UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

SCUOLA POLITECNICA E DELLE SCIENZE DI BASE

DIPARTIMENTO DI FISICA "ETTORE PANCINI"

CORSO DI LAUREA MAGISTRALE IN FISICA

CONSTRUCTING THEORIES FROM METRICS: A STUDY OF CONSERVATIVE AND RADIATIVE PHENOMENA

Relatori
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Abstract

General relativity (GR), formulated by Albert Einstein in 1916 [1], remains a cornerstone of modern physics, excelling in describing large-scale structures and passing numerous precision tests at intermediate energy scales.
However, GR encounters limitations at the quantum scale and in explaining certain cosmological phenomena, such as the accelerated expansion of the universe and the presence of dark matter.
For this reason, physicists are exploring the possibility of modifying gravity in a way that allows us to account for all of these problems.
This thesis explores possible modifications to GR, focusing on Scalar-Tensor theories [2] and the possibility of building the most general solution possible to the inverse problem of gravity. In other words, there is the possibility of verifying if a given metric is a solution for a specific theory starting directly from the metric.
We will investigate potential modifications to General Relativity, particularly emphasising Scalar-Tensor theories [2, 3, 4, 5]. It aims to construct the most general solution possible to the inverse problem of gravity. Specifically, it explores the feasibility of verifying if a given metric is a solution for a specific theory by starting directly from the metric[6]

After this, we will verify the sanity of the theory computing the Quadrupole Formula and the Radiated Power of the pulsar PSR J0737-3039[7].

In summary, this thesis advances the understanding of extended theories of gravity and set the foundation to build a general algorithm that will allow, given a metric, to verify if it's the solution of a theory and then compute Gravitational Waves.

Sommario

La relatività generale (GR), formulata da Albert Einstein nel 1916, rimane un pilastro della fisica moderna, eccellendo nella descrizione delle strutture su larga scala e superando numerosi test di precisione a scale energetiche intermedie.
Tuttavia, la GR presenta limitazioni a livello quantistico e nella spiegazione di certi fenomeni cosmologici, come l'espansione accelerata dell'universo e la presenza di materia oscura.
Per questo motivo, i fisici stanno esplorando la possibilità di modificare la gravità in modo da affrontare tutti questi problemi.
Questa tesi esplora possibili modifiche alla GR, concentrandosi sulle teorie scalari-tensoriali e sulla possibilità di costruire la soluzione più generale possibile al problema inverso della gravità. In altre parole, la possibilità di verificare se una data metrica è una soluzione per una specifica teoria partendo direttamente dalla metrica.

Successivamente, verificheremo la validità della teoria calcolando la formula del quadrupolo e la potenza radiata del pulsar PSR J0737-3039.

In sintesi, questa tesi avanza la comprensione delle teorie estese della gravità e pone le basi per costruire un algoritmo generale che permetterà, data una metrica, di verificare se essa è la soluzione di una teoria e successivamente calcolare le onde gravitazionali.

Contents

Why do we need an extended theory of gravity?

General Relativity, along with quantum mechanics, is one of the pillars of modern physics. This theory was formulated by Albert Einstein in 1916, and it was a groundbreaking way to describe gravity in a completely different way when confronted to Newton's law of universal gravitation formulated in the 17th century.
Einstein's theory has passed all the precision tests [8, 9, 10, 11, 12], most of which are probes of weak field gravity, which means that they probe gravity at intermediate length $(1\mu m \lesssim l \lesssim 10^{11} m)$, and therefore intermediate energy scales.
Despite its successes, GR is not without limitations, particularly when addressing phenomena at the quantum scale, cosmological observations, and the unification of fundamental forces.[13, 5].

A primary motivation for extending General Relativity (GR) stems from the pursuit of unifying gravity with the other fundamental forces—electromagnetic, weak, and strong interactions—into a single theoretical framework. While GR excels at describing large-scale structures, it is incompatible with quantum mechanics, the theory governing the subatomic world [14]. This incompatibility indicates that GR may be an incomplete description of gravity, necessitating extensions that can bridge the gap between quantum mechanics and gravity
Moreover, several astrophysical and cosmological observations challenge the completeness of GR [15, 16]. The accelerated expansion of the universe, inferred from supernova observations and the cosmic microwave background radiation, implies the existence of dark energy—a form of energy that GR cannot adequately explain. Additionally, the behavior of galaxies and galaxy clusters suggests the presence of dark matter, which, differently from ordinary matter, only interacts gravitationally and not electromagnetically. These phenomena indicate that our current understanding of gravity may be missing key components.

How do we modify gravity?

There are countless distinct ways to modify GR, many of which lead to theories that can be designed to agree with current observations. Cosmological observations and fundamental physics considerations suggest that GR must be modified at very low and/or very high energies.

- **$f(R)$ gravity**
  One of the simplest ways to modify GR is to change the Lagrangian of the Hilbert Einstein Action [17, 18]. In GR, the action is

$$S = k \int d^4x \sqrt{-g} R + S_m, \tag{1}$$

  where S_m represents the action of matter field, g is the determinant of the metric tensor $g_{\mu\nu}$ and R is the Ricci scalar. We can modify it with a generic function of R:

$$S = k \int d^4x \sqrt{-g} f\left(R\right) + S_m. \tag{2}$$

  This modification leads to field equations that differ from Einstein's, and contain some additional terms that can account for the accelerated expansion of the universe without introducing Dark Energy.

- **Scalar-Tensor theories**
  This class of theories introduces a scalar field φ coupled with gravity. A known example that we will analyze further in this work is the Brans-Dicke theory [19], where the action has the following expression

$$S = k \int d^4x \sqrt{-g} f\left(\phi R - \frac{1}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi\right) \tag{3}$$

- **Higher dimensional theories**
  Another possible approach to modifying gravity is to assume a higher number of dimensions to our universe [20, 21]. This, of course, leads to a change in the Hilbert-Einstein action, but instead of changing the Lagrangian density, we change the number of variables of integration

$$\int d^4x \to \int d^nx. \tag{4}$$

  Gravitational theory's built-in dimensions other than four have a strong theoretical interest for several reasons, including the formulation of consistent String Theories, like the Bosonic String Theory [21] and Superstring theory [22], that both assume the space-time to have a higher number of dimensions.

