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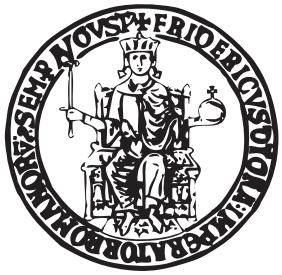

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Semiclassical Approaches in Solving Quantum Field Theory

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Introduction

Quantum Field Theory is the product of decades of attempts to construct a framework within which we can successfully describe the fundamental constituents and interactions of nature. The history of this compelling challenge begins with the unification of Quantum Mechanics and Special Relativity and has come down to the present day preserving the theory's undisputed power and centrality in human understanding of the phenomena underlying the universe, which to this day is encapsulated in the Standard Model of Particles. The conceptual structure of the theory has several milestones that make it one of humanity's greatest scientific achievements, among them we can mention: Perturbations Theory, Gauge Theories and the Rinormalization Group. As has often been the case in Physics, having a fundamental theory is not always synonymous with a total understanding of phenomena; nature is often difficult to understand. For example, we can exactly describe the structure of a hydrogen atom using the Schrodinger Equation, the heart of non-Relativistic Quantum Mechanics, but it becomes hard to study even the simplest molecule with the same tool, and this teaching still holds even in Fundamental Physics. One of the most glaring cases is that of 3-colors QCD, the fundamental theory of the interaction between quarks and gluons, the elementary constituents of hadrons. QCD is a non-Abelian gauge theory and it exhibits several properties that make it one of the phenomenologically richest interactions ever: asymptotic freedom and confinement. The interaction becomes weak, hence perturbative, at high energies, while at low energies it cannot be studied with the usual perturbative techniques, and this greatly complicates the study of the physics of baryons and mesons that are precisely part of the low-energy spectrum of the theory. This complexity forces us to separate the theory

into regimes, each of which can be studied with different techniques and approaches. These difficulties underlie the introduction of Effective Theories that capture the characterizing aspects of the theory in a given regime. In this way one is able to obtain very accurate predictions at the cost of having accurate results in that specific regime only. In the set of effective theories of QCD we can mention the theory of Chiral Perturbations where the fundamental particles are hadrons themselves and the masses of the quarks are considered negligible, or the Effective Theory of Heavy Quarks and so on. In this context, fundamental theories become the nucleus of a constellation of effective theories, which in turn require the introduction of new methods and strategies of theoretical investigation that often open up new lines of research with a life of their own. The aim of this work is to jointly apply some cutting-edge techniques concerning effective theories and QFT more widely. Specifically, we will start by illustrating a generalisation of the well-known Linear Sigma Model and upon this we will carry out a fixing of the charges associated with the global symmetries of the system, this operation has a double value, phenomenological and technical. Firstly, it allows the various phases of a theory to be probed as the system's charge density changes, and secondly it allows new perturbative parameters to be introduced, which are necessary when constructing an effective theory. In a second step we will exploit a fundamental property of such a model, namely the fact that it possesses a fixed point, a value of the coupling constant for which the beta-function vanishes. We will see how this implies the possibility of the system acquiring Conformal Invariance. We will then introduce the fundamental aspects of Conformal Field Theory and see how this theory, applicable to our case, is capable of extending the limits of the perturbative approach even where it is not possible in non-conformal QFT, allowing for example the calculation of the anomalous dimensions of large-charge operators.

Chapter 1

The $SU(2N)$ Linear Sigma Model

1.1 Effective Theories

Solving a QFT means being able to calculate all n-point correlation functions. However, some theories are more complex than others and as already mentioned, it is sometimes necessary to separate the theory into regimes and study them separately. The notion of complexity of a theory in QFT has a very precise meaning. It depends on the behaviour of the degrees of freedom along the RG-flow, the number of independent parameters of the theory and the relationships between these parameters, the exact and approximate symmetries that are realised and the breaking patterns of these symmetries. The interplay of these features can make it difficult not only to solve the theory in the various regimes, but even to correctly identify and split the regimes themselves. This is the case with QCD, where the combination of features such as asymptotic freedom, colour confinement, inhomogeneity between masses of quarks, and approximate symmetries, makes the study of strong interaction very challenging and organised into effective theories, each with its own 'phase' of pertinence. Before moving onto the aspects we want to focus on, it is therefore good to briefly explore the theory and draw an intuitive picture of its phases. The elementary degrees of freedom of the theory are the quarks ψ^a_f and the gluons A^a_μ, where f is the flavour index and $a = 1, 2, 3$ the colour index. The quarks transform into the fundamental representation of the colour group $SU_C(3)$ while the gluons into the adjoint representation. Being f a

flavour index, the Lagrangian of QCD has the form

$$\mathcal{L} = \sum_f \bar{\psi}_f^a (i\bar{D} - m_f)\psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \tag{1.1}$$

where $G^a_{\mu\nu}$ is the field strenght tensor defined as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \ . \tag{1.2}$$

So far, we can make a first important classification among quarks. The masses of the quarks are quite different as the flavors vary and can differ even by several orders of magnitude, this allows us to introduce very well-defined energy scales at which effective theories pertaining to a certain energy scale can be constructed, neglecting the dynamics that take place at other energy scales. We could then ideally consider the mass of the light quarks to be zero and that of the heavy quarks as infinite, leaving the coupling constant the only parameter in the theory. We can then calculate the one-loop β-function which will give us essential information on the behaviour of the theory as the energy scale involved changes. Direct calculation results in

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right) \tag{1.3}$$

It is self-evident that for $N_f < \frac{11}{2}N_c$ the theory is asymptotically free, this implies that the coupling is strong at low energies and weak at high energies. We can emphasise this point by calculating the coupling constant itself, remembering that the β-function is the logarithmic derivative of g with respect to the energy scale q, we have

$$g^2(q^2) = \frac{16\pi^2}{(\frac{11}{3}N_c - \frac{2}{3}N_f)\log\left(\frac{q^2}{\Lambda^2_{QCD}}\right)} \tag{1.4}$$

The only dimensional parameter in the above expression is Λ_{QCD}, which in simple terms sets the scale at which QCD is no longer perturbative. The estimation of this parameter would deserve a rigorous independent treatment, for our purposes we will simply say that approximately the scale of Λ corresponds more or less with the hadronic scale, thus $\simeq 3 - 4 \times 10^2$ MeV. We have introduced a first important distinction between perturbative and non-perturbative regimes, but

it is an inconclusive analysis as long as the symmetries of the theory and the behaviour of the degrees of freedom in these regimes are not analysed. Let us then study the local and global symmetries of the theory. The theory has a local gauge symmetry $SU_C(3)$, let $U(x)$ be a local $SU_C(3)$ transformation,

$$\psi(x) \to U(x)\psi(x) \ ; \ \ A_\mu(x) \to U(x)A_\mu U^\dagger(x) + iU(x)\partial_\mu U^\dagger(x), \tag{1.5}$$

Whether in some cases the masses of quarks can be considered equal, we gain a new global invariance related to rotations in the flavor's space, the Flavor Symmetry

$$\psi_f \to V_{fg}\psi_g \ , \ \ V_{fg} \in SU(3) \tag{1.6}$$

The symmetry is enlarged to $SU_L(3) \times SU_R(3)$ if the masses are set to zero. In fact, in the latter case any coupling between left-handed and right-handed spinors vanishes. Being $\psi_{L,R}$ the chiral components of ψ, namely

$$\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\psi, \tag{1.7}$$

the action of $SU_L(3) \times SU_R(3)$ reads as

$$\psi_{L,f} \to L_{f,g}\psi_g \ , \ \ \psi_{R,f} \to R_{f,g}\psi_g \ ; \ \ L_{f,g}, R_{f,g} \in SU_L(3), \ SU_R(3). \tag{1.8}$$

We know for sure that the masses of the quarks are neither zero nor all equal, but for light quarks we have $m_u, m_d < m_s \ll \Lambda_{QCD}$, and it is certainly licit to make use of the chiral symmetry. The Lagrangian possesses two additional $U(1)$ symmetries :

$$\psi_L(x) \to e^{i\alpha}\psi_L(x) \ ; \ \ \psi_R(x) \to e^{i\alpha}\psi_R(x) \qquad U_V(1), \tag{1.9}$$

$$\psi_L(x) \to e^{i\beta}\psi_L(x) \ ; \ \ \psi_R(x) \to e^{-i\beta}\psi_R(x) \qquad U_A(1). \tag{1.10}$$

The $U_V(1)$ symmetry is exact even when the masses of the quarks are non-zero and is the baryon number conservation symmetry, whereas the $U_A(1)$ symmetry, although classically valid, is quantum-mechanically anomalous. In fact, the associated current has non-zero divergence:

$$\partial_\mu J^\mu_A = \frac{g^2 N_f}{16\pi^2} \operatorname{Tr}(\tilde{F_{\mu\nu}} F^{\mu\nu}) \tag{1.11}$$

In QCD the chiral symmetry is spontaneously broken by the condensate

$$\left\langle \bar{\psi}_f \psi_g \right\rangle = \delta_{fg} \Lambda \tag{1.12}$$

acquiring a non-zero vacuum expectation value that only preserves the $SU_V(3)$ vectorial subgroup of $SU_L(3) \times SU_R(3)$ that we actually know is explicitly broken due to the mass difference between the quarks. Since $m_u, m_d \ll m_s$, the isospin symmetry $SU(2)$ is a better symmetry than the previous one. If we assume these symmetries to be correct, we expect a number of Goldstone bosons to appear from the spontaneous symmetry breaking mechanism :

$$SU_L(3) \times SU_R(3) \to SU_V(3) \to 8 \; Goldstone's, \tag{1.13}$$

$$SU_L(2) \times SU_R(2) \to SU_V(2) \to 3 \; Goldstone's. \tag{1.14}$$

If the breaking is induced by the condensate, it is then natural to set ourselves the goal of constructing a theory in which the condensate corresponds to the VEV of an 'effective' field, Σ, whose transformation properties are exactly inherited from the condensate itself. Being L and R transformations of SU_L and SU_R respectively, the effective field Σ must transform as :

$$\langle \Sigma \rangle \sim \left\langle \bar{\psi}_f \psi_g \right\rangle \Rightarrow \Sigma \to L \Sigma R^\dagger. \tag{1.15}$$

Such a field can be interpreted as the field describing the fluctuations around the condensate, it is the chiral order parameter and the Goldstone bosons are the Σ fluctuations invariant under the coset $(SU_L(N_f) \times SU_R(N_f))/SU_V(N_f)$ elements. This is the first step towards the construction of an 'effective theory', i.e. a theory that reproduces the symmetries of a given fundamental theory but whose degrees of freedom correspond to the 'net' particle content of the microscopic theory at a certain scale. For example, the 3 Goldostones of Σ in the $SU(2)$ case could represent pions. Recalling that the condensate is a matrix, the kinetic part of the lagrangian describing Σ will be of the type :

$$\mathcal{L} \sim \mathrm{Tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger). \tag{1.16}$$

We could of course add potential terms or even terms that take into account the masses of the quarks at the effective level without changing the starting

global symmetries, we also have several choices in the explicit parametrization of the Σ field. The class of theories we obtain in this way is known as σ-models and, depending on the parametrization, we can distinguish between Linear Sigma Models or Non-Linear Sigma Model. For example, in the $SU(3)$ case we can use the exponential parametrization :

$$\Sigma = e^{\frac{i\lambda_a \pi^a}{f_\pi}}, \tag{1.17}$$

where λ_a are the Gell-Mann matrices and f_π is the pion decay constant. The advantage of this parameterisation is that the Lagrangian can be developed to a given order in the powers of $\frac{\partial}{f_\pi}$, so a perturbative scheme is automatically introduced. In the case of isospin invariance, exploiting the homomorphism between $SU(2) \times SU(2)$ and $SO(4)$, knowing that 3 Goldstones are expected from this theory, we can introduce a chiral partner of pions, typically denoted by σ, and use a linear parameterisation

$$\Sigma = \sigma + i\vec{\tau} \cdot \vec{\pi} \tag{1.18}$$

If we wish to retain the interpretation of pions as pseudo-Goldstone bosons, we must ensure that only σ acquires mass and that $\vec{\pi}$ remains massless. We then introduce potential terms and study symmetry breaking pattern

$$\mathcal{L} = \frac{1}{4}\,\mathrm{Tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) + u_0\,\mathrm{Tr}(\Sigma^\dagger \Sigma) + v_0\,\mathrm{Tr}(\Sigma^\dagger \Sigma)^2 \tag{1.19}$$

For $u_0 > 0$, the potential's minimum is obtained for

$$\left\langle \sigma^2 + \vec{\pi}^2 \right\rangle = \frac{u_0}{v_0} \tag{1.20}$$

and we may require that $\langle \vec{\pi} \rangle = 0$, obtaining $\langle \sigma \rangle = \sqrt{\frac{u_0}{v_0}}$. If we assume that parity cannot be spontaneously broken in QCD then $\langle \pi \rangle = 0$ is no longer a choice but a requirement. This result is known as the Vafa-Witten theorem, which states that vector-like global symmetries, including parity in vector-like gauge theories, such as QCD, cannot be spontaneously broken as long as the θ-angle is zero [**12**]. This result is clearly consistent with Goldstone's theorem, the spontaneously broken symmetries of $SO(4)$ are all and only those involving the σ direction, and

therefore the Goldstone bosons are equal in number to the dimension of $SO(4)$, i.e. $\frac{n(n-1)}{2}$, minus the number of pairs of axes involving the σ axis, i.e. 3, from which we obtain 3 massless bosons.

1.2 Charge Fixing

Another important tool that we wish to introduce for further development and subsequent applications is the fixing of charges in a QFT. Assuming that a QFT has a certain number of global symmetries, whose conserved Noether charges are of course associated, we could in principle require that the system shows a certain fixed charge. There are several reasons why this idea can be instructive, and they are both technical and phenomenological. Firstly, in a system of quantum nature where a charge is conserved, assuming that the charge is sufficiently large with respect to a certain scale that depends on the system under examination, we expect the system to 'classicize' so it comes naturally to introduce semi-classical techniques. We will see that this demand in general does not preserve Lorentz invariance. Moreover, a charged system in such a way allows us to introduce the so-called "large charge expansions", an idea that turns out to be particularly useful in the area of strongly coupled theories where the inverse of the fixed charge, in the limit where this is sufficiently large, can be used as a perturbative parameter in terms of which observables are expanded. Lastly, charge fixing is especially fruitful in the study of the 'phases' of a theory, as it is possible to construct actual phase diagrams in which the aforementioned fixed charge is one of the axes, and in this way analyse possible phase transitions. An emblematic case of this latter approach is represented by QCD-like thoeries, which will be at the heart of the subsequent developments of this work. In the remaining of the present chapter we want to develop these ideas, bringing notable examples and preparing ourselves for the subsequent discussion of the QCD-like theories. Let us illustrate the ideas we briefly introduced earlier in more detail, starting with the simplest charge fixing model. Let j^μ be a Noether current for a relativistic

theory, then its conserved charge will be

$$Q = \int dx^{d-1} j^0. \tag{1.21}$$

The charge fixing condition is trivially expressed by imposing that the equation $Q = \tilde{Q}$ is satisfied, where $\tilde{Q}$ is just the charge fixed value. We know that the zero component of the current transforms as a vector, so the charge fixing constraint seems to explicitly break the Lorentz invariance. The Fixed Charge constraint can be implemented at the Lagrangian level by simply introducing a chemical potential term :

$$\hat{\mathcal{L}} = \mathcal{L} + \mu Q. \tag{1.22}$$

An interesting aspect lies in the relationship between fixed charges and the mechanism of spontaneous symmetry breaking. Let $\hat{H}$ be the Hamiltonian operator associated with $\hat{\mathcal{L}}$, the vacuum of the theory $|0\rangle$, is by definition the state which minimizes $\hat{H}$

$$\hat{H} = H - \mu Q \tag{1.23}$$

$$\hat{H}\,|0\rangle = 0 \tag{1.24}$$

It can happen that the vacuum is not an eigenstate of $\tilde{Q}$, for instance when the symmetry related to $\tilde{Q}$ is spontaneously broken by $|0\rangle$ itself. Indeed, there is nothing to prevent the charge-fixing operation from inducing non-zero Vacuum Expectation Value. The first remarkable consequence is that, since the vacuum is no longer eigenstate of charge, it cannot be the eigenstate of H, which implies that the states of the system cannot be classified with the eigenvalues of the relativistic Hamiltonian H, but it is necessary to diagonalise $\hat{H}$, which ,as can be seen, is not relativistic. There are several arguments why the breaking of the Lorentz invariance is not of concern. Indeed, we note that the chemical potential term can be seen as a coupling between the field and the zero component of a background gauge field. Furthermore, the starting theory remains relativistic, so the non-relativistic system can be interpreted as a state of a relativistic theory where the Lorentz-invariance is only spontaneously broken. Thus, the charge

