said that they define a *generalized phase space*, with $U_\kappa(\mathfrak{p})$ defining the *generalized momenta* and $\mathcal{C}_\kappa(P)$ the *generalized coordinates*.

Reconnecting to Section 1.5, it is possible to show that taking the *quasiclassical limit* of this Quntum Group (the procedure is outlined in [48]) one obtains the Poincaré Lie-bialgebra $(\mathfrak{p}, \delta)$, with cocommutator $\delta$ given in [59]. It is, therefore, interesting to note that while at the Quantum Group level we have two dual Poincaré Hopf algebras, at the "classical" level we have two dual Lie-bialgebras.

### 2.2.2 Majid-Ruegg basis and $U_\kappa(\mathfrak{p})$ bicrossproduct structure

At this point we would like to replicate the reasoning made in Section 2.1.4 to find $\mathcal{M}_\kappa$ starting from $U_\kappa(\mathfrak{p})$.

We have already stated that there are multiple possible choices for the generators of the $\kappa$-Poincaré Hopf algebra connected through nonlinear combinations.

Upon performing the following change of varibles (see [48]):

$$P_0 \to -P_0, \tag{2.42a}$$
$$P_j \to -P_j e^{-\frac{\lambda P_0}{2}}, \tag{2.42b}$$
$$N_j \to N_j e^{-\frac{\lambda P_0}{2}} - \frac{\lambda}{2}\epsilon_{jkl} M_k P_l e^{-\frac{\lambda P_0}{2}}, \tag{2.42c}$$
$$M_j \to M_j, \tag{2.42d}$$

we obtain $U_\kappa(\mathfrak{p})$ in the so-called "Majid-Ruegg basis", given by the Hopf algebra structure:

$$[P_\mu, P_\nu] = 0, \tag{2.43a}$$
$$[M_j, P_0] = 0, \tag{2.43b}$$
$$[M_j, P_k] = i\epsilon_{jkl} P_l, \tag{2.43c}$$
$$[N_j, P_0] = iP_j, \tag{2.43d}$$
$$[N_j, P_k] = i\delta_{jk}\left(\frac{1}{2\lambda}(1 - e^{-2\lambda P_0}) + \frac{\lambda}{2}P^2\right) - i\lambda P_j P_k, \tag{2.43e}$$
$$[M_j, M_k] = i\epsilon_{jkl} M_l, \tag{2.43f}$$
$$[M_j, N_k] = i\epsilon_{jkl} N_l, \tag{2.43g}$$
$$[N_j, N_k] = -i\epsilon_{jkl} M_l, \tag{2.43h}$$
$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \tag{2.43i}$$
$$\Delta P_j = P_j \otimes 1 + e^{-\lambda P_0} \otimes P_j, \tag{2.43j}$$
$$\Delta M_j = M_j \otimes 1 + 1 \otimes M_j, \tag{2.43k}$$
$$\Delta N_j = N_j \otimes 1 + e^{-\lambda P_0} \otimes N_j - \lambda\epsilon_{jkl} P_k \otimes M_l, \tag{2.43l}$$
$$S(P_0) = -P_0, \tag{2.43m}$$
$$S(P_j) = -e^{\lambda P_0} P_j, \tag{2.43n}$$
$$S(M_j) = -M_j, \tag{2.43o}$$
$$S(N_j) = -e^{\lambda P_0} N_j + \lambda\epsilon_{jkl} P_k M_l, \tag{2.43p}$$

and trivial counits. Again, taking the limit $\lambda \to 0$, the standard Poincaré algebra is recovered. The fundamental difference between (2.41a-o) and (2.43a-p) is that in the Majid-Ruegg case the Lorentz sector is not deformed, and all the deformations reside in the cross-relations between Lorentz and translational sectors. It is, therefore, possible to express $U_\kappa(\mathfrak{p})$ as a bicrossproduct Hopf algebra, dual to $\mathcal{C}_\kappa(P)$. This construction leads to the decomposition

$$U_\kappa(\mathfrak{p}) = U(so(1,3)) \bowtie T, \tag{2.44}$$

that can be seen to be dual to (2.37) [51], in the sense of (1.46a-e), imposing the relations:

$$\langle a^\mu, P_\nu \rangle = i\delta^\mu{}_\nu, \tag{2.45a}$$

$$\langle \Lambda^\mu{}_\nu, M^{\alpha\beta} \rangle = i(g^{\alpha\mu}\delta^\beta{}_\nu - g^{\beta\mu}\delta^\alpha{}_\nu). \tag{2.45b}$$

Taking the generators in the following form:

$$P_\mu = 1 \otimes p_\mu, \tag{2.46a}$$

$$M_j = m_j \otimes 1, \tag{2.46b}$$

$$N_j = n_j \otimes 1, \tag{2.46c}$$

and recalling (1.62), we find the connection between the right action and the commutators:

$$[X_\mu, X_\nu] = -1 \otimes (x_\nu \triangleleft x_\mu), \tag{2.47}$$

where $X_\mu = 1 \otimes x_\mu$ are $\kappa$-Poincaré generators.

Applying (2.47) to (2.43b-e), we can find the right action explicitly:

$$p_0 \triangleleft m_i = 0, \tag{2.48a}$$

$$p_i \triangleleft m_j = -\epsilon_{ijk} p_k, \tag{2.48b}$$

$$p_0 \triangleleft n_i = -ip_i, \tag{2.48c}$$

$$p_i \triangleleft n_j = -i\delta_{ij}\left(\frac{1}{2\lambda}(1 - e^{-2\lambda p_0}) + \frac{\lambda}{2}p^2\right) + i\lambda p_i p_j. \tag{2.48d}$$

The same way, one can apply (1.108c) to (2.43a-p) (see [51]), to obtain

$$\beta_L(m_i) = 1 \otimes m_i, \tag{2.49a}$$

$$\beta_L(n_i) = e^{-\lambda p_0} \otimes n_i + i\lambda\epsilon_{ijk} p_j \otimes m_k. \tag{2.49b}$$

At this point, following the discussion carried on in [51], this bicrossproduct structure can be seen as an Hopf algebra with algebra sector given by a semidirect product of the undeformed Lorentz group acting in a deformed way on the translational sector (eq.(2.48a-d)), and with coalgebra sector given by a semidirect product of the translational sector coacting on the Lorentz group (eq.(2.49a-b)). Following a reasoning similar to that of Section 1.2, it is now possible to introduce $\mathcal{M}_\kappa$ so that $U_\kappa(\mathfrak{p})$ acts covariantly on it as a deformed symmetry group. This construction is obtained via dualization of the momenta space $T$ as shown in [1] and [51].

## 2.3 Observers, observables and the problem of localizability

We are now interested in discussing the problem of localizability of states in a 4D $\mathcal{M}_\kappa$. Our goal will be to see if there exist states in our Hilbert space that represent sharp events, i.e. that consent to localize spacetime points. As aforementioned, since we are dealing with a noncommutative spacetime, the answer is rather than trivial, and furthermore it turns out that localizability properties depend not only on the noncommutative structure of the spacetime itself, but even on the deformation of the symmetry group, i.e. on the reference frame we are considering. The conclusion is that, unlike the case of other more simple noncommutative spaces that admit undeformed symmetry groups such as the QM phase space, in our noncommutative spaces different observers will not necessarily agree on the localizability properties of the same state.

To perform this analysis we firstly have to choose a suitable realization of $\kappa$-Minkowski commutation relations and of the $\kappa$-Poincaré algebra sector. In Subsection 2.3.1 we start deriving the uncertainty relations causing issues for the localizability of states, then we construct a realization for $\kappa$-Minkowski, solve the eigenvalue problem for the time operator and define two complete sets of operators isometrically related one to another by means of an integral (Mellin) transform. In Subsection 2.3.2 we discuss localization of states as seen by an inertial observer located in the origin of its reference frame; to do this we analyze two relevant classes of states: one centered in the spacial origin, for which we will see perfect localization can be achieved, and the other away from it, showing how the uncertainty relations fully come into play. Subsection 2.3.3, after the introduction of $\kappa$-Poincaré uncertainty relations and the analysis of the resulting constraints on pure transformations, is devoted to the construction of a useful realization of the $\kappa$-Poincaré group, and specializing the discussion to the 1+1D case we show concretely in which sense $\kappa$-Minkowski is the homogeneous subspace obtained by quotienting $\kappa$-Poincaré by the Lorentz subgroup, completing the discourse initiated in 2.1.4. Finally, in Subsection 2.3.4, following [43], we propose an interpretation of the notion of observables' and observers' states, discussing some relevant deformed transformations and their effects on the spacetime localizability, and extending the discussion made in that paper performing explicit calculations of uncertainties coming from these transformations.

### 2.3.1 $\mathcal{M}_\kappa$ coordinate realization

Considering relations (2.1a-b), we expect an uncertainty relation between coordinate operators to occur, given by the *generalized Heisenberg uncertainty relation* [58]. Given $\mathcal{A}$, $\mathcal{B}$ self-adjoint operators, defined their standard deviations $\Delta\mathcal{A} = \sqrt{\langle\mathcal{A}^2\rangle - \langle\mathcal{A}\rangle^2}$, $\Delta\mathcal{B} = \sqrt{\langle\mathcal{B}^2\rangle - \langle\mathcal{B}\rangle^2}$, where $\langle\cdot\rangle$ stands for the expectation value, the following relation holds:

$$\Delta\mathcal{A}\Delta\mathcal{B} \geq \frac{1}{2}\left|\langle[\mathcal{A},\mathcal{B}]\rangle\right| . \tag{2.50}$$

Applying (2.50) to (2.1a-b) one obtains

$$\Delta x^0 \Delta x^i \geq \frac{\lambda}{2} \left| \langle x^i \rangle \right| , \tag{2.51}$$

a relation that tells us the impossibility to localize, in general, spacetime events (i.e. obtain sharp eigenvalues for space and time operators simultaneously).

To begin our exposition, the first thing to do is to set a suitable realization for coordinate operators. Following [43] we choose

$$x^i \psi(x) = x^i \psi(x), \tag{2.52a}$$

$$x^0 \psi(x) = i\lambda \left( \sum_i x^i \partial_{x^i} + \frac{3}{2} \right) \psi(x) = i\lambda \left( r\partial_r + \frac{3}{2} \right) \psi(x), \tag{2.52b}$$

where the factor $\frac{3}{2}$ is needed to have symmetric operators. We are taking $x^i$ as a complete set of observables on the Hilbert space $L^2(\mathbb{R}^3_x)$, defining $x^0$ to be another operator on this space and $\psi(x) \in L^2(\mathbb{R}^3_x)$[6]. At this point an important thing has to be noted. The realization (2.52a-b) is not the only possible choice, and it is not yet clear if the following discussion assumes different forms based on the chosen realization.

Comparing (2.52a-b) with the analogous quantum phase space relations

$$q^i \phi(q) = q^i \phi(q), \tag{2.53a}$$

$$p_i \phi(q) = -i\hbar \frac{\partial}{\partial q^i} \phi(q), \tag{2.53b}$$

where in this case we have chosen a complete set of observables $q^i$ on $L^2(\mathbb{R}^3_q)$, $p_i$ self-adjoint operators on $L^2(\mathbb{R}^3_q)$ and $\phi \in L^2(\mathbb{R}^3_q)$[7], we can see an analogy between $x^i$, $q^i$ and $x^0$, $p_i$.

In our case $x^0$ has the form of a (self-adjoint) dilation operator.

The previous $\kappa$-Minkowski relations can be rewritten in a self-adjoint operatorial polar basis $(r, \cos(\theta), e^{i\phi})$ by means of the substitutions

$$r\cos(\theta) = x^3, \tag{2.54a}$$

$$re^{i\phi} = (x^1 + ix^2), \tag{2.54b}$$

so that (2.1a-b) assume the form

$$[x^0, \cos(\theta)] = 0, \tag{2.55a}$$

$$[x^0, e^{i\phi}] = 0, \tag{2.55b}$$

[6] This is actually not completely true, since $x^i$ and $x^0$ are unbounded operators, and therefore they admit only improper eigenfunctions, so that the notion of the usual Hilbert space must be substituted by that of a rigged Hilbert space.

[7] With the caveat of the improper eigenfunctions.

$$[x^0, r] = i\lambda r. \tag{2.55c}$$

Since $x^0$ commutes with every function of $(\theta, \phi)$, it will be convenient to factorize eigenfunctions in a radial part and an angular part; furthermore, since angular operators commute with everything, we will consider only the radial part.

In these new coordinates (2.51) will be expressed as

$$\Delta x^0 \Delta r \geq \frac{\lambda}{2} \left|\langle r \rangle\right|. \tag{2.56}$$

Posing the eigenvalue problem for monomials in $r$:

$$i\lambda \left( r\partial_r + \frac{3}{2} \right) r^\alpha = i\lambda \left( \alpha + \frac{3}{2} \right) r^\alpha, \tag{2.57}$$

one obtains real eigenvalues $\lambda\tau$ if and only if $\alpha = -\frac{3}{2} - i\tau$, with $\tau \in \mathbb{R}$; the spectrum of $x^0$ is the whole of $\mathbb{R}$ (while that of $r$ is the positive real line) and the nondimensionalized (by the introduction of $\lambda$) eigenvectors assume the form

$$T_\tau = \frac{r^{-\frac{3}{2}-i\tau}}{\lambda^{-i\tau}} = r^{-\frac{3}{2}} e^{-i\tau \log\left(\frac{r}{\lambda}\right)}. \tag{2.58}$$

As plane waves in the case of (2.53b), these are not physical states, being improper eigenfunctions. To give a feeling of the scale $\tau$, notice that from (2.52b) and (2.57) $\tau = \frac{x^0}{\lambda}$; supposing that $\lambda \sim \ell_P$, a $\tau \sim 1$ corresponds to an $x^0 \sim \ell_P$ and so a time $t \sim \frac{\ell_P}{c} = t_P = 5,391247 \times 10^{-44} s$.

