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Entanglement entropy in conformal quantum mechanics

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Platone, *Apologia di Socrate*

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Chapter 1

Introduction

In the last decades of the past century, the study of the black holes has acquired a great importance. A crucial event that changed our past knowledge was Hawking's 1976 discover that black holes emit thermal radiation at a temperature:

$$T = \frac{\kappa}{2\pi}, \tag{1.1}$$

where κ is the horizon surface gravity. More precisely, this phenomenon is a characteristic of any local horizon [1]-[5] and can occur even in a flat spacetime if there are observers with constant acceleration. In fact, the casual structure of the Rindler space is similar to that of the maximally-extended Schwarzschild spacetime of an eternal black hole. The studies on the temperature associated to the Rindler horizon were made for the first time by Unruh in order to understand the physics of the Hawking effect.

Another fundamental discovery was made by Bekenstein, who conjectured that a black hole must have an intrinsic entropy proportional to the area of its event horizon, in order to preserve the second law of thermodynamics [6, 7]. Related to the Hawking temperature (1.1) is the well-known expression of the Bekenstein-Hawking entropy of black holes in terms of their horizon area A:

$$S_{BH} = \frac{A}{4\hbar G} \tag{1.2}$$

where $\hbar$ is the Planck constant and G is the gravitational constant [8].

In [6, 7] it was shown that the event horizon of a black hole hides information and since entropy measures the missing information, we can think that an event horizon has an associated entropy. One can ask what is the nature of

such entropy: some works have suggested that the black hole entropy can be interpreted as entanglement entropy [9, 21].
Since the black hole horizon is a casual boundary, it makes sense to compute a reduced density matrix tracing over the degrees of freedom of a quantum field inside the horizon, as discussed in [22]. In fact, let us suppose to have two regions, I outside the horizon and II inside of the horizon. If we trace over the degrees of freedom behind the horizon we obtain a reduced density matrix which describes a mixed state because the two regions are entangled. This density matrix, computed as $\rho_I \equiv \mathrm{tr}_{II} \left|0\right\rangle \left\langle 0\right|$, can be expressed as $\rho_I = \mathcal{N} \exp(-\beta H)$, where $\mathcal{N}$ is the normalization constant, H is the generator of the time translations outside the horizon and $\beta = 2\pi/\hbar\kappa$. We can see that this matrix is proportional to that one of the canonical ensamble at the Hawking temperature. Thus the entanglement entropy obtained from this density matrix has the nature of a thermal entropy [9].
Hawking's result marked an important change in the study of Quantum Field Theory in curved spacetime and led to the information-loss paradox. When Hawking radiation escapes to infinity, energy goes away from the black hole: the mass shrinks and the surface gravity increases with the temperature. During this process the entire mass evaporates, just Hawking radiation remains and the information used to specify the system before it became a black hole is lost. The attempts to unify Quantum Field Theory with General Relativity seem very hard: in fact, a pillar of the former is unitary time evolution (the information that specifies a state at a certain time is equal to that required to specify the state at next time) which appears to be violated if via the mechanism of Hawking radiation black holes allow the evolution of pure quantum states into mixed states [11]. The impossibility to reconcile these two theories is one of the greatest challenges of theoretical physics.

The focus of the present thesis is on the entanglement entropy of thermofield double states (the same states that are at the basis of the Hawking and Unruh thermal phenomena) and how these can be described in the simplest conformal field theory: Conformal Quantum Mechanics. After this introduction, in Chapter 2 it's shown how the Unruh effect works in a two-dimensional Minkowski spacetime and how thermal effects are possible for an accelerating observer in a vacuum state. In Chapter 3 the focus is on the entropy and its properties, on the concept of entanglement and on what is the entanglement entropy. A brief introduction to the AdS/CFT correspondence is given in Chapter 4 where we also introduce the thermofield double as a vacuum state in Conformal Quantum Mechanics. In the following Chapter

5, starting from the density matrix of a harmonic oscillator, we computed the entanglement entropy and then repeated the calculation for a system of two harmonic oscillators, showing that it has the structure of a thermofield double state. The Chapter 6 is the central chapter of this thesis: considering the thermofield double state of conformal quantum mechanics, evaluating its reduced density matrix, that is thermal, we computed its entanglement entropy following the same steps made for a system of two harmonic oscillators. In this chapter we show the results of our studies. In the final Chapter 7 there are conclusion and future prospects.

Chapter 2

The Unruh effect

In this chapter we want to introduce a known concept in quantum field theory: the notion of vacuum state is observer dependent [10]. This generates a phenomenon called Unruh effect, which was discovered in order to understand the Hawking effect. To describe it, we'll follow Carroll's work [11]. The Unruh effects states that an accelerating observer in the Minkowski vacuum state perceives a thermal spectrum of particles and it has a great importance whenever there is a black hole event horizon, but it's manifest even when there is a local horizon. To explain how this effect works, let us consider a two-dimensional Minkowski space with metric $ds^2 = -dt^2 + dx^2$ and a uniformly accelerating observer which is moving along the x-direction with a trajectory x^μ given by:

$$
\begin{aligned}
t(\tau) &= \frac{1}{\alpha}\sinh(\alpha\tau) \\
x(\tau) &= \frac{1}{\alpha}\cosh(\alpha\tau)
\end{aligned}
\tag{2.1}
$$

where α is the magnitude of the acceleration. According to these coordinates, since $x^2(\tau) = t^2(\tau) + \alpha^2$, the observer will move along an hyperboloid asymptoting to null paths $x = -t$ in the past and $x = t$ in the future. Now we can choose new coordinates (η, ξ), with ranges $-\infty < \eta, \xi < +\infty$, that describe this accelerated motion. If we choose t and x as:

$$
\begin{aligned}
t &= \frac{1}{a}e^{a\xi}\sinh(a\eta) \\
x &= \frac{1}{a}e^{a\xi}\cosh(a\eta),
\end{aligned}
\tag{2.2}
$$

where $x > |t|$, then (2.1) becomes:

$$\begin{aligned} \eta(\tau) &= \frac{\alpha}{a}\tau \\ \xi(\tau) &= \frac{1}{a}\ln\left(\frac{a}{\alpha}\right) \end{aligned} \tag{2.3}$$

and the metric $ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$. The first region of the Minkowski space-time with this metric is called Rindler space. We notice that the metric so expressed is indipendent of η and so ∂_η is a Killing vector, that can be written as:

$$\begin{aligned} \partial_\eta &= \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x = \\ &= e^{a\xi}[\cosh(a\eta)\partial_t + \sinh(a\eta)\partial_x] = a(x\partial_t + t\partial_x). \end{aligned} \tag{2.4}$$

Thus the vector ∂_η is the generator of Lorentz boost in the x-direction and it's space-like in regions II and III and time-like in regions I and IV. The null lines $x = t$ and $x = -t$, that we recall H^+ and H^-, are Killing horizons for this vector field.

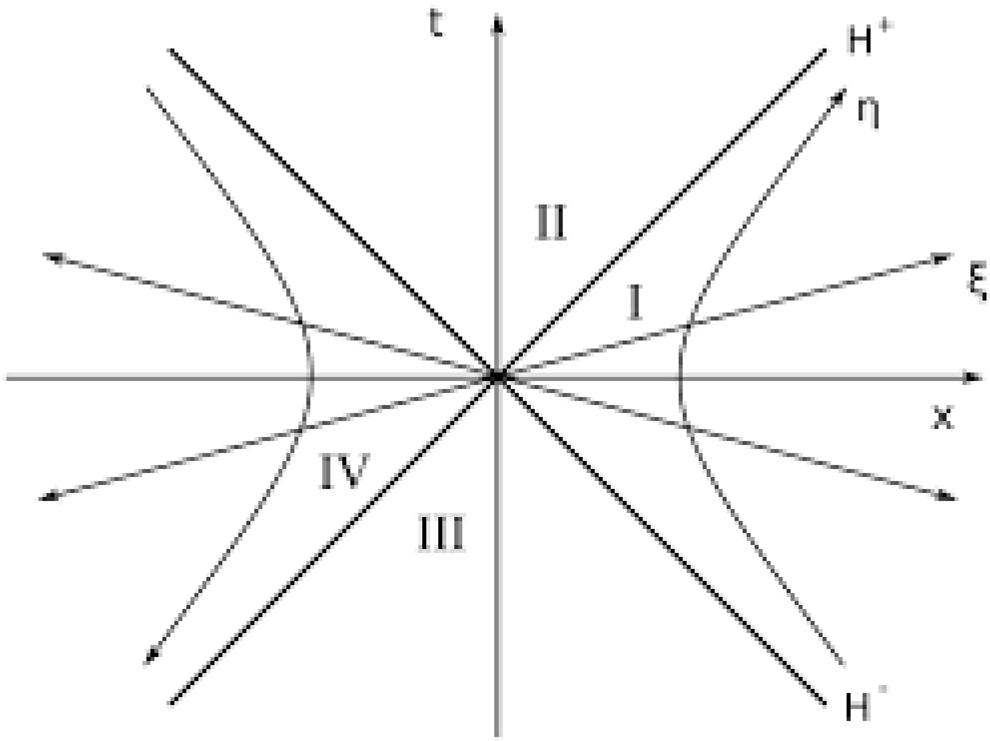

Figure 2.1: Minkowski space-time in Rindler coordinates (η, ξ). The image is taken from S.M. Carroll, *"Spacetime and geometry: an introduction to General Relativity"*, Cambridge University Press.

At this point we can construct two sets of modes, one with positive frequency for the region I and one with negative frequency for the region IV. In fact, if we consider the Klein-Gordon equation in Rindler coordinates

$$\Box\phi = e^{-2a\xi}(-\partial_\eta{}^2 + \partial_\xi{}^2)\phi = 0, \tag{2.5}$$

we can see how the plane wave $g_k = (4\pi\omega)^{-1/2} e^{-i\omega\eta + ik\xi}$, with $\omega = |k|$, solves the equation. This plane wave is divided into two parts:

$$\begin{aligned} g_k^{(1)} &= \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi} & \mathrm{I} \\ 0 \quad \mathrm{IV} \end{cases} \\ g_k^{(2)} &= \begin{cases} 0 \quad \mathrm{I} \\ \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi} & \mathrm{IV} \end{cases} \end{aligned} \tag{2.6}$$

and we can expand the field ϕ in terms of these modes and their conjugates, introducing the annihilation operators $\hat{b}_k^{(1,2)}$:

$$\phi = \int dk \Big(\hat{b}_k^{(1)} g_k^{(1)} + \hat{b}_k^{(1)\dagger} g_k^{(1)\star} + \hat{b}_k^{(2)} g_k^{(2)} + \hat{b}_k^{(2)\dagger} g_k^{(2)\star} \Big). \tag{2.7}$$

Nevertheless we know that, rather than the Rindler modes, it's possible another expansion in terms of the Minkowski modes:

$$\phi = \int dk \Big(\hat{a}_k f_k + \hat{a}_k^\dagger f_k^\star \Big). \tag{2.8}$$

This difference in writing ϕ gives rise to two different notions of vacuum, the Minkowski vacuum $|0_M\rangle$ satisfying $\hat{a}_k |0_M\rangle = 0$ and the Rindler vacuum $|0_R\rangle$ satisfying $\hat{b}_k^{(1)} |0_R\rangle = \hat{b}_k^{(2)} = 0$. The two vacua don't coincide because the Rindler annihilation operators are superpositions of Minkowski creation and annihilation operators, since Rindler modes have support on the half-line. This implies that a Rindler observer in the Minkowski vacuum state detects a certain number of particles, while an inertial observer describes the state as it's empty. To see this, it's necessary to evaluate the Bogoliubov coefficients between Minkowski and Rindler modes and then the expectation value of the Rindler number operator in the Minkowski vacuum: in this way we obtain a Planck spectrum with temperature $T = \frac{a}{2\pi}$. As we'll see soon, we can notice that this temperature has the same form of the Hawking temperature where, instead of a, there is the surface gravity κ. If we rewrite the temperature restoring units, we have:

$$T = \frac{a}{2\pi} \frac{\hbar}{ck_B} \tag{2.9}$$

where $\hbar$ is the reduced Planck constant, c is the speed of light in the vacuum and k_B is the Boltzmann constant. If we make the limit $\hbar \to 0$ and $c \to \infty$ we can notice that the temperature goes to zero. Thus we can conclude that the Unruh effect is a quantum mechanics and a relativistic effect.

2.1 Connection with the entropy

An important aspect we want to focus on in this chapter is the presence of casual horizons because we should associate entropy to them. In fact the observation suggests that they hide information [12]. Using these arguments, we know that it's possible to derive the Einstein's field equations from a relation that holds for all the local Rindler causal horizons:

$$\delta Q = TdS \tag{2.10}$$

where Q is the energy that flows across the horizon, S is the entropy and T is just the Unruh temperature we told about. In quantum field theory the entropy associated to the horizon is divergent but if there is a cutoff length l_c, then the entropy is finite and, we should assume, proportional to the horizon area in units of l_c^2.
These last considerations show that the Unruh effect can have important implications in quantum gravity and that it's connected with the entropy [14].