About this thesis work

This thesis will be divided into three main parts. In the first one, we will review the Article "A family of solutions to the inverse problem in gravitation: building a theory around a metric" written by Arthur G. Suvorov, we will understand the criticalities and the possible ways to improve it.
In the second part, we will derive the quadrupole formula both in General Relativity and for f(R) theories. After this, we will compute the Radiated Power for the Pulsar PSR J0737-3039.
In the third part, we will find new numerical and analytical solutions to the inverse problem of gravity, using different sets of theories and metrics, after this, we will try to compute the quadrupole formula.
The final goal will be to have a method to be able, given a metric, to build a theory that is suitable for that metric, and then compute the quadrupole formula, in the most general way.

–1–
Introduction

In this chapter, We'll provide an algorithm to solve the inverse problem of gravity for a special class of theories, and build a general background to study Gravitational Waves.

1.1 The Inverse Problem, a review of Suvorov's paper

In this first section, we will review the article called "*A family of solutions to the inverse problem in gravitation: building a theory around a metric*" [6]. The idea presented in this article by Arthur G. Suvorov is to build an algorithm which allows to take a parametric metric and build an algorithm to find a scalar-tensor theory of gravity for which the given metric is a solution

$$G_{\mu\nu} \to \delta S \to \mathcal{L}. \tag{1.1}$$

This is particularly useful when studying a parametric metric built via an experimental observation. This method can identify a class of theories that satisfy this metric.

1.1.1 Action and equations of motion

We will study the following Scalar-Tensor theory

$$A = k\int d^4x\sqrt{-g}f\left(F\left(\phi\right)R + V\left(\phi\right) - \omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha\phi\right) = k\int d^4x\sqrt{-g}f\left(X\right). \quad (1.2)$$

Where $k = \left(16\pi G\right)^{-1}$, G is the Newton's constant, R is the scalar curvature for metric tensor g, and F, V and ω are potential functions of the scalar field ϕ. The $f(R)$ theory of gravity is also recovered for constant scalar fields and vanishing potential V. Here, we consider only vacuum solutions for this theory. However, it's still possible to study a system with matter fields.

The first that we need to do is find the equations of motion. To do this, we have to calculate the variation with respect to the metric $\delta g_{\mu\nu}$ and the scalar field ϕ

- Variation with respect to $g_{\mu\nu}$

$$\begin{aligned} \delta A &= k\int d^4x\left[f(X)\delta\left(\sqrt{-g}\right) + \sqrt{-g}\delta\left(f(X)\right)\right] \\ &= k\int d^4x\left[-\frac{f(X)}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}f'(X)\delta X\right]. \end{aligned} \quad (1.3)$$

Let's focus on the second term of the integration:

$$\begin{aligned} \delta X &= \delta\left[F\left(\phi\right)g^{\mu\nu}R_{\mu\nu} + V\left(\phi\right) - \omega\left(\phi\right)g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi\right] \\ &= F(\phi)R_{\mu\nu}\delta g^{\mu\nu} + F(\phi)\delta R_{\mu\nu}g^{\mu\nu} - \omega(\phi)\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu}. \end{aligned} \quad (1.4)$$

We can rewrite the variation as follows:

$$\begin{aligned} \delta A = k\int d^4x\sqrt{-g}\left[f'(X)F(\phi)R_{\mu\nu} - f'(X)\omega(\phi)\nabla_\mu\phi\nabla_\nu\phi - \frac{f(X)}{2}g_{\mu\nu}\right]\delta g^{\mu\nu} \\ +k\int d^4x\sqrt{-g}f'(X)F(\phi)g^{\mu\nu}\delta R_{\mu\nu}. \end{aligned} \quad (1.5)$$

It can be shown that $g^{\mu\nu}\delta R_{\mu\nu}$ can be written as as:

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\partial_\sigma\left[\delta\Gamma^\sigma_{\mu\nu}\right] - g^{\mu\sigma}\partial_\sigma\left[\delta\Gamma^\nu_{\mu\nu}\right] = \partial_\sigma W^\sigma \to \ldots \to \\ W^\sigma = \partial^\sigma\left[g_{\mu\nu}\delta g^{\mu\nu}\right] - \partial^\mu\left[g_{\mu\nu}\delta g^{\nu\sigma}\right], \end{aligned} \quad (1.6)$$

so the second integral of (1.5) become:

$$k\int d^4x\sqrt{-g}f'(X)F(\phi)\partial_\sigma\left[\partial^\sigma\left(g_{\mu\nu}\delta g^{\mu\nu}\right) - \partial^\mu\left(g_{\mu\nu}\delta g^{\nu\sigma}\right)\right]. \quad (1.7)$$

Making an integration by parts and discarding the total divergences we have

$$k\int d^4x\partial_\sigma\left[\sqrt{-g}f'(X)F(\phi)\right]\left[\partial^\mu\left(g_{\mu\nu}\delta g^{\nu\sigma}\right)-\partial^\sigma\left(g_{\mu\nu}\delta g^{\mu\nu}\right)\right]. \tag{1.8}$$

we will now perform another part of the integration

$$\begin{aligned}&k\int d^4x\partial_\sigma\partial^\sigma\left[\sqrt{-g}f'(X)F(\phi)\right]g_{\mu\nu}\delta g^{\mu\nu}-k\int d^4x\partial_\sigma\partial^\mu\left[\sqrt{-g}f'(X)F(\phi)\right]g_{\mu\nu}\delta g^{\nu\sigma}\\&=k\int d^4x\partial_\sigma\partial^\sigma\left[\sqrt{-g}f'(X)F(\phi)\right]g_{\mu\nu}\delta g^{\mu\nu}-k\int d^4x\partial_\mu\partial_\nu\left[\sqrt{-g}f'(X)F(\phi)\right]\delta g^{\mu\nu}\end{aligned} \tag{1.9}$$