At this point, since $x^0$ is self-adjoint, $T_\tau$ form a complete basis and complete sets of observables can be given by $(r, \theta, \phi)$ or $(\tau, \theta, \phi)$, the connection between them be a *Mellin transform*, which take the role of what Fourier transforms were for the quantum phase space case:

$$\psi(r, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau r^{-\frac{3}{2}} e^{-i\tau \log\left(\frac{r}{\lambda}\right)} \tilde{\psi}(\tau, \theta, \phi) = \mathcal{M}^{-1}\left[\tilde{\psi}(\tau, \theta, \phi), r\right], \tag{2.59a}$$

$$\tilde{\psi}(\tau, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dr r^{\frac{1}{2}} e^{i\tau \log\left(\frac{r}{\lambda}\right)} \psi(r, \theta, \phi) = \mathcal{M}\left[\psi(r, \theta, \phi), \frac{3}{2} + i\tau\right]. \tag{2.59b}$$

(2.59a-b) are norm-preserving and satisfy a Parseval identity, so that they are isometries between $L^2$ spaces, and the discussion is coherent.

A last thing to be noted is that if we measure the expectation value of the time operator on a real spherical-symmetric state we obtain 0:

$$\begin{aligned} \langle x^0 \rangle &= 4\pi \int r^2 \bar{\psi}(r) i\lambda \left( r\partial_r + \frac{3}{2} \right) \psi(r) dr = \\ &= 4\pi i\lambda \left( \int \bar{\psi}(r) r^3 \partial_r \psi(r) dr + \frac{3}{2} \int r^2 |\psi(r)|^2 dr \right), \end{aligned} \tag{2.60}$$

but

$$\int \bar{\psi}(r) r^3 \partial_r \psi(r) dr = \cancel{r^3|\psi|^2|_0^\infty} - \int \psi(r) r^3 \partial_r \bar{\psi}(r) dr - 3 \int r^2 |\psi(r)|^2 dr, \tag{2.61}$$

and therefore, since $\bar{\psi} = \psi$ being real,

$$\langle x^0 \rangle = 0; \tag{2.62}$$

this is the analogue of the fact that in QM phase space real functions have 0 mean value for measurements of the momentum.

### 2.3.2 Localized states

We discuss, here, localization of states as seen by an inertial observer located at the origin of its reference frame.

While it is not yet known a general theory regarding all possible states of $\kappa$-Minkowski in the realization given above, we present here two explicit examples of states in relation to their localizability properties.

#### Spatial origin localization and states

The uncertainty relations (2.1a-b) pose a problem in localizability for states in the Hilbert space of the realization we have chosen. A first thing to be noted is that, for points located at the spatial origin of the observer's reference frame, localization should be possible, since the right-hand side of (2.1a) would vanish. To see if that is true we start with a state that saturates eqs.(2.1a-b), akin to Gaussian distributions that saturate QM phase space commutation relations. Such states are normalized *log-normal distributions* [43]:

$$L(r, r_0) = N e^{-\frac{(\log r - \log r_0)^2}{\sigma^2}} = \frac{e^{-\left(\frac{\log\left(\frac{r}{r_0}\right)}{\sigma}\right)^2} e^{-\frac{9}{16}\sigma^2}}{\sqrt{\sigma}(2\pi)^{3/4}\sqrt{r_0^3}}. \tag{2.63}$$

It is possible to show that for (2.63) two localization limits exists:

(i) for $\sigma \to 0$ they localize at the maximum $r = r_0$,

(ii) for $\sigma \geq 0$ and $r_0 \to 0$ they localize at $r = 0$.

Applying the Mellin transform one obtains the state in the $(\tau, \theta, \phi)$ basis:

$$\tilde{L}(\tau, r_0) = \frac{\sqrt{\sigma} e^{-\frac{1}{4}\sigma^2 \tau(\tau - 3i)}}{2\pi^{3/4}\sqrt[4]{2}} \left(\frac{r_0}{\lambda}\right)^{i\tau}. \tag{2.64}$$

Computing the square norm one finds the result that the probability density is a Gaussian independent on $r_0$:

$$|\tilde{L}(\tau, r_0)|^2 = \frac{\sigma e^{-\frac{\sigma^2\tau^2}{2}}}{4\pi^{3/2}\sqrt{2}}. \tag{2.65}$$

Calculating the mean values of coordinate operators one obtains:

$$\langle r^n \rangle = e^{\frac{\sigma^2 n(n+6)}{8}} n r_0, \tag{2.66a}$$

$$\langle (x^0)^n \rangle = \frac{1}{4\pi}\left(\frac{\lambda}{\sigma}\right)^n \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even.} \end{cases} \tag{2.66b}$$

Now, taking the limit $r_0 \to 0$ in (2.66a) it is clear that the state localizes in space at $r = 0$; furthermore, if we take $\sigma \to \infty$, from (2.66b) we see that the state is localized also in time at $\tau = 0$; to localize it at a different time $\tau_0$ a plain shift can be performed by multiplying the function by $r^{i\tau_0}$.

It is possible, then, to define an origin eigenstate $|0\rangle$ and a 1-parameter family of states $|0_\tau\rangle$ localized at the origin of space and at a fixed time $\tau$ that both can be reached as a limit of normalized states of $L^2(\mathbb{R}^3_x)$.

#### Localization away from the spatial origin

To analyze, now, localization properties away from the spacial origin consider the following example. Suppose we have a particle localized in a spherical shell described by the following wavefunction

$$\psi(r) = \frac{\delta(r - r_0)}{r_0^2}; \tag{2.67}$$

its Mellin transform will be

$$\tilde{\psi}(\tau) = \frac{1}{\sqrt{2\pi}} r_0^{-\frac{3}{2}} e^{i\tau \log\left(\frac{r_0}{\lambda}\right)}. \tag{2.68}$$

Computing $|\tilde{\psi}(\tau)|^2$ one notice that it does not depend on $\tau$ and therefore all time values are equally probable. This means, in turn, that space localization implies time delocalization, as in the QM phase space perfect localization in space (given by delta distributions) implies complete delocalization in momenta (given by the Fourier transform of delta distributions: plane waves whose probability density does not depend on momenta) and viceversa.

Note that (2.68) is not normalizable, but it can be approximated smearing (2.67) on a spherical shell as done in [43]. In doing so an even probability density is found, which confirms the result (2.64).

### 2.3.3 $\kappa$-Poincaré realization

In the previous Subsection we have dealt with localization properties of states as seen by an inertial observer located in the origin of its reference frame. The main result we have obtained is that it can measure with sharp precision the origin of its reference frame, while as a result of uncertainty relations (2.51) it cannot perfectly localize spatially-distant states. At this point the existence of a privileged event in spacetime (i.e. the origin) seems at first to invalidate the generalized relativity principle. The problem is that in our noncommutativity spacetime we are not dealing with classical symmetry groups, but rather with deformed Quantum Groups; this imply, as we have exhaustively discussed in the construction of $\kappa$-Poincaré, that the symmetry group itself shows noncommutative properties. Recalling, for instance, eqs. (2.6), (2.8) and (2.30), we can think to define new uncertainty relations based on (2.50):

$$\Delta a^\mu \Delta a^\nu \geq \frac{\lambda}{2}|\delta^\mu{}_0\langle a^\nu\rangle - \delta^\nu{}_0\langle a^\mu\rangle|, \tag{2.69a}$$

$$\Delta\Lambda^\mu{}_\alpha \Delta\Lambda^\nu{}_\beta \geq 0, \tag{2.69b}$$

$$\Delta\Lambda^\mu{}_\nu \Delta a^\rho \geq \frac{\lambda}{2}|\langle\Lambda^\mu{}_0\Lambda^\rho{}_\nu\rangle - \delta^\mu{}_0\langle\Lambda^\rho{}_\nu\rangle + (\langle\Lambda_{0\nu}\rangle - g_{0\nu})g^{\mu\rho}|. \tag{2.69c}$$

(2.69a) of course are non other than (2.51) for the translation parameters, as stated before. $\Lambda$ and $a$ are self-adjoint observable operators in this picture, and the usual measure theory should apply without issues, therefore there are no problems in defining these relations. For the interpretation of a measure of these objects we will wait the next Subsection, in which we will consider them as operators defined on a "Hilbert space of observers", specifying the observer-state we are taking into account. Working, for example, in 1+1D, we can consider measures of the rapidity and the translational parameter in an observer-state and the results will be subject to the uncertainty relations (2.69a-c).

We must, then, reject the classical point of view of Poincaré invariance, embracing the possibility that deformed symmetry groups can bring new counter-intuitive physical features in the picture. In particular we can legitimately expect that performing a $\kappa$-Poincaré transformation from an observer to another, the localizability properties of the same state can change according to (2.69a-c), as we will show in the following.

In a recent work ([4]) it was demonstrated in the Majid-Ruegg basis that pure boosts does not exist in $\kappa$-Poincaré. Inspired by this result, and following [53] we now show that relations (2.69a-c) constrain the possible types of $\kappa$-Poincaré transformations. Let us begin considering a pure $\kappa$-Lorentz transformation. A transformation of this type implies sharp localization of translational parameters in zero, so that $\langle a^\mu\rangle = 0$ and $\Delta(a^\mu) = 0$. From this last condition one can see that the only non-trivial relation coming from (2.69a-c) is (2.69c) that implies the right-hand side to be zero. The condition $|\langle\Lambda^\mu{}_0\Lambda^\rho{}_\nu\rangle - \delta^\mu{}_0\langle\Lambda^\rho{}_\nu\rangle + (\langle\Lambda_{0\nu}\rangle - g_{0\nu})g^{\mu\rho}| = 0$ is satisfied if $\langle\Lambda^\mu{}_0\rangle = \delta^\mu{}_0$ as shown in [53], and this implies that pure boosts do not exist in $\kappa$-Poincaré. Let us analyze the case of a pure translation, for which $\langle\Lambda^\mu{}_\nu\rangle = \delta^\mu{}_\nu$ and $\Delta(\Lambda^\mu{}_\nu) = 0$. The only non-trivial relation is (2.69a), so that, if the translation is sharp in the temporal parameter ($\Delta a^0 = 0$), the constraint coming from the relation is $\langle a^i \rangle = 0$, and one obtains that the only possible sharp time translations are pure ones; in turn, if the space parameter is perfectly localized ($\Delta a^i = 0$) the constraint would be the same $\langle a^i \rangle = 0$, meaning that there cannot be pure space translations sharply localized aside trivial ones. A last case worth to note is that of the identity transformation, for which $\langle a^\mu \rangle = 0$, $\langle \Lambda^\mu{}_\nu \rangle = \delta^\mu{}_\nu$; substituting these relations in the right-hand sides of (2.69a-c), one notices that these transformations are admitted and the parameters can be sharply localized.

Note, also, that from (2.69b) we expect that performing only $\kappa$-Lorentz transformations the localizability properties remain unchanged, while this is not true for translations and mixed transformations.

Let now turn our attention to the construction of a realization of the $\kappa$-Poincaré Quantum Group $\mathcal{C}_\kappa(P)$. At first note that, taking relations (2.6), (2.8), (2.30), the Lorentz sector is undeformed, and so the usual representation theory of the Lorentz group applies. Defining the infinitesimal generators of the Lorentz group $\omega^\mu{}_\nu$, we have that

$$\Lambda^\mu{}_\nu = (\exp \omega)^\mu{}_\nu, \tag{2.70}$$

with the auxiliary antisymmetry condition

$$\omega^\mu{}_\rho g^{\varrho\nu} = -\omega^\nu{}_\varrho g^{\varrho\mu}, \tag{2.71}$$

that reduces the 16 degrees of freedom of $\Lambda^\mu{}_\nu$ to 6.

As in the classical case, $\omega^\mu{}_\nu$ commutes with each other, but given (2.30) we expect they do not commute with $a^\mu$.

A possible vector-field realization of $a^\mu$, coherent with commutation relations (2.6), (2.30), is given in [43] and has the form

$$a^\rho = i\lambda((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho})\Lambda^\beta{}_\gamma \frac{\partial}{\partial \omega^\alpha{}_\gamma}. \tag{2.72}$$

Now that we have a realization of $\mathcal{C}_\kappa(P)$, the action of the fields on wavefunctions $\phi(\omega) \in L^2(SO(1,3))$ will be:

$$\Lambda^\mu{}_\nu \phi(\omega) = (\exp \omega)^\mu{}_\nu \phi(\omega), \tag{2.73a}$$

$$a^\rho \phi(\omega) = i\lambda((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho})\Lambda^\beta{}_\gamma \frac{\partial \phi(\omega)}{\partial \omega^\alpha{}_\gamma}. \tag{2.73b}$$

The issue with this realization is that it is not faithful, as shown in [43]. A possible way to overcome the problem is to enlarge the realization as a direct sum of (2.70), (2.72) with the $\mathcal{M}_\kappa$ realization (2.52a-b), acting on the Hilbert space $L^2(SO(1,3) \times \mathbb{R}^3)$.