fixing has a significant interplay with the mechanism of spontaneous symmetry breaking and this aspect proves to be crucial in the construction of effective theories. Let us analyse a simple case, which will serve as the first basic template for subsequent developments and generalisations, of the application of these QFT methodologies by introducing 2-colour QDC at finite baryonic dentity [**kogut1**] with the purpose of building the Effective Theory. Although this model does not refer to a theory of nature, which would be the 3-colour QCD, it possesses some unique features that make the treatment simpler and more powerful. Suppose then that we want to determine the spectrum of such a theory whose Lagrangian in Euclidean formulation is

$$\mathcal{L} = \sum_{f=1}^{N_f} \bar{\psi}_f \gamma^\mu D_\mu \psi_f + m_q \bar{\psi}_f \psi_f + \mu \bar{\psi}_f \gamma^0 \psi_f, \tag{1.25}$$

where N_f is the number of flavor and f the flavor index. For the sake of simplicity, let us consider the massless case with $N_f = 2$. At $\mu = 0$, with two colors and two flavors, the theory admits a pseudo-real representation of $SU_c(2)$, so we have an $SU(4)$ global flavor symmetry which is spontaneously broken by the condensate into $Sp(4)$ with the creation of 5 Goldstone Bosons. For $\mu \neq 0$ the $SU(4)$ symmetry is explicitly broken into $SU_L(2) \times SU(2)_R \times U(1)$ and spontaneously into $SU_L(2) \times SU_R(2)$ creating a single Goldstone with the other 4 acquiring a mass proportional to $\mu \ll \Lambda_{QCD}$. The latter statement is fundamental and is the reason why Effective Theory is needed: we are studying the 2-color-2-flavors QCD below the Chiral Symmetry Breaking Scale Λ_{QCD}.We will elaborate towards the end on how these methods are related to one of the most important and challenging aspects of QCD: its phases. The fundamental Lagrangian is

$$\mathcal{L} = q_L^\dagger i \sigma_\mu D_\mu q_L + \mathcal{L} = q_R^\dagger i \bar{\sigma}_\mu D_\mu q_R, \tag{1.26}$$

we can now use the color pseudo-reality $T_a^* = T_a^T = -T_2 T_a T_2$, where T_i are Pauli's matrices, then

$$D_\mu^T = -T_2 D_\mu T_2 \to -\vec{\partial_\mu} + A_\mu^a T_a^T, \tag{1.27}$$

defining $\tilde{q} = \sigma_2 T_2 q_R^*$ and $\tilde{q}^\dagger = q_R^T T_2 \sigma_2$, for $N_f = 2$ if $\Psi = (q_1, q_2, \tilde{q}_1, \tilde{q}_2)^T$, then the covariant derivative part of the lagrangian becomes

$$\mathcal{L} = \Psi^\dagger i \sigma_\mu D_\mu \Psi \tag{1.28}$$

As can be easily seen, the total global symmetry is $SU(2N_f) \times U_B(1)$. Whitout pseudo-reality it would have been $SU_L(2) \times SU_R(2) \times U(1)$. We can also rewrite the mass term as

$$q_R^\dagger q_L + q_L^\dagger q_R = \tilde{q}^T \sigma_2 T_2 q + q^\dagger \sigma_2 T_2 (\tilde{q}^\dagger)^T = \frac{1}{2} \Psi^T \sigma_2 T_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Psi, \tag{1.29}$$

while the chemical term becomes

$$q_L^\dagger q_L + q_R^\dagger q_R = q^\dagger q - \tilde{q}^\dagger \tilde{q} = \Psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi. \tag{1.30}$$

The chemical term preserves separately $SU(2) \times SU(2) \times U(1)$. At $\mu = 0$ the non-zero VEV of the mass term breaks $SU(4)$ spontaneously into $Sp(4)$ (5 Goldstones), while for $\mu \neq 0$ the symmetry is $SU_L(2) \times SU_R(2) \times U_B(1)$, spontaneously broken into $Sp(2) \times Sp(2)$ (1 Goldstone). Four of the previous 5 Goldstones acquire mass dependent on the chemical potential that goes to zero when μ goes to zero. This is the general picture of the microscopic theory once the charge-fixing is performed. We want now to construct the Effective Theory in which the degrees of freedom are represented by fluctuations of the condensate in the microscopic theory, namely Σ[1]

$$\Sigma \sim \Psi \Psi^T \sigma_2 T_2 \tag{1.31}$$

Under $SU(4)$ we have $\Psi \to U\Psi$, therefore the Effective Field Σ transforms as $\Sigma \to U \Sigma U^T$. The kinetic effective Lagrangian takes the form

$$\mathcal{L} = f_\pi^2 \, \mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) \tag{1.32}$$

The Σ matrix encodes 5 degrees of freedom corresponding to 5 Goldstones in the microscopic theory for $\mu = 0$, so it has to be unitary and anti-symmetric

[1] io so cosa vuoi dire ma devi introdurre un po' meglio le cose: σ chi sono? T chi sono? U chi è? meglio spendere qualche parola in più per essere più chiari

in order to codify 5 independent real fields. The chemical potential term of the microscopic theory explicitly breaks SU(4). We can first recover this symmetry by also transforming the source coupled to the breaking term, to do so we write the term in μ as

$$\mu\Psi^{\dagger} i\sigma_{\mu} B_{\mu}\Psi \tag{1.33}$$

$$M_{\mu} = \delta_{o\mu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.34}$$

but this time B also transforms under SU(4) as

$$M_{\mu} \to U M_{\mu} U^{\dagger} \tag{1.35}$$

This procedure is just a direct way to introduce at the effective level a potential term through the simplest non-linear term in S and B, i.e. $\mu^2 \,\mathrm{Tr}(\Sigma M^t \Sigma^{\dagger} M)$, this term will allow us to find a minimum and also the masses of the pseudo-Golstones encoded in the second derivatives. The last step is the insertion of a symmetry breaking term and the computation of the Pseudo-Goldostone masses dependence on μ; symmetry will bu our guide. Global symmetries in the microscopic theory can be exstended to local symmetries, statement which requires

$$M_{\mu} \to U M_{\mu} U^{\dagger} + \frac{1}{\mu} U \partial_{\mu} U^{\dagger} \tag{1.36}$$

$$\partial_{\mu}\Sigma \to D_{\mu}\Sigma = \partial_{\mu}\Sigma + \mu(M_{\mu}\Sigma + \Sigma M_{\mu}^{T}) \tag{1.37}$$

We are not really realising a gauge theory, this "gauging" procedure is nothing more than a way of effectively linking the mass term with the kinetic term and fixing the relationship between masses and chemical potential. Explicitly calculating the minimum leads us to the relation:

$$M\Sigma_0 = \Sigma_0 M^T \to \Sigma_0 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{1.38}$$

which can be written recursively for arbitrary powers of M and M^T,

$$M\Sigma_0 = \Sigma_0 M^T \to M^2\Sigma_0 = M\Sigma_0 M^T = \Sigma_0 (M^T)^2 \to ... M^n \Sigma_0 = \Sigma_0 (M^T)^n \tag{1.39}$$

making the following exponential relation valid

$$\Sigma_0 = e^{iMt}\Sigma_0 e^{-iM^T t} \tag{1.40}$$

This minimum is not unique, there is a $U(1)$ degeneracy related to the barion number corresponding to the exponentiation of M. In order to capture the curvature of Σ around its vacuum alignment it is convenient to write Σ as $U\Sigma U^T$, with U near to the Identity. At this level, the best thing to do is to separate generators that leave the vacuum invariant (T_i) from those which do not (χ_i) .To find the former let's write the corresponding condition

$$U\Sigma_o U^T = \Sigma_0 \tag{1.41}$$

$$U = e^{i\Phi_j T^j} \tag{1.42}$$

The T_i generators form the $Sp(4)$ sub-group and solve the equation

$$T_i\Sigma_0 = -\Sigma_0 T_i^T \tag{1.43}$$

whose solutions are

$$T_{1-3} = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \; ; \; T_{4-6} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{1.44}$$

$$T_{7-9} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \; ; \; T_{10} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{1.45}$$

while the χ_j solving $\chi_a\Sigma_0 = \Sigma_0\chi_a$ are

$$\chi_{1-3} = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \; ; \; \chi_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.46}$$

$$\chi_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = M \tag{1.47}$$

It should not be surprising that one of the χ's transforming the vacuum is exactly the M matrix, since they solve the same equation. We can finally conclude that,

writing $U = e^{i\pi_a \chi_a}$, π_5 is the single real Goldstone. The other four π's are Pseudo-Goldostones and we can compute their mass

$$-2\partial^2 \operatorname{Tr}(\Sigma_\mu^T \Sigma^\dagger M_\mu) \to -2\partial^2 \operatorname{Tr} U^4 = 4(\partial_\mu \pi_a)^2 \operatorname{Tr}(\mathbf{I}) \tag{1.48}$$

And, as expected, one gets $m_{PG} = m_{1-4} = 2\mu$. Starting from this specific case, we can make some general considerations on symmetry-breaking patterns that will give us precise indications in the construction of more general effective theories, in the sense that we will not make restrictive assumptions on the number of fermions in the microscopic theory. From the previous discussion we have seen that the T_i generators belong to the global symmetry group of the effective theory that is not spontaneously broken, i.e. $Sp(4)$, while the X_a generators are relative to the subgroup of $SU(4)$ that is obtained by eliminating the $Sp(4)$ component, i.e. the coset $SU(4)/Sp(4)$. Recall that the dimensions of the above groups are

$$\operatorname{Dim}(SU(2N_f)) = 4N_f^2 - 1 \tag{1.49}$$

$$\operatorname{Dim}(Sp(2N_f)) = N_f(2N_f + 1) \tag{1.50}$$

Therefore, when $\mu = 0$, the number of Goldstones, degrees of freedom in the effective theory, is $NG_{\mu=0} = N_f(2N_f - 1) - 1$, which in the case of $N_f = 2$ returns precisely 5-Goldstones. At non-zero chemical potential the pattern is

$$\mu \neq 0 \to SU(2N_f) \xrightarrow{ESB} SU(N_f) \times SU(N_f) \times U(1) \tag{1.51}$$

The equation for the condensate 1.29 shows that this is a rank 2 anti-symmetric tensor, so the pattern becomes

$$\mu \neq 0 \to SU(2N_f) \xrightarrow{ESB} SU(N_f) \times SU(N_f) \times U(1) \xrightarrow{SSB} Sp(N_f) \times Sp(N_f) \tag{1.52}$$

from which we can easily see that the goldstone number goes from $N_f(2N_f - 1) - 1$ to $2(N_f^2 - 1) - N_f(N_f + 1) = N_f(N_f - 1) - 1 = NG_{\mu \neq 0}$. Following the previous reasoning, then the Pseudo-Goldstone number (NPG) is obtained by subtracting from the Goldstone number at $\mu = 0$ that at $\mu \neq 0$, i.e.

$$NPG = NG_{\mu=0} - NG_{\mu \neq 0} = N_f^2 \tag{1.53}$$

which in fact in the case of $N_f = 2$ returns the four P-GBs calculated earlier. A general feature of these models is the fact that, working at generic N_f, the degrees of freedom in the actual theory must be $NG_{\mu=0} = N_f(2N_f - 1) - 1$, consequently it must hold that Σ is antisymmetric, unitary and unimodular.

$$\Sigma = -\Sigma^T \ ; \ \Sigma^\dagger\Sigma = I \ ; \ \det(\Sigma) = 1 \tag{1.54}$$

That it has exactly $NG_{\mu=0} = N_f(2N_f - 1) - 1$ independent components.

1.3 The Sp(N) group

In the latter subsection we explicitly saw the spontaneous symmetry breaking pattern $SU(4) \to Sp(4)$ which arises from the mass term when neglecting the chemical potential term. The same pattern has been then generalized by analogy into $SU(2N_f) \xrightarrow{SSB} Sp(2N_f)$. We now recall the definition of **Symplectic Group** and give a formal proof of the generalized pattern above. For the $N_f = 2$ case, we saw that the pattern is induced by a second rank anti-symmetric tensor and this remains true also for $N_f \neq 2$, the reason is easily understood [**symple**]. We should first note that once the pseudoreality of $SU_C(2)$ is employed, regardless of the number of fermions, every single fermion mass term can be cast in the form 1.29, then N_f only affects the dimensionality of the Ψ multiplet which becomes $\Psi = (q_1, q_2, ...q_{N_f}, \tilde{q_1}, \tilde{q_2}, ...\tilde{q}^T_{N_f})$, therefore the generalized mass term must be :

$$\mathcal{L}_{mass} = \frac{1}{2}\Psi^T\sigma_2 T_2 \begin{pmatrix} 0 & I_{N_f} \\ -I_{N_f} & 0 \end{pmatrix} \Psi \tag{1.55}$$

where σ_2 and T_2 just carry the $SU(2)$-spin and $SU_C(2)$ color indices respectively. Being Ω the Symplectic matrix :

$$\Omega = \begin{pmatrix} 0 & I_{N_f} \\ -I_{N_f} & 0 \end{pmatrix} \tag{1.56}$$

it is self-evident that the $SU(2N_f)$ subgroup preserving $\mathcal{L}_{mass}$ must satisfy the equation

$$U^T\Omega U = \Omega \tag{1.57}$$

The latter is actually the defining equation for the Symplectic Group $Sp(2N_f)$. We could make the following ansatz for an arbitrary element belonging to $Sp(2N_f)$

$$U \in Sp(2N_f) \; ; \; U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.58}$$

where every block is an $N_f \times N_f$ matrix, it turns out that, in order to satisfy the 1.57, the block components above have to solve the following set of matrix equations

$$C^T A = A^T C \; ; \; B^T D = D^T B \; ; \; A^T D - C^T B = I_{N_f} \tag{1.59}$$

The first two equation are independent constraints in the form $X = X^T$, giving $\frac{n^2-n}{2}$ constraints each, the last equation is also and independent constrain in the form $Y - Z = I$, so it gives n^2 constraints, therefor the dimension of the $Sp(2N_f)$ group in equal to $dim\, Sp(2_{Nf}) = 4N_f^2 - (N_f^2 - N_f) - N_f^2 = N_f(2N_f + 1)$. For the case of generic N_f we can construct the effective theory in exactly the same way as before, and the condensate will obviously continue to have the structure 1.31 consequently the equation finding the minimum in the effective theories will have exactly the same structure, with the only prescription.

$$M_\mu = \delta_{0\mu} \begin{pmatrix} I_{N_f} & 0 \\ 0 & -I_{N_f} \end{pmatrix} \tag{1.60}$$

Finding the minimum in the Effective Theory will still lead us to the equation 1.38 and clearly the subgroup leaving the vacuum invariant depends strongly on the choice of Σ_0. We can prove that if Σ_0 is a $2N_f \times 2N_f$ complex matrix satisfying $\Sigma_0 \Sigma_0^\dagger = \Sigma_0^\dagger \Sigma_0 = |c|^2 I_{2N_f}$, where c is a C-number, and the $SU(2N_f)$ generators in the defining representation are given by $\{T_a, X_b\}$, where T_a and X_b are respectively the unbroken and the broken generators,

$$T_a \Sigma_0 = -\Sigma_0 T_a^T \quad \textbf{unbroken} \tag{1.61}$$

$$X_a \Sigma_0 = \Sigma_0 X_a^T \quad \textbf{broken} \tag{1.62}$$

then the T_a's span the unbroken $Sp(2N_f)$ sub-algebra. The first consequence of our demands on Σ_0 is that

$$|c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0 \tag{1.63}$$

as shown in [**symple2**], for every even-dimensional complex anti-symmetric matrix M there exists an unitary matrix U such that $UMU^T = diag(\mathcal{J}_1, \mathcal{J}_2, ...\mathcal{J}_N)$, where $\mathcal{J}_i = \begin{pmatrix} 0 & z_i \\ -z_i & 0 \end{pmatrix}$, with $|z_i|^2$ eigenvalues of $MM^\dagger$. Since $\Sigma_0\Sigma_0^\dagger$ has one $|c|^2$ fully degenerate eigenvalue, thus we can find two unitary matrix such that

$$U_1\Sigma_0 U_1^T = cU_2\Omega U_2^T = diag(c\mathcal{J}, c\mathcal{J}, ..c\mathcal{J}) \tag{1.64}$$

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.65}$$

There will be also a unitary matrix $\tilde{U} = U_2^{-1}U_1$ such that

$$\tilde{U}\Sigma_0\tilde{U}^T = c\Omega \tag{1.66}$$

Finally, replacing the 1.66 into the 1.63 and defining $\tilde{T_a} = \tilde{U}T_a\tilde{U}^{-1}$, one finds

$$\tilde{T_a}^T = \Omega\tilde{T_a}\Omega \tag{1.67}$$

therefore, we canconclude that, since there is a one-to-one correspondence between the $\tilde{T_a}$ set and the T_a set, and since $\tilde{T_a}$ manifestly span the symplectic Lie algebra, then $SU(2N_f)$ is broken into $Sp(2N_f)$ under the aforementioned choices on the vacuum.