If we put together all the terms of δA we obtain:

$$\begin{aligned}\delta A=k\int d^4x\sqrt{-g}\Big[&f'(X)F(\phi)R_{\mu\nu}-f'(X)\omega(\phi)\nabla_\mu\phi\nabla_\nu\phi-\frac{f(X)}{2}g_{\mu\nu}\\&+g_{\mu\nu}\partial_\sigma\partial^\sigma\left[f'(X)F(\phi)\right]-\partial_\mu\partial_\nu\left[f'(X)F(\phi)\right]\Big]\delta g^{\mu\nu}.\end{aligned} \tag{1.10}$$

The variation with respect to the metric is

$$\begin{aligned}F(\phi)f'(X)R_{\mu\nu}-\frac{f(X)}{2}g_{\mu\nu}+g_{\mu\nu}\Box\left[F(\phi)f'(X)\right]-\nabla_\mu\nabla_\nu\left[F(\phi)f'(X)\right]&\\-\omega(\phi)f'(X)\nabla_\mu\nabla_\nu\phi=0.&\end{aligned} \tag{1.11}$$

- Variation with respect to ϕ
  Similarly, it's possible to calculate the variation with respect to the scalar field

$$\begin{aligned}f'(X)\left[2\omega(\phi)\Box\phi+\frac{d\omega(\phi)}{d\phi}\nabla_\alpha\phi\nabla^\alpha\phi+R\frac{dF(\phi)}{d\phi}+\frac{dV(\phi)}{d\phi}\right]&\\+2\omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha f'(X)=0.&\end{aligned} \tag{1.12}$$

1.1.2 Physical conditions and Brans-Dicke choice

We can impose several conditions on these equations:

1. Demanding that the graviton carries a positive energy
2. Demanding that the kinetic energy of the scalar field is non-negative $\rightarrow$the coefficient of $\Box\phi$ in equation (1.13) must be non-negative

After applying all the conditions, we are left with the following request:

1. $F(\phi)>0$,

2. $2F(\phi)\omega(\phi) + 3\left(\frac{dF(\phi)}{d\phi}\right)^2 \geq 0.$

The **Brans-Dicke** choices, $F(\phi) = \phi$, $\omega(\phi) = \frac{1}{\phi}$ satisfy these conditions:

1. $\phi > 0,$
2. $2 + 3\left(\frac{d\phi}{d\phi}\right)^2 \geq 0.$

1.1.3 How do we build a solution to the inverse problem?

Now that we know what conditions we need to impose on the theory,we can start understanding how to build the solution.
Starting from the two equations of motion, we see that they are both satisfied when the function $f(X)$ has a zero for X_0 that is also a critical point for the function,

$$f'(X_0) = f(X_0) = 0. \tag{1.13}$$

What we need to do at this point is solve the equation $X = X_0$ starting from a specific $g_{\mu\nu}$ metric, and finding an expression of ϕ that counterbalances the curvature, allowing the function f and f' to vanish. Starting from our $f(X)$ in (1.2), if the scalar field counterbalance the Ricci curvature in a precise way, the function $f(X)$ can vanish at a local extremum:

$$X = F(\phi)\, R + V(\phi) - \omega(\phi)\, \nabla_\alpha \phi \nabla^\alpha \phi = X_0. \tag{1.14}$$

1.1.4 Example

To make an example, let's suppose that through astrophysical data we can describe a black hole metric with a generalized Kerr metric:

$$\begin{aligned} ds^2 = {} & \frac{a^2 \sin^2(\theta) - \Delta}{\Sigma} dt^2 - \frac{2a \sin^2(\theta)\,(a^2 + r^2 - \Delta)}{\Sigma} dt d\varphi \\ & + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{(a^2 + r^2)^2 - a^2 \sin^2(\theta)\,\Delta}{\csc(\theta)\,\Sigma} d\varphi^2, \end{aligned} \tag{1.15}$$

with $\Delta = r^2 + 2Mr + a^2 + \epsilon \frac{M^3}{r}$, and $\Sigma = r^2 + a^2 \cos^2\theta$

M and a are the **mass** and **spin** of the BH, while ϵ in an **extra hair**. This metric admits an outer event horizon at the largest positive root of $\Delta = 0$ This metric is a generalization of the Kerr metric and presents some notable properties, like the fact that it's asymptotically flat.

The mixed scalar theory we will study is

$$f(X) = X^{1+\delta}, \tag{1.16}$$

the function in analytical for $\delta \in \mathbb{Z}$.As we said before, we need to find a value of $X_0 : f(X_0) = f'(X_0) = 0$. In the case of the theory we choose, this is true for $X_0 = 0$, if a scalar field ϕ that satisfy this condition exists, then the metric we find is a solution for this $f(X)$ theory:

$$0 = F\left(\phi\right)R + V\left(\phi\right) - \omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha\phi. \tag{1.17}$$

With the **Brans-Dicke** choices, $F(\phi) = \phi$, $\omega(\phi) = \dfrac{1}{\phi}$,$V(\phi) = 0$, the equation becomes

$$0 = R\phi - \frac{1}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi. \tag{1.18}$$

Let's assume that the scalar field ϕ is time and azimuth independent

$$\phi = \phi(r, \theta). \tag{1.19}$$

If we want to calculate $\nabla_\alpha\phi\nabla^\alpha\phi$, we will only need to consider the metric elements g_{rr} and $g_{\theta\theta}$

$$\begin{aligned}
\nabla_\alpha\phi\nabla^\alpha\phi &= g^{\mu\alpha}\nabla_\mu\phi\nabla_\alpha\phi \\
&= g^{tt}(\partial_t\phi)^2 + g^{rr}(\partial_r\phi)^2 + g^{\theta\theta}(\partial_\theta\phi)^2 + g^{\varphi\varphi}(\partial_\varphi\phi)^2 + 2g^{t\varphi}\partial_\varphi\phi\partial_t\phi \\
&= g^{rr}(\partial_r\phi)^2 + g^{\theta\theta}(\partial_\theta\phi)^2 = \frac{\Delta}{\Sigma}(\partial_r\phi)^2 + \frac{1}{\Sigma}(\partial_\theta\phi)^2.
\end{aligned} \tag{1.20}$$