While the Lorentz group will still be realized as (2.70), the translation operators acquire the form

$$\begin{aligned}a^\rho =& i\frac{\lambda}{2}((\Lambda^\alpha{}_0-\delta^\alpha{}_0)\Lambda^\rho{}_\beta+(\Lambda_{0\beta}-g_{0\beta})g^{\alpha\rho})\Lambda^\beta{}_\gamma\frac{\partial}{\partial\omega^\alpha{}_\gamma}+\\&-i\frac{\lambda}{2}\left(\delta^\rho{}_0 q^i\frac{\partial}{\partial q^i}+\delta^\rho{}_i q^i\right)+\frac{1}{2}h.c.\end{aligned}\tag{2.74}$$

The action of the fields on wavefunctions $\phi(q,\omega)\in L^2(SO(1,3)\times\mathbb{R}^3)$ will be:

$$\begin{aligned}a^\rho\phi(q,\omega)=&-i\lambda\delta^\rho{}_0\left(\frac{3}{2}\phi(q,\omega)+q^i\frac{\partial\phi(q,\omega)}{\partial q^i}\right)-\delta^\rho{}_i q^i\phi(q,\omega)+\\&+i\frac{\lambda}{2}((\Lambda^\alpha{}_0-\delta^\alpha{}_0)\Lambda^\rho{}_\beta+(\Lambda_{0\beta}-g_{0\beta})g^{\alpha\rho})\Lambda^\beta{}_\gamma\frac{\partial}{\partial\omega^\alpha{}_\gamma}+\\&+i\frac{\lambda}{2}\phi(q,\omega)\frac{\partial}{\partial\Lambda^\mu{}_\nu}((\Lambda^\alpha{}_0-\delta^\alpha{}_0)\Lambda^\rho{}_\beta+(\Lambda_{0\beta}-g_{0\beta})g^{\alpha\rho}),\end{aligned}\tag{2.75a}$$

$$\Lambda^\mu{}_\nu\phi(q,\omega)=(\exp\omega)^\mu{}_\nu\phi(q,\omega).\tag{2.75b}$$

Although realization (2.83) is quite complex to work with, the discussion can be made easier working in 1+1 dimension where one can obtain, following the same discussion as above (see [43] for details), the realization:

$$\Lambda^0{}_0=\Lambda^1{}_1=\cosh\xi,\tag{2.76a}$$

$$\Lambda^0{}_1=\Lambda^1{}_0=\sinh\xi,\tag{2.76b}$$

$$a^0=i\lambda\left(\frac{1}{2}+q\frac{\partial}{\partial q}\right)+i\lambda\left(\frac{1}{2}\cosh\xi+\sinh\xi\frac{\partial}{\partial\xi}\right),\tag{2.76c}$$

$$a^1=q+i\lambda\left(\frac{1}{2}\sinh\xi+(\cosh\xi-1)\frac{\partial}{\partial\xi}\right).\tag{2.76d}$$

Working in 1+1D it is easy to explicit the discussion carried on in 2.2.4, showing that $\kappa$-Minkowski can be recovered from $\kappa$-Poincaré as the homogeneous space obtained quotienting $P_\kappa$ by the Lorentz subgroup. To do so we will start with the realization (2.76a-d) and since we want to demonstrate that any state in $L^2(\mathbb{R})$ can be obtained as a limit of states in $L^2(SO(1,1)\times\mathbb{R})$, we expand it at first order in $\xi$ around $\xi\sim 0$, sending to the identity the Lorentz sector.

$$\Lambda^0{}_0=\Lambda^1{}_1\sim 1,\tag{2.77a}$$

$$\Lambda^0{}_1=\Lambda^1{}_0\sim\xi,\tag{2.77b}$$

$$a^0\sim i\lambda\left(\frac{1}{2}+q\frac{\partial}{\partial q}\right)+i\lambda\left(\frac{1}{2}+\xi\frac{\partial}{\partial\xi}\right)+O(\xi^2),\tag{2.77c}$$

$$a^1\sim q+i\frac{\lambda}{2}\xi+O(\xi^2).\tag{2.77d}$$

We have obtained a realization that is a direct sum of two $\kappa$-Minkowski realizations (2.52a-b), one acting on $q$ and one on $\xi$. We now consider factorized wavefunctions that localize at $\xi \sim 0$ in the form of

$$\psi_{\sigma,\xi_0}(q,\xi) = f(q)Q_{\sigma,\xi_0}(\xi) = f(q)\frac{e^{-\frac{\sigma^2}{16}}}{\sqrt{\sqrt{2\pi}\xi_0\sigma}} e^{\left(\frac{\log(\xi^2)-\log(\xi_0^2)}{2\sigma}\right)^2}. \tag{2.78}$$

$Q_{\sigma,\xi_0}(\xi)$ is a lognormal distribution such that

$$\langle\psi_{\sigma,\xi_0}|\xi^n|\psi_{\sigma,\xi_0}\rangle = \begin{cases} 0 & n>0 \text{ odd}, \\ e^{\frac{n}{8}(n+2)\sigma^2} & n>0 \text{ even}, \end{cases} \tag{2.79a}$$

$$\langle\psi_{\sigma,\xi_0}|(a^i)^n|\psi_{\sigma,\xi_0}\rangle \xrightarrow[\xi_0\to 0,\sigma\to 0]{} \langle f|(x^i)^n|f\rangle = \int dq\bar{f}(q)(x^i)^n f(q), \quad i=0,1, \tag{2.79b}$$

where for $x^i$ is valid the usual realization (2.52a-b) for $q$ and the limit is taken under the condition $e^{c\sigma^2}\xi_0 \to 0,\ \forall c>0$. This shows that every $f \in L^2(\mathbb{R}_x)$ can be obtained by a limit of the product $f(q)Q_{\sigma,\xi_0}(\xi)$, and the expectation values of translational operators on $\psi_{\sigma,\xi_0}(q,\xi)$ reduce in this limit to expectation values of coordinate operators on $f$.

### 2.3.4 Observers and observables

We turn, now, our attention to the discussion of state transformations. To take in consideration the localizability properties of $\kappa$-Poincaré states, as discussed in the incipit of the previous Subsection, along with the ones intrinsically related to the noncommutative features of the spacetime, we will construct a realization of the tensor product of $\kappa$-Poincaré states with $\kappa$-Minkowski spacetime states.

Consider the transformation (2.2), we interpret $x^\mu$ and $x'^\mu$ as coordinate systems associated to two inertial observers $O$, $O'$ $\kappa$-Poincaré transformed with respect to each other. Following the usual measurement theory of QM, we define

(1) the expectation value $\langle x^\mu\rangle$ as the measure (performed by $O$) of a spacetime event,

(2) the variance $\Delta(x^\mu)^2 = \langle(x^\mu - \langle x^\mu\rangle)^2\rangle$ as the (square of the) indetermination on the measure (i.e. how well the spacetime event is localized),

(3) the higher distributional-moments $\langle(x^\mu - \langle x^\mu\rangle)^n\rangle$ as additional finer details on the distribution of probability defining the localization of the event (the skewness, the kurtosis etc.).

The same event described by $O'$ will be defined by moments $\langle(x'^\mu - \langle x'^\mu\rangle)^n\rangle$.

We recall that, while $x^\mu \in \mathcal{M}_\kappa$, $x'^\mu \in \mathcal{C}_\kappa(P)\otimes\mathcal{M}_\kappa$. But we have constructed in Subsection 2.3.1 a realization for $\mathcal{M}_\kappa$ and in 2.4.2 one for $\mathcal{C}_\kappa(P)$, therefore a realization for $\mathcal{C}_\kappa(P)\otimes\mathcal{M}_\kappa$ can be given by the direct sum of the two realizations (2.52a-b) and (2.83). For convenience we lift the elements $x^\mu \in \mathcal{M}_\kappa$ to $\mathbb{1}\otimes\mathcal{M}_\kappa$, where $\mathbb{1}$ is defined by the identity of $\mathcal{C}_\kappa(P)$ given by the counits (2.32a-b). The realization will act on Hilbert space functions $f(\omega, q, x) \in L^2(SO(1,3) \times \mathbb{R}^3_q) \times L^2(\mathbb{R}^3_x) \sim L^2(SO(1,3) \times \mathbb{R}^3_q \times \mathbb{R}^3_x)$ as follows:

$$
\begin{aligned}
x'^\mu f(\omega, q, x) =& - i\lambda\Lambda^\mu{}_\nu(\omega)\left(\delta^\nu{}_0 x^i \frac{\partial f(\omega, q, x)}{\partial x^i} + \delta^\nu{}_i x^i f(\omega, q, x)\right) + \\
& - i\lambda\delta^\rho{}_0\left(\frac{3}{2}f(\omega, q, x) + q^i\frac{\partial f(\omega, q, x)}{\partial q^i}\right) - \delta^\rho{}_i q^i f(\omega, q, x) + \\
& + i\frac{\lambda}{2}((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho})\Lambda^\beta{}_\gamma \frac{\partial f(\omega, q, x)}{\partial \omega^\alpha{}_\gamma} + \\
& + i\frac{\lambda}{2} f(\omega, q, x)\frac{\partial}{\partial \Lambda^\mu{}_\nu}((\Lambda^\alpha{}_0 - \delta^\alpha{}_0)\Lambda^\rho{}_\beta + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho}).
\end{aligned}
\tag{2.80}
$$

Now, our Hilbert space admits non-entangled states of the kind

$$|\phi, \psi\rangle = |\phi\rangle \otimes |\psi\rangle, \tag{2.81}$$

with $|\phi\rangle \in L^2(SO(1,3) \times \mathbb{R}^3_q)$ and $|\psi\rangle \in L^2(\mathbb{R}^3_x)$, normalized according to $\langle\phi|\phi\rangle = 1$, $\langle\psi|\psi\rangle = 1$.

At this point we are ready to give an interpretation of the realization here constructed. We define $L^2(SO(1,3) \times \mathbb{R}^3_q)$ as the space of states of an observer (i.e. the space of $\kappa$-Poincaré states) and $L^2(\mathbb{R}^3_x)$ as the space of observables (i.e. the space of states of $\kappa$-Minkowski spacetime); furthermore we assume that a generic state can be realized as an untangled element $|\phi, \psi\rangle = |\phi\rangle \otimes |\psi\rangle$, a reasonable postulate since it reflects the natural assumption that the relation between two inertial observers does not depend on the observed state.

The thing here is that unlike the commutative case we both have a noncommutative spacetime on which observables are defined and a noncommutative observer state-space, meaning that in general a $\kappa$-Poincaré transformation between different observers could decrease localizability of states.

Taking into account (2.81), the mean value of the coordinates of a transformed observer would be:

$$\langle x'^\mu\rangle = \langle\phi| \otimes \langle\psi|(\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1)|\phi\rangle \otimes |\psi\rangle = \langle\phi|\Lambda^\mu{}_\nu|\phi\rangle\langle\psi|x^\nu|\psi\rangle + \langle\phi|a^\mu|\phi\rangle. \tag{2.82}$$

Before starting with some relevant examples we turn our attention to the uncertainty of transformed states in relation to that of the starting ones. Consider a generic transformation (2.2), we can write

$$\Delta(x'^\mu)^2 = \langle(x'^\mu)^2\rangle - \langle x'^\mu\rangle^2 = \Delta(\Lambda^\mu{}_\nu \otimes x^\nu)^2 + \Delta(a^\mu)^2 + 2cov(\Lambda^\mu{}_\nu, a^\mu)\langle x^\nu\rangle, \tag{2.83}$$

since $\langle a \otimes b\rangle = \langle a\rangle \otimes \langle b\rangle$ and the covariance between elements on different sides of the tensor product is 0.

**Identity transformation state**

A noteworthy example of a state in the enlarged realization of $\mathcal{C}_\kappa(P)$ is the identity state $|i\rangle$, given as follows on functions $f(a,\Lambda) \in \mathcal{C}_\kappa(P)$:

$$\langle i|f(a,\Lambda)|i\rangle = \varepsilon(f). \tag{2.84}$$

By the definition of the countit it is obvious to see that this state gives the identity transformation of the function $f$. This state can be defined, as we have seen analyzing the uncertainty relations of $\kappa$-Poincaré, and can be obtained explicitly considering a sequence of functions converging to a $\delta$ in the diagonal elements of $\Lambda$ and to 0 in the off-diagonal elements of $\Lambda$ and in all the $a$'s.

The following state

$$|i,\psi\rangle = |i\rangle \otimes |\psi\rangle \tag{2.85}$$

can be linked to the $\kappa$-Poincaré transformation between two coincident observers, as one can see working the following calculation:

$$\langle x'^\mu\rangle = \langle i| \otimes \langle\psi|(\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1)|i\rangle \otimes |\psi\rangle = \langle i|\Lambda^\mu{}_\nu|i\rangle\langle\psi|x^\nu|\psi\rangle + \langle i|a^\mu|i\rangle; \tag{2.86}$$

but recalling the counits (2.32a-b):

$$\langle x'^\mu\rangle = \langle\psi|x^\mu|\psi\rangle. \tag{2.87}$$

The same result is achieved for a generic monomial in coordinates $x'^{\mu_1}\cdots x'^{\mu_n}$:

$$\begin{aligned}\langle x'^{\mu_1}\cdots x'^{\mu_n}\rangle =&\langle i| \otimes \langle\psi|x'^{\mu_1}\cdots x'^{\mu_n}|i\rangle \otimes |\psi\rangle = \\ =&\langle o|a^{\mu_1}\cdots a^{\mu_n}|o\rangle + \langle o|\mathcal{O}^{\mu_1\cdots\mu_n}_\nu(a,\Lambda)|o\rangle\langle\psi|x^\nu|\psi\rangle + \\ &+\cdots+\langle o|\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}(a,\Lambda)|o\rangle\langle\psi|x^{\nu_1}\cdots x^{\nu_n}|\psi\rangle,\end{aligned} \tag{2.88}$$

with $\mathcal{O}(a,\Lambda)$ generic monomials in $a$'s and $\Lambda$'s. Since the counit map is an homomorphism, every monomial that contains at least an $a$ vanishes ($\epsilon(a^\mu) = 0$) and the only surviving term is that with an equal number of upper and lower indices, that is a product of $\Lambda$'s only. Again from the homomorphism property one obtains that $\epsilon(\mathcal{O}^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}(a,\Lambda)) = \delta^{\mu_1}{}_{\nu_1}\cdots\delta^{\mu_n}{}_{\nu_n}$, and

$$\langle x'^{\mu_1}\cdots x'^{\mu_n}\rangle = \langle\psi|x^{\mu_1}\cdots x^{\mu_n}|\psi\rangle. \tag{2.89}$$

Since the Lorentz sector is undeformed, we expect localization problems to arise only when taking translations into account, so that coincident observers described by (2.84) are well-defined in $\kappa$-Minkowski and they agree on every measurement they make. Starting from (2.85) and employing the result for monomials (2.87), one easily sees that uncertainties between the initial and final events, in fact, coincide:

$$\Delta(x'^\mu)^2 = \langle(x'^\mu)^2\rangle - \langle x'^\mu\rangle^2 = \langle(\delta^\mu{}_\nu x^\nu)^2\rangle - \langle x^\mu\rangle^2 = \Delta(x^\mu)^2. \tag{2.90}$$

**Origin state transformations**

Suppose, now, wanting to know what a $\kappa$-Poincaré-transformed observer would measure in the origin of its reference frame if it transforms an event $x^\mu$; the state on which we operate this transformation is

$$|\phi, o\rangle = |\phi\rangle \otimes |o\rangle, \tag{2.91}$$

so that

$$\langle x'^\mu \rangle = \langle \phi| \otimes \langle o|(\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1)|\phi\rangle \otimes |o\rangle = \langle \phi|\Lambda^\mu{}_\nu|\phi\rangle\langle o|x^\nu|o\rangle + \langle \phi|a^\mu|\phi\rangle. \tag{2.92}$$

Recalling from the paragraph concerning spacial localization in Subsection 2.3.2 that $\langle o|x^\mu|o\rangle = 0$ we have that

$$\langle x'^\mu \rangle = \langle \phi|a^\mu|\phi\rangle. \tag{2.93}$$

This result entails the fact that the two observers are comparing positions and not directions, so the expectation value is determined only by the mean value of translation operators.