2.2 Hawking effect and Unruh effect

We can now show the connection between Hawking effect and Unruh effect, that we mentioned before [28].
First of all we have to obtain the expression of the Planck length. Let us consider a space region Δx. In according to the Heisenberg's uncertainty principle $\Delta E \Delta t \simeq \hbar/2$, the energy fluctuations in this region will be $\Delta E \simeq \hbar c/(2\Delta x)$. We can consider that Δx is of the order of the Schwarzschild radius associated to the mass ΔM: $R = (2G\Delta M)/c^2$, thus Δx becomes:

$$\Delta x = \frac{2G\Delta M}{c^2} = \frac{2G\Delta E}{c^4} = \frac{G\hbar}{c^3 \Delta x} = \left(\frac{G\hbar}{c^3}\right)^{1/2} \tag{2.11}$$

where we used the fact the $\Delta E = \Delta M c^2$. This length is just the Planck length L_P, that is the scale at which strong fluctuations of the quantum vacuum appear and virtual black holes are created.
Now let us consider a Schwarzschild black hole with radius $R = 2GM/c^2$. A particle with mass m near to the horizon will have a pontential energy:

$$U = \frac{GMm}{R} \tag{2.12}$$

and a potential gradient:

$$U = \frac{GMm}{R}\Delta x \tag{2.13}$$

where Δx is its displacement. The lost potential energy ΔU is equal to the kinetic energy ΔE_c and if this last one is the energy needed to create a pair particle-antiparticle, that is $\Delta E_c = 2mc^2$, we obtain:

$$\frac{GMm}{R^2}\Delta x = 2mc^2 \tag{2.14}$$

and

$$\Delta x = 2c^2\frac{R^2}{GM}. \tag{2.15}$$

If we repeat this process of creation and annihilation, we will have a gas of particle and every particle of this gas will have a mean kinetic energy:

$$E_c = \frac{3}{2}kT \tag{2.16}$$

where T is the temperature of the gas. This energy should be compared to the kinetic energy that particles have when they move in a space Δx and remembering the Heisenberg's uncertainty principle:

$$\Delta E = \frac{\hbar c}{2\Delta x} = \frac{\hbar GM}{4cR^2} = \frac{\hbar c^3}{16GM} \tag{2.17}$$

we get:

$$\frac{3}{2}kT = \frac{\hbar c^3}{16GM} \tag{2.18}$$

and thus:

$$T = \frac{\hbar c^3}{24kGM}. \tag{2.19}$$

This relation is very similar to that one computed by Hawking:

$$T = \frac{\hbar c^3}{8\pi kGM}. \tag{2.20}$$

We can notice that they differ just for a little quantity.
At this point we focus on the temperature perceived by the Unruh effect. Let us suppose to have an Einstein lift uniformly accelerated toward the top. In the inner of this lift there are some electrons, each of which has a kinetic energy:

$$\Delta E_c = mv\Delta v = ma\Delta x \tag{2.21}$$

where a is the acceleration of the lift. As we said before, this energy can be equal to that needed to create a pair particle-antiparticle and thus:

$$ma\Delta x = 2mc^2 \tag{2.22}$$

and:

$$\Delta x = \frac{2c^2}{a}. \tag{2.23}$$

This Δx is just the space in which there are the pairs particle-antiparticle. Every particle feels an uncertainty in the kinetic energy equal to:

$$\Delta E = \frac{\hbar c}{2\Delta x} = \frac{\hbar a}{4c} \tag{2.24}$$

that has a thermal nature because of the agitation of the particles. Thus:

$$\frac{3}{2}kT = \frac{\hbar a}{4c} \tag{2.25}$$

and

$$T = \frac{\hbar a}{6kc}. \tag{2.26}$$

This is the temperature perceived by the particle gas subjected to the accelerated observer. This expression for T is very similar to the one obtained by Unruh:

$$T = \frac{\hbar a}{2\pi kc}. \tag{2.27}$$

Also in this case they differ for a little quantity.
We can now verify if there is a connection between the Hawking effect and the Unruh effect. Since the Unruh effect is manifest also in curved spacetime, we can consider an observer in a gravitational field generated by a mass M, that will measure a gravitational acceleration $g = GM/r^2$. In this gravitational field a particle of mass m, after a displacement Δx, will have a potential energy:

$$\Delta U = gm\Delta x \tag{2.28}$$

which can create a pair particle-antiparticle:

$$gm\Delta x = 2mc^2 \tag{2.29}$$

and thus:

$$\Delta x = \frac{2c^2}{g}. \tag{2.30}$$

Each of this particle has an energy whose uncertainty is:

$$\Delta E = \frac{\hbar c}{2\Delta x} = \frac{\hbar g}{4c}. \tag{2.31}$$

This is again a thermal kinetic energy:

$$T = \frac{\hbar g}{6ck}. \tag{2.32}$$

At this point we can invoke the equivalence principle, according to which the observer in the gravitational field will perceive a temperature(2.2) with $\vec{g} = -\vec{a}$. In the presence of a Schwarzschild black hole, near the horizon, the gravitational acceleration is:

$$g_{SCH} = \frac{GM}{R_{SCH}^2} = \frac{c^4}{4GM} \tag{2.33}$$

and the temperature near the horizon is:

$$T = \frac{\hbar g}{6ck} = \frac{\hbar c^3}{24kGM}, \tag{2.34}$$

which is the Hawking temperature of the black hole. This shows the connection between the Hawking effect and the Unruh effect.

Chapter 3

Entanglement and entropy

At this point we are ready to introduce some important concepts that constitute the base for this thesis.

3.1 Entanglement

Entanglement is one of the most surprising phenomenon of quantum mechanics and one of the most evident that distinguishes it from classical mechanics. Even if the term "entanglement" was introduced by Schroedinger, the concept appeared for the first time in 1935 in a paper by Albert Einstein, Boris Podolsky and Nathan Rosen [13]. To describe it, let us consider two non-interacting systems A and B with respective Hilbert spaces H_A and H_B such that the Hilbert space of the composite system is $H = H_A \otimes H_B$. If the first system is in the sate $|\phi\rangle_A$ and the second is in the state $|\chi\rangle_B$, then the composite system is in the state $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$. When a state can be written as tensorial product of states, we are in the presence of a separable state, but when this is not possible we can say that the state is entangled. This implies that $|\phi\rangle_A$ and $|\chi\rangle_B$ are correlated, that is a measure on the first state isn't independent from the second.

3.2 Entropy

Entropy plays a fundamental role in different fields of physics and our intent is to analyze it from the point of view of statistical mechanics, introducing the Shannon entropy and the von Neumann entropy.

First of all, let us consider a classical system with a discrete state space labelled by a. If $\vec{p}$ is the probability distribution of the states, with $p_a \geq 0$ and $\sum_a p_a = 1$, we can write the Shannon entropy as:

$$S(\vec{p}) =< -\ln p_a >_{\vec{p}}= -\sum_a p_a \ln p_a \tag{3.1}$$

where $0 \ln 0$ is defined to be zero. This entropy is a measure of the indefiniteness of the state. In fact, given an observable O_a, that is a function of a that doesn't depend on $\vec{p}$, and its expectation value:

$$\langle O_a \rangle_{\vec{p}} = \sum_a O_a p_a, \tag{3.2}$$

if $S(\vec{p}) = 0$ then all observable have definite values, i.e. $\langle O_a^2 \rangle = \langle O_a \rangle^2$, otherwise $S(\vec{p}) > 0$.
Now we can move on to quantum mechanics. We know that the expectation value of an observable can be written in terms of the density matrix, which is an operator such that:

$$\begin{aligned} &\rho^\dagger = \rho \\ &\rho \geq 0 \\ &\operatorname{Tr} \rho = 1 \end{aligned} \tag{3.3}$$

and thus:

$$\langle O \rangle_\rho = \operatorname{Tr}(O\rho). \tag{3.4}$$

The density matrix can be diagonalized as:

$$\rho = \sum_a p_a \left| a \right\rangle \left\langle a \right| \tag{3.5}$$

where p_a form a probability distribution. The Shannon entropy of this distribution is the von von Neumann entropy of ρ:

$$S(\rho) =< -\ln \rho >_\rho= -\operatorname{Tr} \rho \ln \rho. \tag{3.6}$$

If a state is pure, that is $\rho = \left| a \right\rangle \left\langle a \right|$, there is only one a such that $p_a \neq 0$ and then $S(\rho) = 0$. Thus the von Neumann entropy is a measure of the mixedness of the states and it's an example of entanglement entropy.

3.3 Entanglement entropy

In recent years entanglement entropy has acquired an important role in the study of quantum mechanical systems because it allows us to treat a large number of information in a simpler way [16]. If we have N classical spins, which can be 0 or 1, we deal with N numbers but in a quantum mechanical system the dimension of the Hilbert space is 2^N. Thus it's clear that the number of degrees of freedom is enormous to manage. What we could do is to slice the Hilbert space to obtain some subsystems more manageable, but if this is possible with systems that interact weakly, we can't tell the same for strong interactions. For all these cases, entanglement entropy can be very useful.
Let us start with a state $|\psi\rangle$ and with a Hilbert space H, that we can break up into two subspaces: $H = A \otimes \bar{A}$. Introducing the formalism of the density matrix, associated to the state $|\psi\rangle$ we have:

$$\rho = |\psi\rangle \langle\psi| \,. \tag{3.7}$$

We can trace out the subfactor $\bar{A}$, obtaining the reduced density matrix:

$$\rho_A = Tr_{\bar{A}} |\psi\rangle \langle\psi| \,. \tag{3.8}$$

Using this reduced density matrix, we can evaluate the von Neumann entropy:

$$S = -Tr\rho_A log\rho_A. \tag{3.9}$$

It can be seen as a measure of the disorder of a system. In fact, if we consider a pure state:

$$|\psi\rangle_{prod} = |\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} \,, \tag{3.10}$$

where we identify the first spin with A and the second with $\bar{A}$, we can calculate the reduced density matrix:

$$\begin{aligned} \rho_{A,prod} &= \langle\uparrow|_{\bar{A}} \Big(|\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} \langle\uparrow|_A \langle\downarrow|_{\bar{A}} \Big) |\uparrow\rangle_{\bar{A}} + \langle\downarrow|_{\bar{A}} \Big(|\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} \langle\uparrow|_A \langle\downarrow|_{\bar{A}} \Big) |\downarrow\rangle_{\bar{A}} = \\ &= |\uparrow\rangle_A \langle\uparrow|_A \,. \end{aligned} \tag{3.11}$$

We can see that we obtain a pure state again and the von Neumann entropy is zero: we don't have loss of information. Nevertheless, if we repeat the same calculation for an entangled state, the result is different. Let us consider the EPR state:

$$|\psi\rangle_{EPR} = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} + |\downarrow\rangle_A |\uparrow\rangle_{\bar{A}}\Big). \tag{3.12}$$

If we compute the reduced density matrix, we obtain:

$$\begin{aligned}
\rho_{A,EPR} = \frac{1}{2}\Big[&\langle\uparrow|_{\bar{A}} \Big(|\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} \langle\uparrow|_A \langle\downarrow|_{\bar{A}} + |\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} \langle\downarrow|_A \langle\uparrow|_{\bar{A}} + \\
&+ |\downarrow\rangle_A |\uparrow\rangle_{\bar{A}} \langle\uparrow|_A \langle\downarrow|_{\bar{A}} + |\downarrow\rangle_A |\uparrow\rangle_{\bar{A}} \langle\downarrow|_A \langle\uparrow|_{\bar{A}}\Big) |\uparrow\rangle_{\bar{A}} + \\
&+ \langle\downarrow|_{\bar{A}} \Big(|\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} \langle\uparrow|_A \langle\downarrow|_{\bar{A}} + |\uparrow\rangle_A |\downarrow\rangle_{\bar{A}} \langle\downarrow|_A \langle\uparrow|_{\bar{A}} + \\
&+ |\downarrow\rangle_A |\uparrow\rangle_{\bar{A}} \langle\uparrow|_A \langle\downarrow|_{\bar{A}} + |\downarrow\rangle_A |\uparrow\rangle_{\bar{A}} \langle\downarrow|_A \langle\uparrow|_{\bar{A}}\Big) |\downarrow\rangle_{\bar{A}}\Big] = \\
&= \frac{1}{2}\Big(|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A\Big)
\end{aligned} \tag{3.13}$$

that isn't that of a pure state; the two spins are correlated, but not in a classical manner: we can say that the two spins are quantum mechanically entangled. The proof of this is given by the von Neumann entropy that in this case is no longer zero but:

$$\begin{aligned}
S = -\frac{1}{2} \langle\uparrow|_A \Big(|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A\Big) |\uparrow\rangle_A + |\downarrow\rangle_A \Big(|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A\Big) |\downarrow\rangle_A = \\
= \log 2.
\end{aligned} \tag{3.14}$$

At this point we can list some properties of the entanglement entropy:

- if we divide a system in two parts, A and $\bar{A}$, then for a pure state

  $$S_A - S_{\bar{A}} = 0 \tag{3.15}$$

  because the eigenvalues of ρ_A and $\rho_{\bar{A}}$ are the same. In fact, let us consider a state:

  $$|0\rangle = \sum_{ia} \psi_{ia} |i\rangle_A |a\rangle_{\bar{A}} \tag{3.16}$$

  with density matrix $(\rho_A)_{ij} = (\psi\psi^\dagger)_{ij}$ and $(\rho_{\bar{A}})_{ab} = (\psi^T\psi*)_{ab}$. Since $\mathrm{Tr}\,\rho_A^k = \mathrm{Tr}\,\rho_{\bar{A}}^k$ for any positive integer k, then ρ_A and $\rho_{\bar{A}}$ have the same eigenvalues. This property doesn't depend on the characteristics of A and $\bar{A}$, but on the fact that the system has divided in two subsystems [15];

- if we divide a system into two parts as before, for a mixed state ρ with entropy $S(\rho)$, the Araki-Lieb inequality holds:

$$|S_A - S_{\bar{A}}| \leq S_\rho \leq S_A + S_{\bar{A}}, \tag{3.17}$$

thus the difference between the entropy of A and of $\bar{A}$ depends on how impure the state ρ is;

- if we have a Hilbert space made of three or more tensor factors, that is $H = \otimes_i H_i$, and S_1 is the entanglement entropy of the first subfactor H_1, S_2 the entanglement entropy of the second, S_{12} of the subfactor $H_1 \otimes H_2$, then we have the strong subadditivity:

$$S_{12} + S_{23} \geq S_2 + S_{123} \qquad S_{12} + S_{23} \geq S_1 + S_3. \tag{3.18}$$

Now we want to see how entanglement entropy works in a field theory. Let us consider a manifold M and a submanifold $A \subset M$. In this way we have elements inside A and outside A, so it makes sense to ask how the region A is entangled with the remaining part of M. Computing entropy in a quantum field theory is not simple, but a general formula is:

$$S_A = \frac{Area(\partial A)}{\epsilon^{d-2}} + ... \tag{3.19}$$

where d is the dimension of M, ∂A is the boundary of A and it's called entangling surface and ϵ is the UV cutoff. This rule is valid only for $d > 2$. For $d = 2$ there is a well-known formula valid for a finite interval:

$$S(L) = \frac{c}{3} \log\Big(\frac{L}{\epsilon}\Big) \tag{3.20}$$

where L is the length of the interval, c the central charge of the CFT [1] and ϵ the cutoff. We can notice the presence of the logarithmic.
Now we want to compute the entanglement entropy of an interval A of length L on the Lorentzian plane with coordinates $x^\pm = t \pm x$, where $x \in \Big[-\frac{L}{2}, \frac{L}{2}\Big]$. Let us consider the coordinate transformation:

$$x^\pm = \frac{L}{2} \tanh\Big(\frac{y^\pm}{2}\Big) \qquad y^\pm = \tau \pm y. \tag{3.21}$$

[1] We can define the centrale charge starting from the product of stress tensors: $T(z_1)T(z_2) = \frac{c/2}{z_1 - z_2}^4 + \frac{2T(z_2)}{z_1 - z_2}^2 + \frac{T(z_2)}{(z_1 - z_2)} +$ {regular terms}. The c is the central charge of CFT and has different interpretation. For CFT_2 it can be seen as: degrees of freedom of the theory, measure of Casimir energy, measure of Weyl anomaly etc.