Chapter 2

Conformal Field Theory

In the previous chapter we preliminarily introduced the scheme by which effective theories with fixed charges can be constructed with the aim of applying these methods on a more general and complex model. As will become clear later, the model we are about to study possesses an interesting property, the conformal symmetry. This additional symmetry, together with the previous techniques, will prove to be enormously good in solving the theory as it will allow us to restore the applicability of perturbative methods where it would not otherwise be possible. We will then introduce the basics on the conformal group by mainly following the approach of [**difra**] and [**slava**], next we will demonstrate the formidable tools of Conformal Field Theory and again show how to employ them on a simple model, as done by [**epsilon**], before seeing how these results can be put together to compute observables in regimes where the usual techniques would fail.

2.1 Conformal Invariance

Conformal Field Theories (CFTs) are a special class, more precisely a subset, of Quantum Field Theory which exibit invariance under an enlarged symmetry group, the Conformal Group. CFTs can emerge in any spacetime dimensions, the case of $d = 2$ is a special one because of the infinite dimension of the Viraroso algebra. In this work we will not focus on the latter case. In any number of dimensions, the Conformal Group is defined by spacetime transformations that

leave angles unchanged, it turns out to be the Poincarè Group in addition with Dilatations and the so called Special Conformal Transformations. The CT Group forms the largest finite dimensional subsgroup of $Diff(R^d)$. CFTs have many applications in QFT that we can briefly report as follows:

1. Dilations invariance requires the absence of any dimensionful parameter, then CFTs are massless theories.

2. The Renormalization Group flow of non-conformal theories can end in a CTF in the IR or UV. This is for instance the fate of QCD.

3. Within the set of renormalizable QFTs, CFTs are a special subset, the one for which the β-function vanishes.

The latter point will be crucial in our discussion. It is that CFTs are a "vanishing mesure" subset of all possible QFTs, despite this the Conformal Invariance is a powerful tool to study some peculiar regimes of the RG flow of QFTs. Let us define the Conformal Group in an arbitrary dimension $D > 2$. Under a general coordinates transformation, the metric tensor transformas as

$$g_{\mu\nu} \to g^{'}_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} \tag{2.1}$$

Conformal Transormations are those which only change the sclae of the metric tensor

$$g^{'}_{\mu\nu}(x') = \omega(x) g_{\mu\nu}(x) \ ; \ \omega(x) > 0 \tag{2.2}$$

The Poincarè Group is the subset defined by $\omega = 1$, and Scale transformations $x \to \lambda x$ are also included in the (562.2. From the defining relation we want to explicitly characterize the conformal transformations in $d \geq 3$ with the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} = diag(1, -1... - 1)$. First we consider infinitesimal transformations on the coordinates

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \ ; \ \frac{\partial x^{\rho}}{\partial x'^{\sigma}} = \delta^{\rho}_{\sigma} - \frac{\partial \epsilon^{\rho}}{\partial x'^{\sigma}} \tag{2.3}$$

We substitute the transformation into 2.1 by imposing that 2.2 is satisfied in the above hypotesis, getting

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu} \tag{2.4}$$

By contracting both sides of the latter equation with $\eta_{\mu\nu}$, we get

$$\partial^\mu \epsilon_\mu = \frac{d}{2} f(x) \tag{2.5}$$

we can also differentiate the 2.4 with respect to ∂_ρ and sum over the three possible permutations of indices, this leads to

$$2\partial_\rho \partial_\mu \epsilon_\mu = \partial_\rho f(x)\eta_{\mu\nu} + \partial_\mu f(x)\eta_{\rho\nu} - \partial_\nu f(x)\eta_{\mu\rho} \tag{2.6}$$

contracting again with $\eta_{\mu\nu}$ we get

$$2\partial^2 \epsilon_\nu = (2-d)\partial_\nu f(x) \tag{2.7}$$

Now it is sufficient to apply ∂^ν on the latter equation and combine with the (59), this results in

$$(1-d)\partial^2 f(x) = 0 \tag{2.8}$$

We could also differentiate the (61) with respect to ∂_μ and use the (60), this leads to

$$(2-d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f \tag{2.9}$$

The equation 2.8 together with the equation 2.9 imply that the f function can only be linear in coordinates

$$f(x) = A + B_\mu x^\mu \tag{2.10}$$

where A and B_μ are constants. The previous equations imply that ϵ_μ is at most quadratic in coordinates

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\lambda}x^\nu x^\lambda \tag{2.11}$$

Since the last relation must be true in any space-time point, we can substitute order by order in the previous equations in order to constrain the parameters. It is straightforward to check that the a_μ parameters are not constrained, therefore the a_μ part defines translations, while using 2.11, 2.4 and 2.5 we get

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\rho_\rho \, \eta_{\mu\nu} \tag{2.12}$$

The 2.12 forces $b_{\mu\nu}$ to be sum of a pure trace part and an $\omega_{\mu\nu}$-part preserving $\eta_{\mu\nu}$

$$b_{\mu\nu} = \lambda\eta_{\mu\nu} + \omega_{\mu\nu} \tag{2.13}$$

The λ part of the latter corresponds to Scale Transformations $x \to \lambda x$, while the $\omega_{\mu\nu}$ part involves rigid rotation in Minkowski space, i.e. Lorentz transformations. Finally, sobsitution of the quadratic term into 2.11 yields

$$c_{\mu\nu\rho} = \eta_{\mu\rho}C_\nu + \eta_{\mu\nu}C_\rho - \eta_{\rho\nu}C_\mu \ ; \ C_\mu = \frac{1}{d}c^\sigma_{\sigma\mu} \tag{2.14}$$

The latter cohefficients define what we call Special Conformal Transformations, which are basically combinations of respectivelly an inversion $x^\mu \to \frac{x^\mu}{\mathbf{x^2}}$, a translation and another inversion. The entire collection of Conformal Transformation is then given by

- (Translations) $x'^\mu = x^\mu + a^\mu$
- (Dilations) $x'^\mu = \lambda x^\mu$
- (Lorentz Transformations) $x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$
- (Special Conformal Transformations) $x'^\mu = \frac{x^\mu - C^\mu x^2}{1-2(C^\rho x_\rho)+C^2x^2}$

If we neglect transformations for fields, then the Conformal Generators read as

- (Translations) $P_\mu = -i\partial_\mu$
- (Dilations) $D = -ix^\mu\partial_\mu$
- (Lorentz Transformations) $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
- (Special Conformal Transformations) $K_\mu = -i(2x_\mu x\cdot\partial - x^2\partial_\mu)$

Therefore, the commutation rules which define the Conformal Algebra are

$$\begin{gathered} i[P_\mu, K_\nu] = -2g_\mu\nu D + 2M_{\mu\nu} \ ; \ i[M_\beta, K_\gamma] = g_{\alpha\gamma}K_\beta - g_{\beta\gamma}K_\alpha \\ i[D,D] = i[D,M_{\mu\nu}] = i[K_\mu, K_\nu] = 0 \\ i[D,P_\mu] = P_\mu \ ; \ i[D,K_\mu] = K_\mu \end{gathered} \tag{2.15}$$

Once the conformal group has been constructed, it is necessary to study the transformation properties of the classical fields under the conformal group and provide examples of representations. Given an infinitesimal conformal transformation parametrized by ω_a, we are searching for generators T_a such that a general multicomponents field transforms as

$$\Phi'(x') = (1 + i\omega_a T_a)\Phi(x) \tag{2.16}$$

In order to find this generators, we can first represent the sonformal subgroup which leaves the origin $x = 0$ invariant and then translate the generator elsewhere. Let $S_{\mu\nu}$ be a matrix representing an internal Lorentz transformation, namely the Spin, and $L_{\mu\nu}$ our Lorentz generator, clearly

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \tag{2.17}$$

By use of the Poincaè algebra and the Hausdorff formula

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2}[[B, A], A] + ... \tag{2.18}$$

we get the following value for the generator away from the origin

$$e^{ix_\mu P^\mu} L_{\mu\nu} e^{-ix_\mu P^\mu} = S_{\mu\nu} - x_\mu P_\nu + x_\mu P_\nu \tag{2.19}$$

Finally, the full action of $L_{\mu\nu}$ must be

$$L_{\mu\nu}\Phi(x) = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x) \tag{2.20}$$

We can proceed the same way for the full conformal group taking care that the subgroup preserving the origin is just made up by rotations, dilations and special conformal transformation. Being $S_{\mu\nu}, \Delta, k_\mu$ respectively the value of the generators $L_{\mu\nu}, D$ and K_μ at the origin we get

$$K^\mu\Phi(x) = \left\{k_\mu + 2x_\mu\Delta - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\delta_\nu + ix^2\partial_\mu\right\}\Phi(x) \tag{2.21}$$

$$D\Phi(x) = (-ix^\mu\partial_\mu + \Delta)\Phi(x) \tag{2.22}$$

with the subalgebra

$$[\Delta, S_{\mu\nu}] = 0 \ ; \ [\Delta, k_\mu] = -ik_\mu \tag{2.23}$$

$$[k_\rho, S_{\mu\nu}] = i(\eta_{\rho\mu}k_\nu - \eta_{\rho\nu}k_\mu) \tag{2.24}$$

The Shur's lemma states that any matrix commutating with all the set of $S_{\mu\nu}$ must be proportional to the identity, this result fixes Δ to be a multiple of the identity and the k's to be zero. So if we require the $\Phi(x)$ field to belong to an irreducible representation of the Lorentz group, then the conformal group representation that we get is nothing but a Lorentz representation augmented by a dilation. We define **quasi − primary** a spinless field which transforms under dilations according to

$$\Phi(x) \to \Phi'(x) = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \Phi(x) \ ; \ \left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-\frac{d}{2}} \tag{2.25}$$

where $\Lambda(x)$ is the usual scale factor. Before moving on the quantum-mechanical side of Conformal Field Theory, it is proper to see how the symmetry properties of a certain theory under the conformal group are encoded in its stress-energy tensor. To convince ourselves we need to calculate the change in action under conformal transformations, as usual we perform an infinitesimal dilation transformation

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} \ ; \ \Phi'(x) = (1 + i\epsilon_{\alpha} T^{\alpha})\Phi(x) \tag{2.26}$$

the infinitesimal change in the action reads as

$$\delta S = -\int d^d x \ \partial_{\mu} T_B^{\mu\nu} \epsilon_{\nu} \tag{2.27}$$

here the stress-energy tensor is assumed to be symmetric, the symmetry can be easily achieved by using the Belinfante form for $T_B^{\mu\nu}$ which is obtained by adding the divergence of an antisymmetric tensor to the canonical stress-energy tensor $T_c^{\mu\nu}$

$$T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_{\rho} B^{\rho\mu\nu} \ ; \ B^{\rho\mu\nu} = -B^{\mu\rho\nu} \tag{2.28}$$

It can be shown [**difra**] thaht $\partial_{\rho} B^{\rho\mu\nu}$ can be cast in the form

$$B^{\rho\mu\nu} = \frac{i}{4}\left\{\frac{\delta \mathcal{L}}{\delta \partial_{\mu}\Phi} S^{\rho\nu}\Phi + \mathbf{Perm.}\right\} \tag{2.29}$$

It follows that for dilations the conserved current is given by the trace of the symmetrized stress-energy tensor.

$$\partial_{\mu} j_D^{\mu} = T_B{}_{\mu}^{\mu} \tag{2.30}$$

Therefore, a conformally invariant theory must have a traceless energy-momentum tensor. The latter result is crucial for many developments and represents a connection between conformal invariance and Quantum Field Theory. By using Ward Identities and the Callan-Symanzik equation one can prove that given a pure Yang-Mills Theory, the trace of the energy-momentum tensor is given by

$$T^{\mu}{}_{\mu} \sim \beta \,\mathrm{Tr}\,\{F_{\mu\nu}F^{\mu\nu}\} \tag{2.31}$$

where β is the β-funcion and $F_{\mu\nu}$ the force tensor.Thus if the β-function vanishes at some value of the coupling constant, then the theory can acquire confrormal invariance. This establishes a strong relation between the Renormalization Group flow and Conformal Field Theory, we could say that Conformal Field Theory are a null measure set of Quantum Field Theories, more specifically, there are QFT's whose renormalization group flow goes into CFT's, this is for instance the case with QCD. The study of the Renormalization Group flows of any Quantum Field Theory is the most powerful theoretical tool to understand the main features of the dynamics of the theory and how it evolves from the UV to the IR. Following the RG flows of a theory it's a way to highlight different "phases" of the theory itself. The richest case is the one of Non-Abelian Gauge Theories. In the simplest case of a pure Yang-Mills theory the β-funtion is negative, this means that the largest is the scale of energy involved, the smallest is the coupling coupling constang g. This was one of the greatest discovery (Gross, Wilczek, Politzer and others) arising from the study of strong interactions, the Asymptotic Freedom. Consider N_f massless quarks described by Dirac fermions and N_c as the number of colors, at $N_f = 3$ and $N_c = 3$ we obtain QCD, the current theory of strong interactions in nature. The β function for such a theory is at one loop

$$\beta(g) = -\frac{g^3}{16\pi^2}(\frac{11}{3}N_c - \frac{2}{3}N_f) \tag{2.32}$$

It is glaring that if $N_f < \frac{11}{2}N_c$, the theory is asymptotically free. Otherwise, we lost asymptotic freedom and the theory becomes IR-free, similar to the QED with massless electrons. Therefore, $\frac{11}{2}N_c$ is a critical value discerning between completely different behaviors. We can deepen this question and go to two-loops,

then the β will be

$$\beta(g) = -\frac{g^3}{16\pi^2}(\frac{11}{3}N_c - \frac{2}{3}N_f) - \frac{g^5}{(16\pi^2)^2}(\frac{17}{3}{N_c}^2 - \frac{N_f}{6N_c}(13{N_c}^2 - 3)) + \mathcal{O}(g^7) \quad (2.33)$$

At small g, the β_0 the one-loop contribution, namely $-\beta_0\frac{g^3}{16\pi^2}$,is dominant and β behaves as above. If $N_f \simeq \frac{11}{3}N_c$, then we can neglect β_0, β develops a zero due to the $-\beta_1\frac{g^5}{(16\pi^2)^2}$

$$\frac{g^*}{2\pi} = \frac{\beta_0}{\beta_1} \quad (2.34)$$

The effects of higer order corrections are suppressed by $\frac{1}{N_c}$ powers. The zero of β is the Banks-Zaks IR fixed point in which the theory is in the so called Conformal Phase. For $N_f \ll N_c$ we are in the Chiral Symmetry Breaking phase. Since for $N_f = \frac{11}{2}N_c$ we certainly are in the Conformal Phase, there should exist a critical value N_f^* so that in the $N_f^* \leq N_f \leq \frac{11}{2}N_c$ the theory is in the Conformal Phase. The latter interval is known as the Conformal Window. Sometimes it happes that the nth order, β-function with fixed n,does not have a fixed point in four dimensions but it has in $d = 4 - \epsilon$, this is the so called Wilson-Fisher fixed point. This results are of enormous interest from both a phenomenological and a purely technical point of view,in fact, on the one hand, a CFT must be either massless or have a continuous mass spectrum, while on the other hand acquiring a new symmetry, such as symmetry by dilations, can be useful in gaining more control over perturbative expansions of observables. The last point will be precisely the one on which we will focus part of the subsequent discussion; in fact, we will show how scaling dimensions and anomalous field sizes at the fixed point become observables and how they can be computed through a systematic approach that can be employed in conjunction with charge fixing with the aim of boosting the applicability of perturbation theory.