It can be shown by an analogous computation that the result remains true also for a generic monomial in coordinates $x'^{\mu_1} \cdots x'^{\mu_n}$; in fact we have shown in Subsection 2.3.2 that $\langle o|x^{\mu_1} \cdots x^{\mu_n}|o\rangle = 0 \ \forall n$, and therefore:

$$\langle x'^{\mu_1} \cdots x'^{\mu_n} \rangle = \langle \phi| \otimes \langle o|x'^{\mu_1} \cdots x'^{\mu_n}|\phi\rangle \otimes |o\rangle = \langle \phi|a^{\mu_1} \cdots a^{\mu_n}|\phi\rangle. \tag{2.94}$$

In this case, taking into account the previous result, the uncertainty of the transformed event coincide with that of the translation operator:

$$\Delta(x'^\mu)^2 = \langle (x'^\mu)^2 \rangle - \langle x'^\mu \rangle^2 = \langle (a^\mu)^2 \rangle - \langle a^\mu \rangle^2 = \Delta(a^\mu)^2. \tag{2.95}$$

**Translations**

Another interesting case is that of a pure translation $x'^\mu = 1 \otimes x^\mu + a^\mu \otimes 1$ of a generic state (note that in Subsection 2.3.3 we have shown that pure 1D translations aside of temporal ones are not possible, but a general 3D translation is compatible with the uncertainty relations).

To demonstrate that states corresponding to a translation do exist in $L^2(SO(1,3) \times \mathbb{R}^3_q)$ it is needed to take a sequence of functions which converge to a $\delta$ for the diagonal elements of $\Lambda$ and to 0 for off-diagonal ones. Taking such states and (co)acting with the usual coaction (2.2) it is the same thing as (co)acting with $x'^\mu = 1 \otimes x^\mu + a^\mu \otimes 1$ on a generic state of the enlarged realization (2.80).

The expectation value would be

$$\langle x'^\mu \rangle = \langle \phi| \otimes \langle \psi|(1 \otimes x^\mu + a^\mu \otimes 1)|\phi\rangle \otimes |\psi\rangle = \langle \psi|x^\mu|\psi\rangle + \langle \phi|a^\mu|\phi\rangle, \tag{2.96}$$

while the variance

$$\begin{aligned}\Delta(x'^{\mu})^2 &= \langle (x^{\mu})^2 + (a^{\mu})^2 + x^{\mu}a^{\mu} + a^{\mu}x^{\mu}\rangle - \langle x^{\mu}\rangle^2 - \langle a^{\mu}\rangle^2 - 2\langle x^{\mu}\rangle\langle a^{\mu}\rangle = \\ &= \Delta(x^{\mu})^2 + \Delta(a^{\mu})^2,\end{aligned} \tag{2.97}$$

recalling that the covariance between elements in different sides of the tensor product vanishes.

Since

$$\Delta(x'^{\mu})^2 = \Delta(x^{\mu})^2 + \Delta(a^{\mu})^2 \geq \Delta(x^{\mu})^2, \tag{2.98}$$

a pure translation always increases the uncertainty of the state, except if the translational parameter has 0 uncertainty. This last condition is fulfilled in two cases: if we consider the identity transformation or if we perform a pure temporal translation (this latter case is easily seen by noting that in the uncertainty relation (2.69a), if $\langle a^i \rangle = 0$, $i = 1, 2, 3$, there is not uncertainty in $\Delta(a^{\mu})$, this is in analogy with the $\kappa$-Minkowski uncertainty relations (2.51)). In particular this entails the fact that doing a spacetime translation followed by an inverse equal one, one does not revert to the original state having increased the uncertainty, unlike in the classical Poincaré case.

While in the case of a pure translation the uncertainty in the localization of a state can only increase (or rest unmodified), one could ask if there exist $\kappa$-Poincaré transformations that lead to a decrease in uncertainty. Surprisingly the answer is positive, and an explicit example was worked out in the 1+1D case in [43]. The physical implications of this result still have to be investigated.

# Chapter 3

# $\varrho$-deformation

We turn, now, our attention to the analysis of another type of noncommutative Minkowski spacetime in 4D, called *$\varrho$-Minkowski*, associated to a Poincaré Quantum Group called *$\varrho$-Poincaré*. Although lesser studied than $\kappa$-Minkowski, this noncommutative spacetime was recently considered in a series of works in relation to quasinormal modes of Reissner–Nordström black holes [19, 20, 21], and as a possible testing ground for phenomenological effects as the *dual-curvature lensing* [2]. A field theory on it was constructed starting from the twist approach we have discussed in Subsection 1.6.3, in [18]. In [45] it was found, studying the spectrum of the time operator, that in this spacetime time is a quantized variable. Localizability, however, was never discussed before in this case.

The main difference with the more studied $\kappa$-deformation is in the angular nature of the commutation relations, that we choose to be of the form

$$[x^0, x^i] = i\varrho\epsilon^i{}_{j3}x^j, \quad i,j = 1,...,3, \tag{3.1a}$$

$$[x^i, x^j] = 0. \tag{3.1b}$$

Taking $i = 3$ in (3.1a-b) shows that the coordinate $x^3$ is a central element of the algebra, meaning that it commutes with every other coordinate.

In Section 3.1 we derive the $\varrho$-Poincaré Quantum Group $\mathcal{C}_\varrho(P)$, obtained via the introduction of a classical $r$-matrix and following the same discussion carried on for the $\kappa$-Minkowski case in Subsections 2.1.2-2.1.3 [45]. The main difference between the two spacetimes is that the new $r$-matrix satisfies the CYBE, an important feature that allows us to apply the twist procedure to deform the universal enveloping algebra of Poincaré to obtain the dual Quantum Group $U(\mathfrak{p})_\varrho$, as we will see in Section 3.2. In Section 3.3, finally, we pose for the first time the bases to the analysis of the problem of localizability in $\varrho$-Minkowski as done before in the $\kappa$-Minkowski case.

## 3.1 Quantum Poincaré Group $\mathcal{C}_\varrho(P)$

Following the discussion made in Subsection 2.1.2 we will derive the commutation relations (i.e. the algebra sector) of the $\mathcal{C}_\varrho(P)$ Quantum Group starting from the classical $r$-matrix of $\varrho$-Minkowski.

At first note that being this case another type of Poincaré deformation, left and right invariant vector fields retain the same expressions (2.25a-d). The only difference with $\kappa$-Poincaré is in the $r$-matrix, that in this case assumes the form (cfr. [50, 45] and see Appendix B)

$$r = -i\varrho(P_0 \wedge M_{12}), \tag{3.2}$$

where again $P_\mu$, $M_{\mu\nu}$ are linked with invariant vector fields through (2.26a-b).

Note that, unlike the case of the classical $r$-matrix of $\kappa$-Minkowski that satisfies a MYBE, (3.2) satisfies the CYBE, in fact computing the Schouten Bracket:

$$[r_{12}, r_{13}] = \varrho^2[P_0, P_0] \wedge M_{12} \wedge M_{12} = 0, \tag{3.3a}$$

$$[r_{12}, r_{23}] = \varrho^2 P_0 \wedge [M_{12}, P_0] \wedge M_{12} = i\varrho^2 P_0 \wedge (g_{20}P_1 - g_{10}P_2) \wedge M_{12} = 0, \tag{3.3b}$$

$$[r_{13}, r_{23}] = \varrho^2 P_0 \wedge P_0 \wedge [M_{12}, M_{12}] = 0, \tag{3.3c}$$

and thus $[[r, r]] = 0$.

The Sklyanin bracket (1.52) assumes, now, the form

$$\{f, g\} = -\varrho(X^R_{12} \wedge X^R_0 - X^L_{12} \wedge X^L_0)(df, dg), \tag{3.4}$$

so that we can compute the brackets between Poincaré coordinates as done before:

$$\begin{aligned}\{\alpha^\mu, \alpha^\nu\} &= -\varrho\left[\left(\Lambda_{2\sigma}\frac{\partial}{\partial \Lambda^1{}_\sigma} - \Lambda_{1\sigma}\frac{\partial}{\partial \Lambda^2{}_\sigma} + a_2\frac{\partial}{\partial a^1} - a_1\frac{\partial}{\partial a^2}\right) \wedge \frac{\partial}{\partial a^0}\right](a^\mu, a^\nu) = \\ &= -\varrho[\delta^\nu{}_0(a_2\delta^\mu{}_1 - a_1\delta^\mu{}_2) - \delta^\mu{}_0(a_2\delta^\nu{}_1 - a_1\delta^\nu{}_2)],\end{aligned} \tag{3.5a}$$

$$\begin{aligned}\{\Lambda^\mu{}_\nu, \Lambda^\varrho{}_\sigma\} &= -\varrho\left[\left(\Lambda_{2\lambda}\frac{\partial}{\partial \Lambda^1{}_\lambda} - \Lambda_{1\lambda}\frac{\partial}{\partial \Lambda^2{}_\lambda} + a_2\frac{\partial}{\partial a^1} - a_1\frac{\partial}{\partial a^2}\right) \wedge \frac{\partial}{\partial a^0}\right](\Lambda^\mu{}_\nu, \Lambda^\varrho{}_\sigma) = \\ &= 0,\end{aligned} \tag{3.5b}$$

$$\begin{aligned}\{\Lambda^\mu{}_\nu, a^\varrho\} &= -\varrho\left[\left(\Lambda_{2\sigma}\frac{\partial}{\partial \Lambda^1{}_\sigma} - \Lambda_{1\sigma}\frac{\partial}{\partial \Lambda^2{}_\sigma} + a_2\frac{\partial}{\partial a^1} - a_1\frac{\partial}{\partial a^2}\right) \wedge \frac{\partial}{\partial a^0}\right](\Lambda^\mu{}_\nu, a^\varrho) = \\ &= -\varrho\left[\delta^\varrho{}_0(\Lambda_{2\sigma}\delta^\mu{}_1\delta^\sigma{}_\nu - \Lambda_{1\sigma}\delta^\mu{}_2\delta^\sigma{}_\nu) - \Lambda^\sigma{}_0\delta^\varrho{}_\sigma(\Lambda^\sigma{}_1\delta_\sigma{}^\mu g_{2\nu} - \Lambda^\sigma{}_2\delta_\sigma{}^\mu g_{1\nu})\right] = \\ &= -\varrho\left[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})\right];\end{aligned} \tag{3.5c}$$

therefore the commutators are:

$$[a^\mu, a^\nu] = -i\varrho[\delta^\nu{}_0(a_2\delta^\mu{}_1 - a_1\delta^\mu{}_2) - \delta^\mu{}_0(a_2\delta^\nu{}_1 - a_1\delta^\nu{}_2)], \tag{3.6a}$$

$$[\Lambda^\mu{}_\nu, \Lambda^\varrho{}_\sigma] = 0, \tag{3.6b}$$

$$[\Lambda^\mu{}_\nu, a^\varrho] = -i\varrho\left[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})\right]. \tag{3.6c}$$

Again it is easy to see that the commutation relations between $a^\mu$ and $a^\nu$ reproduce eqs.(3.1a-b), and $\varrho$-Minkowski can therefore be recovered by the momenta sector of $\mathcal{C}_\varrho(P)$.

For the coalgebra sector and the antipode, as in the case of $\kappa$-Poincaré, since the groupal laws must be compatible with the Poisson brackets, they are undeformed and equal to (2.31a-b), (2.32a-b), (2.36a-b). It is, then, trivial to see that taking the limit $\varrho \to 0$ the classical commutative case is recovered. Furthermore, properties (2.33)-(2.35) are satisfied the same way as in the $\kappa$-Poincaré case, so that, following a calculation similar to that of Subsection 2.1.1, it is possible to see that the transformation law (2.2) is a covariant left action: $\mathcal{M}_\varrho \to \mathcal{C}_\varrho(P) \otimes \mathcal{M}_\varrho$, and $\mathcal{C}_\varrho(P)$ (co)acts covariantly on $\varrho$-Minkowski being its group of simmetries.

## 3.2 Quantum Universal Enveloping Algebra $U_\varrho(\mathfrak{p})$

We would like, now, to introduce $U_\varrho(\mathfrak{p})$, the Quantum Group dual to $\mathcal{C}_\varrho(P)$, obtained via the deformation of the universal enveloping algebra of Poincaré. Unlike the $\kappa$-Poincaré case, we can follow the discussion carried on in Subsection 1.6.3 (see [45]), based on the introduction of the Drinfel'd twist to obtain the deformed cosector (as in (1.95)) while keeping undeformed the algebra classical.

The Drinfel'd twist for the $\varrho$-Minkowski case was first found in [50] and can be written in the form (cfr. Appendix B)

$$\mathcal{F} = e^{-\frac{i\varrho}{2}[\partial_0 \wedge (x^2\partial_1 - x^1\partial_2)]}; \tag{3.7}$$

it reproduces, defining a $\star$-product by (1.83), the commutation relations (3.1a-b) on the algebra of functions defined over $\mathbb{R}^4$. Note that, according to the result found in Subsection 1.6.3, expanding the inverse of (3.7)

$$\mathcal{F}^{-1} \approx 1 + \frac{i\varrho}{2}[\partial_0 \wedge (x^2\partial_1 - x^1\partial_2)] + \dots, \tag{3.8}$$

one obtains the classical $r$-matrix as 2 times the first order term:

$$r = i\varrho[\partial_0 \wedge (x^2\partial_1 - x^1\partial_2)] = -i\varrho[P_0 \wedge M_{12}], \tag{3.9}$$

that is exactly (3.2), obtained resolving the CYBE.