This change allows us to trace out the degrees of freedom outside A and to consider only the region $D[A]$, that is the inner region. The density matrix associated to a state with these $y^{\pm}$ coordinates is hermitian and positive semidefinite and can be expressed as [31, 32]:

$$\rho_{(y^{\pm})} = \exp(-\pi H_{\tau}) \tag{3.22}$$

where H_{τ} is the generator of time evolution. Using this generator we can construct the unitary operator:

$$U(s) = \rho^{is} = e^{-\pi i H_{\tau} s} \tag{3.23}$$

that generates a symmetry of the system. The transformations $U(s)$ form a one-parameter group and if they are extended to complex parameters, the KMS periodicity relation in imaginary time is respected [33]. Thus the coordinate τ has the periodic form:

$$\tau \sim \tau + 2i\pi. \tag{3.24}$$

From this we deduce that the system is at a temperature $T = \frac{1}{2\pi}$, so $\rho_{(y^{\pm})}$ is a thermal density matrix. To determine the von Neumann entropy we need the reduced density matrix that is related to $\rho_{(y^{\pm})}$ by the following equation:

$$\rho_{(y^{\pm})} = U \rho_{A,(x^{\pm})} U^{\dagger} \tag{3.25}$$

where U is a unitary map that transform the operators in $x^{\pm}$ coordinates in operators in $y^{\pm}$ coordinates. Since the von Neumann entropy is invariant under unitary trasformations, we need only $\rho_{(y^{\pm})}$. In a 2d CFT with a finite temperature we can use the Cardy formula:

$$S = \frac{\pi c}{3} R T \tag{3.26}$$

when $R \gg T^{-1}$. As usual, c is the centrale charge of the CFT and, as we said, $T = \frac{1}{2\pi}$. R is the length of the interval which, according to (3.21), when $y \to \infty$, is infinite. Thus it's clear that the entanglement entropy is UV divergent and we need to regulate it introducing a UV cutoff ϵ such that now $x \in \left[-\frac{L}{2} + \epsilon, \frac{L}{2} - \epsilon\right]$ and R becomes finite:

$$R = 2\log\left(\frac{L}{\epsilon}\right), \tag{3.27}$$

from which we obatain (3.20).
We have obtained the formula of the entanglement entropy on a plane, but we can repeat the procedure for a infinite cylinder in $2d$ CFT because it can be conformally mapped to the plane by the exponential map $z = e^w$. Let us suppose that time interval is compact and that it has a periodicity β: as we have seen before, the system is at a finite temperature and the von Neumann entropy is:

$$S(L)_\beta = \frac{c}{3}\log\Big(\frac{\beta}{\pi\epsilon}\sinh\Big(\frac{\pi L}{\beta}\Big)\Big) \tag{3.28}$$

where L is the length of the interval and β is the inverse of the temperature. When $L \ll \beta$, we reproduce (4.10), but when $L \gg \beta$ we obtain:

$$S(L \gg \beta)_\beta \sim \frac{\pi c}{3}\frac{L}{\beta} \tag{3.29}$$

which reproduces the Cardy formula (3.26).
In this chapter we have seen that there isn't a general formula to compute the entanglement entropy and the problem is often difficult. The only known expression is in $2d$ CFT but our purpose is to study entanglement entropy in a $1d$ CFT.

3.4 Entropy of a free massless scalar field

In a famous paper [15], Srednicki has shown how, given a reduced density matrix for a massless free field, the resulting entropy is proportional to the area of a sphere of radius R. Since entropy is an extensive quantity, we should expect that $S \sim R^3$ but, as we said in the previous section, the entropy S inside the sphere is equal to the entropy S' outside the sphere because they have the same eigenvalues. This suggests that the entropy should be proportional to the region that S and S' have in common, that is the boundary $A = 4\pi R^2$ but, since entropy is dimensionless we need the presence of some dimensional parameters: one is the ultraviolet cutoff M and the other is the infrared cutoff μ. We should expect that the physics in the inner region, and so also S, are independent of μ and conclude that S is proportional to $M^2 A$. Following Srednicki argument, we can start with the Hamiltonian of two coupled oscillators:

$$H = \frac{1}{2}[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2] \tag{3.30}$$

and with the normalized ground state wave function:

$$\psi_0(x_1, x_2) = \pi^{-1/2}(\omega_+\omega_-)^{1/4} \exp\left[-(\omega_+ x_+^2 + \omega_- x_-^2)/2\right] \tag{3.31}$$

where $x_\pm = (x_1 \pm x_2)/\sqrt{2}$, $\omega_+ = k_0^{1/2}$ and $\omega_- = (k_0 + 2k_1)^{1/2}$. Now we can compute the ground state density matrix and then the reduced density matrix tracing over the first oscillator, obtaining:

$$\begin{aligned} \rho_2(x_2, x_2') &= \int_{-\infty}^{\infty} dx_1 \psi_0(x_1, x_2)\psi_0^*(x_1, x_2') \\ &= \pi^{1/2}(\gamma - \beta)^{1/2} \exp\left[-\gamma(x_2^2 + x_2'^2)/2 + \beta x_2 x_2'\right] \end{aligned} \tag{3.32}$$

where $\beta = \frac{1}{4}(\omega_+ - \omega_-)^2/(\omega_+ + \omega_-)$ and $\gamma - \beta = 2\omega_+\omega_-/(\omega_+ + \omega_-)$. At this point we have to find the eigenvalues p_n of ρ_2, in order to know the entropy which is $S = -p_n \log p_n$. The eigenvalues can be obtained by the equation:

$$\int_{-\infty}^{\infty} dx' \rho_2(x, x') f_n(x') = p_n f_n(x). \tag{3.33}$$

For the solution we can write p_n and f_n as:

$$\begin{aligned} p_n &= (1 - \xi)\xi^n \\ f_n(x) &= H_n(\alpha^{1/2} x) \exp\left(-\alpha x^2/2\right) \end{aligned} \tag{3.34}$$

where H_n is a Hermite polynomial, $\alpha = (\gamma^2 - \beta^2)^{1/2} = (\omega_+\omega_-)^{1/2}$, $\xi = \beta/(\gamma + \alpha)$ and n goes from zero to infinity. Thus the entropy is:

$$\begin{aligned} S(\xi) &= -(1-\xi)\xi^n \log(1-\xi)\xi^n = -(1-\xi)\xi^n \log(1-\xi) - (1-\xi)\xi^n \log \xi^n = \\ &= -\log(1-\xi) - (1-\xi)\xi^n n \log \xi = -\log(1-\xi) - \frac{\xi}{1-\xi} \log \xi. \end{aligned} \tag{3.35}$$

We can repeat the same procedure for a system of N coupled harmonic oscillators, whose Hamiltonian is:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \tag{3.36}$$

where K is a real symmetric matrix with positive eigenvalues. Again the normalized ground state wave function is:

$$\psi_0(x_1, ..., x_N) = \pi^{-N/4}(det\Omega)^{1/4} \exp[-x \cdot \Omega \cdot x/2]. \tag{3.37}$$

We can write K as $U^T K_D U$ where K_D is a diagonal matrix and U is an orthogonal matrix and Ω as $U^T K_D^{1/2} U$. Now we can write the reduced density matrix tracing over the first n oscillators:

$$\begin{aligned} &\rho_{n+1,...,N}(x_{n+1},...,x_N;x'_{n+1},...,x'_N) = \\ &\int \prod_{i=1}^{n} dx_i \psi_0(x_1,...,x_n,x_{n+1},...,x_N)\psi^*(x_1,...,x_n,x'_{n+1},...,x'_N). \end{aligned} \tag{3.38}$$

To solve these integrals, we can see that Ω can be written as:

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{3.39}$$

where A is a $n \times n$ matrix and C is a $(N-n) \times (N-n)$ matrix. The reduced density matrix becomes:

$$\rho_{n+1,...,N}(x.x') \sim \exp[-(x \cdot \gamma \cdot x + x' \cdot \gamma \cdot x')/2 + x \cdot \beta \cdot x'] \tag{3.40}$$

where $\beta = \frac{1}{2} B^T A^{-1} B$ and $\gamma = C - \beta$. To find the final expression for the entropy we can try to reproduce (3.32) because the total entropy is $S = \sum_i S(\xi_i)$. The first step is to find the eigenvalues of $\rho_{n+1,...,N}$, so if we consider that $\gamma = V^T \gamma_D V$ and $x = V^T \gamma_D^{-1/2} y$, we obtain:

$$\rho_{n+1,...,N}(y, y') \sim \exp[-(y \cdot y + y' \cdot y')/2 + y \cdot \beta' \cdot y'] \tag{3.41}$$

where $\beta' = \gamma_D^{-1/2} V \beta V^T \gamma_D^{-1/2}$. Setting $y = Wz$ where W is orthogonal and $W^T \beta' W$ is diagonal, we find the expression of the redued density matrix:

$$\rho_{n+1,...,N}(z, z) \sim \prod_{i=n+1}^{N} \exp\left[-(z_i^2 + z_i'^2)/2 + \beta'_i z_i z'_i\right] \tag{3.42}$$

where β'_i is an eigenvalue of β'. Comparing this equation with (3.32), we can see that they are identical if $\gamma \to 1$ and $\beta \to \beta'_i$. We can get the expression of the entropy looking at the examples of two coupled harmonic oscillators, considering that $\xi_i = \beta'_i/[1 + (1 - \beta_i'^2)^{1/2}]$.

Now we can consider the Hamiltonian of a free massless scalar field in $3+1$ dimensions:

$$H = \frac{1}{2} \int d^3x \Big(\pi^2(x) + (\boldsymbol{\nabla} \phi(x))^2 \Big). \tag{3.43}$$

Let us introduce the partial wave expansion:

$$\begin{aligned} \phi_{lm}(r) &= r \int d\Omega \phi(x) Y_{lm}(\theta, \phi) \\ \pi_{lm}(r) &= r \int d\Omega \pi(x) Y_{lm}(\theta, \phi) \end{aligned} \tag{3.44}$$

where $Y_{lm}(\theta, \phi)$ are spherical harmonics and ϕ_{lm} and π_{lm} obey the canonical commutation relations:

$$[\phi_{lm}(x), \pi_{l'm'}(x')] = i\delta_{ll'}\delta_{mm'}\delta(x - x'). \tag{3.45}$$

The Hamiltonian becomes $H = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H_{lm}$ where:

$$H_{lm} = \frac{1}{2} \int_0^\infty dr \left(\pi_{lm}^2(r) + r^2 \left[\frac{\partial}{\partial r} \left(\frac{\phi_{lm}(r)}{r} \right) \right]^2 + \frac{l(l+1)}{r^2} \phi_{lm}^2(r) \right). \tag{3.46}$$

Instead of the continuous radial coordinate x we use a lattice of discrete points with spacing a, so we can use a^{-1} as an ultraviolet cutoff. On the other side, if we consider that the system is in a sphere of radius $L = (N+1)a$, where N is a large integer, we can choose L^{-1} as an infrared cutoff. In these terms, the Hamiltonian becomes:

$$H_{lm} = \frac{1}{2a} \sum_{j=1}^{N} \left[\pi_{lm,j}^2 + \left(j + \frac{1}{2} \right)^2 \left(\frac{\phi_{lm,j}}{j} - \frac{\phi_{lm,j+1}}{j+1} \right)^2 + \frac{l(l+1)}{j^2} \phi_{lm,j}^2 \right] \tag{3.47}$$

where $\phi_{lm,N+1} = 0$ and again:

$$[\phi_{lm,j}, \pi_{l'm',j'}] = i\delta_{ll'}\delta_{mm'}\delta_{jj'}. \tag{3.48}$$

This Hamiltonian is that of a system of N coupled harmonic oscillators, that is H_{lm} is the general form of:

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i,j=1}^{N} x_i K_{ij} x_j. \tag{3.49}$$

For a fixed value of N we can compute the entropy summing over l and m:

$$S(n, N) = \sum_{lm} S_{lm}(n, N). \tag{3.50}$$

Since H_{lm} is independent of m and summing over m we obtain a factor $(2l+1)$, then the entropy becomes:

$$S(n,N) = \sum_l (2l+1) S_l(n,N). \tag{3.51}$$

If $l >> N$ the term that depends on l is dominant and we can compute S_l perturbatively:

$$S_l(n,N) = \xi_l(n)[-\log \xi_l(n) + 1] \tag{3.52}$$

where

$$\xi_l(n) = \frac{n(n+1)(2n+1)^2}{64 l^2 (l+1)^2} + O(l^{-6}). \tag{3.53}$$

If we consider a radius $R = (n + \frac{1}{2})a$, it's possible to fit the computed values of $S(n,N)$ as a functions of R^2 with $N = 60$ and $1 \leq n \leq 30$ finding that $S = 0,30 M^2 R^2$ with $M = a^{-1}$.

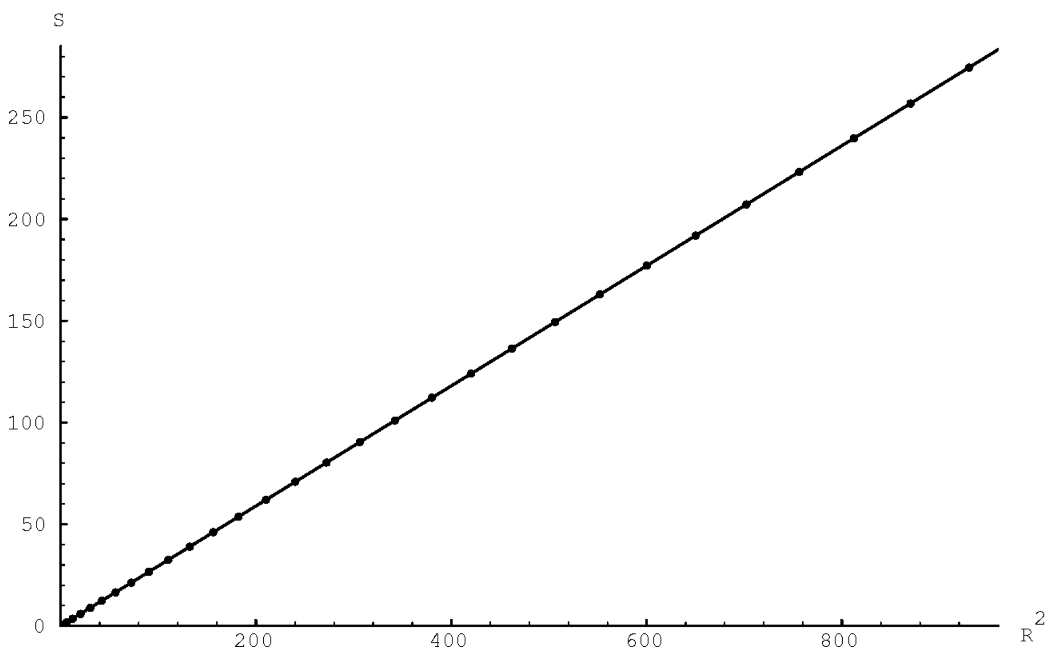

Figure 3.1: Entropy as a function of R^2. We can see that the points are perfectly fit by a straight line.