The scalar curvature is

$$R = -\frac{2M^3\epsilon}{r^3\Sigma}, \tag{1.21}$$

so equation 1.17 become:

$$0 = -\frac{2M^3\epsilon}{r^3\Sigma}\phi - \frac{1}{\phi}\left(\frac{\Delta}{\Sigma}(\partial_r\phi)^2 + \frac{1}{\Sigma}(\partial_\theta\phi)^2\right). \tag{1.22}$$

Let's try solving the equation separating the variables

$$\begin{aligned}
&\frac{2M^3\epsilon}{r^3\Sigma} \quad + \frac{1}{R^2(r)\Theta^2(\theta)}\left[\frac{\Delta}{\Sigma}\Theta^2(\theta)\left(\partial_r R\right)^2 + \frac{1}{\Sigma}R(r)^2\left(\partial_\theta\Theta\right)^2\right] = 0 \rightarrow \\
&\rightarrow \quad \frac{2M^3\epsilon}{r^3} + \Delta\frac{\left(\partial_r R\right)^2}{R^2(r)} = -\frac{\left(\partial_\theta\Theta\right)^2}{\Theta^2(\theta)},
\end{aligned} \tag{1.23}$$

so we have

$$\begin{cases}
\dfrac{2M^3\epsilon}{r^3} + \Delta\dfrac{\left(\partial_r R\right)^2}{R^2(r)} = c^2, \\
\\
\left(\partial_\theta\Theta\right)^2 = -c^2\Theta^2(\theta).
\end{cases} \tag{1.24}$$

We can solve the second one analytically

$$\Theta(\theta) = A\left[e^{ic\theta} + e^{-ic\theta}\right] = 2A\cos[c\theta]. \tag{1.25}$$

On this solution, we can impose the following boundary condition

$$\Theta(0) = \Theta(\pi), \tag{1.26}$$

this leads to

$$2A\cos(c*0) = 2A\cos(c\pi) \to c = 0. \tag{1.27}$$

We have demonstrated that $\Theta(\theta) = 0$, and that scalar field ϕ is independent from the variable θ.

We are left with a function in the sole variable r:

$$0 = 2M^3\epsilon\phi(r)^2 + r^3\Delta(r)\left(\frac{d\phi(r)}{dr}\right)^2. \tag{1.28}$$

We can now calculate a numerical solution for this equation 1.15:

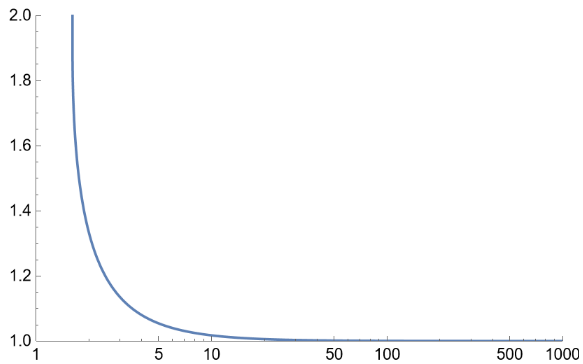


Figure 1.1: Radial ϕ solution for $M = 1$, $a = 0.9$, $\epsilon = -0.2$

In all cases considered, the scalar field ϕ is short ranged, well behaved, and asymptotes to the Newtonian value $\phi_{\text{inf}} = 1$, as expected of physical black hole geometries.

1.1.5 Limitations of the approach

This approach has several limitations. We will go through them, making a distinction between $f(R)$ theories and scalar tensor $f(X)$ theories.

In the case of $f(R)$ theories, it's easier to verify this. For any $f(R)$ theory, the equation of motion is:

$$\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}f(R) + R_{\mu\nu}f'(R) + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R). \tag{1.29}$$

In this case, the solution is trivial. If we can find an $R_0 : f(R_0) = f'(R_0)$ the equation of motion is automatically satisfied.
But this is only possible for theories like

$$f(R) = (R - R_0)^2, \tag{1.30}$$

or

$$f(R) = R + \alpha R^2 + \frac{1}{3}\alpha^2 R^3, \tag{1.31}$$

while this does not work on theories like

$$f(R) = R + \alpha R^2, \tag{1.32}$$

because in this last case, it's impossible to find a common zero for the function f and its first derivative. For Scalar-Tensor theories, the situation is pretty similar, for example, we can't work on theories like

$$f(X) = X, \tag{1.33}$$

this can be easily proved substituting $f(X) = X$ and $f'(X) = 1$ in the equations of motion (1.12),(1.13) calculated before.

When doing this, it's impossible to find a specific X_0 that solves both equations. Like in the case of $f(R)$ theories, this does not work for all the theories that do not allow X_0 to be both a zero and a local extremum for $f(X)$, like $f(X) = X + \alpha X^2$.

1.2 Gravitational Waves in General Relativity

In this section, we will write a short summary of the theory of Gravitational Waves in General Relativity[23, 24, 25]. This will be useful in studying gravitational waves in modified theories. The gravitational interaction can often be considered weak when confronted to others fundamental interactions. So it can be useful to develop a perturbative theory to describe the metric $g_{\mu\nu}$ as a perturbation of the Minkowski metric $\eta_{\mu\nu}$.

$$g_{\mu\nu} \sim \eta_{\mu\nu} + h_{\mu\nu}, \tag{1.34}$$

where

$$\|h_{\mu\nu}\| \ll 1. \tag{1.35}$$

This condition requires both the gravitational field to be weak and the coordinate system to be approximately Cartesian.
As we will see later, this approximation has the advantage of linearizing gravity, since all the quadratic terms of the Scalar Tensor are negligible.