Now, while the Poincaré algebra sector remains undeformed (see eqs.(1.24a-c)), for the coproduct we will use the prescription (1.95), so that the Leibniz rule will twist and $\forall X \in \mathfrak{p}$ :

$$X \triangleright f \star g \doteq \mu_\star \circ \Delta_\mathcal{F}(X)(f \otimes g). \tag{3.10}$$

Since the twist operator is an exponential, to find the coproducts we can use the result (obtained by induction by expanding the exponential):

$$e^{tX} Y e^{-tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!}[X, Y]^{(n)}, \tag{3.11}$$

where $[X,Y]^{(n)}$ is an iteration of Lie brackets such that $[X,Y]^{(0)} = Y$, $[X,Y]^{(1)} = [X,Y]$, $[X,Y]^{(2)} = [X,[X,Y]]$ and so on. Formally $[X,Y]^{(n)}$ stands for the n-th power of the adjoint action of $X$ on $Y$. In this way, every coproduct of a generator $Y \in \mathfrak{p}$ will be deformed as

$$\Delta_{\mathcal{F}} Y = \mathcal{F}(Y \otimes 1 + 1 \otimes Y)\mathcal{F}^{-1} = \sum_{n=0}^{\infty} \frac{(i\varrho)^n}{2^n n!}[P_0 \wedge M_{12}, (Y \otimes 1 + 1 \otimes Y)]^{(n)}. \quad (3.12)$$

To fix the idea we give, now, the first two examples of the calculation. We start from the 0th order of a general coproduct:

$$\Delta_{\mathcal{F}}^{(0)} Y = [P_0 \wedge M_{12}, (Y \otimes 1 + 1 \otimes Y)]^{(0)} = Y \otimes 1 + 1 \otimes Y; \quad (3.13)$$

then the first order would be

$$\Delta_{\mathcal{F}}^{(1)} Y = \frac{i\varrho}{2}[P_0 \wedge M_{12}, (Y \otimes 1 + 1 \otimes Y)]. \quad (3.14)$$

Considering the case $Y = P_0$, the 0th order is the undeformed coproduct $P_0 \otimes 1 + 1 \otimes P_0$, while for the first we must compute $[P_0 \wedge M_{12}, P_0 \otimes 1] - [P_0 \wedge M_{12}, 1 \otimes P_0]$. Recalling that $[A \otimes B, X \otimes 1] = [A,X] \otimes B$, the expression becomes $[P_0, P_0] \wedge M_{12} - P_0 \wedge [M_{12}, P_0]$, but from (1.24a-c) we know the two commutators vanish, so that the first order is 0. Since the following orders are reiterated commutators of $P_0 \wedge M_{12}$ with 0, we obtain the result

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0. \quad (3.15)$$

Note that this result applies to all the generators commuting with $P_0$ and $M_{12}$, and therefore we expect undeformed coproducts for $P_3$ and $M_{12}$ as well.

Since this first case was trivial, we move on to the next, to show a concrete deformation. Consider $Y = P_1$. The 0th order is given by the undeformed coproduct $P_1 \otimes 1 + 1 \otimes P_1$. For the first order we have

$$\begin{aligned} \Delta_{\mathcal{F}}^{(1)} P_1 &= \frac{i\varrho}{2}[P_0 \wedge M_{12}, (P_1 \otimes 1 + 1 \otimes P_1)] = \\ &= \frac{i\varrho}{2}([P_0, P_1] \wedge M_{12} + P_0 \wedge [M_{12}, P_1]) = \frac{\varrho}{2} P_0 \wedge P_2, \end{aligned} \quad (3.16)$$

while for the second:

$$\begin{aligned} \Delta_{\mathcal{F}}^{(2)} P_1 = {} & \frac{i\varrho^2}{8}[P_0 \wedge M_{12}, P_0 \wedge P_2] = \\ = {} & \frac{i\varrho^2}{8}(P_0P_0 \otimes M_{12}P_2 - P_0P_2 \otimes M_{12}P_0 - M_{12}P_0 \otimes P_0P_2 + M_{12}P_2 \otimes P_0P_0 + \\ & - P_0P_0 \otimes P_2M_{12} + P_0M_{12} \otimes P_2P_0 + P_2P_0 \otimes P_0M_{12} - P_2M_{12} \otimes P_0P_0) = \\ = {} & \frac{i\varrho^2}{8}(P_0^2 \odot [M_{12}, P_2] - P_0P_2 \otimes M_{12}P_0 - M_{12}P_0 \otimes P_0P_2 + \\ & + P_0M_{12} \otimes P_2P_0 + P_2P_0 \otimes P_0M_{12}) = \end{aligned} \quad (3.17)$$

$$
\begin{aligned}
&= \frac{i\varrho^2}{8}(P_0^2 \odot [M_{12}, P_2] - P_0P_2 \otimes M_{12}P_0 - M_{12}P_0 \otimes P_0P_2 + \\
&\qquad + P_0M_{12} \otimes P_2P_0 + P_2P_0 \otimes P_0M_{12}),
\end{aligned}
$$

where $\odot$ is the tensor product symmetrized ($A \odot B = A \otimes B + B \otimes A$); summing and subtracting $P_2P_0 \otimes M_{12}P_0$ and $P_0M_{12} \otimes P_0P_2$:

$$
\begin{aligned}
\Delta_{\mathcal{F}}^{(2)} P_1 = &\frac{i\varrho^2}{8}(P_0^2 \odot [M_{12}, P_2] + [P_2, P_0] \otimes M_{12}P_0 + [P_0, M_{12}] \otimes P_0P_2 + \\
&\quad + P_0M_{12} \otimes [P_2, P_0] + P_2P_0 \otimes [P_0, M_{12}]) = \\
= &\frac{i\varrho^2}{8}(P_0^2 \odot [M_{12}, P_2] + [P_0, M_{12}] \otimes P_0P_2 + P_2P_0 \otimes [P_0, M_{12}]) = \\
= &\frac{-\varrho^2}{8} P_0^2 \odot P_1.
\end{aligned}
\tag{3.18}
$$

Computing the higher orders one obtains

$$
\begin{aligned}
\Delta_{\mathcal{F}} P_1 &= \sum_{n=0}^{\infty} \left[ \frac{(i\varrho)^{2n}}{2^{2n} 2n!} P_1 \odot P_0^{2n} + \frac{(i\varrho)^{2n+1}}{2^{2n+1}(2n+1)!} iP_2 \wedge P_0^{2n+1} \right] = \\
&= P_1 \odot \sum_{n=0}^{\infty} \frac{(i\varrho)^{2n}}{2^{2n} 2n!} P_0^{2n} + P_2 \wedge \sum_{n=0}^{\infty} \frac{(i\varrho)^{2n+1}}{2^{2n+1}(2n+1)!} iP_0^{2n+1} = \\
&= P_1 \odot \cos\left(\frac{\varrho}{2}P_0\right) + P_2 \wedge \sin\left(\frac{\varrho}{2}P_0\right).
\end{aligned}
\tag{3.19}
$$

The result follows identical for $P_2$ working the substitutions $P_1 \to P_2$, $P_2 \to P_1$ and noting that since the commutators of $P_2$ with $M_{12}$ have different sign with respect to that of $P_1$ and $M_{12}$, a minus shows up in the antisymmetric part.

The same calculation can be performed for all the other coproducts, and the cosector will then be given by [19]

$$
\Delta_{\mathcal{F}} P_0 = P_0 \otimes 1 + 1 \otimes P_0, \tag{3.20a}
$$

$$
\begin{aligned}
\Delta_{\mathcal{F}} P_1 = &P_1 \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes P_1 + \\
&+ P_2 \otimes \sin\left(\frac{\varrho}{2}P_0\right) - \sin\left(\frac{\varrho}{2}P_0\right) \otimes P_2,
\end{aligned}
\tag{3.20b}
$$

$$
\begin{aligned}
\Delta_{\mathcal{F}} P_2 = &P_2 \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes P_2 + \\
&- P_1 \otimes \sin\left(\frac{\varrho}{2}P_0\right) + \sin\left(\frac{\varrho}{2}P_0\right) \otimes P_1,
\end{aligned}
\tag{3.20c}
$$

$$
\Delta_{\mathcal{F}} P_3 = P_3 \otimes 1 + 1 \otimes P_3, \tag{3.20d}
$$

$$
\begin{aligned}
\Delta_{\mathcal{F}} M_{10} = &M_{10} \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes M_{10} + \\
&+ M_{20} \otimes \sin\left(\frac{\varrho}{2}P_0\right) - \sin\left(\frac{\varrho}{2}P_0\right) \otimes M_{20},
\end{aligned}
\tag{3.20e}
$$

$$
\begin{aligned}
\Delta_{\mathcal{F}} M_{20} = &M_{20} \otimes \cos\left(\frac{\varrho}{2}P_0\right) + \cos\left(\frac{\varrho}{2}P_0\right) \otimes M_{20} + \\
&- M_{10} \otimes \sin\left(\frac{\varrho}{2}P_0\right) + \sin\left(\frac{\varrho}{2}P_0\right) \otimes M_{10},
\end{aligned}
\tag{3.20f}
$$

$$\Delta_{\mathcal{F}} M_{30} = M_{30} \otimes 1 + 1 \otimes M_{30} - \frac{\varrho}{2} P_3 \otimes M_{12} + \frac{\varrho}{2} M_{12} \otimes P_3, \tag{3.20g}$$

$$\Delta_{\mathcal{F}} M_{12} = M_{12} \otimes 1 + 1 \otimes M_{12}, \tag{3.20h}$$

$$\begin{aligned}\Delta_{\mathcal{F}} M_{31} = & M_{31} \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes M_{31} + \\ & + M_{32} \otimes \sin\left(\frac{\varrho}{2} P_0\right) - \sin\left(\frac{\varrho}{2} P_0\right) \otimes M_{32} + \\ & - P_1 \otimes \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) + \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) \otimes P_1 + \\ & - P_2 \otimes \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) - \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) \otimes P_2,\end{aligned} \tag{3.20i}$$

$$\begin{aligned}\Delta_{\mathcal{F}} M_{32} = & M_{32} \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes M_{32} + \\ & - M_{31} \otimes \sin\left(\frac{\varrho}{2} P_0\right) + \sin\left(\frac{\varrho}{2} P_0\right) \otimes M_{31} + \\ & - P_2 \otimes \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) + \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) \otimes P_2 + \\ & + P_1 \otimes \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) + \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) \otimes P_1.\end{aligned} \tag{3.20j}$$

These structures, along with undeformed trivial counits and undeformed antipodes (1.47c-d) form the Quantum Group $U_\varrho(\mathfrak{p})$.

Analogously to the $\kappa$-Poincaré case, it is possible to demonstrate that the commutation relations (3.1a-b) (seen in the $\star$-commutation picture) are covariant under the action of $\varrho$-Poincaré generators through the coproducts given above, as one can see noting that from (3.10) it follows:

$$X \triangleright [f, g]_\star = \mu_\star \circ \Delta_{\mathcal{F}}(X)(f \otimes g - g \otimes f), \tag{3.21}$$

and working the explicit calculations.

## 3.3 Localizability in $\varrho$-Minkowski

Inspired by the analysis carried on in [43], we now mimic the discussion made in Section 2.3 for $\kappa$-Minkowski applying it to our $\varrho$-deformation case. In Subsection 3.3.1, following [45], we derive the uncertainty relations for $\varrho$-Minkowski, construct a useful realization of the spacetime event operators and solve the eigenvalue problem for the time operator. In Subsection 3.3.2 we present some states in the chosen realization to analyze their localizability properties in the origin. In 3.3.3 we derive the novel Quantum Group uncertainty relations and analyze the resulting constraints on pure transformations, then we construct the first known $\varrho$-Poincaré realization, working in analogy with the discussion made for $\kappa$-Poincaré. Finally, in Subsection 3.3.4 we discuss for the first time localizability in $\varrho$-Minkowski in relation to observers and observables, as done before for the $\kappa$-case.

### 3.3.1 $\mathcal{M}_\varrho$ coordinate realization

Firstly we rewrite the algebra (3.1a-b) in a tensorial way as

$$[x^\mu, x^\nu] = i\varrho\epsilon^{\mu\nu}{}_{j3}x^j, \tag{3.22}$$

with $\epsilon^{\mu\nu}{}_{j3}$ the 4D Levi-Civita symbol such that $\epsilon^{0i}{}_{j3} = \epsilon^i{}_{j3} = -\epsilon^{i0}{}_{j3}$; then we compute the uncertainty relations (2.50) for (3.22):

$$\Delta x^\mu \Delta x^\nu \geq \frac{\varrho}{2}\left|\epsilon^{\mu\nu}{}_{j3}\langle x^j\rangle\right|, \tag{3.23}$$

the only nontrivial relation being

$$\Delta x^0 \Delta x^i \geq \frac{\varrho}{2}\left|\epsilon^i{}_{j3}\langle x^j\rangle\right|. \tag{3.24}$$

Note that, by the centrality of $x^3$, this coordinate can be determined with absolutely precision.

A realization of $\varrho$-Minkowski is given by [45]

$$x^i\psi(x) = x^i\psi(x), \tag{3.25a}$$
$$x^0\psi(x) = -i\varrho(x^1\partial_2 - x^2\partial_1)\psi(x), \tag{3.25b}$$

with $x^i$ a complete set of observables on the Hilbert space $L^2(\mathbb{R}^3)$, $x^0$ a self-adjoint operator on $L^2(\mathbb{R}^3)$ acting like an angular momentum along the 3 axis, and $\psi(x)$ a state in the Hilbert space. The same considerations made for the $\kappa$-case apply unchanged.