An interesting result is obtained when we consider a one-dimensional system: in this case the numerical results suggest that $S = \kappa_1 \log(MR) + +\kappa_2 \log(\mu R)$ with κ_1 and κ_2 numerical constants. We can notice the presence of the infrared cutoff μ.
This area law recalls the Bekenstein-Hawking entropy formula:

$$S_{BH} = (1/4)A/L_{Pl}^2, \tag{3.54}$$

where A is the area of the black hole horizon and $L_{Pl} = \sqrt{G\hbar/c^3}$ is the Planck length, but it's a general property of quantum field theory [26]. Since entanglement entropy has acquired a great importance during these last years, another area law has been found, i. e. Ryu-Takayanagi formula, using the correspondence AdS/CFT [27]. We could think that the study of black holes and entanglement entropy will allow us to make important discoveries in quantum field theory.

3.5 Modular Hamiltonian

In this last section we want to introduce a concept that will be very useful later.
Let us suppose to have a Hilbert space H_A and a reduced density matrix ρ_A, then we can define a formal Hamiltonian H_A, called modular Hamiltonian, such that:

$$\rho_A = \frac{1}{Z} e^{-\beta H_A} = \frac{1}{Z} \sum_a e^{-\beta E_a} \left|a\right\rangle_A \left\langle a\right|_A . \tag{3.55}$$

From eq. (3.6) we have:

$$S(A) =< H_A > + \ln Z. \tag{3.56}$$

If the Hilbert space H_B is a copy of H_A, we can construct the state:

$$\left|\psi\right\rangle = \frac{1}{\sqrt{Z}} \sum_a e^{-\beta E_a/2} \left|a\right\rangle_A \otimes \left|a\right\rangle_B . \tag{3.57}$$

This state is known as thermofield double state.

Chapter 4

Thermofield double in conformal quantum mechanics

In [17] is shown how conformal quantum mechanics can be seen as a $0+1$-dimensional conformal field theory dual to AdS_2. Using this evidence, we'll see that the vacuum state in Minkowski space-time has the structure of a thermofield double.
Let us consider a region of Minkowski space-time defined by the intersection of past and future light cones of two events called casual diamond. There's a connection between these casual diamonds and the Rindler space, given by the two-point function. We can show this starting from the fact that there's a correspondence between the generators of radial conformal symmetries in Minkowski space-time and the generators of time evolution in conformal quantum mechanics [18].

4.1 Radial conformal Killing vectors in Minkowski space-time

Let us consider the metric of Minkowski space-time in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{4.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The most general radial conformal Killing vector [19] is such that $L_\xi \eta_{\mu\nu} \propto \eta_{\mu\nu}$ where $L_{\mu\nu}$ is the Lie derivative and $\eta_{\mu\nu}$

is the Minkowski metric, and so we can write it as:

$$\xi = aK_0 + bD_0 + cP_0. \tag{4.2}$$

K_0, D_0 and P_0 are respectively the generators of special conformal transformations, dilations and time translations; they close the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra:

$$\begin{aligned} [P_0, D_0] &= P_0 \\ [K_0, D_0] &= -K_0 \\ [P_0, K_0] &= 2D_0. \end{aligned} \tag{4.3}$$

These generators can be written in terms of ∂_r and ∂_t:

$$\begin{aligned} P_0 &= \partial_t \\ D_0 &= r\partial_r + t\partial_t \\ K_0 &= 2tr\partial_r + (t^2 + r^2)\partial_t \end{aligned} \tag{4.4}$$

and thus the generator ξ becomes:

$$\xi = [a(t^2 + r^2) + bt + c]\partial_t + r(2at + b)\partial_r. \tag{4.5}$$

If we evaluate the quantity $\Delta = b^2 - 4ac$, we obtain three different types of generators:

- $\Delta < 0$: the radial conformal Killing vector is everywhere time-like;
- $\Delta = 0$: the radial conformal Killing vector is everywhere time-like except for $t = -(\frac{b}{2a}), r = 0$;
- $\Delta > 0$: the radial conformal Killing vector is null on the light cones and for $t = t_\pm, r = 0$ where $t_\pm = \frac{-b \pm \sqrt{\Delta}}{2a}$, time-like inside either lightcone or outside both ligth cones and space-like everywhere else.

Now let us consider that in conformal quantum mechanics the most general form of the Hamiltonian, that is the generator of time evolution, is:

$$G = uH + vD + wK \tag{4.6}$$

where H, D and K close the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra:

$$\begin{aligned} [H, D] &= iH \\ [K, D] &= -iK \\ [H, K] &= 2iD. \end{aligned} \tag{4.7}$$

We can see that the generators H, D and K in quantum mechanics correspond to the generators P_0, K_0 and D_0, which represent the Killing vectors of AdS_2 in Poincaré coordinates, if we make the identification $H = iP_0$, $K = iK_0$, $D = iP_0$ and $u = c$, $v = b$ and $w = a$. If we consider the invariance of the $\mathfrak{sl}(2,\mathbb{R})$ Casimir:

$$\mathcal{C} = \frac{1}{2}(HK + KH) - D^2 \tag{4.8}$$

we can classify three different types of Hamiltonian, evaluating the quantity $\Delta = b^2 - 4ac$ as before:

- $\Delta < 0$: these are the generators of elliptic transformations. An example of this type is the generator of rotations R:

$$R = \frac{1}{2}\Big(\alpha H + \frac{K}{\alpha}\Big); \tag{4.9}$$

- $\Delta = 0$: these are the generators of parabolic transformations, like H and K;

- $\Delta > 0$: these are the generators of hyperbolic transformations, like the generator of dilations D and the following generator:

$$S = \frac{1}{2}\Big(\alpha H - \frac{K}{\alpha}\Big). \tag{4.10}$$

The presence of α (it has dimensions of length) is necessary to make R and S dimensionless and α is also the radius of the casual diamond. In Minkowski space the generator S maps a casual diamond into itself , while in conformal quantum mechanics S is the generator of time evolution restricted to the domain $t \in (-\alpha, \alpha)$: in the first case α represents the finite lifetime of a static observer in the origin.
At this point we can verify if there are thermal effect in conformal quantum mechanics, that can be seen as a 0+1-dimensional field theory, in which there are translations, dilations and special conformal transformations. This field theory is constructed starting from a set of eigenstates $|n\rangle$ of the generator L_0 [20]:

$$\begin{aligned} &L_0\,|n\rangle = r_n\,|n\rangle \\ &r_n = r_0 + n, \quad r_0 > 0, \quad n = 0, 1... \\ &\langle n|n'\rangle = \delta_{nn'} \\ &L_\pm\,|n\rangle = \sqrt{r_n(r_n \pm 1) - r_0(r_0 - 1)}\,|n\pm\rangle \end{aligned} \tag{4.11}$$

where

$$L_0 \equiv \frac{1}{2}\Big(\frac{K}{\alpha} + \alpha H\Big) \tag{4.12}$$

$$L_\pm \equiv \frac{1}{2}\Big(\frac{K}{\alpha} - \alpha H\Big) \pm iD \tag{4.13}$$

and r_0 is related to the eigenvalue of the Casimir (4.8):

$$\mathcal{C} = r_0(r_0 - 1). \tag{4.14}$$

The operators L_0 and $L_\pm$ satisfy the following commutation relations:

$$[L_-, L_+] = 2L_0, \quad [L_0, L_\pm] = \pm L_\pm. \tag{4.15}$$

From the $|n = 0\rangle$ "vacuum state" we can construct $|\tau\rangle$ states on which H acts as generator of translations [17], i.e. $H|\tau\rangle = -i\partial_\tau |\tau\rangle$:

$$|\tau\rangle = N(\tau)\exp(-\omega(\tau)L_+)\,|n = 0\rangle \tag{4.16}$$

where

$$N(\tau) = [\Gamma(2r_0)]^{\frac{1}{2}}\Big(\frac{\omega(\tau)+1}{2}\Big)^{2r_0}, \quad \omega(\tau) = \frac{a + i\tau}{a - i\tau}. \tag{4.17}$$

For completness we report the action of the generators D and K on $|\tau\rangle$ states:

$$D|\tau\rangle = -i\Big(\tau\frac{d}{d\tau} + r_0\Big)|\tau\rangle \tag{4.18}$$

$$K|\tau\rangle = -i\Big(\tau^2\frac{d}{d\tau} + 2r_0\tau\Big)|\tau\rangle\,. \tag{4.19}$$

Returning to (4.16), we notice that, for $\tau = 0$, $\omega(\tau) = 1$ and $|\tau\rangle$ becomes:

$$|\tau = 0\rangle = \Gamma(2r_0)^{\frac{1}{2}}\exp(-L_+)\,|n = 0\rangle\,. \tag{4.20}$$

Now we can introduce the two point function of this theory, that can be identified with the inner product between τ-states [18]:

$$G_2(\tau_1, \tau_2) \equiv \langle\tau_1|\tau_2\rangle = \frac{\Gamma(2r_0)\alpha^{2r_0}}{[2i(\tau_1 - \tau_2)]^{2r_0}} \tag{4.21}$$

where r_0 is the conformal weight. From this expression we can see that $|\tau\rangle$-states are not orthonormal because $G_2(\tau_1, \tau_2)$ is divergent for coincident points.

For $r_0 = 1$ this two-point function is proportional to the two-point function of a free massless scalar field in Minkowski space-time evaluated along the trajectory of a static inertial observer in the origin, thus the two-point function in Minkowski space-time are in correspondence with the two-point function of conformal quantum mechanics for states $|\tau\rangle$.

4.2 The thermofield double state

Starting from the state (4.20), we want to derive the diamond temperature, considering that [23, 24]:

$$\begin{aligned} L_+ &= a_L^\dagger a_R^\dagger \\ L_- &= a_L a_R \\ L_0 &= \frac{1}{2}\Big(a_L^\dagger a_L + a_R^\dagger a_R + 1\Big). \end{aligned} \tag{4.22}$$

From (4.20), setting $r_0 = 1$, we can write the τ-vacuum as:

$$|\tau = 0\rangle = \exp\Big[-a_L^\dagger a_R^\dagger\Big] \, |0\rangle_L \otimes |0\rangle_R \tag{4.23}$$

where we used the decomposition of the vacuum state $|n = 0\rangle$:

$$|n = 0\rangle = |0\rangle_L \otimes |0\rangle_R \, . \tag{4.24}$$

Expanding the exponential function and considering that $(a^\dagger)^n \, |0\rangle = n! \, |n\rangle$, we obtain the expression for the τ-vacuum:

$$|\tau = 0\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Big(a_L^\dagger a_R^\dagger\Big)^n \, |0\rangle_L \otimes |0\rangle_R = \sum_{n=0}^{\infty} (-1)^n \, |n\rangle_L \, |n\rangle_R \tag{4.25}$$

where $|n\rangle_L$ and $|n\rangle_R$ are the eigenstates of the number operators $N_L = a_L^\dagger a_L$ and $N_R = a_R^\dagger a_R$. We can notice the bipartite structure of the system and, precisely, its thermofield double structure. Considering the expression (4.10) for S, we can make the identification $L_0 = iS$. So the $|n\rangle$ states are eigenstates of the Hamiltonian S and the vacuum $|n = 0\rangle = |0\rangle_L \otimes |0\rangle_R$ is its "ground state". Now let us rewrite the τ-vacuum in a more useful way. Recalling Euler's formula, we obtain:

$$|\tau = 0\rangle = \sum_{n=0}^{\infty} e^{i\pi n} |n\rangle_L |n\rangle_R \tag{4.26}$$

and using the expression of L_0 in terms of creation and annihilation operators and its relationship with S, we have:

$$\begin{aligned} |\tau = 0\rangle &= \sum_{n=0}^{\infty} e^{i\pi(n+\frac{1}{2}-\frac{1}{2})} |n\rangle_L |n\rangle_R = -i \sum_{n=0}^{\infty} e^{i\pi L_0} |n\rangle_L |n\rangle_R = \\ &= -i \sum_{n=0}^{\infty} e^{-\pi S} |n\rangle_L |n\rangle_R . \end{aligned} \tag{4.27}$$

As we can see in literature [29] this state has the same structure of a thermofield double of a bosonic oscillator. If we evaluate the density matrix $\rho = |\tau = 0\rangle \langle \tau = 0|$ and then the reduced density matrix obtained tracing over one set of L or R degrees of freedom, we find a thermal density matrix at temperature $T = \frac{1}{2\pi}$. As we said before, it's necessary to include a factor α to give the right dimensions to S: in this way the Hamiltonian of the system is $\frac{S}{\alpha}$ and the temperature becomes $T = \frac{1}{2\pi\alpha}$.
What we have shown is that, even in a very simple system, we are able to find thermal effects using the correspondence between the generators of time evolution in conformal quantum mechanics and the radial conformal symmetries in Minkowski space-time, when time evolution is restricted to a finite domain.

Chapter 5

Entanglement entropy of quantum oscillators

Starting from the von Neumann entropy of a single harmonic oscillator, we want to compute the entanglement entropy of a thermofield double state constructed form a system of two coupled harmonic oscillators [26].