2.2 Radial Quantization and Weyl Map

For a scalar field of scaling dimension Δ, under conformal transformations one has

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta}{d}} \phi(x) \quad (2.35)$$

A field with a given Δ, which transforms as 2.35, it's called "Primary", otherwise it is called "descendant". For example, $\partial_\mu \phi$ is descendant. The main consequence of the covariance property above is that $\langle \phi_1(x_1)...\phi_n(x_n)\rangle$ transforms like

$$\langle \phi_1(x_1)...\phi_n(x_n)\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\frac{\Delta_1}{d}} ... \left|\frac{\partial x_n'}{\partial x_n}\right|^{\frac{\Delta_n}{d}} \langle \phi_1'(x_1')...\phi_n'(x_n')\rangle \tag{2.36}$$

This result imposes severe restrictions on 2- and 3-pt functions of primaries. In order to identify invariants on which n-pt functions we need to construct some invariant of the CG in d-dimensions. Translation invariance tells us that n-pt functions cannot depend on x_i, but only on the differences $x_i - x_j$, we have $d(n-1)$ different choices. For scalar objects, rotational invariance (at large enough d) imposes thee dependence on $r_{ij} = |x_i - x_j|$, so we have $n(n-1)/2$ invariants. Scale invariance forces the dependence on $\frac{r_{ij}}{r_{hk}}$, and finally the SCT's invariants are the cross ratios $\frac{r_{ij}r_{lm}}{r_{km}r_{lj}}$. We can conclude that our n-pt functions will depend on cross ratios only, due to the fact that they surely includes all the previous invariants. This fixes the 2-pt function form as follows

$$\langle \phi_1(x_1)...\phi_n(x_n)\rangle = \begin{cases} c_{12}\ r_{12}^{-\Delta_1-\Delta_2} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2. \end{cases} \tag{2.37}$$

So far we have not justified the existence of primary operators, in order to do that we should first construct an Hilbert space and introduce the so called "Radial Quantization". In CFT it is convenient to foliate the space-time in S^{d-1} spheres of r radius, the Dilations generator D will evolve states from one leaf to another. Basically, D takes the place of the usual P^0 of Quantum Mechanics. Hence, the scaling is the energy in the radial quantization.

$$P^0 \to D \ ; \ U = e^{iP^0\Delta t} \to U = e^{iD\Delta t} \tag{2.38}$$

where in the right-hand side of the last equation $t = \log r$. States living on a certain sphere will be classified by their scaling dimension

$$D\,|\Delta\rangle = i\Delta\,|\Delta\rangle \tag{2.39}$$

Since angular momentum $M_{\mu\nu}$ commutes with D, then $|\Delta, l\rangle$ is also a basis for $M_{\mu\nu}$. Let us now insert operators on the sphere. The simplest case is $|0\rangle$, the

vacuum state, that corresponds to insert nothing, the dilation eigenvalue is zero for $|0\rangle$. If we insert operators O_Δ at the origin $x = 0$, the generated state $|\Delta\rangle = O_\Delta(0)\,|0\rangle$ will have

$$D\,|\Delta\rangle = DO_\Delta\,|0\rangle = [D, O_\Delta]\,|0\rangle + O_\Delta D\,|0\rangle = i\Delta O_\Delta\,|0\rangle = i\Delta\,|\Delta\rangle \tag{2.40}$$

Finally, if we insert on Operator $O_\Delta(x)$ in $x \neq 0$, $|\psi\rangle = O_\Delta(x)\,|0\rangle >$ is no longer D's eigenstate. Anyway

$$|\psi\rangle = O_\Delta(x)\,|0\rangle >= e^{iPx}O_\Delta(0)e^{-iPx}\,|0\rangle = e^{iPx}\,|\Delta\rangle \tag{2.41}$$

since $[D, P_\mu] = iP_\mu$, P_μ raises Δ by unit (K_μ does the opposite). We can go backword: given a state of defined Δ we can construct primary operators of scaling dimension Δ by the "state-operator correspondence"

$$\langle\phi_1(x_1)\phi_2(x_2)...O_\Delta(0)\rangle = \langle 0|\,\phi_1(x_1)\phi_2(x_2)...\,|\Delta\rangle \tag{2.42}$$

It can be fruitful to introduce a new one set of coordinates, so far we have worked in radial coordinates

$$r > 0 \qquad ;\vec{n} \in S^{D-1} \subset R^D \tag{2.43}$$

where $\vec{n}$ is the normal vector on the S^{D-1} sphere surface. Let τ be $\tau = \log r$, under dilations one gets

$$r \to e^\lambda r \;\;;\;\; \tau \to \tau + \lambda \tag{2.44}$$

In terms of these coordinates, the n-pt funtions become

$$\langle\phi_1(r_1, \vec{n_1})...\phi_n(r_n, \vec{n_n})\rangle = \frac{1}{r_1^{\Delta_1}}...\frac{1}{r_2^{\Delta_n}}...f(\tau_i - \tau_j, \{\vec{n_i}\}) \tag{2.45}$$

We can define fields on the cylinder by

$$\phi_{cyl.}(\tau, \vec{n}) = r^\Delta\phi(r, \vec{n}) \tag{2.46}$$

Whit this in mind, it is easy to see

$$\langle\phi_{cyl,1}(\tau_1, \vec{n_1})...\phi_{cyl,n}(\tau_n, \vec{n_n})\rangle = f(\tau_i - \tau_j, \{\vec{n_i}\}) \tag{2.47}$$

The latter suggests that the dynamics on the cylinder is invariant under τ translations. The new and old metrics are connected by Weyl transformations

$$ds^2_{flat} = dr^2 + r^2 d\vec{n}^2 \;\;;\;\; ds^2_{cyl} = d\tau^2 + d\vec{n}^2 = \frac{1}{r^2}ds^2_{flat} \tag{2.48}$$

We deduce that, if the theory is conformal, then form any metric connected by Weyl transformations the correlation functions are related by simple rescaling factor as in 2.46. So ϕ_{cyl} is exaxtly the same thing as ϕ_{flat}, but its correlators are measured in a different geometry. We have mentioned products of operators at different points and have corresponded them with the states of the theory. We now want to show that the product of two or more operators at different points can be expressed as a sum of operators at the same point. This is the core idea of the OPE and it applies to sufficiently local operators. For simplicity, consider a scalar field and some state

$$|\psi\rangle = \phi_1(x_1)\phi_2(0)\,|0\rangle \tag{2.49}$$

our state can be expanded in a basis of D's eigenstates

$$|\psi\rangle = \sum_n c_n\,|\Delta_n\rangle \tag{2.50}$$

$|\Delta_n\rangle$ can be created by primaries and descendants which come from acting on primaries with momenta. We can re-organize the sum in multiplets

$$|\psi\rangle = \phi_1(x_1)\phi_2(0)\,|0\rangle = \sum_{\phi,I} C_{\Delta,I}(\partial, x)\phi_I(0)\,|0\rangle \tag{2.51}$$

where ϕ_I is fixed to be primary, ∂ gives the descendants fields and I labels the representation (multiplets). So far we took it for granted that the ϕ fields that transform as in 2.35 are basis of irreducible representations of the Conformal Group. Note that each term of the previous series includes a certain primary ϕ_I and all of its descendants $P^{n_1}...P^{n_k}\,|\phi_I\rangle$. Promoting the latter equation to an operator identity, one gets

$$\phi_1(x_1)\phi_2(0) = \sum_{\phi,I} C_{\Delta,I}(\partial, x)\phi_I(0) \tag{2.52}$$

This equality is true only inside correlation functions and works in a certain neighbor-ought of x. From dimensional analysis we can guess the first term of the sum, let $I = 0$ be the trivial representation

$$C_\Delta(\partial, x)\phi(0) = \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0) + ...) \tag{2.53}$$

In fact the Δ dimension of $\phi(0)$ cancels out the $-Delta$ in the denominator's power giving the right form of the 2-pt function. we can corroborate this result by acting directly with the D operator

$$D\phi_1(x)\phi_2(0)\left|0\right\rangle = i(\Delta_1 + x\cdot\partial)\phi_1(x)\phi_2(0)\left|0\right\rangle + i\Delta_2\phi_1(x)\phi_2(0)\left|0\right\rangle \tag{2.54}$$

if we suppose 3.64 to be true, then we get

$$D\phi_1(x)\phi_2(0)\left|0\right\rangle = i(\Delta_1+\Delta_2-(\Delta_1+\Delta_2-\Delta)\frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}}(\phi(0)+...) \tag{2.55}$$

so the guess is good. The second term, the one involving the first descendant, is fixed by C.I

$$C_\Delta(\partial,x)\phi(0) = \frac{C}{|x|^{\Delta_1+\Delta_2-\Delta}}(1+\alpha x^\mu\partial_\mu+...)\phi(0) \tag{2.56}$$

the α parameters is fixed by Special Conformal Transformations. The claim is that Conformal Invariance entirely fixes the O.P.E. Consider

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \sum_{\Delta'} C_{12\Delta'}C_{\Delta'}(x,\partial_y)\langle\phi_{\Delta'}(y)\phi_\Delta(z)\rangle_{y=0} \tag{2.57}$$

We can first perform the OPE oh the first two operators and then get a 2-pt function, which imposes $\Delta' = \Delta$, so that

$$\langle\phi_1(x)\phi_2(y)\phi_\Delta(z)\rangle = C_{12\Delta}C_\Delta(x,\partial_y)\langle\phi_\Delta(y)\phi_\Delta(z)\rangle_{y=0} \tag{2.58}$$

Since we already know

$$\langle\phi_\Delta(y)\phi_\Delta(z) = \frac{1}{|y-z|^{2\Delta}} \tag{2.59}$$

then the 3-pt function becomes

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \frac{C_{12\Delta}}{|x|^{\Delta_1+\Delta_2-\Delta}|z|^{\Delta_+\Delta_2-\Delta_1}|x-z|^{\Delta_1+\Delta-\Delta_2}} \tag{2.60}$$

We can now expand the 3-pt function for small x and choose $C_\Delta(x,\partial)$ to match term by term.

2.3 Semiclassics for the U(1) model

CFT can be successfully applied to explore regimes where the ordinary Perturbation Theory breaks down in order to solve Quantum Field Theory. We can show this remarkable result in the $U(1)$ theory example brilliantly explored in [**epsilon**]. It is a fact that, even in a weakly coupled theory, the Perturbation Theory eventually fails when the number n of legs grows. This represents a noticeable difficulty in solving multilegged observables. In the next example we will consider a $U(1)$ theory with quartic interactions in $d = 4 - \epsilon$ dimensions at the Wilson-Fisher fixed point to ensure the Conformal Invariance and show how it is possible to compute the Scaling Dimension Δ_{ϕ^n} for the large charge operator ϕ^n. The computation will be done in the double limit

$$\lambda_0 \to 0 \ , \qquad \lambda_0 n = const \tag{2.61}$$

We the above hypothesis in mind, let us consider the theory

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \frac{\lambda_0}{4}(\bar{\phi}\phi)^2 \tag{2.62}$$

Up to two-loops corrections, the beta function gets the form

$$\beta(\lambda) = -\epsilon\lambda + 5\frac{\lambda^2}{(4\pi)^2} - 15\frac{\lambda^3}{(4\pi)^4} + O(\lambda^4) \tag{2.63}$$

which means that the theory posses a fixed point in

$$\frac{\lambda_*}{(4\phi)^2} = \frac{\epsilon}{5} + \frac{3}{25}\epsilon^2 + O(\epsilon^3) \tag{2.64}$$

the fixed point depends on the dimensionality, so the theory is weakly coupled for $\epsilon \ll 1$. Let $[\phi^n]$ be the rinormalized field and Z_ϕ the rinormalization factor of the field

$$\phi^n = Z_{\phi^n}[\phi^n] \tag{2.65}$$

the anomalous dimension γ_{ϕ^n} is then given by

$$\gamma_{\phi^n} = \frac{\partial \log(Z_{\phi^n})}{\partial\lambda}\beta(\lambda) \tag{2.66}$$

Solving the Callan-Symanzik equation, one can show that the operator's physical dimension at the fixed point is

$$\Delta_{\phi^n} = n(\frac{d}{2} - 1) + \gamma_{\phi^n}(\lambda*) \tag{2.67}$$

The anomalous dimension is scheme dependent but it becomes oservable at the Wilson-Fisher fixed point. A first attempt to calculate the scaling dimension could therefore involve the perturbations theory to obtain γ.The calculation can be performed in two ways, both perturbative and divergente as n grows even when λ is small, the first one is a direct diagrammatical computation which up to 3-loops (whose details are collected in the Appendix of [**epsilon**]) working in the Minimal Subtraction Scheme gives us

$$Z_{\phi^n} == 1 - \frac{\lambda n(n-1)}{(16\pi^2)2\epsilon} - \frac{\lambda}{16\pi^2}\left[\frac{n^4 - 2n^3 - 9n^2}{9\epsilon^2} + \frac{2n^3 - 2n^2 - n}{2\epsilon}\right] \tag{2.68}$$

which implies

$$\gamma_{\phi^n} = n\left[\frac{\lambda}{16\pi^2}\frac{n-1}{2} - \left(\frac{\lambda}{16\pi^2}\right)^2 \frac{2n^2 - 2n - 1}{4}\right] \tag{2.69}$$

and one can easily compute the scaling dimension at the fixed point

$$\Delta_{\phi^n} = n\left[\left(\frac{d}{2} - 1\right) + \frac{\epsilon}{10}(n-1) - \frac{\epsilon^2}{100}(2n^2 - 8n + 5)\right] \tag{2.70}$$

The second way is to work directly on the correlation function and computing it via a semi-classical expansion on the path integral around a non-trivial trajectory. As mentioned before, the Perturbations Theory breaks down at large n. We therefore want to use a calculation scheme that naturally allows us to perform the limit with large n, not only when λn is small, but even when λn grows. Let's focus directly on the correlation function

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \frac{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\bar{\phi}^n(x_f)\phi^n(x_i)\, exp[-\int \mathcal{L}]}{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\, exp[-\int \mathcal{L}]} \tag{2.71}$$

Let's re write the above integral as follows

$$\phi \to \frac{\phi}{\sqrt{\lambda_0}} \;\; ; \;\; -\int \mathcal{L} \to -\frac{1}{\lambda_0}\int \mathcal{L} \tag{2.72}$$

and exponentiate the operators insertions

$$\langle\bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \frac{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\, e^{[-\frac{1}{\lambda_0}\int \partial\bar{\phi}\partial\phi+\frac{1}{4}(\bar{\phi}\phi)^2-\lambda_0 n(\ln\left(x_f\right)+\ln(x_i))]}}{\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\, e^{[-\frac{1}{\lambda_0}\int \partial\bar{\phi}\partial\phi+\frac{1}{4}(\bar{\phi}\phi)^2]}} \tag{2.73}$$

The path integral in the denominator can be computed by the saddle point expansion around the minima $\phi = \bar{\phi} = 0$, while the one in the numerator is stationary around non-trivial values of the fields $\phi, \bar{\phi} \neq 0$ due to the insertions, which spontaneously breaks the U(1) symmetry so that the minimal value is no longer null. The most general form for the whole path-integral is

$$Z^2_{\phi^n}\langle[\bar{\phi}^n(x_f)][\phi^n(x_i)]\rangle = {\lambda_0}^{-\frac{1}{2}} e^{\frac{1}{\lambda_0}\Gamma_{-1}(\lambda_0 n, x_{fi})+\Gamma_0(\lambda_0 n, x_{fi})+\lambda_0\Gamma_1(\lambda_0 n, x_{fi})+\ldots} \tag{2.74}$$

The Γ cohefiicients will have a finite and a divergent part, namely Γ^{div}_K and Γ^{ren}_K, then we can write $\langle[\bar{\phi}^n(x_f)][\phi^n(x_i)]\rangle = n! e^{\sum \lambda_k \Gamma^{ren}_k}$. At small fixed $\lambda_0 n$ the path integral can still be computed in the usual saddle-point scheme expanding around $\phi = \bar{\phi} = 0$, in this case the insertion does not change anything and working at the order λ one finds

$$\langle\bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \frac{n!\left[1-\frac{\lambda n(n-1)}{2(4\pi)^2}\left(\frac{2}{\epsilon}+\ln x^2_{fi}1+\gamma+\ln\pi\right)+\mathcal{O}(\lambda^2)\right]}{\Omega_{d-1}(d-2)(x^2_{fi})^{(\frac{d}{2}-1)}} \tag{2.75}$$

where $\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ and γ is the Eulero-Mascheroni constant. The latter equation points out how quantum corrections become significant as long as $\lambda_0 n$ grows, thus a new calculation method is required. Turning back to the path-integral expression, we can treat the operator insertions as sources which obviously modify the equation of motion into

$$\begin{aligned}\partial^2\phi(x) - \frac{1}{2}\bar{\phi}^2(x)\phi(x) &= -\frac{\lambda_0 n}{\phi(\bar{x}_f)}\delta^{(d)}(x-x_f)\\ \partial^2\phi(\bar{x}) - \frac{1}{2}\phi^2(x)\phi(\bar{x}) &= -\frac{\lambda_0 n}{\phi(x_i)}\delta^{(d)}(x-x_i)\end{aligned} \tag{2.76}$$

Solving the previous equations for sufficiently small $\lambda_0 n$ should hopefully lead to the same two-point function computed via diagrammatic calculations. This can be done using perturbation theory, i.e. expanding the solution as

$$\phi = (\lambda n)^{1/2}\left[\phi^{(0)}+\phi^{(1)}+\ldots\right] \tag{2.77}$$

where $\phi^{(k)} = \mathcal{O}(\lambda^k n^k)$ and assuming that $\phi^{(0)}$ solves the equation

$$\begin{aligned} \partial^2 \phi^{(0)}(x) &= -\frac{1}{\bar{\phi}^{(0)}(x_f)} \delta^{(d)}(x - x_f) \\ \partial^2 \bar{\phi}^{(0)}(x) &= -\frac{1}{\phi^{(0)}(x_f)} \delta^{(d)}(x - x_f) \end{aligned} \tag{2.78}$$