Linearized gravity can be interpreted as a field theory where the tensor field $h_{\mu\nu}$ "lives" in the Minkowski space.

This theory is Lorentz invariant, the tensor $\eta_{\mu\nu}$ is invariant, while $h_{\mu\nu}$ trans"forms as:

$$h_{\mu'\nu'} = \Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'}h_{\mu\nu}. \tag{1.36}$$

Now, we can derive all the elements needed to describe a theory of gravitation:

$$\Gamma^{\rho}_{\mu\nu} \sim \frac{1}{2}\eta^{\rho\lambda}\left[\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right], \tag{1.37}$$

$$R^{\mu}_{\nu\rho\sigma} \sim \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\rho\nu} = \frac{1}{2}\eta^{\mu\lambda}\left[\partial_\rho\partial_\nu h_{\lambda\sigma} - \partial_\rho\partial_\lambda h_{\sigma\nu} - \partial_\sigma\partial_\nu h_{\lambda\rho} + \partial_\sigma\partial_\lambda h_{\rho\nu}\right], \tag{1.38}$$

$$R_{\nu\sigma} \sim \frac{1}{2}\eta^{\mu\lambda}\left[\partial_\mu\partial_\nu h_{\lambda\sigma} - \partial_\mu\partial_\lambda h_{\sigma\nu} - \partial_\sigma\partial_\nu h_{\lambda\mu} + \partial_\sigma\partial_\lambda h_{\mu\nu}\right], \tag{1.39}$$

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h. \tag{1.40}$$

With these elements, we can build the Einstein tensor $G_{\mu\nu}$ and write the Einstein equation:

$$\frac{1}{2}\left[\partial_\mu\partial_\nu h^{\mu}_{\sigma} + \partial^\mu\partial_\sigma h_{\mu\nu} - \Box h_{\sigma\nu} - \partial_\sigma\partial_\nu h - \eta_{\nu\sigma}\partial_\mu\partial^\alpha h^{\mu}_{\alpha} + \eta_{\nu\sigma}\Box h\right] = 8\pi G T_{\nu\sigma}. \tag{1.41}$$

It is possible to clean up this expression with a few changes rather than working with $h_{\mu\nu}$ we can use the trace reversed perturbation [26]

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{1.42}$$

Replacing $h_{\mu\nu}$ with $\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h$.
With this substitution, the Einstein equation becomes:

$$\frac{1}{2}\left[\partial_\rho\partial_\nu \bar{h}^{\rho}_{\mu} + \partial^\rho\partial_\mu \bar{h}_{\rho\nu} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial^\sigma \bar{h}^{\rho}_{\sigma}\right] = 8\pi G T_{\mu\nu}, \tag{1.43}$$

this expression can be further simplified by choosing an appropriate coordinate system, or Gauge.

1.2.1 Linearized gravity Gauge

Einstein's equations are solved by a metric tensor defined up to diffeomorphism that specifies its coordinates. For this reason, there is not a single decomposition of the metric tensor. We will start by considering an infinitesimal coordinate change:

$$x'^{\mu} = x^{\mu} - \xi^{\mu}. \tag{1.44}$$

If we apply the transformation to the metric tensor with (1.36) we obtain:

$$\begin{aligned} g_{\alpha'\beta'}(x') = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}(x) &= \left(\delta^\mu_\alpha + \partial_\alpha \xi^\mu\right)\left(\delta^\nu_\beta + \partial_\beta \xi^\nu\right) g_{\mu\nu}(x) \\ &= g_{\alpha\beta}(x) + \partial_\alpha \xi^\mu g_{\mu\beta}(x) + \partial_\beta \xi^\nu g_{\alpha\nu}(x). \end{aligned} \tag{1.45}$$

We can expand

$$g_{\mu\nu}(x) = g_{\mu\nu}(x' + \xi) \sim g_{\mu\nu}(x') + \partial_\sigma g_{\mu\nu}(x')\xi^\sigma, \tag{1.46}$$

$$g_{\alpha'\beta'}(x') \sim g_{\alpha\beta}(x') + \partial_\sigma g_{\mu\nu}(x')\xi^\sigma + \partial_\alpha \xi^\mu g_{\mu\beta}(x') + \partial_\beta \xi^\nu g_{\alpha\nu}(x'). \tag{1.47}$$

At this point, we can explicit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and then disregard the the terms like $\xi^\mu h_{\mu\nu}$, since both $h_{\mu\nu}$ and ξ are small:

$$h_{\alpha'\beta'} = h_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha. \tag{1.48}$$

This is similar to the Gauge transformation of the four-potential, that leaves unaltered the physical quantities. In the same way, (1.48) leaves the Riemann tensor unaltered.
We will apply the following Gauge condition called Lorentz Gauge to the trace reversed perturbation:

$$\partial^\mu \bar{h}_{\mu\nu} = 0. \tag{1.49}$$

We want to verify that this Gauge does not alter the physical observables of the theory. To verify this, we start with writing (1.48) in terms of $\bar{h}_{\mu\nu}$:

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu'\nu'} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho, \tag{1.50}$$

applying the condition (1.49) we obtain:

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^{\mu'} \bar{h}_{\mu'\nu'} = \partial^\mu \bar{h}_{\mu\nu} + \Box \xi_\nu. \tag{1.51}$$

Therefore, if the initial configuration $h_{\mu\nu}$is such that $\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x)$, to obtain $\partial^{\nu'} \bar{h}_{\mu'\nu'} = 0$, we must choose $\xi_\mu(x)$ so that

$$\Box \xi_\mu = f_\mu(x). \tag{1.52}$$

When imposing the condition (1.49) to the Einstein tensor, we obtain:

$$G_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu}. \tag{1.53}$$

So the linearized Einstein equation is:

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{1.54}$$

1.2.2 The transverse-traceless gauge

To study the propagation of gravitational waves, we are interested in studying the Einstein equation outside the source, where $T_{\mu\nu} = 0$:

$$\Box \bar{h}_{\mu\nu} = 0 \tag{1.55}$$

For such space-times, one can, along with choosing the Lorentz gauge, further specialize the gauge to make the metric perturbation purely spatial:

$$h_{tt} = h_{ti} = 0 \tag{1.56}$$

and traceless

$$h_i^i = 0. \tag{1.57}$$

From the traceless condition, we obtain

$$\bar{h}_{\mu\nu} = h_{\mu\nu}$$

and the Lorentz condition becomes:

$$\partial^i h_{ij} = 0 \tag{1.59}$$

This is called the transverse traceless gauge (TT gauge). A metric The perturbation put into TT gauge will be written $h_{\mu\nu}^{TT}$.
It's important to note that the TT gauge cannot be chosen inside the source, since in this case $\Box \bar{h}_{\mu\nu} \neq 0$. Inside the source, once gaugeve chosen the Lorentz gauge, we still have the freedom to perform a transformation with $\Box \xi_\mu = 0$. Equation (1.55) has plane waves solution:

$$h_{ij}^{TT}(x) = C_{ij}(k) e^{ikx}, \tag{1.60}$$

with $k^\mu = \left(\omega, \vec{k}\right)$.

The tensor C_{ij} is called the polarization tensor and, given the condition imposed on h_{ij}^{TT}, it has to be symmetric and traceless. Given a single plane wave with a given vector $\vec{k}$, we define $\hat{n} = \frac{\vec{k}}{\left|\vec{k}\right|}$ we choose $\hat{n}$ along the z axis, and imposing that h_{ij}^{TT} be symmetric and traceless, we have:

$$h_{ij}^{TT} = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega\left(t - z\right)\right]. \tag{1.61}$$

The elements h_+ and h_x are called the amplitudes of the "plus" and "cross" polarization of the wave.

Given a plane wave solution $h_{\mu\nu}(x)$ propagating in the direction $\hat{n}$, outside the sources, already in the Lorentz gauge but not in the TT gauge, we can find the form of the wave in the TT gauge as follows. First, we introduce the tensor

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j. \tag{1.62}$$

P_{ij} is transverse, a projector and its trace is $P_{ii} = 2$. With this tensor, we can build

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \tag{1.63}$$

With this tensor, it is possible to show that, given a plane wave $h_{\mu\nu}$ in the Lorentz gauge, it is possible to obtain the GW in the TT gauge with

$$h_{ij}^{TT} = \Lambda_{ij,kl}h_{kl} \tag{1.64}$$

1.2.3 Deriving the quadrupole formula

We start with the linearized Einstein equation with a source matter:

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{1.65}$$

This equation can be solved by using a Green's function:

$$\Box G(t,x,t',x') = \delta^{(3)}(x-x')\delta(t-t'), \tag{1.66}$$

$$\bar{h}_{\mu\nu} = -16\pi\int dt'd^3x'G(t,x,t',x')T_{\mu\nu}. \tag{1.67}$$

The Green function associated with the wave operator is:

$$G(t,x,t',x') = -\frac{\delta\left(t' - [t-|x-x'|]\right)}{|x-x'|}. \tag{1.68}$$

The quantity $t-|x-x'|$ is the retarded time, it take into consideration that the metric $\bar{h}_{\mu\nu}$ that we observe at (t,x) is generator by the source $T_{\mu\nu}$ that is located at a distance $|x-x'|$ and in a precedent instant of time $t-|x-x'|$.
We can now compute the convolution between the Green function and the source:

$$\bar{h}_{\mu\nu}(t,x) = 4\int d^3x'\frac{T_{\mu\nu}\left(t-|x-x'|,x'\right)}{|x-x'|}. \tag{1.69}$$

We can start considering only the spacial element of the tensor:

$$\bar{h}_{ij}(t,x) = 4\int d^3x'\frac{T_{ij}\left(t-|x-x'|,x'\right)}{|x-x'|}, \tag{1.70}$$

we will now evaluate this integral at large distance from the source $|x| \gg |x'| \to |x| = r$ so we can write:

$$T_{ij}\left(t - |x - x'|, x'\right) \approx T_{ij}\left(t - r, x'\right), \tag{1.71}$$

$$\bar{h}_{ij}(t, x) = \frac{4}{r} \int d^3x' T_{ij}\left(t - r, x'\right). \tag{1.72}$$

We can split the stress-energy tensor using the property $\partial_\mu T^{\mu\nu} = 0$. We can break up this condition into time and space components:

$$\begin{aligned} \partial_t T^{tt} + \partial_i T^{it} &= 0, \\ \partial_t T^{it} + \partial_j T^{ij} &= 0. \end{aligned} \tag{1.73}$$

We can now derive the first equation in t equation in i and subtract them:

$$\partial_t^2 T^{tt} + \partial_t \partial_i T^{it} - \partial_i \partial_t T^{it} - \partial_i \partial_j T^{ij} = 0. \tag{1.74}$$

From this, it follows that

$$\partial_t^2 T^{tt} = \partial_i \partial_j T^{ij}, \tag{1.75}$$

we can now multiply both sides of the equation by $x^k x^l$ and manipulate them:

$$\left[\partial_t^2 T^{tt}\right] x^k x^l = \partial_t^2 \left[T^{tt} x^k x^l\right]. \tag{1.76}$$