5.1 Entanglement entropy of a single harmonic oscillator

Let us consider the Hamiltonian of a single harmonic oscillator:

$$H = \frac{1}{2}\Big(\pi^2 + \omega^2\phi^2\Big) \tag{5.1}$$

where

$$\phi = \frac{1}{\sqrt{2\omega}}\Big(a^\dagger + a\Big) \tag{5.2}$$

$$\pi = \sqrt{\frac{\omega}{2}}i\Big(a^\dagger - a\Big) \tag{5.3}$$

and $a^\dagger$ and a are the creation and annihilation operators. Of course, inserting (5.2) and (5.3) in (5.1) we obtain:

$$\begin{aligned} H &= \frac{1}{2}\Big[\frac{\omega}{2}(a^\dagger - a)^2 + \omega^2\frac{1}{2\omega}(a^\dagger + a)^2\Big] = \\ &= \frac{1}{2}\Big(a^\dagger a^\dagger + aa - a^\dagger a - aa^\dagger\Big) + \frac{\omega}{2}\Big(a^\dagger a^\dagger + aa + a^\dagger a + aa^\dagger\Big) = \\ &= \frac{1}{2}\Big[\omega(a^\dagger a + aa^\dagger)\Big] = \frac{1}{2}\Big[\omega(a^\dagger a + 1 + a^\dagger a)\Big] = \omega\Big(a^\dagger a + \frac{1}{2}\Big) \end{aligned} \tag{5.4}$$

where we used the commutator $[a, a^\dagger] = 1$.
We can find the eigenstates of the number operator $a^\dagger a$ using the following equation:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle \,. \tag{5.5}$$

For this harmonic oscillator we can prepare a Boltzmann thermal ensemble with temperature $kT = \frac{1}{\beta}$ and we can define a density matrix:

$$\rho_{nm} = \delta_{nm}\frac{1}{Z}e^{-\beta\omega(n+\frac{1}{2})} \tag{5.6}$$

where Z is the partition function of the harmonic oscillator:

$$Z = \sum_{n=0}^{\infty} e^{-\beta\omega(n+\frac{1}{2})} = \frac{e^{\frac{-\beta\omega}{2}}}{1 - e^{-\beta\omega}} = \frac{1}{2\sinh\Big(\frac{\beta\omega}{2}\Big)}. \tag{5.7}$$

At this point we can compute the von Neumann entropy:

$$\begin{aligned} &S(\rho) = -\operatorname{tr}(\rho\log\rho) = \\ &-\sum_{n=0}^{\infty}\Big(1 - e^{-\beta\omega}\Big)e^{-\beta\omega n}\log\Big[\Big(1 - e^{-\beta\omega}\Big)e^{-\beta\omega n}\Big] = \\ &= -\sum_{n=0}^{\infty}\Big(1 - e^{-\beta\omega}\Big)e^{-\beta\omega n}\log\Big(1 - e^{-\beta\omega}\Big) + \\ &-\sum_{n=0}^{\infty}\Big(1 - e^{-\beta\omega}\Big)e^{-\beta\omega n}\log\Big(e^{-\beta\omega n}\Big) = \\ &= -\log\Big(1 - e^{-\beta\omega}\Big) - \sum_{n=0}^{\infty} ne^{-\beta\omega n}\log\Big(e^{-\beta\omega}\Big) = \\ &= -\log\Big(1 - e^{-\beta\omega}\Big) + \beta\omega\sum_{n=0}^{\infty} ne^{-\beta\omega n} \end{aligned} \tag{5.8}$$

where we used the fact that: $\log\Big(e^{-\beta\omega n}\Big) = n\log\Big(e^{-\beta\omega}\Big)$.
Remembering that:

$$\sum_{n=0}^{\infty} n e^{-\beta\omega n} = \frac{e^{-\beta\omega}}{(1-e^{-\beta\omega})^2} \tag{5.9}$$

we obtain the final expression for the entropy:

$$S = -\log\Big(1-e^{-\beta\omega}\Big) + \beta\omega\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}. \tag{5.10}$$

5.2 Entanglement entropy of a system of two harmonic oscillators

Now we want to repeat the same procedure for a system of two coupled harmonic oscillators. The Hamiltonian of this system will be:

$$H = \frac{1}{2}\sum_{i=1}^{2}\pi_i^2 + \frac{1}{2}\sum_{i,j=1}^{2}\phi_i K_{ij}\phi_j \tag{5.11}$$

where K_{ij} is real and symmetric. Using $\tilde{\pi}_i = O_{ij}\pi_j$ and $\tilde{\phi}_i = O_{ij}\phi_j$, we can write an equivalent Hamiltonian:

$$H = \frac{1}{2}\sum_{i=1}^{2}\Big(\tilde{\pi}_i{}^2 + \omega_i^2\tilde{\phi}_i{}^2\Big). \tag{5.12}$$

Now let us introduce the notation: $\phi_i = \phi_L, \phi_R$ and write the matrix K_{ij} explicitly:

$$K_{ij} = \omega^2\begin{pmatrix} 1+2\tan^2\theta & \frac{2\tan\theta}{\cos\theta} \\ \frac{2\tan\theta}{\cos\theta} & 1+2\tan^2\theta \end{pmatrix}. \tag{5.13}$$

The Hamiltonian becomes:

$$H = \frac{1}{2}\Big\{\pi_L^2 + \pi_R^2 + \omega^2\Big[\phi_L^2(1+2\tan^2\theta) + 2\phi_L\frac{2\tan\theta}{\cos\theta}\phi_R + \phi_R^2(1+2\tan^2\theta)\Big]\Big\}. \tag{5.14}$$

In order to find the eigenvalues that diagonalize K_{ij}, we have to solve the equation $det(K_{ij} - \omega_\pm^2 I)$. Explaining the calculation:

$$\begin{aligned}
&\omega^2\Big[(1+2\tan^2\theta-\omega_\pm^2)(1+2\tan^2\theta-\omega_\pm^2)-\Big(2\frac{\tan\theta}{\cos\theta}\Big)\Big(2\frac{\tan\theta}{\cos\theta}\Big)\Big]=\\
&=\omega^2\Big[1+2\tan^2\theta-\omega_\pm^2+2\tan^2\theta+4\tan^4\theta-2\omega_\pm^2\tan^2\theta+\\
&-\omega_\pm^2-2\omega_\pm^2\tan^2\theta+\omega_\pm^4-4\frac{\tan^2\theta}{\cos^2\theta}\Big]=\\
&=\omega^2\Big[\omega_\pm^4-2\omega_\pm^2-4\omega_\pm^2\tan^2\theta+1+4\tan^2\theta+4\tan^4\theta-4\frac{\tan^2\theta}{\cos^2\theta}\Big].
\end{aligned} \tag{5.15}$$

We can solve the equation for $\omega_\pm^2$:

$$\begin{aligned}
&\omega_\pm^2=\frac{\omega^2}{2}\Big[2+4\tan^2\theta\pm\\
&\pm\Big((-2-4\tan^2\theta)^2-4\Big(1+4\tan^2\theta+4\tan^4\theta-4\frac{\tan^2\theta}{\cos^2\theta}\Big)\Big)^{\frac{1}{2}}\Big]=\\
&=\frac{\omega^2}{2}\Big(2+4\tan^2\theta\pm\\
&\pm\Big(4+16\tan^4\theta+16\tan^2\theta-4-16\tan^2\theta-16\tan^4\theta+16\frac{\tan^2\theta}{\cos^2\theta}\Big)^{\frac{1}{2}}\Big)=\\
&=\frac{\omega^2}{2}\Big(2+4\tan^2\theta\pm4\frac{\tan\theta}{\cos\theta}\Big)=\\
&=\omega^2\Big(1+2\tan^2\theta\pm2\frac{\tan\theta}{\cos\theta}\Big)=\omega^2\Big(\frac{\cos^2\theta+2\sin^2\theta\pm2\sin\theta}{\cos^2\theta}\Big)=\\
&=\omega^2\Big(\frac{\cos^2\theta+\sin^2\theta-\sin^2\theta+2\sin^2\theta\pm2\sin\theta}{\cos^2\theta}\Big)=\\
&=\omega^2\Big(\frac{1+\sin^2\theta\pm2\sin\theta}{\cos^2\theta}\Big)=\omega^2\Big[\frac{(1\pm\sin\theta)^2}{\cos^2\theta}\Big].
\end{aligned} \tag{5.16}$$

Having found the eigenvalues of K_{ij} we can write the diagonalized Hamiltonian in the following way:

$$\begin{aligned}
H &= \frac{1}{2}[\tilde{\pi}_+^2 + \tilde{\pi}_-^2 + \omega_+^2\tilde{\phi}_+^2 + \omega_-^2\tilde{\phi}_-^2] = \\
&= \frac{1}{2}\left[\frac{1}{2}(\pi_L+\pi_R)^2 + \frac{1}{2}(\pi_L-\pi_R)^2 + \omega_+\left(\frac{1}{2}(\phi_L+\phi_R)^2\right) + \omega_-^2\left(\frac{1}{2}(\phi_L-\phi_R)^2\right)\right] = \\
&= \frac{1}{2}\left[(\pi_L^2+\pi_R^2) + \right. \\
&\left. + \frac{\omega^2(1+\sin\theta)^2}{2\cos^2\theta}(\phi_L^2+\phi_R^2+2\phi_L\phi_R) + \frac{\omega^2(1-\sin\theta)^2}{2\cos\theta}(\phi_L^2+\phi_R^2-2\phi_L\phi_R)\right] = \\
&= \frac{1}{2}\left\{\pi_L^2+\pi_R^2 + \frac{\omega^2}{\cos^2\theta}[\phi_L^2+\phi_R^2+\sin^2\theta(\phi_L^2+\phi_R^2)+4\phi_L\phi_R\sin\theta]\right\}.
\end{aligned} \tag{5.17}$$

At this point we are ready to introduce the vacuum state of the coupled system using the ansatz:

$$|0\rangle = \sqrt{1-A^2}e^{Aa_L^\dagger a_R^\dagger}|0\rangle_L|0\rangle_R. \tag{5.18}$$

The factor A can be computed requiring that $|0\rangle$ is annihilated by the operator:

$$\begin{aligned}
a_\pm &= \frac{1}{\sqrt{2}}\left(\sqrt{\omega_\pm}\tilde{\phi}_\pm + \frac{i}{\sqrt{\omega_\pm}}\tilde{\pi}_\pm\right) \\
a_\pm|0\rangle &= 0.
\end{aligned} \tag{5.19}$$

After the calculation, contained in appendix C, we should obtain:

$$A = \frac{\tan\frac{\theta}{2}}{\sqrt{1-\tan^2\frac{\theta}{2}}}. \tag{5.20}$$

This expression of A is arbitrary because it depends on how we construct the matrix K_{ij}. As we'll soon see, the angle θ is related to the temperature: we can consider different values of θ, for example if we choose $\theta \to 0$, that means $T = \frac{1}{\beta} \to 0$, we find the decoupled system:

$$|0\rangle = |0\rangle_L|0\rangle_R \tag{5.21}$$

while if we choose the angle $\theta \to \frac{\pi}{2}$ we obtain divergent values of A. Returning to the state (5.18) we can rewrite it expanding the exponential:

$$|0\rangle = \sqrt{1-A^2}\sum_{n=0}^{\infty}\frac{A^n}{n!}(a_L^\dagger)^n(a_R^\dagger)^n\,|0\rangle_L\,|0\rangle_R\,. \tag{5.22}$$

Remembering that $|n\rangle_L\,|n\rangle_R = \frac{1}{n!}(a_L^\dagger)^n(a_R^\dagger)^n\,|0\rangle_L\,|0\rangle_R$, the state becomes:

$$|0\rangle = \sqrt{1-A^2}\sum_{n=0}^{\infty}A^n\,|n\rangle_L\,|n\rangle_R\,. \tag{5.23}$$

We can notice that the state has a thermofield double structure, even if it's not a thermal state but a pure state with entanglement between right and left part. If we multiply and divide for $|A|$ and make the substitution $|A| = e^{-\frac{\beta\omega}{2}}$ we obtain:

$$\begin{aligned}|0\rangle &= \sqrt{1-A^2}\frac{1}{|A|}\sum_{n=0}^{\infty}(-1)^n A^n|A|\,|n\rangle_L\,|n\rangle_R \\ &= \sqrt{1-A^2}\frac{1}{e^{-\frac{\beta\omega}{2}}}\sum_{n=0}^{\infty}(-1)^n A^n e^{-\frac{\beta\omega}{2}}\,|n\rangle_L\,|n\rangle_R\,.\end{aligned} \tag{5.24}$$

We can use the phase convention: $|n\rangle_R' = (-1)^n\,|n\rangle_R$. Introducing the partition function of the harmonic oscillator:

$$Z = \sum_{n=0}^{\infty}e^{-\beta\omega(n+\frac{1}{2})} = \frac{e^{-\frac{\beta\omega}{2}}}{1-e^{-\beta\omega}} \tag{5.25}$$

and thus $\sqrt{1-A^2}\frac{1}{\sqrt{A}} = \sqrt{(1-e^{-\beta\omega})e^{\frac{\beta\omega}{2}}} = \frac{1}{\sqrt{Z}}$.
Finally the vacuum state becomes:

$$|0\rangle = \frac{1}{\sqrt{Z}}\sum_{n=0}^{\infty}e^{-\frac{\beta\omega n}{2}}e^{-\frac{\beta\omega}{2}}e^{\frac{\beta\omega}{4}}\,|n\rangle_L\,|n\rangle_R = \frac{1}{\sqrt{Z}}\sum_{n=0}^{\infty}e^{-\frac{\beta\omega}{2}(n+\frac{1}{2})}\,|n\rangle_L\,|n\rangle_R \tag{5.26}$$

where we renamed $|n'\rangle_R$ with $|n\rangle_R$. The density matrix of the state is:

$$\rho_{RL} = |0\rangle\,\langle 0| = \frac{1}{Z}\sum_{n=0}^{\infty}e^{-\beta\omega(n+\frac{1}{2})}\,|n\rangle_L\,|n\rangle_R\,\langle n|_L\,\langle n|_R\,. \tag{5.27}$$

Considering the partial trace, we obtain the reduced density matrix:

$$\begin{aligned}\rho_L &= \mathrm{tr}_R \left|0\right\rangle \left\langle 0\right| = \\ &= \left\langle n\right|_R \left|0\right\rangle \left\langle 0|n\right\rangle_R = \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\beta\omega(n+\frac{1}{2})} \left|n\right\rangle_L \left\langle n\right|_R \left|n\right\rangle_R \left\langle n\right|_R \left|n\right\rangle_R \left\langle n\right|_L .\end{aligned} \tag{5.28}$$

If we recall the expression of Z, we obtain:

$$\rho_L = \left(1 - e^{-\beta\omega}\right) \sum_{n=0}^{\infty} \left(e^{-\beta\omega n}\right) \left|n\right\rangle_L \left\langle n\right|_L . \tag{5.29}$$

In order to obtain the expression for the entropy we can repeat the same procedure we have done before:

$$\begin{aligned}S &= -\,\mathrm{tr}(\rho_L \log \rho_L) = \\ &= -\sum_{n=0}^{\infty}\left(1 - e^{-\beta\omega}\right)\left(e^{-\beta\omega n}\right) \log\left[\left(1 - e^{-\beta\omega}\right)e^{-\beta\omega n}\right] \left\langle n\right|_L \left|n\right\rangle_L \left\langle n\right|_L \left|n\right\rangle_L \\ &= -\log\left(1 - e^{-\beta\omega}\right) + \beta\omega \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}}.\end{aligned} \tag{5.30}$$

Now we are interested in the case with $T \to \infty$, that is $\beta \to 0$. First of all we can rewrite (5.30) in the following way:

$$\begin{aligned}S &= -\log\left(1 - e^{-\beta\omega}\right) + \beta\omega e^{-\frac{\beta\omega}{2}} \frac{e^{-\frac{\beta\omega}{2}}}{1 - e^{-\beta\omega}} = \\ &= -\log\left(1 - e^{-\beta\omega}\right) + \beta\omega e^{-\frac{\beta\omega}{2}} \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)},\end{aligned} \tag{5.31}$$

then we can expand the function $\sinh\left(\frac{\beta\omega}{2}\right)$:

$$S \simeq -\log\left(1 - e^{-\beta\omega}\right) + \beta\omega e^{-\frac{\beta\omega}{2}} \frac{1}{2\frac{\beta\omega}{2}} = -\log\left(1 - e^{-\beta\omega}\right) + e^{-\frac{\beta\omega}{2}} \tag{5.32}$$

and finally the exponential function:

$$S \simeq -\log(1 - (1 - \beta\omega)) + 1 - \frac{\beta\omega}{2} = -\log(\beta\omega) + 1 - \frac{\beta\omega}{2}. \tag{5.33}$$

We can identify $\beta\omega$ with ϵ and obtain the final expression for the entropy:

$$S \simeq -\log \epsilon + 1 - \frac{1}{2}\epsilon. \tag{5.34}$$

Even in this case we can notice the logarithmic dependence of the entropy, as we already shown in Chapter 3.
We have seen that thermofield double states are built entangling two copies of a conformal field theory and starting from them we obtain a density matrix which is a thermal density matrix at a temperature $T = \beta^{-1}$ [29]. This systems allow us to study many aspects of entanglement, black holes and quantum information.