Whose solutions are trivially given by

$$\phi^{(0)}(x) = \frac{c_0}{\Omega_{d-1}(d-2)} \frac{1}{|x - x_f|^{d-2}} \tag{2.79}$$

Therefore we could compute the first correction $\phi^{(1)}$ and compute the effective action, which corresponds to the leading term of the 2.74. The general solution without restrictions about λn is an aside quest in which studying the theory at the fixed point in order to gain conformal invariance will be crucial to get enhanced symmetries to constrain the form of the observables for our large charge operators. The advantage to work on fixed point is that we can exploit the power of conformal invariance. First of all, we shall map our theory from the plane to the cylinder in order to map the eigenvalues of the Dilation D into the eigenvalues of H_{cyl}, the energy spectrum on the cylinder. The action on the cylinder reads

$$S_{cyl} = \int d^d x \sqrt{g} \big[g^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \phi + m^2 \bar{\phi} \phi + \frac{\lambda_0}{4} (\bar{\phi}\phi)^2 \big] \tag{2.80}$$

The mass term arises from the request of conformal invariance which can be gained just by coupling the field to the Ricci's scalar into a quadratic term as $R(g)\bar{\phi}\phi$, therefore the m mass is just a rewriting of $m^2 = (\frac{d-2}{2R})^2$. As said before, the theory on the cylinder is fully equivalent to the one on flat space by the correspondence

$$\langle \mathcal{O}^\dagger(x_f) \mathcal{O}(x_i) \rangle_{cyl} = |x_f|^{\Delta_\mathcal{O}} |x_i|^{\Delta_\mathcal{O}} \langle \mathcal{O}^\dagger(x_f) \mathcal{O}(x_i) \rangle_{flat} \tag{2.81}$$

Computing the limit for $x_i \to 0$ corresponding to $\tau_i \to -\infty$, one gets

$$\langle \mathcal{O}^\dagger(x_f) \mathcal{O}(x_i) \rangle_{cyl} \overset{\tau_i \to -\infty}{=} e^{-E_\mathcal{O}(t_f - t_i)} \tag{2.82}$$

Where $E_\mathcal{O} = \frac{\Delta_\mathcal{O}}{R}$. We can read the previous equations as follows making use of the state/operator correspondence: the action of $\mathcal{O}$ at $-\infty$ creates a state of energy $E_\mathcal{O}$ that carries all the quantum numbers of the operator. Consequently,

we can get the same expression just by replacing $\mathcal{O}$ with ϕ^n, let be $t_f, i = \pm T/2$, we get

$$\langle \bar{\phi}(x_f)\phi(x_i)\rangle = \mathcal{N} e^{-E_{\phi^n} T} \tag{2.83}$$

where the $\mathcal{N}$ cohefficient is T-independent and divergent. As above, the structure og the 2-pt function will have the form of the 2.74. The expontent of 2.74 as the form $\lambda_0^k \Gamma_k$, where each Γ_k has a finite and a divergent part. By comparison with 2.83 we deduce that the divergent part of the Γ will be T-independent and will fix the $\mathcal{N}$ coheficcient, while the finite part will necessary depend on T. Similarly to the expression $\langle [\bar{\phi}^n(x_f)][\phi^n(x_i)]\rangle = n! e^{\sum \lambda_k \Gamma_k^{ren}}$, we can write :

$$RE_{\phi^n} = \frac{1}{\lambda_0} e_{-1}(\lambda_0 n, d) + e_0(\lambda_0 n, d) + \lambda_0 e_1(\lambda_0 n, d) + ... \tag{2.84}$$

$$= \frac{1}{\lambda} \bar{e}_{-1}(\lambda n, RM, d) + \bar{e}_0(\lambda n, RM, d) + \lambda \bar{e}_1(\lambda n, RM, d) + ... \tag{2.85}$$

where the $\bar{e}_i$ coefficients are analogous with the e_i but renormalized and therefore they depend on a sliding scale, namely M. Thus, if the coupling is evaluated at the fixed point $\lambda = lambda_*$, the dependence on such a scale must disappear by scale invariance giving a result of the form : By fixing $\lambda = \lambda_*$ we expect to be able to compute at least the leading terms of the following semiclassical expansion for the scaling dimension

$$\Delta_{\phi^n} = \frac{1}{\lambda_*} \Delta_1(\lambda_* n) + \Delta_0(\lambda_* n) + ... \tag{2.86}$$

The computation of Δ_0 can be set this way; we can make use of the state/operator correspondence to introduce a state $|\psi_n\rangle$ of charge n such that 2.83 reads as follow

$$\langle \psi_n | \, e^{-HT} \, |\psi_n\rangle =_{T\to\infty} = \mathcal{N} e^{-E_{\phi^n} T} \tag{2.87}$$

we could also write ϕ as $\phi = \frac{\rho(\vec{n})}{\sqrt{2}} e^{-i\chi(\vec{n})}$, being $\vec{n}$ a general coordinate on the $(d-1)$-Sphere, it is good to recall that, as long as we are in the Schrodinger picture, fields will not depend on the coordinate of the quantization axis. In order to find the $|\psi_n\rangle$ state, we cant explicitly build the charge operator $\hat{Q}$ and figure out the general form for its eigenstates.The computation of the charge operator is

straightforward once we know the Noether current related to the $U(1)$ symmetry, namely

$$j_\mu = \bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi} \tag{2.88}$$

we know that the charge is just

$$Q = \int d\Omega_{d-1} R^{d-1} j_0 = \int d\Omega_{d-1} R^{d-1}\rho^2\partial_\tau\chi \tag{2.89}$$

If our Hilbert Space is made up by functionals $|\psi\rangle = \Psi[\rho,\chi]$, then iit is easy to convince ourselves that the operator corrisponding to the classical charge (2.89) is just

$$\hat{Q} = \int d\Omega_{d-1} R^{d-1} - i\hat{\rho}^2\frac{\partial}{\partial\chi} \tag{2.90}$$

thus, a quite general state of charge n is given by

$$|\psi_n\rangle = \int \mathcal{D}\chi \exp\left\{ i\frac{n}{\Omega_{d-1}R^{d-1}}\int d\Omega_{d-1}\chi \right\} |\rho,\chi\rangle \tag{2.91}$$

if we perform an homogeneity ansatz on $\rho = f$, where f is a constant, then our amplitude is given by

$$\langle\psi_n| e^{-HT} |\psi_n\rangle = \mathcal{Z}^{-1}\int \mathcal{D}\chi_i\mathcal{D}\chi_f \; e^{i\frac{n}{\Omega_{d-1}R^{d-1}}\int d\Omega_{d-1}(\chi_f-\chi_i)} \int_{\rho=f,\chi=\chi_i}^{\rho=f,\chi=\chi_f} \mathcal{D}\chi\mathcal{D}\rho e^{-\mathcal{S}} \tag{2.92}$$

where

$$\mathcal{Z} = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi e^{-\mathcal{S}} \tag{2.93}$$

using that

$$\int d\Omega_{d-1}(\chi_f - \chi_i) = \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \int d\Omega_{d-1}\dot{\chi} \tag{2.94}$$

we can writhe the amplitude as a simple path-integral in the form

$$\langle\psi_n| e^{-HT} |\psi_n\rangle = \mathcal{Z}^{-1}\int_{\rho=f}^{\rho=f} \mathcal{D}\rho\mathcal{D}\chi e^{-\mathcal{S}_{eff}} \tag{2.95}$$

where the effective action $\mathcal{S}_{\rceil\{\{}$ is given by

$$\mathcal{S}_{eff} = \int_{\frac{T}{2}}^{\frac{T}{2}} d\tau \int d\Omega_{d-1}\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2 + \frac{m^2}{2}\rho^2 + \frac{\lambda_0}{16}\rho^4 + i\frac{n}{R^{d-1}\Omega_{d-1}}\dot{\chi} \tag{2.96}$$

The equation of motion with respect to our ansatz give the following solutions

$$\rho = f \;\; ; \;\; \chi = -i\mu\tau + const \tag{2.97}$$

where μ is called "chemical potential", which plays the same role of the chemical potential introduced in the previous section, in fact the presence of this term is related to the ansatz of homogeneity on the vacuum and reflects the fact that we are dealing with a fixed charge density $j_0 = \mu f^2$. The effective action evaluated upon this solutions gives us the (-1)-th order of the expansion 2.74

$$\frac{e_{-1}(\lambda_0 n, d)}{R}\frac{1}{\lambda_0} = \frac{S_{eff}}{T} \tag{2.98}$$

Finally, if we consider the classical value for the chemical potential arising from the equation of motion and put $\lambda = \lambda_*$ with $d = 4$ we get the leading-order-scaling dimension

$$\frac{\Delta_{-1}}{\lambda_*} = \begin{cases} n\left[1 + \frac{1}{2}\frac{\lambda_* n}{16\pi^2} - \frac{1}{2}\frac{(\lambda_* n)^2}{(16\pi^2)^2} + \mathcal{O}\left(\frac{(\lambda_* n)^3}{(16\pi^2)^3}\right)\right] & \text{for } \lambda_* n \ll 4\pi^2 \\ \frac{8\pi^2}{\lambda_*}\left[\frac{3}{4}\left(\frac{\lambda_* n}{8\pi^2}\right)^{\frac{4}{3}} + \frac{1}{2}\left(\frac{\lambda_* n}{8\pi^2}\right)^{\frac{2}{3}} + \mathcal{O}(1)\right] & \text{for } \lambda_* n \gg 4\pi^2 \end{cases} \tag{2.99}$$

This computation is a proof of how CFT can provide some remarkable tools to compute observables restoring the perturbation theory which, as we can see in 2.75 can fail even if we are dealing with weakly-coupled theories. We will use the latter calculation as a template to discuss the scaling dimension for the $SU(2N_f)$ Linear Sigma Model extending this techniques to the computation of next-to-leading-order corrections to Δ. The next-to-leading-order computation is similar but much more complicated, the idea is to use the same solution for the equation of motion as seen for Δ_{-1} but with an additional perturbative term dependent on coordinates both for ρ and χ and compute the second order Action and the functional determinant of the resulting differential operator in the action. The first thing we need to compute is the second order action. In order to capture fluctuations we write fields as :

$$\rho = f + r(x) \quad ; \quad \chi = -i\mu\tau + \frac{1}{f\sqrt{2}}\pi(x). \tag{2.100}$$

The one-loop action written in the above mentioned fluctuations reads :

$$S^{(2)} = \int_{\frac{T}{2}}^{\frac{T}{2}} d\tau \int d\Omega_{d-1}\frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial\pi)^2 - 2i\mu\tau r\partial_\tau\pi + (\mu^2 - m^2)r^2. \tag{2.101}$$

Computing zeros of the determinant of the inverse propagator in the action and solving them with respect to the energy ω, we find

$$\omega_\pm^2 = J_l^2 + 3\mu^2 - m^2 \pm \sqrt{4J_l^2\mu^2 + (3\mu^2 - m^2)^2}, \tag{2.102}$$

where J_l^2 is the squared momentum or, equivalently, the eigenvalues of the Laplacian on the sphere. So far, we just need to write the one-loop extended version of the 2.95, which is :

$$\langle\psi_n|\, e^{-HT}\, |\psi_n\rangle = e^{\frac{e_{-1}(\lambda_o n,d)T}{\lambda_o R}} \frac{\int \mathcal{D}r\mathcal{D}\pi\ e^{-S^{(2)}}}{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\exp\Big\{-\int_{\frac{T}{2}}^{\frac{T}{2}}\left(\partial\bar{\phi}\partial\Phi + m^2\bar{\phi}\phi\right)\Big\}} = \tag{2.103}$$

$$\tilde{\mathcal{N}}\exp\Big\{-\left[\frac{e_{-1}(\lambda_o n,d)}{\lambda_0} + e_0(\lambda_0 n,d)\right]\frac{T}{R}\Big\}. \tag{2.104}$$

The latter shows ho we can identify the e_0 coefficient with fluctuations of the Gaussian integral in (2.105), explicitly :

$$??\frac{T}{R}e_0 = \log\left\{\frac{\sqrt{\det(S^{(2)})}}{\det\left(-\partial_t^2 - \Delta_{S_{d-1}} + m^2\right)}\right\} = \tag{2.105}$$

$$\frac{T}{2}\sum_{l=0}^{\infty} n_l \int \frac{d\omega}{2\pi}\log\frac{(\omega^2 + w_-^2(l))(\omega^2 + w_+^2(l))}{\omega^2 + \omega_0^2(l)}, \tag{2.106}$$

where n_l is the multiplicity of the Laplacian on the $(d-1)-$dimensional sphere, and ω_0 is the energy of the free theory :

$$n_l = \frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(l+1)\Gamma(d-1)} \quad ; \quad \omega_0^2 = J_l^2 + m^2. \tag{2.107}$$

Computing explicitly the integral in (2.106) and neglecting the sum of zero point energy, we find :

$$e_0(\lambda_0 n,d) = \frac{R}{2}\sum_{l=0}^{\infty} n_l\left[\omega_+(l) + \omega_-(l)\right]. \tag{2.108}$$

This result has to be regularized, therefore, we first need to isolate the divergent part of it. We could think of computing the latter in a large momenta limit, which can be intuitively identified as a large l limit. A simple asymptotic analysis of the right hand side of the last equation can show that the large l behavior of the summand in (2.108) is :

$$n_l\left[\omega_+(l) + \omega_-(l)\right] \sim \sum_{n=1}^{\infty} l^{d-n}. \tag{2.109}$$

In $d = 4$ dimensions, the first five terms of this sum carry a divergent contribution. In order to separate the divergent part from the regular one, we can compute

$d = 4 - \epsilon$ expansion of the first five coefficients. By comparison one gets :

$$c_1 = \frac{2}{R} + \mathcal{O}(\epsilon) \ ; \ c_2 = \frac{6}{R} + \mathcal{O}(\epsilon) \ ; \ c_3 = 2\mu^2 R + \frac{4}{R} + \mathcal{O}(\epsilon) \ ; \ c_4 = 2\mu^2 R + \mathcal{O}(\epsilon) \tag{2.110}$$

$$c_5 = \frac{5(\mu^2 R^2 - 1)^2}{4R} + \epsilon \frac{[225\mu^4 R^4 + 50\mu^2 R^2 + 150\gamma(\mu^2 R^2 - 1) + 113]}{120R} + \mathcal{O}(\epsilon^2). \tag{2.111}$$

We can now rewrite the (2.108) as a sum of a divergent part and a finite part, this results in :

$$\sum_{l=0}^{\infty} n_l \left[\omega_+(l) + \omega_-(l)\right] = \frac{1}{2} \sum_{n=1}^{5} c_n \sum_{l=1}^{\infty} l^{d-n} + \frac{1}{2} \sum_{l=1}^{\infty} \bar{\sigma}(l), \tag{2.112}$$

where $\bar{\sigma}(l)$ is just the original summand $n_l \left[\omega_+(l) + \omega_-(l)\right]$ without the divergent part of the first five coefficient computed above. The divergent contribution can be even more manipulated remembering the definition of the Riemann Zeta, $\sum_{l=0}^{\infty} l^x = \zeta(-x)$, and recalling $\zeta(1 - x) \sim \frac{1}{x}$ for enough small x. Thus

$$\frac{1}{2} \sum_{n=1}^{5} c_n \sum_{l=1}^{\infty} l^{d-n} = -\frac{5(\mu^2 R^2 - 1)^2}{8R\epsilon} - \frac{15\mu^4 R^4 - 6\mu^2 R^2 + 7}{16R}. \tag{2.113}$$

So far we have just computed the first and second terms in (2.112), separating the divergent part from the finite one in the second term of the (2.113) expansion. However, we have to recall that our true goal is to compute the regularized version of e_0. Details of a complete regularization scheme can be found in [**epsilon**], we will show how to define and compute the regularized scaling dimension in Chapter 4.

$$\Delta_0 = \bar{e}_0(\lambda n, RM, 4) + \frac{\partial}{\partial_\epsilon} \left[\frac{1}{\lambda} \bar{e}_{-1}(\lambda n, RM, 4 - \epsilon)\right] \Big|_{\epsilon=0}, \tag{2.114}$$

where we can compute the $\bar{e}_{-1}$ as

$$\frac{1}{\lambda} \bar{e}_{-1}(\lambda n, RM, 4 - \epsilon) = \frac{1}{\lambda M^\epsilon} e_{-1}(\lambda n R M^\epsilon, 4 - \epsilon). \tag{2.115}$$