Analogously to the case of $\kappa$-Minkowski, we can choose a more convenient way of writing commutators and uncertainty relations, given by the fact that the $\varrho$-deformation is of angular nature. We define

$$r = \sqrt{(x^1)^2 + (x^2)^2}, \tag{3.26a}$$
$$z = x^3, \tag{3.26b}$$
$$\varphi = \arctan\frac{x^2}{x^1}. \tag{3.26c}$$

As in the previous case, we take $e^{i\varphi}$ instead of $\varphi$, for the latter is a multivalued function and it cannot be promoted to a self-adjoint operator, so that (3.1a-b) becomes:

$$[x^0, r] = 0, \tag{3.27a}$$
$$[x^0, z] = 0, \tag{3.27b}$$
$$[x^0, e^{i\varphi}] = \varrho e^{i\varphi}. \tag{3.27c}$$

In this way, we have two complete sets of operators given by $(r, z, \varphi)$ and $(r, z, x^0)$, with $r, z, \varphi$ acting as multiplication operators and $x^0$ as an angular momentum along the 3 axis

$$x^0\psi(r, z, x^0) = -i\varrho\partial_\varphi\psi(r, z, x^0). \tag{3.28}$$

Since $x^0$ acts as an angular momentum we already know how to solve the eigenvalue problem

$$-i\varrho\partial_\varphi\chi(\varphi) = n\varrho\chi(\varphi); \tag{3.29}$$

$n\varrho$ are eigenvalues associated to eigenstates

$$\chi(\varphi) = e^{in\varphi}. \tag{3.30}$$

The connection between the two bases can be given, therefore, by a Fourier expansion of the angular term:

$$\psi(r, z, \varphi) = \sum_{n=-\infty}^{\infty} \psi_n(r, z)e^{in\varphi}. \tag{3.31}$$

As pointed out in [45], we have obtained the interesting result that in this case the spectrum of time is discrete being the whole of $\mathbb{Z}$. A first conclusion seems to be that there exists a unique clock that counts universally the discretized time instants. We may ask, then, if an observer could identify an exact point in the universal time, for example the origin. At first we note that the operator $i\partial_\varphi$ is self-adjoint in other domains aside of periodic functions $\chi$. A possibility is to consider

$$\chi_\alpha(\varphi) = e^{i(n+\alpha)\varphi}, \tag{3.32}$$

with $e^{i\alpha}$ a generic phase. In this case the spectrum of time operator would be $\varrho(n+\alpha)$ and while the eigenvalues of measured time depend on the basis, the difference between them is unchanged. An observer, thus, could only measure quantized intervals instead of a universal quantized time, not knowing the phase $\alpha$.

### 3.3.2 Localized states

Eigenstates of $\varphi$ are given by a Fourier superposition

$$\delta(\varphi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\varphi}. \tag{3.33}$$

Suppose, now, to measure sharply an eigenvalue $\varrho\bar{n}$ of the time operator. The system would be in an eigenstate of time $\bar{\chi}(\varphi) = e^{i\bar{n}\varphi}$, so that we would have complete delocalization in $\varphi$. If the measure has instead some degree of uncertainty in time, we would have a finite sum in (3.33) over the available elements of the basis, and this would give, in turn, a degree of uncertainty in $\varphi$, as in the ordinary QM angular momentum theory.

From (3.24) we expect, however, sharp spacetime localization be possible in the case $x^1 = x^2 = 0$. In our cylindrical coordinates this corresponds to perfect localization in $r = 0$. Since $r$ commutes with $z$ and $x^0$ (eq.(3.27a)) we can find without issues a state that is both localized in $r$ as well as in $z$ and $x^0$. A state of this kind can be constructed as follows:

$$\psi(r,z,x^0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-n_0)\varphi} d\varphi \phi(r,z), \tag{3.34}$$

where the first term is a $\delta(n-n_0)$, that gives a state localized in time at $n_0$ and $\phi(r,z)$ are functions of $r$ and $z$ localized around values $r_0$, $z_0$. For $\phi(r,z)$ it is sufficient to take a factorized product of two states in the Hilbert space (e.g. Gaussian distributions) that tend to delta distributions in the limit of their amplitudes$\rightarrow 0$ (e.g. the Gaussian variances$\rightarrow 0$). At this point, from (3.26), $x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$, but $\varphi$ is completely undetermined since we are in an eigenstate of $x^0$. Computing the mean values on the state, $\langle x^1\rangle = r_0\cos\varphi$ and $\langle x^2\rangle = r_0\sin\varphi$, and perfect localization in $x^\mu$ is possible only if $r_0 = 0$. We obtain, then, a 2-parameters localized family of states $|o_{n,z}\rangle$. In the particular case of $n_0 = z_0 = 0$ we can define a localized origin state $|o\rangle$.

### 3.3.3 $\varrho$-Poincaré realization

Since also in the $\varrho$-case the symmetry group is deformed, we expect localization problems to arise also in observer transformations. In analogy with eqs.(2.69a-c) we propose the introduction of novel uncertainty relations for $\varrho$-Poincaré in the form:

$$\Delta a^\mu \Delta a^\nu \geq \frac{\varrho}{2}|\delta^\nu{}_0(\langle a_2\rangle\delta^\mu{}_1 - \langle a_1\rangle\delta^\mu{}_2) - \delta^\mu{}_0(\langle a_2\rangle\delta^\nu{}_1 - \langle a_1\rangle\delta^\nu{}_2)|, \tag{3.35a}$$

$$\Delta\Lambda^\mu{}_\alpha \Delta\Lambda^\nu{}_\beta \geq 0, \tag{3.35b}$$

$$\Delta\Lambda^\mu{}_\nu \Delta a^\rho \geq \frac{\varrho}{2}|\delta^\varrho{}_0(\langle\Lambda_{2\nu}\rangle\delta^\mu{}_1 - \langle\Lambda_{1\nu}\rangle\delta^\mu{}_2) - \langle\Lambda^\varrho{}_0\Lambda^\mu{}_1\rangle g_{2\nu} + \langle\Lambda^\varrho{}_0\Lambda^\mu{}_2\rangle g_{1\nu}|. \tag{3.35c}$$

We note, again, that while performing only $\varrho$-Lorentz transformations new uncertainties do not show up, in the case of translations or mixed transformations we must take into account localizability issues coming from eqs.(3.35a,c).

We perform, now, the analysis of relations (3.35a-c) to see if pure $\varrho$-Poincaré transformations can be made. We start with the case of pure $\varrho$-Lorentz transformations; from $\langle a^\mu\rangle = 0$, $\Delta a^\mu = 0$, the relevant constraint is $\delta^\varrho{}_0(\langle\Lambda_{2\nu}\rangle\delta^\mu{}_1 - \langle\Lambda_{1\nu}\rangle\delta^\mu{}_2) - \langle\Lambda^\varrho{}_0\Lambda^\mu{}_1\rangle g_{2\nu} + \langle\Lambda^\varrho{}_0\Lambda^\mu{}_2\rangle g_{1\nu} = 0$ that, in a way similar to the case of $\kappa$-Poincaré discussed in [53], admits a solution for $\langle\Lambda^\mu{}_0\rangle = \delta^\mu{}_0$, $\langle\Lambda^\mu{}_1\rangle = \delta^\mu{}_1$, $\langle\Lambda^\mu{}_2\rangle = \delta^\mu{}_2$, so that the only free parameter is $\Lambda^3{}_3$, but since $\Lambda \in SO(1,3)$ have determinant$= 1$, it follows that the only allowable form for $\Lambda^\mu{}_3$ is $\delta^\mu{}_3$, and so the only pure $\varrho$-Lorentz transformation is the identical one. For the case of pure translations $\langle\Lambda^\mu{}_\nu\rangle = \delta^\mu{}_\nu$ and $\Delta\Lambda^\mu{}_\nu = 0$; substituting in (3.35c) we see that the relation is satisfied, and the only relevant condition is (3.35a). Since $a^3$ is central in the algebra, pure translations along the 3-axis do exist without issues; considering a pure time translation the conditions to impose on (3.35a) are that $\langle a^i \rangle = 0$ and $\Delta a^i = 0$ and the equation is trivially satisfied. For pure translations along the 1- and 2-axis the result is different since considering, for example, the first case one would have $\langle a^2 \rangle = 0$, that is compatible with $\Delta x^0 = 0$, but this last condition impose also that $\langle a^1 \rangle = 0$, the same being true inverting $a^1$ and $a^2$. This means that $\varrho$-Poincaré admits only pure time translations and pure space translations along the 3-axis as pure translations. Turning our attention to the identical transformation $\langle a^\mu \rangle = 0$, $\langle \Lambda^\mu{}_\nu \rangle = \delta^\mu{}_\nu$, $\Delta a^\mu = 0$, $\Delta\Lambda^\mu{}_\nu = 0$, we see that the uncertainty relations are satisfied, therefore, as we expected, there exists the identity in $\varrho$-Poincaré.

We want now to find a realization for the $\varrho$-Poincaré group. This problem, to our knowledge, was never discussed in literature about the $\varrho$-deformation, therefore we try to mimic the approach used in Section 2.3.3.

We start noting that also in this case the $\Lambda$'s commute with each other, so for the Lorentz sector the standard realization (2.70) can be considered. For the $a$'s, we start considering the commutation relation (3.6c) and formulate the ansatz

$$a^\varrho = i\varrho\left[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})\right]\frac{\partial}{\partial\Lambda^\mu{}_\nu}. \tag{3.36}$$

(3.36) satisfies automatically (3.6c), while (2.70) is in accordance with (3.6b). To have a realization of the group, therefore, we must show that (3.36) is coherent with (3.6a).

Let us start making more explicit the form of the translational commutators imposing (3.36) on (3.6a):

$$\begin{aligned}[a^\mu, a^\nu] &= -i\varrho[\delta^\nu{}_0(a_2\delta^\mu{}_1 - a_1\delta^\mu{}_2) - \delta^\mu{}_0(a_2\delta^\nu{}_1 - a_1\delta^\nu{}_2)] = \\ &= i\varrho(\delta^\mu{}_0 a_2\delta^\nu{}_1 - \delta^\mu{}_0 a_1\delta^\nu{}_2 - \delta^\nu{}_0 a_2\delta^\mu{}_1 + \delta^\nu{}_0 a_1\delta^\mu{}_2),\end{aligned} \tag{3.37}$$

now, being the metric tensor diagonal, the term in $\delta$ in (3.36) vanishes for $a_1$ and $a_2$, so that we are left with

$$\begin{aligned}[a^\mu, a^\nu] = &- \varrho^2(-\delta^\mu{}_0\Lambda_{20}\delta^\nu{}_1 + \delta^\mu{}_0\Lambda_{10}\delta^\nu{}_2 + \delta^\nu{}_0\Lambda_{20}\delta^\mu{}_1 - \delta^\nu{}_0\Lambda_{10}\delta^\mu{}_2)\times \\ &\times (\Lambda^\alpha{}_1 g_{2\beta} - \Lambda^\alpha{}_2 g_{1\beta})\frac{\partial}{\partial\Lambda^\alpha{}_\beta}.\end{aligned} \tag{3.38}$$

We compute, now, the commutators employing (3.36), starting from $a^\varrho a^\sigma$ acting on a function $\phi(\omega)$:

$$\begin{aligned}a^\varrho a^\sigma\phi(\omega) = &- \varrho^2\left[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})\right]\left[\delta^\sigma{}_0(g_{2\mu}\delta^\nu{}_\lambda\delta^\delta{}_1 + \right. \\ &\left. - g_{1\mu}\delta^\nu{}_\lambda\delta^\delta{}_2) + \delta^\sigma{}_\mu\delta^\nu{}_0(\Lambda^\delta{}_2 g_{1\lambda} - \Lambda^\delta{}_1 g_{2\lambda}) + \Lambda^\sigma{}_0(\delta^\delta{}_\mu\delta^\nu{}_2 g_{1\lambda} - \delta^\delta{}_\mu\delta^\nu{}_1 g_{2\lambda})\right] \times \\ &\times \frac{\partial\phi}{\partial\Lambda^\delta{}_\lambda} - \varrho^2\left[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})\right] \times \\ &\times \left[\delta^\sigma{}_0(\Lambda_{2\lambda}\delta^\delta{}_1 - \Lambda_{1\lambda}\delta^\delta{}_2) - \Lambda^\sigma{}_0(\Lambda^\delta{}_1 g_{2\lambda} - \Lambda^\delta{}_2 g_{1\lambda})\right]\frac{\partial^2\phi}{\partial\Lambda^\mu{}_\nu\partial\Lambda^\delta{}_\lambda}.\end{aligned} \tag{3.39}$$

The same calculation can be made for $a^\sigma a^\varrho$, and since $\mu, \nu, \delta, \lambda$ are dummy indices, one obtains the same result of (3.39) with the exchange of $\varrho$ and $\sigma$. This implies that terms symmetric in those indices cancel out, and after a bit of manipulation we are left with

$$\begin{aligned}[a^\varrho, a^\sigma] = \varrho^2 \big[&\delta^\varrho{}_0(\Lambda_{20}\delta^\sigma{}_1 - \Lambda_{10}\delta^\sigma{}_2)(\Lambda^\delta{}_2 g_{1\lambda} - \Lambda^\delta{}_1 g_{2\lambda}) + \\ &- \delta^\sigma{}_0(\Lambda_{20}\delta^\varrho{}_1 - \Lambda_{10}\delta^\varrho{}_2)(\Lambda^\delta{}_2 g_{1\lambda} - \Lambda^\delta{}_1 g_{2\lambda}\big] \frac{\partial\phi}{\partial\Lambda^\delta{}_\lambda}.\end{aligned} \tag{3.40}$$

Comparing (3.40) to (3.38), we note that the two expressions coincide and therefore (2.70),(3.36) give a true realization of $\varrho$-Poincaré.

Finally, although the faithfulness of (3.36) has not yet been proven, we enlarge the realization in analogy with the $\kappa$-case adding the realization (3.25a-b) of $\varrho$-Minkowski:

$$\begin{aligned}a^\varrho = i\frac{\varrho}{2}\left[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})\right] \frac{\partial}{\partial\Lambda^\mu{}_\nu} + \\ + i\frac{\varrho}{2}[\delta^\varrho{}_i q^i - \delta^\varrho{}_0(q^1\partial_2 - q^2\partial_1)] + \frac{1}{2}h.c.,\end{aligned} \tag{3.41}$$

defined on an Hilbert space $L^2(SO(1,3)\times\mathbb{R}^3)$.