Chapter 6

Entanglement entropy of the τ-vacuum state

In the previous chapter we computed the entanglement entropy of a coupled system of two harmonic oscillators, seen as a thermofield double state, and we found that the entropy has a logarithmic dependence in the limit $T \to \infty$. Now we have all the tools to develop the main argument of this thesis, that is the calculation of the entanglement entropy of the τ-vacuum state, introduced in Chapter 4. We want to do it reproducing the same procedure that allowed us to obtain the entropy of the harmonic oscillators.
Let us recall the expression for the vacuum state of two coupled harmonic oscillators reviewed in the previous chapter (5.18):

$$|0\rangle = \sqrt{1-A^2} e^{A a_L^\dagger a_R^\dagger} |0\rangle_L |0\rangle_R . \tag{6.1}$$

This expression can be compared to the τ-vacuum state:

$$|\tau = 0\rangle = e^{-a_L^\dagger a_R^\dagger} |0\rangle_L |0\rangle_R = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (a_L^\dagger a_R^\dagger) |0\rangle_L |0\rangle_R = \sum_{n=0}^{\infty} (-1)^n |n\rangle_L |n\rangle_R . \tag{6.2}$$

This state is not normalizable because, as we can easily see, there is an infinite sum, which is divergent. Thus, as a consequence, if we compute the entanglement entropy, this will be divergent.

Looking at the state (5.18) we can notice that when $A = -1$, that is the case of the τ-vacuum state (6.2), the normalization constant $N = \frac{1}{1-A^2} \to \infty$, thus we can add an infinitesimal factor σ in the exponential to regularize it:

$$\begin{aligned}
|\tau=0\rangle &= \frac{1}{\sqrt{N}} e^{(-1+\sigma)a_L^\dagger a_R^\dagger} |0\rangle_L |0\rangle_R = \\
&= \frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} \frac{(-1+\sigma)^n}{n!} (a_L^\dagger a_R^\dagger) |0\rangle_L |0\rangle_R = \\
&= \frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} (-1+\sigma)^n |n\rangle_L |n\rangle_R .
\end{aligned} \tag{6.3}$$

Considering that:

$$N = \frac{1}{1-(-1+\sigma)^2} \tag{6.4}$$

neglecting higher order infinitesimal terms, we obtain:

$$\frac{1}{\sqrt{N}} = \sqrt{2\sigma}. \tag{6.5}$$

Let us rewrite the state:

$$|\tau=0\rangle = \sqrt{2\sigma} \sum_{n=0}^{\infty} (-1+\sigma)^n |n\rangle_L |n\rangle_R . \tag{6.6}$$

We can notice that, in the limit $\sigma \to 0$, the factor N diverges.
Now we can introduce the density matrix:

$$\rho_{RL} = |\tau=0\rangle \langle\tau=0| = 2\sigma \sum_{n=0}^{\infty} (-1+\sigma)^{2n} |n\rangle_L |n\rangle_R \langle n|_L \langle n|_R , \tag{6.7}$$

and considering the partial trace:

$$\begin{aligned}
\rho_L = \mathrm{tr}_R\, \rho_{RL} &= \langle n|_R \left(2\sigma \sum_{n=0}^{\infty} (-1+\sigma)^{2n} |n\rangle_L |n\rangle_R \langle n|_L \langle n|_R \right) |n\rangle_R = \\
&= 2\sigma \sum_{n=0}^{\infty} (-1+\sigma)^{2n} |n\rangle_L \langle n|_L ,
\end{aligned} \tag{6.8}$$

we can compute the von Neumann entropy:

$$
\begin{aligned}
&S(\rho_L) = -\operatorname{tr}(\rho_L \log \rho_L) = \\
&= -\sum_{n=0}^{\infty} 2\sigma(-1+\sigma)^{2n} \log\Big(2\sigma(-1+\sigma)^{2n}\Big) \left\langle n\right|_L \left|n\right\rangle_L \left\langle n\right|_L \left|n\right\rangle_L = \\
&= -\sum_{n=0}^{\infty} 2\sigma(-1+\sigma)^{2n} \log\Big(2\sigma(-1+\sigma)^{2n}\Big) = \\
&= -\log(2\sigma) - \sum_{n=0}^{\infty} 2\sigma(-1+\sigma)^{2n} \log(-1+\sigma)^{2n} = \\
&= -\log(2\sigma) - \sum_{n=0}^{\infty} 2\sigma(-1+\sigma)^{2n} n \log(-1+\sigma)^{2} = \\
&= -\log(2\sigma) - \frac{(1-2\sigma)}{2\sigma} \log(1-2\sigma) = -\log(2\sigma) + \Big(1 - \frac{1}{2\sigma}\Big) \log(1-2\sigma)
\end{aligned}
\tag{6.9}
$$

where we used the fact that:

$$
\sum_{n=0}^{\infty} n(-1+\sigma)^{2n} = \frac{(-1+\sigma)^2}{[1-(-1+\sigma)^2]^2} \tag{6.10}
$$

and developed the argument of the logarithm neglecting the term σ^2. Since σ is a little quantity, we can expand the logarithm in (6.9):

$$
\begin{aligned}
S &= -\log(2\sigma) + \Big(\frac{\sigma}{2\sigma} + \frac{\sigma - 1}{2\sigma}\Big) \log(1-2\sigma) \simeq \\
&\simeq -\log(2\sigma) + \Big(\frac{\sigma}{2\sigma} + \frac{\sigma - 1}{2\sigma}\Big)(-2\sigma) = \\
&= -\log(2\sigma) + 1 - \sigma
\end{aligned}
\tag{6.11}
$$

where we neglected the term in σ^2 again. This entropy has the same form of the entropy of the two coupled harmonic oscillators (5.34), but in this case the argument of the logarithm is $\sigma = \beta\omega/2$ instead of $\epsilon = \beta\omega$.

In order to gain some physical insight in the regularization we just introduced let us consider again the state

$$
\left|\tau = 0\right\rangle = \frac{1}{\sqrt{N}} e^{-a_L^\dagger a_R^\dagger} \left|0\right\rangle_L \left|0\right\rangle_R . \tag{6.12}
$$

Notice that we can consider a time translation in the parameter ϵ using the Hamiltonian H

$$|\epsilon\rangle = e^{iH\epsilon}|\tau=0\rangle = \frac{1}{\sqrt{N}} e^{iH\epsilon} e^{-a_L^\dagger a_R^\dagger} |0\rangle_L |0\rangle_R = \frac{1}{\sqrt{N}} e^{(\alpha+i\epsilon)H} |0\rangle_L |0\rangle_R \,, \tag{6.13}$$

where we used the identification $H = -\frac{L_+}{\alpha}$. We can write such state as follows:

$$\begin{aligned} |\epsilon\rangle &= \frac{1}{\sqrt{N}} e^{-L_+(\frac{i\epsilon}{\alpha}+1)} |0\rangle_L |0\rangle_R = \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Big(\frac{i\epsilon}{\alpha}+1\Big) |0\rangle_L |0\rangle_R = \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} (-1)^n \Big(\frac{i\epsilon}{\alpha}+1\Big) |n\rangle_L |n\rangle_R = \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} e^{\pi(i-\frac{\epsilon}{\alpha})n} |n\rangle_L |n\rangle_R \end{aligned} \tag{6.14}$$

where we have used Euler's formula. The time-translated state $|\epsilon\rangle$ is normalizable and one can evaluate the normalization constant:

$$N = \frac{1}{1-A^2} = \frac{1}{1-\Big(e^{\pi(i-\frac{\epsilon}{\alpha})}\Big)^2} = \frac{1}{1-e^{-2\pi\frac{\epsilon}{\alpha}}}. \tag{6.15}$$

We can choose $\epsilon << 1$ and verify that the state (6.6) represents in fact the same regularization given by the state (6.14), if we rewrite the latter as follows:

$$\begin{aligned} |\tau=0\rangle &= \sqrt{2\sigma} \sum_{n=0}^{\infty} (-1+\sigma)^n |n\rangle_L |n\rangle_R = \sqrt{2\sigma} \sum_{n=0}^{\infty} (e^{i\pi} - \sigma e^{i\pi})^n |n\rangle_L |n\rangle_R = \\ &= \sqrt{2\sigma} \sum_{n=0}^{\infty} e^{i\pi n} (1-\sigma)^n |n\rangle_L |n\rangle_R \,. \end{aligned} \tag{6.16}$$

Now we can identify the quantity $(1-\sigma)$ with $e^{-\pi\frac{\epsilon}{\alpha}}$ and the quantity $(1-\sigma)^2$, which is in (6.4), with $e^{-2\pi\frac{\epsilon}{\alpha}}$ and thus reproduce (6.14).
We can now write down the density matrix associated to the pure state $|\epsilon\rangle$:

$$\begin{aligned} \rho_{LR} = |\epsilon\rangle\langle\epsilon| &= \Big(1-e^{-2\pi\frac{\epsilon}{\alpha}}\Big) \sum_{n,n'=0}^{\infty} e^{\pi(i-\frac{\epsilon}{\alpha})n} e^{\pi(-i-\frac{\epsilon}{\alpha})n'} |n\rangle_L |n\rangle_R \langle n'|_L \langle n'|_R = \\ &= \Big(1-e^{-2\pi\frac{\epsilon}{\alpha}}\Big) \sum_{n=0}^{\infty} e^{-2\pi\frac{\epsilon}{\alpha}n} |n\rangle_L |n\rangle_R \langle n|_L \langle n|_R \end{aligned} \tag{6.17}$$

and the reduced density matrix:

$$\rho_L = \mathrm{tr}_R\, \rho_{LR} = \left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right) \sum_{n=0}^{\infty} e^{-2\pi\frac{\epsilon}{\alpha}n} \left|n\right\rangle_L \left\langle n\right|_L . \tag{6.18}$$

The calculation of the entanglement entropy will follow the same procedure of the Chapter 5 referred to the entanglement entropy of a system of two harmonic oscillators and, if we identify the factor $\frac{-\pi\epsilon}{\alpha}$ with $\frac{-\beta\omega}{2}$, we should obtain a similar result:

$$\begin{aligned} S = -\,\mathrm{tr}\,\rho_L \log\rho_L &= \sum_{n=0}^{\infty}\left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right)e^{-2\pi\frac{\epsilon}{\alpha}n}\log\left[\left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right)e^{-2\pi\frac{\epsilon}{\alpha}n}\right] = \\ &= -\log\left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right) + 2\pi\frac{\epsilon}{\alpha}\left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right)\frac{e^{-2\pi\frac{\epsilon}{\alpha}}}{(1 - e^{-2\pi\frac{\epsilon}{\alpha}})^2}. \end{aligned} \tag{6.19}$$

The expression of the entropy can be easily rewritten as:

$$S = -\log\left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right) + 2\pi\frac{\epsilon}{\alpha}e^{-\pi\frac{\epsilon}{\alpha}}\frac{1}{2\sinh\left(\frac{\pi\epsilon}{\alpha}\right)}. \tag{6.20}$$

At this point, recalling the limit $T \to \infty$, that is $\beta \to 0$ in the case discussed in Chapter 5, we can make the limit for $\epsilon \to 0$. First of all we expand the function $\sinh\left(\frac{\pi\epsilon}{\alpha}\right)$ and then the exponential:

$$S \simeq -\log\left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right) + 2\pi\frac{\epsilon}{\alpha}e^{-\pi\frac{\epsilon}{\alpha}}\frac{1}{2\pi\frac{\epsilon}{\alpha}} \tag{6.21}$$

$$S \simeq -\log\left[1 - \left(1 - 2\pi\frac{\epsilon}{\alpha}\right)\right] + 1 - \pi\frac{\epsilon}{\alpha} = -\log\left(2\pi\frac{\epsilon}{\alpha}\right) + 1 - \pi\frac{\epsilon}{\alpha}. \tag{6.22}$$

To conclude this chapter we can make some considerations about the expression of the entanglement entropy we have obtained. The first thing to notice is that (6.22) can be compared to (5.34) with the right substitution. Another comparison to do is between the reduced density matrix of the harmonic oscillator (5.29) and the reduced density matrix of the τ-vacuum state

(6.18):

$$\begin{aligned}\rho_L &= \left(1 - e^{-\beta\omega}\right) \sum_{n=0}^{\infty} \left(e^{-\beta\omega n}\right) |n\rangle_L \langle n|_L \\ \rho_L &= \left(1 - e^{-2\pi\frac{\epsilon}{\alpha}}\right) \sum_{n=0}^{\infty} e^{-2\pi\frac{\epsilon}{\alpha}n} |n\rangle_L \langle n|_L \,.\end{aligned} \tag{6.23}$$

From this two equations it's clear that the entanglement entropy (6.22) can be seen as the entropy of a harmonic oscillator with $\omega = \frac{1}{\alpha}$ and $T = \frac{1}{2\pi\epsilon}$. The second important result is that (6.22) recalls the entropy of CFT_2 (3.20):

$$S(L) = \frac{c}{3} \log\left(\frac{L}{\epsilon}\right). \tag{6.24}$$

In fact ϵ, that we used to make the time translation, coincides with the UV regulator and $L = \frac{\alpha}{2\pi}$ with the dimension of the entangling region.