Chapter 3

Charging the $U(2N_f)$ model

3.1 Classical Analysis

Using tools developed in the previous section, we now want to focus on a fairly general model, whose interesting features and phenomenological implications are already known [**wilczek**]. Our analysis will mainly focus on applying methods and results of the large charge approach in conjunction with a semi-classical study of the fixed-point system, using techniques shown in the section about CFT. The original purpose of this work will be to show how, under the aforementioned hypotheses, it is possible to introduce a systematic calculation scheme, that makes use of the observations previously explored in order to characterise the system under investigation in its crucial features such as: non-relativistic dispersion relations for the modes of the theory, ground state energy, quantum corrections to the energy and symmetry breaking patterns. The system we have chosen to study in detail consists of a linear realisation of the Sigma Model for an N_f-flavors underlying theory with an Higgs-like potential, whose Lagrangian results in

$$\mathcal{L} = \mathrm{Tr}\left(\partial_\mu H^\dagger \partial^\mu H\right) - u\,\mathrm{Tr}\left(H^+ H\right)^2 - v\left(\mathrm{Tr}\,H^+ H\right)^2 + m^2\,\mathrm{Tr}(H^\dagger H) \qquad (3.1)$$

The H field is in perfect analogy with the Σ field of the two-flavor model set out in section 2. The notable difference is that it is a $U(N_f) \times U(N_f)$ field transforming in the adjoint-antisymmetric representation, but our real goal will be to use pseudo-reality to study an analogous Lagrangian for a field $U(2N_f)$,

which we can realize setting $N_C = 2$. There are several reasons why this theory is an interesting case study. Firstly, the Linear realisation of this model implies that the theory is renormalizable, but above all that it exhibits a well-defined fixed point, so that the conformality of the theory is ensured and does not require any special precautions when applying the Weyl map to the cylinder. The theory, without any prescription on N_C, describes at the effective level the condensate fluctuations corresponding to a microscopic theory of N_f massless quarks in the fundamental representation of the $SU(N_C)$ gauge group. Such a theory has a global invariance $SU(N_f) \times SU(N_f) \times U_B(1) \times U_A(1)$, where $U_A(1)$ (axial) can be classically valid, but quantum-mechanically anomalous. One should then notice that with these prescriptions we obtain a theory that has exactly the global starting symmetries set out in the first chapter and the same would apply if we took $U(2N_f)$ as the starting global symmetry. This allows us to apply many of the results of Chapter 2 without loss of generality. Before introducing the $U(2N_f)$ case, let us first make some general remarks on the $SU(N_f) \times SU(N_f) \times U(1)$ case studied in [**safecft**], which we will then adapt. Let's write the Noether's currents and charges associated to the $SU_{L,R}(N_f)$ symmetry:

$$J_L = \frac{i}{2}\left(dHH^\dagger - HdH^\dagger\right) \quad ; \quad J_R = -\frac{i}{2}\left(H^\dagger dH - dH^\dagger H\right) \tag{3.2}$$

and

$$Q_L = \int dx^3 J_L^0 \quad ; \quad Q_R = \int dx^3 J_R^0 \tag{3.3}$$

The problem of characterising the vacuum state in our theory is not as trivial as in the previous case and it is not taken for granted that the vacuum is spatially homogeneous, we will then use the last relation derived from a simpler case as a hint to introduce an ansatz on the general form of the vacuum that we will seek homogeneous, we will then impose that the $Q_{L,R}$ charges are still preserved. Let $H_0(t)$ be our vacuum;

$$H_0(t) = e^{iM_L t} B e^{-iM_R^T t}. \tag{3.4}$$

By comparison with the previous case, B has the same meaning as Σ_0 and the $M_{L,R}$ matrices are analogues of th M in the previous section and live in the

Cartan subalgebra of $SU(N_f)$. The form of the M matrices depends strongly on the charge-fixing configuration, whereas for B it is sufficient to be a self-adjoint matrix. Recall that, since we have not determined the vacuum state using the equations of motion, we must impose that the charges are preserved. These two constraints will in fact be crucial in fixing some important relations between the matrices introduced in our ansats. Let V be the volume of the spatial manifold, namely M_3, over which we integrate the zero-components of the currents

$$\dot{Q_L} = -iVe^{iM_Lt}\left([M_L^2, BB^\dagger] - 2[M_L, BM_RB^\dagger]\right)e^{-iM_Lt} = 0 \tag{3.5}$$

$$\dot{Q_R} = +iVe^{iM_Rt}\left([M_R^2, B^\dagger B] - 2[M_R, B^\dagger M_L B]\right)e^{-iM_Rt} = 0 \tag{3.6}$$

The resulting relationships give an idea of the interplay between the configuration of fixed charges and the vacuum state. Since charge conservation imposes constraints on the vacuum state, a configuration of $M_{L,R}$ and B consistent with this constraints is found, the vacuum state is therefore fixed and this in turn determines the shape of charges. A simple solution could be the one in which M and B are diagonal matrices united with $M_{L,R} = M$, which reduces to

$$H_0 = e^{2iMt}B \tag{3.7}$$

We would like to use the same form as the above one for the vacuum in the $U(2N_f)$ theory, but with a less trivial choice for the M and B matrices. We will in fact set B to be anti-symmetric and M to be diagonal. Before moving on, we have to show that such a choice is a practicable one and in order to do that, we have just to replace $U(N_f) \times U(N_f)$ with our $U(2N_f)$. The first consequence is that we loose the distinction between left and right-handed spinors, so the current takes the form

$$J = 4i\,\mathrm{Tr}\left\{\left(H\partial_\mu H^\dagger\right)t_a\right\}. \tag{3.8}$$

where t_a are generalized Gell-Mann matrices. The conserved charge is

$$Q = 4i\int dx^3\left(H\partial_0 H^\dagger\right)t_a \tag{3.9}$$

and, requiring that the latter is preserved with the (61) form for the vacuum, we get that the following choice for M and B are perfeclty coherent.

$$B = ib \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} \;\; ; \;\; M = \mu \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \tag{3.10}$$

The choice of the B matrix is dictated by the defining equation of minima, which can be achieved just by differentiating with respect to the field the potential part of the Lagrangian. There are several choices of the vacuum, corresponding to different regimes of the theory, but this will be subsequently discussed with the due care. Our goal for the moment is just to show that the ansatz in (3.7) is practicable, since it is coherent with charge conservation. We can first write the equations of motion varying with respect to $H^\dagger$ and substitute the ansats for the vacuum. This will provide us a further constraint between the Lagrangian parameters and the defining parameters of the M and B matrices

$$-\partial^2 H + 2uHH^\dagger H + 2vH\,\mathrm{Tr}(H^\dagger H) + m^2 H = 0. \tag{3.11}$$

If the ansatz is $H_0(t) = e^{2iMt}B$, the equation of motion takes the form

$$4M^2 + 2uBB^\dagger + 2v\,\mathrm{Tr}(B^t B) + m^2 = 0. \tag{3.12}$$

Before proceeding, we must note that, as with the theory in section 2, it is the relationship between the system's charge configuration and vacuum that sets the symmetry-breaking pattern of the model. This will soon be the subject of study, but it requires an in-depth study of the charges, which, as already mentioned, are related to the particular choice of the vacuum and of the independent degrees of freedom of the system. At this stage, we can already draw some very important and characterising preliminary information on the system. Now that we have put all the fundamental pieces of the theory in place and specialised the treatment with the previous hypotheses, we can, for example, think about deriving the dispersion relations of the theory and the classical energy. This results will be the starting point for the semi-classical analysis of the following chapters. Since the theory is non-relativistic, we can derive them by proceeding with an expansion to the quadratic order of the Lagrangian and calculating the zeros in the determinant

of the inverse propagators, which of course will lead us to have to specify a block decomposition for the H field as well. Without specifying the particular form of Φ,we write the most general form for H, enucleating its vacuum state

$$H(x) = \exp(2iMt)(B + \Phi(x)). \tag{3.13}$$

Substituting H into the Lagrangian and expanding it to $\mathcal{O}(H^2)$ is a fairly simple but laborious task, the salient results of which we report here, leaving out the details in the Appendix.

- The kinetic terms takes the form :

$$\mathrm{Tr}(\partial_\mu H \partial^\mu H^\dagger) \simeq \mathrm{Tr}(\partial_\mu \Phi \partial^\mu \Phi^\dagger) + 4\mu^2 \,\mathrm{Tr}\left\{\left(\Phi^\dagger + B\right)(\Phi + B)\right\} \tag{3.14}$$

$$- 2i\mu \,\mathrm{Tr}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\left(\Phi^\dagger + B\right)\partial_o \Phi - \partial_o \Phi^\dagger \left(B + \Phi\right)\right] \right\} \tag{3.15}$$

- While the the potential takes the form :

$$\mathrm{Tr}(H^\dagger H) = \mathrm{Tr}(B^\dagger B) + \mathrm{Tr}\left\{B\left(\Phi + \Phi^\dagger\right)\right\} + \mathrm{Tr}(\Phi^\dagger \Phi) \tag{3.16}$$

$$\mathrm{Tr}(H^\dagger H H^\dagger H) = 4b^2 \,\mathrm{Tr}(\Phi\Phi^\dagger) + 2b^2 \,\mathrm{Tr}\left\{B\left(\Phi^\dagger + \Phi\right)\right\} + \tag{3.17}$$

$$+ \mathrm{Tr}\left(\Phi^\dagger B \Phi^\dagger B\right) + \mathrm{Tr}\left(\Phi B \Phi B\right) \tag{3.18}$$

$$\mathrm{Tr}^2(H^\dagger H) = 4b^2 N \,\mathrm{Tr}\left(\Phi\Phi^\dagger\right) + \mathrm{Tr}^2\left\{B\left(\Phi + \Phi^\dagger\right)\right\} + 4b^2 N \,\mathrm{Tr}\left\{B\left(\Phi + \Phi^\dagger\right)\right\} \tag{3.19}$$

We can get rid of constant terms and by using the constraint of the equations of motion (3.12), the Lagrangian is greatly simplified :

$$\begin{aligned} \mathcal{L} = & \,\mathrm{Tr}\left(\partial_\mu \Phi \partial^\mu \Phi^\dagger\right) + \\ & - 2i\mu \,\mathrm{Tr}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left[\left(\Phi^\dagger + B\right)\partial_o \Phi - \partial_o \Phi^\dagger \left(B + \Phi\right)\right] \right\} + \\ & - 2ub^2 \,\mathrm{Tr}\left(\Phi\Phi^\dagger\right) - u \,\mathrm{Tr}\left(\Phi^\dagger B \Phi^\dagger B\right) - u \,\mathrm{Tr}\left(\Phi B \Phi B\right) + \\ & - v \,\mathrm{Tr}^2\left\{B\left(\Phi + \Phi^\dagger\right)\right\}. \end{aligned} \tag{3.20}$$

Once we get the second order Lagrangian, we want to study the whole set of degrees of freedom encoded in the Φ field. In order to find dispersion relations, we have to expand the field in the $u(2N)$ algebra, formally :

$$\Phi = (Q_a + iP_a)T_a \ \ ; \ \ \mathrm{Tr}\,(T_a T_b) = \frac{\delta_{ab}}{2}. \tag{3.21}$$

This form leads to :

$$\mathcal{L} = \frac{1}{2}\sum_a (\partial Q_a)^2 + (\partial P_a)^2 - 2ub^2 \sum_a (Q_a^2 + P_a^2) \tag{3.22}$$

$$-v\,\mathrm{Tr}^2\left\{\sum_a 2Q_a B T_a\right\} - 2u\sum_{a,b}\mathrm{Tr}\left\{T_a B T_b B\right\}(Q_a Q_b - P_a P_b) \tag{3.23}$$

$$-2i\mu\sum_{a,b}\mathrm{Tr}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} T_a T_b\right\}\mathcal{H}_{ab}\left(Q,\dot{Q},P,\dot{P}\right), \tag{3.24}$$

where the $\mathcal{H}_{ab}$ coefficient has a symmetric part and an anti-symmetric part under the exchange of a and b indices, given by

$$\mathcal{H}_{ab} = \begin{cases} Q_a\dot{Q}_b - \dot{Q}_a Q_b + P_a\dot{P}_b - \dot{P}_a P_b \ \ ; \ \ antisymm. \\ i\left(Q_a\dot{P}_b - \dot{P}_a Q_b + P_a\dot{Q}_b - \dot{Q}_a P_b\right) \ \ ; \ \ symm. \end{cases} \tag{3.25}$$

Before we move forward, we should try to introduce the most convenient orthogonal basis in the $u(2N)$-algebra. To identify this basis, it is good to look at the B and M matrices, which are $2N \times 2N$ matrices, organized in four $N \times N$ blocks. Once we notice this, it is clear that a convenient basis should have generators organized in the same way. Details of the construction of such a basis can be found in the Appendix, here we just recall that the $u(2N)$-algebra can be spanned by four class of generators, namely : T_{a_i}, T_{b_i}, T_{x_i}, T_{y_i}, given by:

$$T_{a_i} = \frac{1}{2}\begin{pmatrix}U_i & 0\\ 0 & U_i\end{pmatrix} \ \ ; \ \ T_{b_i} = \frac{1}{2}\begin{pmatrix}U_i & 0\\ 0 & -U_i\end{pmatrix} \tag{3.26}$$

$$T_{x_i} = \frac{1}{2}\begin{pmatrix}0 & U_i\\ U_i & 0\end{pmatrix} \ \ ; \ \ T_{y_i} = \frac{i}{2}\begin{pmatrix}0 & U_i\\ -U_i & 0\end{pmatrix}, \tag{3.27}$$

where the $U_i^{N\times N}$ block within generators spans a $u(N)$-algebra, therefore the i-index runs on the $\{1, 2, ..., N^2\}$ interval. It is straightforward to check that this

is an allowed basis of the $u(2N)$-algebra, which is made up by $4N^2$ self-adjoint generators. The orthogonality condition on the $u(N)$-blocks is fully inherited by the $u(2N)$ generators :

$$\mathrm{Tr}(U_i U_j) = \frac{\delta_{ij}}{2} \implies \mathrm{Tr}\left(T_{h_i} T_{k_j}\right) = \frac{\delta_{ij}}{2}\delta_{hk}, \tag{3.28}$$

where the h and k indices just specify the class, $h, k = a, b, x, y$. Now we have to compute the three terms which give us the mass matrix and the couplings induced by the chemical potential. Let's start with the couplings term :

$$\mu\,\mathrm{Tr}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T_h T_k\right\} = M\,\mathrm{Tr}\left\{M T_h T_k\right\}. \tag{3.29}$$

By explicit computations, it can be shown that this quantity is not null only for the following generators pairs :

$$\mathrm{Tr}\left\{M T_{a_i} T_{b_j}\right\} = \mathrm{Tr}\left\{M T_{b_i} T_{a_i}\right\} = \mathrm{Tr}\left\{-iM T_{x_i} T_{y_j}\right\} = \mathrm{Tr}\left\{iM T_{y_i} T_{x_j}\right\} = \mu\frac{\delta_{ij}}{2}. \tag{3.30}$$

The latter equation shows how degrees of freedom are organized into independent pairs. We should also notice how the $ab-$type pairs just involve the symmetric part of the $\mathcal{H}_{hk}$ coefficient, while the $xy-$pairs just involve the anti-symmetric part. With this in mind, one can see that the coupling term equals:

$$-2i\sum_{h,k}^{4N^2}\mathrm{Tr}\left\{M T_h T_k\right\}\mathcal{H}_{hk} = -2\mu\sum_{i=1}^{N^2}\left(Q_{a_i}\dot{P}_{b_i} - \dot{P}_{a_i}Q_{b_i} + P_{a_i}\dot{Q}_{b_i} - \dot{Q}_{a_i}P_{b_i}\right) \tag{3.31}$$

$$-2\mu\sum_{i=1}^{N^2}\left(Q_{x_i}\dot{Q}_{y_i} - \dot{Q}_{x_i}Q_{y_i} + P_{x_i}\dot{P}_{y_i} - \dot{P}_{x_i}P_{y_i}\right). \tag{3.32}$$

The first mass term is given by $-2u\sum_{h,k}\mathrm{Tr}\left\{T_h B T_k B\right\}(...)$, and the non zero contributions to the latter are:

$$\mathrm{Tr}\left\{B T_{x_i} B T_{x_j}\right\} = -\,\mathrm{Tr}\left\{B T_{y_i} B T_{y_j}\right\} = \tag{3.33}$$

$$-\,\mathrm{Tr}\left\{B T_{a_i} B T_{a_j}\right\} = \mathrm{Tr}\left\{B T_{b_i} B T_{b_j}\right\} = -\frac{\delta_{ij}}{2} \tag{3.34}$$

This result show how the mass matrix is diagonal in the basis we choose. The u-proportional mass term is then:

$$-ub^2\sum_{i=1}^{N^2}\left(Q_{y_i}^2 - P_{y_i}^2 - Q_{x_i}^2 + P_{x_i}^2 + Q_{a_i}^2 - P_{a_i}^2 - Q_{b_i}^2 + P_{b_i}^2\right) \tag{3.35}$$

We are left with one last term to be studied; $-v\,\mathrm{Tr}^2\left\{\sum_a 2Q_aBT_a\right\}$. In order to compute this term, it is sufficient to notice that the B matrix is nothing but one of our generators, specifically $B \sim T_{y_1}$, once we state that $U_1 = I$. Since we are working with an orthogonal basis, there will be just one generator T_h giving contribution to the latter trace, and this generator can be no other than T_{Y_1} itself, thus

$$-v\,\mathrm{Tr}^2\left\{\sum_a 2Q_aBT_a\right\} = -4vb^2N_fQ_{y_1}^2. \tag{3.36}$$