For the right side of the equation, we can start with:

$$\begin{aligned} \partial_i \partial_j \left[T^{ij} x^k x^l\right] &= \partial_i \left[x^k x^l \partial_j T^{ij} + T^{ij} \partial_j \left(x^k x^l\right)\right] \\ &= \partial_i \left(x^k x^l\right) \partial_j T^{ij} + x^k x^l \partial_i \partial_j T^{ij} + \partial_i T^{ij} \partial_j \left(x^k x^l\right) + T^{ij} \partial_i \partial_j \left(x^k x^l\right) \\ &= 2\partial_i \left(x^k x^l\right) \partial_j T^{ij} + x^k x^l \partial_i \partial_j T^{ij} + T^{ij} \partial_i \partial_j \left(x^k x^l\right) \implies \\ \implies& \; x^k x^l \partial_i \partial_j T^{ij} = \partial_i \partial_j \left[T^{ij} x^k x^l\right] - 2\partial_i \left(x^k x^l\right) \partial_j T^{ij} - T^{ij} \partial_i \partial_j \left(x^k x^l\right). \end{aligned} \tag{1.77}$$

Let's focus on the last two terms

$$\begin{aligned} 2\partial_i \left(x^k x^l\right) \partial_j T^{ij} &= 2\left[x^k \delta_i^l + x^l \delta_i^k\right] \partial_j T^{ij} + T^{ij}\left[\delta_i^k \delta_j^l + \delta_j^k \delta_i^l\right] \\ &= 2\left[x^k \partial_j T^{lj} + x^l \partial_j T^{kj}\right] + 2T^{kl} \\ &= 2\left[\partial_j \left(x^k T^{lj}\right) - \delta_j^k T^{lj} + \partial_j \left(x^l T^{kj}\right) - \delta_j^l T^{kj}\right] \\ &= 2\partial_j \left(x^k T^{lj} + x^l T^{kj}\right) - 4T^{kl}, \end{aligned} \tag{1.78}$$

$$T^{ij} \partial_i \partial_j \left(x^k x^l\right) = T^{ij}\left[\delta_i^k \delta_j^l + \delta_j^k \delta_i^l\right] = T^{kl} + T^{lk} = 2T^{kl}. \tag{1.79}$$

Putting everything together, we obtain

$$x^k x^l \partial_i \partial_j T^{ij} = \partial_i \partial_j \left[T^{ij} x^k x^l\right] - 2\partial_j \left(x^k T^{lj} + x^l T^{kj}\right) + 2T^{kl} \tag{1.80}$$

We can substitute everything in the equation 1.75:

$$\partial_t^2 \left[T^{tt}x^k x^l\right] = \partial_i\partial_j \left[T^{ij}x^k x^l\right] - 2\partial_i \left[T^{ik}x^l + T^{il}x^k\right] + 2T^{kl}. \tag{1.81}$$

So equation (1.72) becomes:

$$\begin{aligned} &\frac{4}{r}\int d^3x' T_{kl}\left(t-r,x'\right) \\ &= \frac{4}{r}\int d^3x' \left[\frac{1}{2}\partial_t^2\left(T^{tt}x^k x^l\right) + \partial_i\left(T^{ik}x^l + T^{il}x^k\right) - \frac{1}{2}\partial_i\partial_j\left(T^{ij}x^k x^l\right)\right] \\ &= \frac{2}{r}\int d^3x' \partial_t^2\left(T^{tt}x^k x^l\right). \end{aligned} \tag{1.82}$$

We used that the second and third terms under the integral are divergences. Using Gauss's Theorem, they can be transformed into surface integrals. Taking the integral outside the source, their contribution is 0.

$$\frac{2}{r}\int d^3x' \partial_t^2\left(T^{tt}x^k x^l\right) = \frac{2}{r}\partial_t^2 \int d^3x' \rho x^k x^l. \tag{1.83}$$

We define the integral:

$$I_{ij}(t) = \int d^3x' \rho x^k x^l, \tag{1.84}$$

so equation (1.72) becomes:

$$\bar{h}_{ij}(t,x) = \frac{2}{r}\frac{d^2 I_{ij}}{dt^2}. \tag{1.85}$$

From equation (1.64) we obtain

$$h_{ij}^{TT}(t,x) = \Lambda_{ij,kl}\bar{h}_{ij}(t,x) = \frac{2}{r}\Lambda_{ij,kl}\frac{d^2 I_{ij}}{dt^2}. \tag{1.86}$$

1.2.4 The energy of Gravitational Waves

The next step is to understand the energy carried by gravitational waves [27].
To understand whether GWs curve the background space-time, we need to change the framework in which we study them. Until now, we have studied the metric tensor $h_{\mu\nu}$ as a tensor that lives in the flat Minkowski space $\eta_{\mu\nu}$ and it cause the space-time curvature. However, we cannot continue with the same background, because, otherwise, we exclude from the beginning the possibility that GWs curve the background space-time. For this reason, we must allow the background space-time to be dynamical:

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{1.87}$$

where $|h_{\mu\nu}| \ll 1$.
The next problem that arises is to decide which part of $g_{\mu\nu}$ is the background, which is the fluctuation. As we will see, this analysis will allow us to understand some critical properties of GWs, such as their energy-momentum tensor.
A natural splitting between background and GWs arises when there is a clear separation of scales. For example, the separation occurs if $\bar{g}_{\mu\nu}$ has frequencies up to f_B while $h_{\mu\nu}$ is peaked around a frequency f such that:

$$f \gg f_B. \tag{1.88}$$

In this case $h_{\mu\nu}$ is a high frequency perturbation of a slowly varying background.

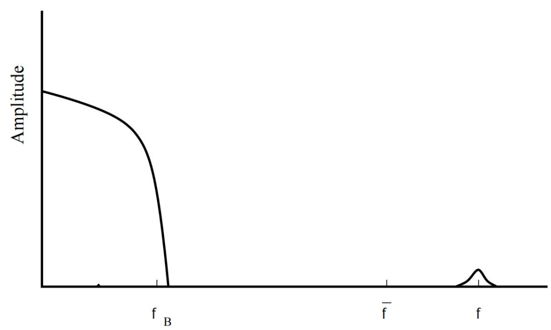


Figure 1.2: A situation that allows us to separate the metric into a low-frequency background and a small high-frequency perturbation.