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Chapter 7

Conclusion

Since 1927, when entanglement entropy was introduced for the first time by von Neumann, many studies pointed out its importance and its numerous applications. In this thesis we focused only on one aspect of the entanglement entropy, starting from the correspondence AdS/CFT, conjectured for the first time by Maldacena in 1997 [35]. Precisely, we considered a $0+1$-conformal field theory, that is the conformal quantum mechanics, and noticed that, even in this simple case, thermal effects are possible. This happens because the state associated to the inertial vacuum in Minkowski space-time, that we called $|\tau = 0\rangle$, has the structure of a thermofield double and in fact, computing the reduced density matrix, we discovered a thermal density matrix. The most evident analogy is with a system of two harmonic oscillators, in which the thermofield double formalism is used to treat the mixed state $\rho = e^{-\beta H}$. These two oscillators are correlated because the state is entangled, but we can restrict to one or the other, obtaining a thermal state. At this point we computed the entanglement entropy and then repeated the calculation for the state we constructed in quantum conformal mechanics. One of the results of this thesis is that the entanglement entropy of the τ-vacuum state can be seen as the entropy of a harmonic oscillator with $\omega = \frac{1}{\alpha}$ and $T = \frac{1}{2\pi\epsilon}$ where α is related to the size of the time interval and ϵ is the UV regulator. The other result is the analogy with the entanglement entropy in CFT_2. We know that, given an arbitrary state, computing the entanglement entropy is not simple, but for the case $d = 2$ there is the famous formula (3.20). In this thesis we found an expression for CFT_1, that is similar to (3.20), in which there is the logarithmic dependence from the factor ϵ and the dimension of the entangling region $L = \frac{\alpha}{2\pi}$.

As we said before, entanglement entropy plays an important role in different fields of physics and in particular in quantum gravity. Starting from the EPR paradox, that gave rise to the concept of entanglement, much progress has been made. In fact, in order to understand the nature of black holes, a great amount of work was made in the last decades on computing the entanglement entropy in field theory and we cannot fail to mention the results of Srednicki [15], Jacobson [5], Bekenstein [6] and Hawking [8]. Some studies suggest that the black hole entropy should be interpreted as entanglement entropy. In this regard we started this work illustrating the main aspect of a phenomenon central for quantum gravity, that is the Unruh effect, but we analyzed it in the particular case of $d = 2$, showing that this is possible even in a flat space-time when there is a local horizon. We focused on the fact that the variation of the local horizon entropy is proportional to the inverse of the Unruh temeperature [12] and that this last one is connected with the Hawking temperature of the black hole [28]. Other discoveries must be made if we want to have a more complete picture: in this sense the correspondence AdS/CFT, in which it's argued that quantum gravity in anti-de Sitter spacetime AdS_{d+1} is a conformal field theory at the boundary of AdS spacetime, should be useful, in particular to understand some aspects of quantum gravity formulated in terms of string theory. In recent years CFT is studied from the point of view of quantum information [36] and we know that the entanglement entropy is the central object of this theory.

In this thesis we have discussed the entanglement entropy in the contest of CFT, in the hope to add a little brick to the knowledge in the field. Recent studies show how entanglement entropy has become an important tool in other fields of physics. For example, it can be applied to condensed matter physics: tracing out part of the degrees of freedom of correlated quantum system, obtaining a reduced density matrix, whose eigenvalues constitute the entanglement spectrum, it's possible to study the bipartite structure of clean and disordered system [34]. Another example we can make is quantum computing, that uses the qubit as the basic unit. Qubits is a two-level quantum-mechanical system and can be expressed as a superposition of states. Unlike classical bits, they can exhibit quantum entanglement. These are just two of the many contexts in which entanglement entropy is fundamental. We hope that future research will allow us to better understand how the Universe works.

Appendix A

Quantum field theory in flat spacetime

Let us consider a real scalar field $\phi(x^\mu)$ in flat space-time and the Klein-Gordon Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{A.1}$$

The equation of motion is the Klein-Gordon equation:

$$\Box\phi - m^2\phi = 0. \tag{A.2}$$

We can get the expression of the conjugate momentum using the following relation:

$$\pi = \frac{\partial L}{\partial(\partial_0\phi)} = \dot{\phi}. \tag{A.3}$$

By a Legendre transformation we can relate the Lagrangian density to the Hamiltonian density:

$$\mathcal{H}(\phi,\pi) = \pi\dot{\phi} - \mathcal{L}(\phi,\partial_\mu\phi) = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2, \tag{A.4}$$

where $(\nabla\phi)^2 = \delta^{ij}(\partial_i\phi)(\partial_j\phi)$. It's easy to see that a good solution for the equation (A.2) is a plane wave:

$$\phi(x^\mu) = \phi_0 e^{ik_\mu x^\mu} = \phi_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \tag{A.5}$$

where $k^\mu = (\omega, \mathbf{k})$ and the dispersion relation holds:

$$\omega^2 = \mathbf{k}^2 + m^2. \tag{A.6}$$

Since the frequency depends on $\mathbf{k}$, instead of a single solution, we have a set of them, but we can construct the most general solution using a complete and orthonormal set of modes, thus we have to introduce an inner product:

$$(\phi_1, \phi_2) = -i \int_{\Sigma_t} (\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1) d^{n-1}x \tag{A.7}$$

where Σ_t is a constant-time hypersurface. Applying Stoke's theorem we can see that the inner product is independent of Σ_t and considering (A.5) referred to two plane waves, we can do the following calculus:

$$\begin{aligned}
&(e^{ik_1^\mu x_\mu}, e^{ik_2^\mu x_\mu}) = \\
&= -i \int_{\Sigma_t} (e^{-i\omega_1 t + i\mathbf{k}_1 \cdot x} \partial_t e^{i\omega_2 t + i\mathbf{k}_2 \cdot x} \partial_t - e^{i\omega_2 t + i\mathbf{k}_2 \cdot x} \partial_t \partial_t e^{-i\omega_1 t + i\mathbf{k}_1 \cdot x}) d^{n-1}x = \\
&= (\omega_2 + \omega_1) e^{-i(\omega_1 - \omega_2)t} \int_{\Sigma_t} e^{i(\mathbf{k}_1 - \mathbf{k}_2)\cdot \mathbf{x}} d^{n-1}x = \\
&= (\omega_2 + \omega_1) e^{-i(\omega_1 - \omega_2)t} (2\pi)^{n-1} \delta^{n-1}(\mathbf{k}_1 - \mathbf{k}_2)
\end{aligned} \tag{A.8}$$

where we used tha fact that:

$$\int e^{i\mathbf{k}\cdot\mathbf{x}} d^{n-1}x = (2\pi)^{n-1} \delta^{(n-1)}(\mathbf{k}). \tag{A.9}$$

From the expression of the inner product we can see that an orthonormal set of modes is in the form:

$$f_{\mathbf{k}}(x^\mu) = \frac{e^{ik_\mu x^\mu}}{[(2\pi)^{n-1} 2\omega]^{1/2}} \tag{A.10}$$

and thus:

$$(f_{\mathbf{k}_1}, f_{\mathbf{k}_2}) = \delta^{(n-1)}(\mathbf{k}_1 - \mathbf{k}_2). \tag{A.11}$$

These modes satisfy the relation:

$$\partial_t f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}, \qquad \omega > 0. \tag{A.12}$$

In this set of modes we consider also the complex conjugates $f^*_{\mathbf{k}}(x^\mu)$ which satisfy the relation:

$$\partial_t f^*_{\mathbf{k}} = i\omega f^*_{\mathbf{k}}, \qquad \omega > 0. \tag{A.13}$$

The following relations are valid:

$$(f_{\mathbf{k}_1}, f^*_{\mathbf{k}_2}) = 0 \tag{A.14}$$

$$(f^*_{\mathbf{k}_1}, f^*_{\mathbf{k}_2}) = -\delta^{n-1}(\mathbf{k}_1 - \mathbf{k}_2). \tag{A.15}$$

Through $f_{\mathbf{k}}$ and $f^*_{\mathbf{k}}$ we can expand the solutions of the Klein-Gordon equation, in fact:

$$\phi(t,\mathbf{x}) = \int d^{n-1}xk[\hat{a}_{\mathbf{k}} f_{\mathbf{k}}(t,\mathbf{x}) + \hat{a}^\dagger_{\mathbf{k}} f^*_{\mathbf{k}}(t,\mathbf{x})] \tag{A.16}$$

where $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ are the annihilation and the creation operators. At this point we should remember the canonical commutation relations on equal-time hypersurfaces for $\phi(t,\mathbf{x})$ and $\pi(t,\mathbf{x})$:

$$\begin{aligned} &[\phi(t,\mathbf{x}), \phi(t,\mathbf{x'})] = 0 \\ &[\pi(t,\mathbf{x}), \pi(t,\mathbf{x'})] = 0 \\ &[\phi(t,\mathbf{x}), \pi(t,\mathbf{x'})] = i\delta^{(n-1)}(\mathbf{x} - \mathbf{x'}). \end{aligned} \tag{A.17}$$

If we insert (A.16) into (A.17) we obtain the commutation relations for $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$:

$$\begin{aligned} &[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k'}}] = 0 \\ &[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k'}}] = 0 \\ &[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k'}}] = \delta^{(n-1)}(\mathbf{k} - \mathbf{k'}). \end{aligned} \tag{A.18}$$

The annihilation operator is such that, if we apply it to a vacuum state, we obtain:

$$\hat{a}^\dagger_{\mathbf{k}} |0\rangle = 0 \tag{A.19}$$

for all $\mathbf{k}$; at the same time, if we apply many times the creation operator to a vacuum state, we obtain:

$$|n_{\mathbf{k}}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}}}} \left(\hat{a}^\dagger_{\mathbf{k}}\right)^{n_{\mathbf{k}}} |0\rangle \,. \tag{A.20}$$

With these two operators we can define the number operator:

$$\hat{n}_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \tag{A.21}$$

and the states that are eigenstates of the number operator form the **Fock basis**. At this point we can write the Hamiltonian (A.4) in terms of creation and annihilation operators. Let us start rewriting the term with ϕ^2:

$$\begin{aligned}
&\frac{1}{2}m^2 \int d^{n-1}x\phi^2 = \\
&= \frac{1}{2}m^2 \int d^{n-1}x d^{n-1}k d^{n-1}k' \left(\hat{a}_{\mathbf{k}} f_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}} f^*_{\mathbf{k}}\right)\left(\hat{a}_{\mathbf{k'}} f_{\mathbf{k'}} + \hat{a}^\dagger_{\mathbf{k'}} f^*_{\mathbf{k'}}\right) = \\
&= \frac{1}{2}m^2 \int d^{n-1}x d^{n-1}k d^{n-1}k' \Big(\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k'}} f_{\mathbf{k}} f_{\mathbf{k'}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k'}} f^*_{\mathbf{k}} f_{\mathbf{k'}} + \\
&+ \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k'}} f_{\mathbf{k}} f^*_{\mathbf{k'}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k'}} f^*_{\mathbf{k}} f^*_{\mathbf{k'}}\Big).
\end{aligned} \tag{A.22}$$

We first consider the piece:

$$\begin{aligned}
&\int d^{n-1}x d^{n-1}k' \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k'}} f_{\mathbf{k}} f_{\mathbf{k'}} \int d^{n-1}x d^{n-1}k' \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k'}} \frac{e^{-i(\omega+\omega')t} e^{i(\mathbf{k}+\mathbf{k'})\cdot\mathbf{x}}}{2(2\pi)^{n-1}\sqrt{\omega\omega'}} = \\
&= \int d^{n-1}k' \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k'}} \frac{e^{-i(\omega+\omega')t}}{2\sqrt{\omega\omega'}} \delta^{(n-1)}(\mathbf{k}+\mathbf{k'}) = \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{-k}} \frac{e^{-2i\omega t}}{2\omega}
\end{aligned} \tag{A.23}$$

where we used (A.9) and (A.10). In the same way we can find all the other terms. The final expression for the potential energy is:

$$\begin{aligned}
&\frac{1}{2}m^2 \int d^{n-1}x\phi^2 = \\
&= \frac{1}{2}m^2 \int d^{n-1}k \left(\frac{1}{2\omega}\right)\left[\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{-k}} e^{-2i\omega t} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{-k}} e^{2i\omega t}\right].
\end{aligned} \tag{A.24}$$

We can repeat the same procedure for all the terms of the Hamiltonian, obtaining:

$$\begin{aligned}
&\frac{1}{2}\int d^{n-1}x\dot{\phi}^2 = \frac{1}{2}\int d^{n-1}k \left(\frac{\omega}{2}\right)\left[-\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{-k}} e^{-2i\omega t} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{-k}} e^{2i\omega t}\right] \\
&\frac{1}{2}\int d^{n-1}x(\nabla\phi)^2 = \frac{1}{2}\int d^{n-1}k \left(\frac{\mathbf{k}^2}{2\omega}\right)\left[\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{-k}} e^{-2i\omega t} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{-k}} e^{2i\omega t}\right].
\end{aligned} \tag{A.25}$$

Using (A.6) we can write the final expression for the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\int d^{n-1}k \left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\right]\omega = \int d^{n-1}k \left[n^\dagger_{\mathbf{k}} + \frac{1}{2}\delta^{(n-1)}(0)\right]\omega \tag{A.26}$$

where we used the commutation relations (A.18).