Now it's time to take a breath and recap what we have done so far. We expanded the Φ field in the $u(2N)$-algebra, using the aforementioned basis in order to go along with the form of the M and B matrices and get a diagonal mass matrix. This procedure allow us to get much simpler computations. However, we overlooked the fact that our field is parameterized by an anti-symmetric matrix, hence we worked with more degrees of freedom than necessary. There is no need to worry about that, it is sufficient to get rid of the unwanted generators. It is easy to see that an anti-symmetric generator can be obtained just by requiring that the U_i matrices inside the T_{a_i}, T_{b_i} and T_{x_i} are anti-symmetric, while the U_i block inside the T_{y_i} generators has to be symmetric. Therefore, our true anti-symmetric basis is made up by:

$$T_{a_i} = \frac{1}{2}\begin{pmatrix} U_i & 0 \\ 0 & U_i \end{pmatrix} \;;\; U_i = -U_i^T \;;\; i = 1, 2, ..., \frac{N(N-1)}{2} \tag{3.37}$$

$$T_{b_j} = \frac{1}{2}\begin{pmatrix} U_j & 0 \\ 0 & -U_j \end{pmatrix} \;;\; U_j = -U_j^T \;;\; j = 1, 2, ..., \frac{N(N-1)}{2} \tag{3.38}$$

$$T_{x_k} = \frac{1}{2}\begin{pmatrix} 0 & U_k \\ U_k & 0 \end{pmatrix} \;;\; U_k = -U_k^T \;;\; k = 1, 2, ..., \frac{N(N-1)}{2} \tag{3.39}$$

+

$$T_{y_h} = \frac{i}{2}\begin{pmatrix} 0 & U_h \\ -U_h & 0 \end{pmatrix} \;;\; U_k = U_k^T \;;\; k = 1, 2, ..., \frac{N(N+1)}{2} \tag{3.40}$$

This specification only affects the number of fundamental real fields and the couplings term, since there is no longer way to get a coupling between x and y modes. In fact, the U_i inside T_{x_i} and the U_j inside the T_{y_j} are now forced to be

different, canceling out the δ_{ij} in the (3.28). With this in mind, the coupling term becomes :

$$-2i\sum_{h,k}\mathrm{Tr}\left\{MT_hT_k\right\}\mathcal{H}_{hk}=-2\mu\sum_{i=1}^{\frac{N(N-1)}{2}}\left(Q_{a_i}\dot{P}_{b_i}-\dot{P}_{a_i}Q_{b_i}+P_{a_i}\dot{Q}_{b_i}-\dot{Q}_{a_i}P_{b_i}\right). \tag{3.41}$$

We are left with $4N^2-2N$ fundamental fields, $2N^2-N$ of which are coupled in independent $ab-$pairs, while the remaning $2N^2$ x and y fields are free. Everything is now ready to compute dispersion relations. As we said before, we have some free real fields, whose dispersion relations are trivial. In fact, being ω the energy and p the 3-momentum, we have :

$$\omega^2_{Q_{x_i}}=p^2\ ;\ \omega^2_{P_{x_i}}=p^2+2ub^2\ ;\ \frac{N(N-1)}{2}\ d.o.f.\ each \tag{3.42}$$

$$\omega^2_{P_{y_i}}=p^2\ ;\ \omega^2_{Q_{y_i}}=p^2+2ub^2\ ;\ \frac{N(N+1)}{2}-1\ d.o.f.\ each \tag{3.43}$$

$$\omega^2_{Q_{y_1}}=p^2+4\mu^2-m^2\ ;\ \omega^2_{P_{y_1}}=p^2\ ;\ 1\ d.o.f.\ each \tag{3.44}$$

For the a and b modes, we first need to collect their inverse propagators, which, in the Fourier space, read as :

$$\mathcal{D}^{-1}_{Q_aP_b}=\begin{pmatrix}\omega^2-p^2-2ub^2 & 4i\mu\omega\\ -4i\mu\omega & \omega^2-p^2-2ub^2\end{pmatrix} \tag{3.45}$$

$$\mathcal{D}^{-1}_{Q_bP_a}=\begin{pmatrix}\omega^2-p^2 & 4i\mu\omega\\ -4i\mu\omega & \omega^2-p^2\end{pmatrix}. \tag{3.46}$$

The corresponding dispersion relations are get by solving, with respect to ω, the zeros of the determinant. Giving :

$$\omega_{QaPb}=\pm2\mu+\sqrt{p^2+4\mu^2}\ ;\ \frac{N(N-1)}{2}\ d.o.f\ each \tag{3.47}$$

$$\omega_{PaQb}=\pm2\mu+\sqrt{p^2+4\mu^2+8\alpha\mu^2-2\alpha m^2}\ ;\ \frac{N(N-1)}{2}\ d.o.f\ each \tag{3.48}$$

where we used the equation of motion to rewrite b^2 as a function of μ, defining $\alpha=\frac{u}{u+2N_fV}$. The spectrum that arises from fixing the baryonic charge and the particular choice of vacuum we made, exhibits a certain number of degrees of freedom that are not at all affected by the chemical potential, while all the other

modes just display a shift in the energy, which is proportional to the chemical potential. From this we deduce that the former correspond to degrees of freedom that carry no baryonic charge, while the latter do. The reason this happens, is that the vacuum we have chosen is invariant under the rotation generated by the baryonic charge. We can, likewise, choose a vacuum state that does not display this symmetry. If this is the case, we expect a completely different spectrum, the typical one of a superfluid phase. One possible choise is:

$$B = ib\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \;\; ; \;\; J = \begin{pmatrix} 0 & I_{\frac{N_f}{2}\times\frac{N_f}{2}} \\ -I_{\frac{N_f}{2}\times\frac{N_f}{2}} & 0 \end{pmatrix}. \tag{3.49}$$

This choice provides the same second-order Lagrangian of the previous case and the coupling term is still given by the (3.41). Once again, the vacuum is either one of our generators, or a linear combination of generators of the $a-$type. The equation (3.36) still holds, but the only surviving generator will be a certain $T_{\tilde{a}}$, thus we have to replace Q_{y_1} with $Q_{\tilde{a}}$. The last step is to check how the $-2u\sum_{hk}\mathrm{Tr}\left\{BT_hBT_k\right\}(...)$ term changes. From direct computation, it results:

$$\mathrm{Tr}\left\{BT_{h_i}BT_{k_j}\right\} = -2\delta_{hk}\,\mathrm{Tr}\left\{JU_iJU_j\right\}. \tag{3.50}$$

Since we can always find a $\{U_i\}$ orthogonal basis within which J is contained, the right hand side of the latter equation can be written as :

$$-2\delta_{hk}\,\mathrm{Tr}\left\{JU_iJU_j\right\} = -2\delta_{hk}\,\mathrm{Tr}\left\{U_aU_iU_aU_j\right\} = \delta_{hk}\delta_{ij}. \tag{3.51}$$

Replacing this result in the u proportional mass term, we get :

$$-ub^2\sum_k\left(Q_k^2 - P_k^2\right). \tag{3.52}$$

This result, conjoined with the $-2ub^2\,\mathrm{Tr}(\Phi^{dagger}\Phi)$ term, tells us that all the P_h's have a zero mass term. For the x and y modes, dispersion relations are :

$$\omega^2_{P_{x_i}} = p^2 \;\; ; \;\; \omega^2_{Q_{x_i}} = p^2 + 2ub^2 \;\; ; \;\; \frac{N(N-1)}{2}\;\; d.o.f.\;\; each \tag{3.53}$$

$$\omega^2_{P_{y_i}} = p^2 \;\; ; \;\; \omega^2_{Q_{y_i}} = p^2 + 2ub^2 \;\; ; \;\; \frac{N(N+1)}{2}\;\; d.o.f.\;\; each. \tag{3.54}$$

On the other hand, the inverse propagators in the momentum space are equal to :

$$\mathcal{D}^{-1}_{Q_a P_b} = \mathcal{D}^{-1}_{P_a Q_b} = \mathcal{D}^{-1}_{P_{\tilde{a}} Q_{\tilde{b}}} \begin{pmatrix} \omega^2 - p^2 - 2ub^2 & 4i\mu\omega \\ -4i\mu\omega & \omega^2 - p^2 \end{pmatrix} \tag{3.55}$$

$$\mathcal{D}^{-1}_{Q_{\tilde{a}} P_{\tilde{b}}} \begin{pmatrix} \omega^2 - p^2 - 8\mu^2 - 2m^2 & 4i\mu\omega \\ -4i\mu\omega & \omega^2 - p^2 \end{pmatrix} \tag{3.56}$$

whose dispersion relations for large μ and small momenta are:

$$\omega_{ab} = \omega_{\tilde{a}\tilde{b}} = \sqrt{\frac{u}{3u + 4N_f v}} p \;\; ; \;\; \omega_{ab} = \omega_{\tilde{a}\tilde{b}} = \sqrt{\frac{8(3u + 4vN_f)}{u + 2N_f v}} \mu + \mathcal{O}(p^2). \tag{3.57}$$

Thus, the (3.55) kind of inverse propagator corresponds to $N(N-1) - 1$ Goldstone Bosons and the same number of massive modes, while the latter inverse propagator gives rise, in the large μ and small p limit, to a Goldstone boson and a massive mode with energy :

$$\omega = \frac{p}{\sqrt{3}} + ... \;\; ; \;\; \omega = 2\sqrt{6}\mu + \frac{5p^2}{12\sqrt{6}\mu} + ... \tag{3.58}$$

The first one is a universal sector for any fixed charge and scale invariant theory, known as the Conformal Mode.

3.2 Symmetry Breaking Patterns

The theory we chose to study has a global $U(2N_f) \simeq SU(2N_f) \times U_A(1)$ symmetry. Given the potential part of the Lagrangian,

$$\mathcal{V}(H^\dagger H) = u_0 \operatorname{Tr}\left(H^\dagger H\right)^2 + v_0 \left(\operatorname{Tr} H^\dagger H\right)^2 + (m^2 - 4\mu^2) \operatorname{Tr}(H^\dagger H), \tag{3.59}$$

we can see that $\mathcal{V}(H^\dagger H)$ is minimized on a manifold of B vacua defined by :

$$\frac{\partial \mathcal{V}}{\partial H^\dagger} = 2u \operatorname{Tr}\left(H^\dagger H H\right) + 2v \operatorname{Tr}\left(H^\dagger H\right) \operatorname{Tr}\left(H\right) + (m^2 - 4\mu^2) \operatorname{Tr}(H) \Big|_{H=B} = 0 \tag{3.60}$$

$$\frac{\partial^2 \mathcal{V}}{\partial H^{\dagger 2}} = 2u \operatorname{Tr}(H^2) + 2v \operatorname{Tr}^2(H) \Big|_{H=B} \geq 0 \tag{3.61}$$

We can easily check that both minima we chose belong to this manifold. Since we are fixing the baryon charge, we can distinguish vacua in two classes: those that

preserves the baryon charge and those which don't. These two circumstances characterize two phases of the theory, that can be respectively defined as the **Normal Phase** and the **Broken Phase**, depending on the ratio between the mass and the chemical potential. Intuitively, we can figure out that, if $m^2 > 4\mu^2$, the vacuum is not driven by the chemical potential, and therefore stays invariant under the baryon generator. Otherwise, if $m^2 < 4\mu^2$, the ground state is set by the same chemical potential which fixes the baryon charge, so it must change under baryon rotations. It is a consequence that the pattern of symmetry braking is strongly dependent on the phase and we want to check out how symmetries are broken, both explicitly and spontaneously, in the two phases. The normal phase is defined by the vacuum :

$$H_0 = B = b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad ; \quad b \in R. \tag{3.62}$$

It is self-evident that without any chemical potential, the group preserving this vacuum under the adjoint representation;

$$U H_o U^T = H_o \Big|_{H_o = B}, \tag{3.63}$$

is, by definition, the symplectic group $Sp(2N)$, resulting in the trivial pattern:

$$SU(2N_f) \times U_A(1) \xrightarrow{SSB} Sp(2N_f) \tag{3.64}$$

In order to get the full Symmetry Breaking Pattern we can proceed in two ways. If the ansatz concerning the vacuum is:

$$H_0(t) = e^{2iMt} B, \tag{3.65}$$

with

$$B = b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad ; \quad b \in R, \tag{3.66}$$

$$M = \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \mu \in R, \tag{3.67}$$

we get an addition term at the lagrangian level, which reads as

$$-2i\mu \,\mathrm{Tr}\left\{ M \left(\, \phi^\dagger \partial_0 \phi - \phi \partial_0 \phi^\dagger \right) \right\}. \tag{3.68}$$

The latter term explicitly breaks the starting symmetry and it is invariant under the $\mathcal{C}(M)$ group, where $\mathcal{C}(M)$ is defined by :

$$UMU^\dagger = M \implies [U, M] = 0. \tag{3.69}$$

By direct computations, one can check that the latter defines :

$$\mathcal{C}(M) = U_L(N_f) \times U_R(N_f). \tag{3.70}$$

The insertion of a non-zero chemical potential not only gives rise to the term in (3.68), but also affects the vacuum, turning the (3.62) into the (3.65). Thus, there must be a spontaneous breaking from $\mathcal{C}(M) \times U_A(1)$ to another group, namely $\mathcal{G}$, due to a new configuration of the vacuum when the chemical potential is turned on.

$$... \ \mathcal{C}(M) \xrightarrow{SSB} \mathcal{G}. \tag{3.71}$$

We can figure out how $\mathcal{G}$ looks like just by requiring H_0 to be invariant under a certain group. Since for $\mu = 0$, the pattern must reduce to the one in (3), then $\mathcal{G}$ is nothing but $\mathcal{C}(M)$ when "embedded" into $Sp(2N_f)$. This argument is fully equivalent to find the $U(2N_f)$-subgroup such that (3.63) is satisfied. Thus, given the general form of the $Sp(2N_f)$ Algebra,

$$T_a = \begin{pmatrix} A_a & Ba \\ B_a^\dagger & -A_a^T \end{pmatrix}, \tag{3.72}$$

with $A_a = A_a^\dagger$, we should write the (243) as :

$$MT_a - Ta^\dagger M = 0, \tag{3.73}$$

which leads to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A_a & Ba \\ B_a^\dagger & -A_a^T \end{pmatrix} - \begin{pmatrix} A_a & Ba^\dagger \\ B_a & -A_a^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0, \tag{3.74}$$

whose solutions are :

$$B_a = 0 \ \hookrightarrow \ T_a = \begin{pmatrix} A_a & 0 \\ 0 & -A_a^T \end{pmatrix}. \tag{3.75}$$

Taking into account the never broken $U_A(1)$, the latter equation defines $U(N_f)$. This procedure must be read as a way to enlighten and distinguish the role of M and the one of B. There is no good reason to not writing the (3.63) up directly. The problem of such an approach is that the pattern one gets requiring (3.63) from the very beginning, shows both explicit and spontaneous breaking in once and this is due to the twofold nature of H_0, which carries information about both charge fixing and vacuum state at $\mu = 0$. The previous way of finding the symmetry breaking pattern can also be cast as follow :

- Starting from the theory $\mathcal{L}(H)$, we can redefine

$$H \to e^{2iMt}\tilde{H} \;\; ; \quad \mathcal{L}(H) \to \mathcal{L}(\tilde{H}), \tag{3.76}$$

  where, in principle, $\tilde{H}$ carries the same symmetries of the old H. The $\mathcal{L}(\tilde{H})$ theory will be the same of the old $\mathcal{L}(H)$ except from additional explicit breaking terms in M.