### 3.3.4 Observers and observables

We turn, now, our attention to the problem of localizability in relation to the enlarged states of a realization of the tensor product $C_\varrho(P)\otimes\mathcal{M}_\varrho$. We can find the action of elements $x'^\mu \in C_\varrho(P)\otimes\mathcal{M}_\varrho$ on functions $f(\omega,q,x)\in L^2(SO(1,3)\times\mathbb{R}^3_q)\times L^2(\mathbb{R}^3_x)\sim L^2(SO(1,3)\times\mathbb{R}^3_q\times\mathbb{R}^3_x)$ by means of the direct sum of realizations (3.25a-b),(3.41):

$$\begin{aligned}x'^\varrho f(\omega,q,x) =& i\varrho\Lambda^\varrho{}_\sigma(\delta^\sigma{}_i x^i - \delta^\sigma{}_0(x^1\partial_{x_2} - x^2\partial_{x_1})]f(\omega,q,x) + \\ &+ i\frac{\varrho}{2}\left[\delta^\varrho{}_0(\Lambda_{2\nu}\delta^\mu{}_1 - \Lambda_{1\nu}\delta^\mu{}_2) - \Lambda^\varrho{}_0(\Lambda^\mu{}_1 g_{2\nu} - \Lambda^\mu{}_2 g_{1\nu})\right] \frac{\partial}{\partial\Lambda^\mu{}_\nu} f(\omega,q,x) + \\ &+ i\frac{\varrho}{2}[\delta^\varrho{}_i q^i - \delta^\varrho{}_0(q^1\partial_{q_2} - q^2\partial_{q_1})]f(\omega,q,x) + \frac{1}{2}h.c.\end{aligned} \tag{3.42}$$

Our Hilbert space admits the same decomposition as (2.81), allowing the same interpretation in terms of the tensorial product of an Hilbert space of observers and a Hilbert space of observables.

Mean values and variances of $\varrho$-Poincaré-transformed observers in relation to generic states retains the same forms of eqs.(2.82),(2.83).

#### Identity transformation state

We define the identity state $|i\rangle$ in $\mathcal{C}_\varrho(P)$, given as follows on functions $f(a,\Lambda)\in\mathcal{C}_\varrho(P)$:

$$\langle i|f(a,\Lambda)|i\rangle = \varepsilon(f). \tag{3.43}$$

The state corresponding to this transformation can be found with the same methods outlined in Subsection 2.3.4. As noted in the $\kappa$-deformation case, this state gives the identity transformation of the function $f$. We can, therefore, define

$$|i, \psi\rangle = |i\rangle \otimes |\psi\rangle, \tag{3.44}$$

as the $\varrho$-Poincaré transformation between two coincident observers. In fact, being the counits of $\mathcal{C}_\varrho(P)$ the same as those of $\mathcal{C}_\kappa(P)$, the discussion made in Subsection 2.3.4 applies invariate, and one obtains the result:

$$\langle x'^\mu \rangle = \langle \psi | x^\mu | \psi \rangle. \tag{3.45}$$

The same applies to monomials in coordinates $x'^{\mu_1} \cdots x'^{\mu_n}$, leading to eq.(2.89). Calculating the uncertainties we obtain, again, that

$$\Delta(x'^\mu)^2 = \Delta(x^\mu)^2, \tag{3.46}$$

the calculation be the same of that of $\kappa$-Poincaré.

We have recovered the result that since localization problems arise only when taking translations into account, coincident observers described by (2.84) are well-defined in $\varrho$-Minkowski and they agree on their measurements.

**Origin state transformations**

We have seen in Subsection 3.3.2 that there exist in $\varrho$-Minkowski states perfectly localized in the origin. We can, therefore, analyze $\varrho$-Poincaré transformations of the origin. The state on which we perform this transformation is, again,

$$|\phi, o\rangle = |\phi\rangle \otimes |o\rangle, \tag{3.47}$$

so that we recover the same result valid in $\kappa$-Poincaré:

$$\langle x'^\mu \rangle = \langle \phi | a^\mu | \phi \rangle. \tag{3.48}$$

The result remains true for a generic monomial in coordinates $x'^{\mu_1} \cdots x'^{\mu_n}$ (as shown in Subsection 2.3.4), and we can compute the uncertainty of the transformed event:

$$\Delta(x'^\mu)^2 = \langle (x'^\mu)^2 \rangle - \langle x'^\mu \rangle^2 = \langle (a^\mu)^2 \rangle - \langle a^\mu \rangle^2 = \Delta(a^\mu)^2. \tag{3.49}$$

**Translations**

Since the Lorentz sector of $\varrho$-Poincaré is undeformed as that of $\kappa$-Poincaré, states corresponding to pure translations can be found with an identical procedure as that outilned in Subsection 2.3.4. If we consider pure translations $x'^\mu = 1 \otimes x^\mu + a^\mu \otimes 1$ of a generic state $|\phi\rangle \otimes |\psi\rangle$, we obtain the usual results

$$\langle x'^\mu \rangle = \langle \phi | \otimes \langle \psi | (1 \otimes x^\mu + a^\mu \otimes 1) | \phi \rangle \otimes | \psi \rangle = \langle \psi | x^\mu | \psi \rangle + \langle \phi | a^\mu | \phi \rangle, \tag{3.50a}$$

$$\Delta(x'^{\mu})^2 = \langle (x^{\mu})^2 + (a^{\mu})^2 + x^{\mu}a^{\mu} + a^{\mu}x^{\mu}\rangle - \langle x^{\mu}\rangle^2 - \langle a^{\mu}\rangle^2 - 2\langle x^{\mu}\rangle\langle a^{\mu}\rangle = \\ = \Delta(x^{\mu})^2 + \Delta(a^{\mu})^2. \tag{3.50b}$$

We recover the inequality

$$\Delta(x'^{\mu})^2 = \Delta(x^{\mu})^2 + \Delta(a^{\mu})^2 \geq \Delta(x^{\mu})^2, \tag{3.51}$$

from which one sees that acting with a pure translation leads in general with an increase in the state uncertainty. The difference from the $\kappa$-case is in the situations in which the translational parameter has zero uncertainty and pure translations do not decrease localizability. While this certainly occurs in the identity transformation case, the second case is when the right-hand side of eq.(3.35a) vanishes, i.e. when $\langle a^1\rangle = \langle a^2\rangle = 0$. Differently to $\kappa$-Poincaré where we had that only pure temporal translations did not decrease the localizability of a state, in our case, since $a^3$ is a central coordinate, we have also the case of pure translations along $a^3$ or even mixed translations in $a^0$, $a^3$.

It would be interesting to see if also in $\varrho$-Poincaré there exists some transformation that leads to a decrease in uncertainty as in the $\kappa$-case.

# Chapter 4

# Conclusions

Here we draw our conclusions about the studied models. In Section 4.1 we recall the main results obtained in the $\kappa$- and $\varrho$-Minkowski spacetimes, and compare them to highlight their similarities and differences in relation to their definitory properties. In Section 4.2 we present some possible future research prospectives starting from the analysis we have worked out.

## 4.1 Comparing the models

In this work we have dealt with two particular kind of Lie algebra-type noncommutative spacetimes and deformations of their relative Poincaré groups, $\kappa$-Minkowski and the lesser-known $\varrho$-Minkowski. It is now interesting to compare the two models employing the results obtained.

We start noting that the main difference between them is in the nature of the commutation relations. While eqs.(1.27a-b) are clearly of radial nature, (1.28a-b) are explicitly of an angular one. In the first case there are not central coordinates, while in the latter $x^3$ commutes with every other, so that it is legitimate to think that this coordinate can be determined without any uncertainty and will not pose any problem in localizability. Another fundamental difference, this time between the Poincaré Quantum Groups, is the bicrossproduct structure of the $\kappa$'s ones versus the quasitriangularity of the $\varrho$'s; this reflects in two very different procedures to obtain the Quantum Universal Enveloping Algebras, since the Drinfel'd twist method is employable only in the case of a quasitriangular Hopf algebra.

Turning our attention to the problem of localizability, we have shown that the deformed nature of Poincaré Quantum Groups leads to the unintuitive feature of having uncertainties arising from deformed Poincaré transformations. This imply that two different observers will in general not agree on localizability properties of the same state both in the $\kappa$- as in the $\varrho$-case. The localizability properties of the Quantum Groups can be intuitively seen by writing uncertainty relations between the noncommutative group parameters. These relations, surprisingly, pose constraints on the possible deformed-Poincaré transformations; for example we have seen that pure spacial translations and pure boosts do not exist in $\kappa$-Poincaré and the same is true for pure translation along the 1- and 2-axis and pure Lorentz transformations in $\varrho$-Poincaré. The physical consequences of these results, we feel, are worth to be studied.

In the first spacetime model we have shown that perfect localization of observable states can be achieved in the spacial origin, coherently with the fact that in the given realization the only nonmultiplicative operator ($x^0$) was of dilational nature, while in the second model the "special position" is at $x^1 = x^2 = 0$, in accordance with the angular nature of the only nonmultiplicative operator ($x^0$) that acts as an angular momentum along the 3-axis.

Surprisingly, turning our attention to the Quantum Groups, we have found that a realization of $\varrho$-Poincaré can be constructed in the same way as that of $\kappa$-Poincaré discussed in Section 2.3.3. Starting from these realizations and constructing realizations for elements $x'^\mu$ useful to deal with and interpret coaction transformations $x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$, we have analyzed in both cases localizability properties of particular kind of states, i.e. Poincaré identity transformations, Minkowski origin transformations and Poincaré translations. In the first case, since the dependence on the Hopf algebra structure was only in the counits, the results were the same in both $\kappa$- and $\varrho$- cases: identity transformations do not increase uncertainties, as we expected. Even origin transformations deal the same results in both cases, since there exist states that perfectly localize in the spacetime origin for both the spacetimes. While from the undeformed nature of the Lorentz sector in both cases pure deformed Lorentz transformations do not cause problems, translations works differently in the two cases. Although translations in general increase the uncertainty, in the $\kappa$-case a pure temporal one does not, while in the $\varrho$-case this is true for pure temporal ones, pure spacial ones along $x^3$ and mixed ones of $x^0$ and $x^3$, due to the angular symmetry of the commutation relations. The general case of mixed Lorentz-translation transformations requires more study, and as said before it would be interesting to see if also in the $\varrho$-case there exist transformations that lead to a decrease in state uncertainty, and how this feature can be physically interpreted.

The last thing we note is that it is possible to consider many other types of noncommutative spacetimes such as the angular one defined by commutation relations

$$[x^3, x^i] = i\sigma\epsilon^i{}_{j3}x^j, \quad i, j = 1, 2, 3, \tag{4.1a}$$

$$[x^0, x^i] = 0, \tag{4.1b}$$

$$[x^1, x^2] = 0, \tag{4.1c}$$

where $\sigma$ plays the role of the deformation length, that are easily related to the models we have studied. In particular, this specific case is a variation of $\varrho$-Minkowski with the roles of $x^0$ and $x^3$ interchanged. The same analysis can be performed on this spacetime and we expect the same formal results as in the $\varrho$-case, but with a different interpretation being the noncommutativity purely spacial in contrast to the mixed one of $\varrho$-Minkowski.

## 4.2 Future prospectives

In Section 1.1 we have argued that one of the reasons for choosing a noncommutative spacetime instead of a commutative one is to avoid the generation of singularities from localization at the Planck scale, arising from the combined prescriptions of GR and QM. We have, however, shown that both $\kappa$-Minkowski and $\varrho$-Minkowski have particular points that can be localized with absolutely precision by an observer; this is in full contrast with the motivation to introduce them, as was for the first time pointed out in [17]. One could argue that the origins of these spaces ultimately loose their preferred status, since different observers will in general not see them localized, but still for a single observer would be possible to sharply localize a spacetime point in its reference frame origin. This could be an evidence that points towards the choice of other types of noncommutative spacetimes, or maybe it is possible, deepening the analysis, to reconcile these features with a deeper understanding of localizability properties.

In Section 1.8 we have given an heuristic definition of the notion of an observer, in contrast with the more formal ones regarding functional states and observables. The notion of an observer is at least problematic even in ordinary QM, so we feel the need to formalize it in a way that is suitable to both commutative and noncommutative spaces. In this regard it would be interesting to try extending the tensorial definition given for example in [52].

In Subsections 2.3.3,3.3.3 we have dealt with uncertainty relations for Poincaré group parameters, showing how some constraints on the possible deformed transformations arise from them. It would be interesting to generalize the discussion to cover wider classes of noncommutative spacetimes whose Quantum Groups are known, such as those described in [50].

In Subsection 2.3.4 and 3.3.4 we have presented some specific cases of deformed finite Poincaré transformations. In particular we have analyzed for both the $\kappa$ and the $\varrho$ cases the uncertainties coming from translations. The next step would be that of considering general Lorentz and mixed transformations. It is interesting to note that in [4] was demonstrated in the Majid Ruegg basis that pure boost transformations do not exists in $\kappa$-Poincaré; an analogous investigation in the infinitesimal $\varrho$-Poicaré case would be interesting.

Note, also, that throughout the entire work we have dealt only with finite transformations, operated by means of the $C_q(P)$-type Quantum Groups. Recently, in [45], was proposed to describe active (or observer) transformations by means of $C_q(P)$, and passive (or observable) transformations by means of the dual $U_q(\mathfrak{p})$, inspired by the fact that in the classical case active transformations are more naturally considered in terms of finite transformations while passive in terms of infinitesimal ones. An analysis of localizability in the dual picture could, therefore, lead to interesting new features, maybe related to the different choices of bases.

Another interesting feature to study is the dependence of the discussion on the chosen realization. In Sections 2.3,3.3 we have worked in specific realizations of noncommutative spacetimes and Quantum Groups to derive localization properties; while we did not expect the results obtained to be dependent on the chosen realization, this question requires further analysis. In particular, to our knowledge, the realizations of $\varrho$-Minkowski and $\varrho$-Poincaré here derived are the only ones known in literature, so that further study in this direction can be crucial to fully understand localizability in that spacetime.

As noted before, in this thesis we have worked at a "pure kinematical level", avoiding to introduce momenta and dynamical effects. Connection with theories such as the aforementioned relative locality and the curved momenta spaces could make the picture more significant.