Appendix B

Bogoliubov transformation

In order to obtain the Unruh temperature, it's necessary to know the Bogoliubov coefficients. We'll briefly introduce quantum field theory in curved space-time to compute these coefficients.
Let us start with the Lagrange density of a scalar field:

$$\mathcal{L} = \sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\Big) \tag{B.1}$$

where g is the determinant of the metric tensor $g_\mu\nu$, R is the curvature scalar and ξ is the parameter involved in the conformal coupling and it's equal to $\frac{1}{6}$ in four dimensions. Starting from this Lagrangian we can compute the conjugate momentum:

$$\pi = \frac{\partial\mathcal{L}}{\partial(\nabla_0\phi)} = \sqrt{-g}\nabla_0\phi. \tag{B.2}$$

The canonical commutation relations are:

$$\begin{aligned}
&[\phi(t,\mathbf{x}),\phi(t,\mathbf{x'})] = 0\\
&[\pi(t,\mathbf{x}),\pi(t,\mathbf{x'})] = 0\\
&[\phi(t,\mathbf{x}),\pi(t,\mathbf{x'})] = \frac{i}{\sqrt{-g}}\delta^{(n-1)}(\mathbf{x}-\mathbf{x'}).
\end{aligned} \tag{B.3}$$

The equation of motion for ϕ is:

$$\Box\phi - m^2\phi - \xi R\phi = 0. \tag{B.4}$$

and the inner product between ϕ_1 and ϕ_2 is:

$$(\phi_1, \phi_2) = -i \int_{\Sigma} (\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1) n^\mu \sqrt{\gamma} d^{n-1} x \tag{B.5}$$

where Σ is a spacelike hypersurface, γ is the metric and n^μ the unit normal vector. We can expand the field ϕ in terms of a complete and orthonormal set of solutions $f_i(x^\mu)$, that is:

$$\phi = \sum_i \left(\hat{a}_i f_i + \hat{a}_i^\dagger f_i^* \right) \tag{B.6}$$

where:

$$\begin{aligned} &(f_i, f_j) = \delta_{ij} \\ &(f_i^*, f_j^*) = -\delta_{ij}. \end{aligned} \tag{B.7}$$

The known commutation relations for the annihilation and creation operators are:

$$\begin{aligned} &[\hat{a}_i, \hat{a}_j] = 0 \\ &[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \\ &[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \end{aligned} \tag{B.8}$$

If we apply the annihilation operator to a vacuum state we have:

$$\hat{a}_i^\dagger |0_f\rangle = 0 \tag{B.9}$$

for all i. If we apply many times the creation operator to this vacuum state we can construct the **Fock basis**:

$$|n_i\rangle = \frac{1}{\sqrt{n_i!}} \left(a_i^\dagger \right)^{n_i} |0_f\rangle . \tag{B.10}$$

We remember that the number operator is $n_{f_i}^\dagger = \hat{a}_i^\dagger \hat{a}_i$. The modes $f_i(x^\mu)$ are not the unique choice, in fact we can consider the set of modes $g_i(x^\mu)$ and expand ϕ in terms of them:

$$\phi = \sum_i \left(\hat{b}_i g_i + \hat{b}_i^\dagger g_i^* \right); \tag{B.11}$$

the new operators $\hat{b}_i$ and $\hat{b}_i^\dagger$ have the same commutation relations (B.8). In the same way there will be a vacuum state $|0_g\rangle$ such that:

$$\hat{b}_i\,|0_g\rangle = 0 \tag{B.12}$$

for all i. We can build the Fock basis as before and define the number operator $\hat{n}_{gi} = \hat{b}_i^\dagger \hat{b}_i$. These two sets of modes originate two different notions of vacuum because two observers will see a different number of particles. We can understand this expanding each set of modes in terms of the other:

$$\begin{aligned} g_i &= \sum_j \Big(\alpha_{ij} f_j + \beta_{ij} f_j^*\Big) \\ f_i &= \sum_j \Big(\alpha_{ji}^* f_j - \beta_{ji} g_j^*\Big). \end{aligned} \tag{B.13}$$

These transformations are known as a **Bogoliubov transformation** and α_{ij} and β_{ij} are the **Bogoliubov coefficients**. We can express the operators $\hat{a}_i$ and $\hat{b}_i$ in terms of these coefficients:

$$\begin{aligned} \hat{a}_i &= \sum_i \Big(\alpha_{ji} \hat{b}_j + \beta_{ji}^* \hat{b}_j^\dagger\Big) \\ \hat{b}_i &= \sum_j \Big(\alpha_{ij}^* \hat{a}_j - \beta_{ij}^* \hat{a}_j^\dagger\Big). \end{aligned} \tag{B.14}$$

In the f-vacuum the observer will not perceive any f-particles, but what about an observer which uses the g-modes? To answer this question we have to calculate the expectation value of the g number operator in the f-vacuum:

$$\begin{aligned} &\langle 0_f|\,\hat{n}_{gi}\,|0_f\rangle = \langle 0_f|\,b_i^\dagger b_i\,|0_f\rangle = \langle 0_f| \sum_j k\Big(\alpha_{ij}\hat{a}_j^\dagger - \beta_{ij}\hat{a}_j\Big)\Big(\alpha_{ik}^*\hat{a}_k - \beta_{ik}^*\hat{a}_k^\dagger\Big)\,|0_f\rangle = \\ &= \sum_{jk}(-\beta_{ij})(-\beta_{ik}^*)\,\langle 0_f|\,\hat{a}_j\hat{a}_k^\dagger\,|0_f\rangle = \sum_{jk}\beta_{ij}\beta_{ik}^*\,\langle 0_f|\,\Big(\hat{a}_k^\dagger\hat{a}_j + \delta_{jk}\Big)\,|0_f\rangle = \\ &= \sum_{jk}\beta_{ij}\beta_{ik}^*\delta_{jk}\,\langle 0_f|0_f\rangle = \sum_j \beta_{ij}\beta_{ik}^*. \end{aligned} \tag{B.15}$$

This quantity differs from zero. This shows that the notion of vacuum is not absolute: if an observer perceives a vacuum state, we cant' say the same for another one.

Appendix C

The value of A in the ground state

Considering the operator a_+ (5.19), we have to rewrite it using (5.16):

$$
\begin{aligned}
a_+ = \frac{1}{\sqrt{2}}\Bigg\{&\sqrt{\frac{(1+\sin\theta)\omega}{\cos\theta}}\Big[\frac{1}{2\omega}(a_L^\dagger + a_L) + \frac{1}{2\omega}(a_R^\dagger + a_R)\Big] \\
&+ i\sqrt{\frac{\cos\theta}{(1+\sin\theta)\omega}}\Big[\frac{\sqrt{\omega}}{2}i(a_L^\dagger - a_L) + \frac{\sqrt{\omega}}{2}i(a_R^\dagger + a_R)\Big]\Bigg\}.
\end{aligned}
\tag{C.1}
$$

Now we can apply it to the ground state of the system of two coupled harmonic oscillators (5.18), obtaining:

$$
\begin{aligned}
&a_+\left|0\right\rangle = \\
&= \frac{1}{2\sqrt{2}}\Bigg[\sqrt{\frac{1+\sin\theta}{\cos\theta}}(a_L^\dagger + a_L + a_R^\dagger + a_R) - \sqrt{\frac{\cos\theta}{1+\sin\theta}}(a_L^\dagger - a_L + a_R^\dagger - a_R)\Bigg] \\
&\sqrt{1-A^2}e^{Aa_L^\dagger a_R^\dagger}\left|0\right\rangle_L\left|0\right\rangle_R .
\end{aligned}
\tag{C.2}
$$

We know that $a_+^\dagger = \frac{1}{\sqrt{2}}(a_L^\dagger + a_R^\dagger)$ and that $a_+ = \frac{d}{da_+^\dagger}$, thus $a_+ = \frac{1}{\sqrt{2}}(\frac{d}{da_L^\dagger} + \frac{d}{da_R^\dagger})$. We can apply again the operator to the vacuum state:

$$
\frac{1}{\sqrt{2}}\Big(\frac{d}{da_L^\dagger} + \frac{d}{da_R^\dagger}\Big)\left|0\right\rangle = \frac{1}{\sqrt{2}}\sqrt{1-A^2}e^{a_L^\dagger a_R^\dagger}A(a_L^\dagger + a_R^\dagger)\left|0\right\rangle_L\left|0\right\rangle_R . \tag{C.3}
$$

This equation can be compared to (C.2), remembering that $a_L \left|0\right\rangle_L \left|0\right\rangle_R = a_R \left|0\right\rangle_L \left|0\right\rangle_R = 0$:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\sqrt{1-A^2}e^{a_L^\dagger a_R^\dagger}A(a_L^\dagger + a_R^\dagger)\left|0\right\rangle_L \left|0\right\rangle_R = \\
&= \frac{1}{2\sqrt{2}}\Big[\sqrt{\frac{1+\sin\theta}{\cos\theta}}(a_L^\dagger + a_L + a_R^\dagger + a_R) + \\
&- \sqrt{\frac{\cos\theta}{1+\sin\theta}}(a_L^\dagger - a_L + a_R^\dagger - a_R)\Big]\sqrt{1-A^2}e^{Aa_L^\dagger a_R^\dagger}\left|0\right\rangle_L \left|0\right\rangle_R .
\end{aligned} \tag{C.4}$$

The equation for A is:

$$A = \frac{1}{2}\Big[\sqrt{\frac{1+\sin\theta}{\cos\theta}} - \sqrt{\frac{\cos\theta}{1+\sin\theta}}\Big]. \tag{C.5}$$

Squaring both sides we obtain:

$$\begin{aligned}
A^2 &= \frac{1}{4}\Big(\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} - 2\Big) = \\
&= \frac{1}{4}\Big(\frac{\cos^2\theta + (1+\sin\theta)^2 - 2\cos\theta(1+\sin\theta)}{(1+\sin\theta)\cos\theta}\Big) = \\
&= \frac{1}{2}\Big(\frac{1+\sin\theta - \cos\theta(1+\sin\theta)}{(1+\sin\theta)\cos\theta}\Big) = \frac{1}{2}\Big(\frac{1-\cos\theta}{\cos\theta}\Big).
\end{aligned} \tag{C.6}$$

Thus the value of A is:

$$A = \pm\frac{1}{\sqrt{2}}\sqrt{\frac{1-\cos\theta}{\cos\theta}}. \tag{C.7}$$

Considering that:

$$\pm\sqrt{\frac{1-\cos\theta}{2}} = \sin\frac{\theta}{2} \tag{C.8}$$

and that:

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \tag{C.9}$$

we obtain:

$$A = \frac{\sin\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta}{2}(1-\tan^2\frac{\theta}{2})}} = \frac{\tan\frac{\theta}{2}}{\sqrt{1-\tan^2\frac{\theta}{2}}}. \tag{C.10}$$

Thus if we choose the angle $\theta = 0$ we find the decoupled system:

$$|0\rangle = |0\rangle_L \, |0\rangle_R \tag{C.11}$$

while if we choose, for example, the angle $\theta = \frac{\pi}{2}$ we obtain divergent values of A.

Appendix D

Another way to compute the entanglement entropy

In order to obtain a well-known expression for the entropy, we can rewrite the state:

$$|\tau = 0\rangle = \sum_{n=0}^{\infty} (-1+\sigma)^n \, |n\rangle_L \, |n\rangle_R \tag{D.1}$$

in another way. Using Euler's formula, we obtain:

$$\begin{aligned}|\tau = 0\rangle &= \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} (e^{i\pi} - \sigma e^{i\pi})^n \, |n\rangle_L \, |n\rangle_R = \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} e^{i\pi n} (1-\sigma)^n \, |n\rangle_L \, |n\rangle_R = \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} e^{i\pi n} e^{-\delta n} \, |n\rangle_L \, |n\rangle_R\end{aligned} \tag{D.2}$$

where we used the identification $(1-\sigma) = e^{-\delta}$. In fact, in the limit $\sigma \to 0$ we obtain 1, that is the same result of the limit $\delta \to 0$. We can notice that there is a connection between σ that regularizes (6.2) and δ that regularizes the state:

$$|\tau = 0\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} e^{i\pi n} \, |n\rangle_L \, |n\rangle_R \, . \tag{D.3}$$

Making the limit $\sigma \to 0$ in (D.1) we reproduce (6.2), at the same time, if we make the limit $\delta \to 0$ we reproduce (D.3).

Let us consider the normalization constant:

$$N = \frac{1}{1-A^2} = \frac{1}{1-e^{-2\delta}}. \tag{D.4}$$

The state becomes:

$$|\tau = 0\rangle = \left(\sqrt{1-e^{-2\delta}}\right)\sum_{n=0}^{\infty} e^{i(\pi+i\delta)n} |n\rangle_L |n\rangle_R . \tag{D.5}$$

First of all we want to compute the entanglement entropy for this state and then check if it coincides with (6.9). The density matrix $\rho = |\tau = 0\rangle \langle \tau = 0|$ will be:

$$\begin{aligned} \rho &= \frac{1}{N}\sum_{n,n'=0}^{\infty} e^{i\pi(n-n')}e^{-\delta(n+n')} |n\rangle_L |n\rangle_R \langle n'|_L \langle n'|_R = \\ &= \frac{1}{N}\sum_{n=0}^{\infty} e^{-2\delta} |n\rangle_L |n\rangle_R \langle n|_L \langle n|_R \end{aligned} \tag{D.6}$$

where

$$N = \frac{1}{1-e^{-2\delta}}. \tag{D.7}$$

The reduced density matrix is obtained computing $\rho_L = \mathrm{tr}_R |\tau = 0\rangle \langle \tau = 0|$:

$$\rho_L = \left(1-e^{-2\delta}\right)\sum_{n=0}^{\infty} e^{-2\delta n} |n\rangle_L \langle n|_L . \tag{D.8}$$

At this point we are ready to compute the entropy, using the fact that $S = -\operatorname{tr}\rho_L \log \rho_L$:

$$\begin{aligned}
S &= -\sum_{n=0}^{\infty}\left(1-e^{-2\delta}\right)e^{-2\delta n}\log\left[\left(1-e^{-2\delta}\right)e^{-2\delta n}\right] = \\
&= -\log\left(1-e^{-2\delta}\right) - \sum_{n=0}^{\infty}\left(1-e^{-2\delta}\right)e^{-2\delta n}\log e^{-2\delta n} = \\
&= -\log\left(1-e^{-2\delta}\right) + \sum_{n=0}\left(1-e^{-2\delta}\right)e^{-2\delta n} n 2\delta = \\
&= -\log\left(1-e^{-2\delta}\right) + \left(1-e^{-2\delta}\right)\frac{e^{-2\delta}}{\left(1-e^{-2\delta}\right)^2}2\delta = \\
&= -\log\left(1-e^{-2\delta}\right) + \frac{e^{-2\delta}}{\left(1-e^{-2\delta}\right)}2\delta = \\
&= -\log\left(1-e^{-2\delta}\right) + e^{-\delta}\frac{1}{2\sinh\delta}2\delta.
\end{aligned} \tag{D.9}$$

First of all we can expand the function $\sinh\delta$ and then the exponential $e^{-2\delta}$, when $\delta \to 0$:

$$\begin{aligned}
S &\simeq -\log\Big(1-\big(1-2\delta\big)\Big) + \Big(1-\delta\Big)\frac{1}{2\delta}2\delta = \\
&= -\log(2\delta) + 1 - \delta.
\end{aligned} \tag{D.10}$$

We can notice that when $\delta \to 0$, the term $\log(2\delta) \to \infty$. Now if we consider the substitutions:

$$\begin{aligned}
&1-\sigma = e^{-\delta} \\
&(1-\sigma)^2 \sim 1-2\sigma = e^{-2\delta}
\end{aligned} \tag{D.11}$$

and if we use in (6.9) we can easily verify that:

$$\begin{aligned}
S &= -\log\big(1-e^{-2\delta}\big) + \log\big(e^{-2\delta}\big)\frac{1-e^{-2\delta}-1}{1-e^{-\delta}} = \\
&= -\log\big(1-e^{-2\delta}\big) + 2\delta\frac{e^{-2\delta}}{1-e^{-2\delta}}
\end{aligned} \tag{D.12}$$

that is exactly (D.9).

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