- We can now set $\tilde{H}_0$ to be $\tilde{H}_0 = B$, in other words

$$\mathcal{L}(H) \xrightarrow[\mu\neq 0]{ESB} \mathcal{L}\left(e^{2iMt}\tilde{H}\right) \xrightarrow{SSB} \mathcal{L}\left(e^{2iMt}(B+\Phi)\right) \tag{3.77}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \tag{3.78}$$

$$U(2N_f) \xrightarrow[\mu\neq 0]{ESB} \mathcal{C}(M) \xrightarrow{SSB} U(N_f) \times U_B(1) \tag{3.79}$$

$$\downarrow \mu = 0 \tag{3.80}$$

$$Sp(2N_f) \tag{3.81}$$

The explicit symmetry breaking due to the insertion of a baryon chemical potential is fixed and does not depend on the subsequent choice of the ground state, therefore, in order to capture the symmetry breaking pattern in the broken phase we just have to ask what is the $\mathcal{C}(M)$ subgroup preserving the B vacuum as follows :

$$B = ib \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \tag{3.82}$$

where J is a $N_f \times N_f$ symplectic matrix. As already stated, $\mathcal{C}(M)$ is nothing but $U_L(N_f) \times U_R(N_f)$, consequently the generic $U \in \mathcal{C}(M)$ must have the form

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad AA^\dagger = I, \qquad BB^\dagger = I. \tag{3.83}$$

If we require that $U \in \mathcal{C}(M)$ also preserves B, we obtain

$$AJA^T = J \ ; \ BJB^T = J, \tag{3.84}$$

this two equation crearly define the $Sp(N_f) \times Sp(N_f)$ group. Finally, the symmetry breaking pattern in the broken phase results in :

$$U(2N_f) \xrightarrow[\mu \neq 0]{ESB} \mathcal{C}(M) \xrightarrow{SSB} Sp(N_f) \times Sp(N_f). \tag{3.85}$$

The number of spontaneously broken generators N_{SBG} in both phases equals the difference between $\dim(\mathcal{C}(M))$ and $\dim(\mathcal{G}(B))$, which is nothing but the dimension of the coset $\mathcal{C}(M)/\mathcal{G}(B)$, where in both cases $\mathcal{G}$ is the group preserving the vacuum. Thus, we get

$$N_{SBG=} \dim \left(\frac{\mathcal{C}(M)}{\mathcal{G}(B)} \right) \Big|_{B=i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = 2N^2 - N^2 = N^2 \tag{3.86}$$

$$N_{SBG} = \dim \left(\frac{\mathcal{C}(M)}{\mathcal{G}(B)} \right) \Big|_{B=i \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}} = 2N^2 - 2\frac{N(N+1)}{2} = N^2 - N. \tag{3.87}$$

As a consequence of the Goldstone Theorem, we expect for each case a certain number of massless mode that is bigger or equal to the number of broken generators. From the dispersion relations computed in the previous section it is easy to check that such an inequality is true in both cases. For sake of completeness, we should refer to the non-relativistic version of the Goldstone Theorem, details of which can be found in [**GBcount**], here we just recall that, in a non-relativistic scenario, Goldstone Bosons can be classified in two categories, namely Type I GB and Type II GB, for the former the energy ω is proportional to the momentum, $\omega \sim p$, while for the latter $\omega \sim p^2$. The theorem aslo states that, in order to

get a Type II Goldstone, at least two generator have to be broken, wherease for the Type I Goldstones a one-to-one correspondence with broken generators still holds. At the end of the day, in a non-relativistic theory, the number of massless particle must satisfy the inequality :

$$N_{Type\,I} + 2N_{Type\,II} \geq N_{SBG}. \tag{3.88}$$

3.3 The non-relativistic Goldstone Theorem

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Chapter 4

Scaling Dimension

4.1 Next to Leading Order

As we already said, our true purpose is to compute the scaling dimension Δ as the computed in Chapter 1 for the $U(1)$ case. We sow that, in order to capture quantum fluctuation, we need first to compute the leading order energy E_{LO} and then a sum over dispersion relations as in **??**. Let's now compute the Leading Order Energy of the system. First of all, it is necessary to evaluate the action on a classical trajectory. We could, for example, chose the broken phase define by the block-diagonal vacuum in (3.82), on which the classical energy results in :

$$8N_f\mu^2b^2 + 2N_fu_0b^4 + 4N_f^2v_0b^4 + 2N_fm^2b^2 = \mathcal{L}(H_0) \tag{4.1}$$

We can use the equation of motion as a constrain. Evaluating them on the same trajectory, this yields to

$$4M^2 + 2u_0BB^\dagger + 2v_0\,\mathrm{Tr}(B^\dagger B) + m^2 = 0 \tag{4.2}$$

which in terms of the μ and b parameters, reads as

$$4\mu^2 + 2u_0b^2 + 4N_fv_0b^2 + m^2 = 0 \tag{4.3}$$

In order to compute the classical energy E_{LO}, since we are fixing the charges, we can proceed via the Legendre transform :

$$U \to \hat{U} = \mu Q - U. \tag{4.4}$$

This expression would have been good even if we didn't have a single charge, but a charge for each generator of the algebra $Q \equiv \left\{Q_1, Q_2, ..., Q_{4N_f^2-1}\right\}$. For the latter case, Legendre transform is not so trivial. However, we could have taken advantage of the fact that the charges transform together with the field :

$$H \to UHU^T \to Q \to UQU^\dagger \tag{4.5}$$

so we could have ideally $SU(2N_f)$-rotate the charges into $Q = \tilde{Q}\{1, 0, ...0\}$. It is easy to see that the quantity $\mathrm{Tr}\left(Q^\dagger Q\right)$ is invariant, so computing $\tilde{Q}$ is now trivial and we just need to normalize the generators into $\mathrm{Tr}(t_a t_b) = \delta_{ab}$. Finally we get $\mu Q = -16\mu^2 b^2 N_f$. Now it is possible to explicitly compute the Energy and the direct calculation leads to :

$$E_{LO} = \frac{mN}{2u_o + 4v_0 N_f}\left(48\frac{\mu^4}{m^4} + 8\frac{\mu^2}{m^2} - 1\right), \tag{4.6}$$

coherently with what found in [**nonabelian**]. The last formula can be rewritten by reversing the relationship between μ and Q. Using the equations of motion it is easy to see that $\mu \sim Q^{\frac{1}{3}}$ and that therefore $E_{LO} \sim Q^{\frac{4}{3}}$. These are the first classical steps of the semiclassical analysis that we are going to complete with a full calculation of the scaling dimension Δ next to leading order. In order to capture the quantum fluctuations it the scaling dimension, we need to replicate the computation scheme involved in section 2. It is easy to understand that the right generalization of the scaling dimension in the $U(1)$ which matches with our model must be in the form :

$$\frac{T}{R}\Delta_0 = \log\left\{\frac{\sqrt{\det(S^{(2)})}}{\det\left(-\partial_t^2 - \Delta_{S_{d-1}} + m^2\right)}\right\} = \frac{T}{2}\sum_{l=0}^{\infty} n_l \int \frac{d\omega}{2\pi}\sum_i g_i \log\left\{\frac{\omega^2 + \omega_i^2(l)}{\omega^2 + \omega_0(l)}\right\}, \tag{4.7}$$

where g_i the degeneracy of the ω_i dispersion relation, i.e. the numer of modes having that dispersion relation. One can note that the whole list of this number has already been computed via the algebrical characterization of the previous section. The integral in the last equation can be directly computed, leading to:

$$\Delta_0 = \frac{R}{2}\sum_{l=0}^{\infty} n_l \left\{\sum_i g_i(N)\omega_i(N) - \tilde{N}\omega_0(l)\right\}. \tag{4.8}$$

The $\tilde{N}$ is just the total number of fundamental fields in the theory. It is easy to convince ourselves that the second part of the summand in (4.8) is a sum over ground state energies, therefore we will neglect it in upcoming computations. As it as be done previously, Δ_0 is definitely a divergent quantity and, again, we want to capture the divergent part and then regularize it. We do expect that the divergent part is related to the UV behavior of the scaling dimension, and it is hence encoded it the large l asymptotic dependence of the sum over dispersion relation. We can show that the following relation holds for sufficiently large momenta :

$$\frac{R}{2}\sum_{l=0}^{\infty} n_l \left\{ \sum_i g_i(N)\omega_i(N) \right\} \xrightarrow{l\gg 1} \sum_{n=1}^{\infty} c_n l^{d-n}. \tag{4.9}$$

Crearly the g_i do not depend on l, while n_l and ω_i do. Recalling the degeneracy of the laplacian on a $d-$sphere is given by (2.107), we can use the Stirling formula to get the large l behavior of the Euler's Gamma functions in n_l, thus :

$$\Gamma(l+h+1) = (l+h)! \xrightarrow{l\gg 1} (l+h)^{(l+h)} e^{-(l+h)} \implies \tag{4.10}$$

$$n_l \sim (2l+d-2)\frac{\Gamma(l+d-2)}{\Gamma(l+1)} \xrightarrow{h+1=d-2} (2l+h+1)\frac{(l+h)^{(l+h)}e^{-(l+h)}}{l^l e^{-l}}. \tag{4.11}$$

The n_l coefficient is also multiplied for $\omega_i(l)$, whose large l behavior is an $l-$linear function. At the end of the day it is an easy task to show that the asymptotic dependence of the summand in (282) is :

$$\mathcal{O}(l)n_l \sim l^{h+1}e^{\frac{1}{l}} = \sum_{n=1}^{\infty} c_n l^{d-n}. \tag{4.12}$$

Once again, the divergent part of (4.9) must be encoded in the first five terms, therefore we are interested in fixing the firts five coefficients of the latter expression. In order to get $c_1, ..., c_5$ it is sufficient to compute the large l expansion of Δ_0, which is easily achievable once we know the whole set of dispersion relations.

A direct calculation shows that :

$$c_1 = \frac{N(-1+3N)}{R} + \mathcal{O}(\epsilon) \;\; ; \quad c_2 = \frac{5N(-1+3N)}{R} + \mathcal{O}(\epsilon) \;\; ; \tag{4.13}$$

$$c_3 = \frac{2Nv\left[32R^2\mu^2 - N(15+16R^2\mu^2) + N^2(45+16R^2\mu^2)+\right]}{8R(u+2Nv)} + \tag{4.14}$$

$$\frac{u\left[(16-32R^2\mu^2) - N(7+48R^2\mu^2) + N^2(33+64R^2\mu^2)\right]}{8R(u+2Nv)} + \mathcal{O}(\epsilon) \;\; ; \tag{4.15}$$

$$c_4 = \frac{2Nv\left[96R^2\mu^2 + 3N^2(5+16R^2\mu^2) - N(5+48R^2\mu^2)\right]}{16R(u+2Nv)} + \tag{4.16}$$

$$\frac{u\left[48-96R^2\mu^2 + N(19-144R^2\mu^2) + 3N^2(-7+64R^2\mu^2)\right]}{16R(u+2Nv)} + \mathcal{O}(\epsilon) \;\; ; \tag{4.17}$$

$$c_5 = ... \tag{4.18}$$

As we can see, the dependence of this coefficients on the theory's parameters, such μ or R, exactly matches the one shown in (112) of [**epsilon**] in the $N = 1$ limit. Having isolated the divergent part and computed it coefficients we can now rewrite the analogous of (2.112), which leads to :

$$\sum_{l=0}^{\infty} n_l \left\{ \sum_i g_i(N)\omega_i(N) \right\} = \frac{1}{2}\sum_{n=1}^{5} c_n \sum_{l=1}^{\infty} l^{d-n} + \frac{1}{2}\sum_{l=1}^{\infty} \bar{\sigma}(l) \tag{4.19}$$

Finally, we regularize the (4.19) expression using that $\sum_{l=0}^{\infty} l^x = \zeta(-x)$, and recalling $\zeta(1-x) \sim \frac{1}{x}$ for enough small x. Thus

$$\frac{1}{2}\sum_{n=1}^{5} c_n \sum_{l=1}^{\infty} l^{d-n} = \frac{1}{\epsilon}\mathcal{X}(\mu, N, R, u, v) + \mathcal{Y}(\mu, N, R, u, v). \tag{4.20}$$

Where the exact expressions of $\mathcal{X}$ and $\mathcal{Y}$ can be found in the Appendix. Here we're just interested in isolating, within the latter sum, a divergent quantity in the $\epsilon \to 0$ limit.

4.2 Regularization

As anticipated in the last section of Chapter 2, the scaling dimension we have computed needs to be regularized, and so far we have only shown how to isolate divergences, but that is only half the job. We now want to show how, in a semiclassical framework, it is possible to define and compute finite quantities,

which we can call **dressed**, from divergent ones, which we will refer to as **bare** instead. First we write the bare g_i couplings as a function of those dressed $\tilde{g}_i$:

$$g_i = M^{\epsilon}\tilde{g}_i Z_{g_i}(\vec{g}) \tag{4.21}$$

where Z_g defines the $\beta-$functions of the theory as

$$\beta(g_i, \epsilon) = -\epsilon g_i + {g_i}^2 \frac{\partial Z_{g_i}}{\partial g_i}(\vec{g}). \tag{4.22}$$

We can imagine of writing the scaling dimension, or any other quantity, as

$$\Delta = \sum_{j=-1}^{\infty} g^j e_j(\mathcal{A}_o, d) = \sum_{j=-1}^{\infty} \tilde{g}^j \bar{e}_j(\mathcal{A}, d, RM), \tag{4.23}$$

where M is the renormalization scale, the e_i are the bare coefficients and the $\bar{e}_i$ are the dressed ones, while $\mathcal{A}_0$ and $\mathcal{A}$ are just vector variables which encode a set of bare and dressed parameters respectively on which we don't want to focus. The latter expression exhibits just one coupling, while we have two couplings, but it is easy to figure out a natural generalization involving a multi-variable Taylor Series. Also the following statements can be generalized the same way. Equation 4.21, together with 4.23 show how the dressed coefficients of a given order always mix the bare coefficients of different orders, therefore, for the leading order energy e_{-1}, we could write :

$$\frac{e_{-1}(\mathcal{A}_o, d)}{g} = \frac{e_{-1}(\mathcal{A}, d)}{\tilde{g}} + \sum_{j=0} g^j f_j(\mathcal{A}, d, RM). \tag{4.24}$$

Since the regularized next to leading order scaling dimension $\bar{e}_0$ is of the same order of f_0, we have :

$$\bar{e}_0(\mathcal{A}, d, RM) = e_0(\mathcal{A}, d) + f_o(\mathcal{A}, d, RM). \tag{4.25}$$

We should point out that the next to leading order scaling dimension Δ_0 that we computer so far, corresponds to the e_0 in the right hand side of 4.25, and our true purpose is to compute the one on the left hand side. Equation 4.23 together with 4.24 also suggests that :

$$\bar{e}_{-1}(\mathcal{A}, d) = e_{-1}(\mathcal{A}, d). \tag{4.26}$$

Thus, in order to get the f_0 coefficient, we should expand the leading order energy E_{LO}, which corresponds to the e_{-1} coefficient, in powers of the couplings but working in $d = 4 - \epsilon$, The ϵ dependence of such quantities is ensured by the V volume, which obviously depends on the dimensionality, moreover, couplings depend on ϵ when evaluated at the fixed point where the $\beta-$functions vanish. Hence, in order to fully characterize the f_o coefficient, we need to know the $\beta-$functions. The problem of computing the β's in absolutely not trivial and it will be matter of future developments. Anyway, we can figure out what should happen to the next to leading order scaling dimension even in absence of this crucial information. As we can see in 4.6, there is a $1/g$ dependence, where g specifically is $u + 2vN$. We can think that, If we perhaps knew the expression of this coupling at the fixed point, making use of 4.21 for the leading order scaling dimension E_{LO},we could expand it the couplings. As we said before, with the purpose of obtaining the regularization coefficient f_0, we should expand the leading order energy in the couplings, then in ϵ, since couplings at the fixed point are ϵ-dependent. We cannot know the exact expression of couplings at the fixed point, but we can at most imagine that they should be some polynomial in ϵ. Whit this in mind, we could compute an $\epsilon-$ dependent E_{LO} by supposing the aforementioned polynomial takes the place of couplings, therefore:

$$E_{LO} = \frac{64\mu^2\pi^{\frac{4-\epsilon}{2}} M^{-\epsilon} R^{3-\epsilon} \left(4\mu^2 - \frac{(d-\epsilon-2)^2}{4R^2}\right)}{\Gamma\left(\frac{4-\epsilon}{2}\right)(B\epsilon + C\epsilon^2 + ...)}, \tag{4.27}$$

Here we have merely replaced the expression of mass and volume as functions of radius. The polynomial $-B\epsilon + C\epsilon^2 + ...$ that appears in the denominator is simply a formal expression of what should be the couplings at the fixed point; in this sense, knowing the beta functions is equivalent to fixing the coefficients of the polynomial. So, by virtue of the latter expression, computing f_0 is equivalent to expanding into ϵ and retaining only those terms which do not vanish when ϵ is set to be zero. With this in mind, we get

$$f_0 = -\frac{64\pi^2\mu^2 R\,(4\mu^2R^2 - 1)}{B\epsilon} + \mathcal{Z}(\mu, R, M, B, C) + \mathcal{O}(\epsilon) \tag{4.28}$$

where the $\mathcal{Z}(\mu, R, M, B, C)$ quantity is finite. One should notice that the f_0 coefficient has the same structure of 4.20. This is a satisfying result because

we expect that once f_0 is substituted into 4.25 the divergent part in f_0 exactly cancels out the divergent part of Δ_0, i.e. the $\mathcal{X}\epsilon^{-1}$ in 4.20. This suppression of divergences solely depends on the B coefficient, and since we do not know B, due to the lack of β, we can solely read this procedure as a way of computing B by requiring the regularization a priori. We also expect that the dependence on the renormalization scale M drops at the fixed point, as expected from conformal invariance.

4.3 Conclusions and Outlooks

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