At last, our investigation does not lead directly toward a general theory of localizability in $\kappa$-Minkowski and $\varrho$-Minkowski, since as stated before we do not know a general theory of localized states in those spacetimes neither on the observer spaces. We think, however, that this is a good starting point to the future development of a full theory of localizability in general classes of noncommutative Minkowski spacetimes.

# Appendices

# Appendix A

# Hopf algebras over rings and their deformation

To show more explicitly the duality expressed by eqs.(1.41a-b), we introduce a more categorical way of representing algebras and coalgebras, extending their construction over rings rather than over fields.

Category theory deals with triples called *categories* $C(ob(C), mor(C), \circ)$, where:

(a) $ob(C)$ is a class whose elements are called *objects*,

(b) $mor(C) : ob(C) \to ob(C)$ is a class whose elements are called *morphisms* or *maps*,

(c) $\circ : mor(C) \times mor(C) \to mor(C)$ is a binary associative operation with unity that realizes the composition of morphisms.

A categorical property is usually stated by means of *commutative diagrams*, diagrams made associating every object with a vertex and every morphism with an arrow between objects. The composition of morphisms is realized by taking directed paths along arrows in the diagram between two non adjacent vertices. The commutation of such a diagram means that all directed paths with the same starting vertex and endpoint lead to the same result.

Consider for example the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow{\delta} & D \end{array}$$

$A, B, C, D$ are objects, $\alpha, \beta, \gamma, \delta$ are morphisms, $\beta \circ \alpha : A \to D$ and $\delta \circ \gamma : A \to D$ are compositions of morphisms. The diagram is commutative since the path $A \to B \to D$ gives the same result as $A \to C \to D$.

Recall, now, the definition of an algebra $\mathcal{A}$ (1.29a-b)-(1.30a-d). A generalized characterization can be given by means of modules on commutative rings.

A *ring* $(\mathcal{R}, +, \cdot)$ is a set $\mathcal{R}$ endowed with two binary maps $+ : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ that satisfy the properties $(a, b, c \in \mathcal{R})$:

(1) $(\mathcal{R}, +)$ is an abelian group with a neutral element 0:

(a) $(a + b) + c = a + (b + c)$,

(b) $a + b = b + a$,

(c) $\exists 0 \in \mathcal{R} : 0 + a = a + 0$,

(d) $\forall a \in \mathcal{R}, \exists -a \in \mathcal{R} : a + (-a) = (-a) + a = 0$;

(2) $(\mathcal{R}, \cdot)$ is a semigroup:

(e) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(3) $\cdot$ is distributive with respect to $+$:

(f) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

(g) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

$\mathcal{R}$ is said to be commutative if

(h) $a \cdot b = b \cdot a$,

and unitary if it admits an element $1 \in \mathcal{R}$ such that

(i) $a \cdot 1 = 1 \cdot a = a$.

Given a ring $\mathcal{R}$, a *left $\mathcal{R}$-module* $M$ is an abelian group (cfr. properties (1)) $(M, +)$ endowed with an operation $\times : \mathcal{R} \times M \to M$ such that $\forall a, b \in \mathcal{R}$, $\forall x, y \in M$:

(i) $a \times (x + y) = a \times x + a \times y$,

(ii) $(a + b) \times x = a \times x + b \times x$,

(iii) $a \cdot b) \times x = a \times (b \times x)$,

(iv) $1 \times x = x$.

A *right $\mathcal{R}$-module* is a structure identical to a left $\mathcal{R}$-module but with the ring acting on the right through a multiplication map $\times : M \times \mathcal{R} \to M$. If $\mathcal{R}$ is commutative, left $\mathcal{R}$-modules are the same as right $\mathcal{R}$-modules.

If $\mathcal{R}$ is a unitary commutative ring, and $M$ an $\mathcal{R}$-module, there exists an isomorphism: $M \otimes \mathcal{R} \sim M$, $\mathcal{R} \otimes M \sim M$.

An (associative, unital) *algebra over a commutative ring* $\mathcal{R}$ is a $\mathcal{R}$-module[1] $\mathcal{A}$ equipped with two maps

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \tag{A.1a}$$

[1]Note that in this case as mentioned above left and right modules are equal.

$$\eta : \mathcal{R} \to \mathcal{A}, \tag{A.1b}$$

such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{R} \arrow[r, "id\otimes\eta"] \arrow[d, "\sim"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\
\mathcal{A} \arrow[r, "id"] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \otimes \mathcal{A} \arrow[r, "\eta\otimes id"] \arrow[d, "\sim"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\
\mathcal{A} \arrow[r, "id"] & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu"] & \mathcal{A}
\end{tikzcd}$$

The first two diagrams express the properties of the unit map, while the third the associativity of the product map. An algebra is said to be *commutative* if the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \arrow[r, "\tau"] \arrow[d, "\mu"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu"] \\
\mathcal{A} \arrow[r, "id"] & \mathcal{A}
\end{tikzcd}$$

with $\tau$ the usual flip map defined in (1.31).

A *coalgebra* $\mathcal{C}$ *over a commutative ring* $\mathcal{R}$ is an $\mathcal{R}$-module endowed with maps

$$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}, \tag{A.2a}$$

$$\varepsilon : \mathcal{C} \to \mathcal{R}, \tag{A.2b}$$

such that the following diagrams commute

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{R} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "id\otimes\varepsilon"'] \\
\mathcal{C} \arrow[u, "\sim"] & \mathcal{C} \arrow[l, "id"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R} \otimes \mathcal{C} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "\varepsilon\otimes id"'] \\
\mathcal{C} \arrow[u, "\sim"] & \mathcal{C} \arrow[l, "id"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "\mu\otimes id"'] \\
\mathcal{C} \otimes \mathcal{C} \arrow[u, "id\otimes\Delta"] & \mathcal{C} \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

It is, then, clear in what sense algebras and coalgebras are categorical dual structures; take, for example, the commutative algebra diagrams, reverse all the arrows working the substitutions:

$$\mathcal{A} \to \mathcal{C}, \tag{A.3a}$$

$$\eta \to \varepsilon, \tag{A.3b}$$

$$\mu \to \Delta, \tag{A.3c}$$

and the result will be the three commutative coalgebra diagrams.

Dual to the notion of commutativity of an algebra there is that of the *cocommutativity* of a coalgebra, described by the commutative diagram

$$\begin{tikzcd} \mathcal{C}\otimes\mathcal{C} & \mathcal{C}\otimes\mathcal{C} \arrow[l, "\tau"'] \\ \mathcal{C} \arrow[u, "\Delta"] & \mathcal{C} \arrow[l, "id"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

At this point we are ready to define an Hopf algebra in categorical terms. An *Hopf algebra* $\mathcal{H}$ *over a commutative ring* $\mathcal{R}$ is a $\mathcal{R}$-module such that:

(i) $\mathcal{H}$ is an algebra and a coalgebra over $\mathcal{R}$,

(ii) $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ and $\varepsilon : \mathcal{H} \to \mathcal{R}$ are homomorphisms of algebras,

(iii) $\mu : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$ and $\eta : \mathcal{R} \to \mathcal{H}$ are homomorphisms of coalgebras,

(iv) $\mathcal{H}$ is endowed with a bijective map $S : \mathcal{H} \to \mathcal{H}$ (the antipode) that makes commuting the following diagrams:

$$\begin{tikzcd} \mathcal{H}\otimes\mathcal{H} \arrow[r, "S\otimes id"] & \mathcal{H}\otimes\mathcal{H} \arrow[d, "\mu"] & \mathcal{H}\otimes\mathcal{H} \arrow[r, "id\otimes S"] & \mathcal{H}\otimes\mathcal{H} \arrow[d, "\mu"] \\ \mathcal{H} \arrow[u, "\Delta"] \arrow[r, "\eta\circ\varepsilon"] & \mathcal{H} & \mathcal{H} \arrow[u, "\Delta"] \arrow[r, "\eta\circ\varepsilon"] & \mathcal{H} \end{tikzcd}$$

Omitting condition (iv) one would have obtained the definition of a *bialgebra over a commutative ring*.

Now we want to to formalize the notion of a deformation of an Hopf algebra. We start noting that the set of formal power series in $q$ with coefficients in a commutative ring $\mathcal{R}$ is itself a ring $\mathcal{R}[[q]]$ called the *ring of formal power series in* $q$ *over* $\mathcal{R}$.

An *ideal* $\mathcal{I}$ of a commutative ring $\mathcal{R}$ is a subset of $\mathcal{R}$ such that $(\mathcal{I}, +)$ is a subgroup of $(\mathcal{R}, +)$ and $\forall a \in \mathcal{R}$ and $\forall x \in \mathcal{I}$, $ax = xa \in \mathcal{I}$. Two elements $a, b \in \mathcal{R}$ are said to be *congruent modulo* $\mathcal{I}$ if $a - b \in \mathcal{I}$.

A *topological Hopf algebra* is an Hopf algebra $(\mathcal{H}, \eta, \mu, \varepsilon, \Delta, S)$ with $\mathcal{H}$ a topological space.

A deformation $(\mathcal{H}_q, \eta_q, \mu_q, \varepsilon_q, \Delta_q, S_q)$ of an Hopf algebra $(\mathcal{H}, \eta, \mu, \varepsilon, \Delta, S)$ over a field $\mathbb{K}$ is a topological Hopf algebra over the ring $\mathbb{K}[[q]]$ of formal power series in $q$ over $\mathbb{K}$ such that:

(i) $\mathcal{H}_q$ is isomorphic to the algebra $\mathcal{H}[[q]]$ of formal power series in $q$ with coefficients in $\mathcal{H}$ as a $\mathbb{K}[[q]]$-module,

(ii) $\mu_q \equiv \mu \pmod q$,

(iii) $\Delta_q \equiv \Delta \pmod{q}$.

Two Hopf algebra deformations are equivalent if there exists an isomorphism between them that is the identity modulo $q$.

Note that in the previous definition we have not mentioned the unit, counit and antipode of the Hopf algebra; this is because it can be shown that any deformation is equivalent to another one in which the unit and counit are undeformed. Furthermore any deformation of a bialgebra is an Hopf algebra, so that we can ignore the antipode.

# Appendix B

# General $r$-matrices and twists

We now present the general form of a classical $r$-matrix for noncommutative Minkowski spacetimes following the discussion carried on in [50].

We start with a general 4D noncommutative Minkowski spacetime of the form (1.26), rewritten as

$$[x^\mu, x^\nu] = \lambda^2 \theta^{\mu\nu}(\lambda^{-1}x), \tag{B.1}$$

with $\lambda$ a length parameter and $\theta^{\mu\nu}$ given by the expansion

$$\theta^{\mu\nu}(\lambda^{-1}x) = \theta^{\mu\nu(0)} + \theta^{\mu\nu(1)}_{\varrho}\lambda^{-1}x^\varrho + \theta^{\mu\nu(2)}_{\varrho\tau}\lambda^{-2}x^\varrho x^\tau + \dots. \tag{B.2}$$

A general classical $r$-matrix for (B.1), satisfying the CYBE, can be written in the form

$$r = \frac{\lambda^2}{2}\theta^{\mu\nu(0)}P_\mu \wedge P_\nu + \frac{\lambda}{2}\theta^{\mu\nu\varrho(1)}P_\mu \wedge M_{\nu\varrho} + \frac{1}{2}\theta^{\mu\nu\varrho\sigma(2)}M_{\mu\nu} \wedge M_{\varrho\sigma}. \tag{B.3}$$

At this point various $r$-matrices can be found imposing conditions on the $\theta$ parameter. In the present work we analyze two types of Lie-algebra spacetimes, therefore our discussion will specialize to that class of noncommutativity. In this case the general conditions on the $\theta$ parameter are:

$$\theta^{\mu\nu(0)} = \theta^{\mu\nu\varrho\sigma(2)} = 0, \qquad \theta^{\mu\nu\varrho(1)} = \epsilon^{\mu\nu\varrho\tau}v_\tau, \tag{B.4}$$

with $\nu$, $\varrho$ fixed indices and $v_\tau$ a 4-vector with two nonvanishing components. The twist operator assumes the form:

$$\mathcal{F} = e^{\lambda f} = e^{\frac{i\lambda}{2}(\zeta^\lambda P_\lambda \wedge M_{\alpha\beta})}, \tag{B.5}$$

with $\alpha, \beta$ fixed, $\lambda \neq \alpha, \beta$, and where $\zeta^\lambda = \theta^{\lambda\alpha\beta(1)}$ have vanishing components $\zeta^\alpha$, $\zeta^\beta$.

In [50] the general expression for the twisted coproduct is given, according to the deformation defined by (1.95). It is noteworthy to point out that for two of the momenta, corresponding to the two nonvanishing components of $\zeta^\lambda$, the coproduct remains undeformed:

$$\Delta_{\mathcal{F}}(\zeta^\lambda P_\lambda) = \Delta(\zeta^\lambda P_\lambda). \tag{B.6}$$

An important aspect, not discussed before in Subsection 1.6.3, is the fact that in general a twisted deformation of an Hopf algebra deforms also the antipode map along with the coproduct, in accordance with the prescription

$$S_{\mathcal{F}} = USU^{-1}, \tag{B.7}$$

with $U$ given by $U = \mathrm{f}^{\alpha} S(\mathrm{f}_{\alpha})$ and $U^{-1} = S(\bar{\mathrm{f}}^{\alpha})\bar{\mathrm{f}}_{\alpha}$. The fundamental result according to which we omitted this feature in Subsection 1.6.3, is that in the Lie-algebra noncommutativity case the antipodes remain always classical, as shown in [50].

Let us apply the reasoning to the $\varrho$-Minkowski case. Comparing eq.(B.1) with (3.1a-b) one obtains the following identifications:

$$\lambda = \varrho, \qquad \theta^{\mu\nu}{}_{\varrho} = \theta^{0i}{}_{j} = i\epsilon^{(0)i}{}_{j3} v^3, \qquad v^{\mu} = (0,0,0,1)^T. \tag{B.8}$$

Substituting in eq.(B.3) one obtains the stated result (3.2) and by (B.5) the twist operator (3.7).

According to (B.6) note that the two coproduct-undeformed 4-momenta are $P_0$ and $P_3$, as shown by (3.20a,d).

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