To study how GWs curve the background, we start by expanding the Einstein equations around the background metric $\bar{g}_{\mu\nu}$. In this expansion, we have two small parameters: one is the amplitude h, and the second is $\frac{f_B}{f}$.
It is convenient to cast the Einstein equations in the form

$$R_{\mu\nu} = 8\pi \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right], \tag{1.89}$$

now we can expand the Ricci tensor to $O\left(h^2\right)$,

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ..., \tag{1.90}$$

where $\bar{R}_{\mu\nu}$ depends only on $\bar{g}_{\mu\nu}$, $R^{(1)}_{\mu\nu}$ is linear in $h_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$. The quantity $\bar{R}_{\mu\nu}$ contains only low frequency modes. $R^{(1)}_{\mu\nu}$ contains only high frequency modes, while $R^{(2)}_{\mu\nu}$ contains both high and low frequencies mode. Therefore, we can split the Einstein equations into two equations for high and low frequencies:

$$\bar{R}_{\mu\nu} = - \left[R^{(2)}_{\mu\nu} \right]^{Low} + 8\pi \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]^{Low} \tag{1.91}$$

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{High} + 8\pi\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^{High} \tag{1.92}$$

The explicit expression of $R^{(1)}_{\mu\nu}$ is:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\bar{D}^{\alpha}\bar{D}_{\mu}h_{\nu\alpha} + \bar{D}^{\alpha}\bar{D}_{\nu}h_{\nu\alpha} - \bar{D}^{\alpha}\bar{D}_{\alpha}h_{\mu\nu} - \bar{D}_{\nu}\bar{D}_{\mu}h\right], \tag{1.93}$$

while $R^{(2)}_{\mu\nu}$ is:

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Bigg[&\frac{1}{2}\bar{D}_{\mu}h_{\rho\alpha}\bar{D}_{\nu}h_{\sigma\beta} + \left(\bar{D}_{\rho}h_{\nu\alpha}\right)\left(\bar{D}_{\sigma}h_{\mu\beta} - \bar{D}_{\beta}h_{\mu\sigma}\right) \\ &+h_{\rho\alpha}\left(\bar{D}_{\nu}\bar{D}_{\mu}h_{\sigma\beta} + \bar{D}_{\beta}\bar{D}_{\sigma}h_{\mu\nu} - \bar{D}_{\beta}\bar{D}_{\nu}h_{\mu\sigma} - \bar{D}_{\beta}\bar{D}_{\mu}h_{\nu\sigma}\right) \\ &+\left(\frac{1}{2}\bar{D}_{\alpha}h_{\rho\sigma} - \bar{D}_{\rho}h_{\alpha\sigma}\right)\left(\bar{D}_{\nu}h_{\mu\beta} + \bar{D}_{\mu}h_{\nu\beta} - \bar{D}_{\beta}h_{\mu\nu}\right)\Bigg]. \end{aligned} \tag{1.94}$$

To continue this analysis, we will go from using the scale of frequencies to the one of amplitudes. For the background, the length scale is L_B, while for the GWs it's λ, where $L_B \gg \lambda$.
We introduce a scale $\bar{l}$ that is an intermediate scale:

$$\lambda \ll \bar{l} \ll L_B. \tag{1.95}$$

We can average the elements of $\bar{R}_{\mu\nu}$ over a spatial volume with side $\bar{l}$. In this way, modes with wavelength L_B remain unaffected because they are constant over $\bar{l}$, while modes with the wavelength λ are oscillating very fast and their average is 0. We can, therefore, write

$$\bar{R}_{\mu\nu} = -\left\langle R^{(2)}_{\mu\nu}\right\rangle + 8\pi\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle. \tag{1.96}$$

Now, we define an effective energy-momentum tensor

$$\left\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right\rangle = \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}, \tag{1.97}$$

By definition, $\bar{T}_{\mu\nu}$ is purely low frequency. We also define the quantity $t_{\mu\nu}$as

$$t_{\mu\nu} = -\frac{1}{8\pi}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle, \tag{1.98}$$

where $R^{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$, and we define the trace as

$$\begin{aligned} t &= \bar{g}^{\mu\nu}t_{\mu\nu} = -\frac{1}{8\pi}\bar{g}^{\mu\nu}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle \\ &= -\frac{1}{8\pi}\left\langle \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{g}_{\mu\nu}R^{(2)}\right\rangle = \frac{1}{8\pi}\left\langle R^{(2)}\right\rangle, \end{aligned} \tag{1.99}$$

we used the property of $\bar{g}^{\mu\nu}$ that $\bar{g}^{\mu\nu}\left\langle R^{(2)}_{\mu\nu}\right\rangle = \left\langle \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}\right\rangle$ because $\bar{g}^{\mu\nu}$ is a purely low frequency quantity. We can now put everything together in (1.98):

$$t_{\mu\nu} = -\frac{1}{8\pi}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle = -\frac{1}{8\pi}\left\langle R^{(2)}_{\mu\nu}\right\rangle + \frac{1}{2}\bar{g}_{\mu\nu}t, \tag{1.100}$$

From this, we obtain the following:

$$-\left\langle R^{(2)}_{\mu\nu}\right\rangle = 8\pi\left[t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right]. \tag{1.101}$$

Now, we can rewrite (1.96) as

$$\bar{R}_{\mu\nu} = 8\pi\left[t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right] + 8\pi\left[T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}T\right], \tag{1.102}$$

or equivalently,

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = 8\pi\left[\bar{T}_{\mu\nu} + t_{\mu\nu}\right]. \tag{1.103}$$

1.2.5 The energy-momentum tensor of Gravitational Waves

We want to compute the tensor $t_{\mu\nu}$ using equation (1.100). To do this, we also need the expression of $R^{(2)}_{\mu\nu}$ (1.94). Since we are interested in the energy and momentum carried by the GWs at large distances from the source, we can approximate the background as flat so that we can substitute all covariant derivatives with simple derivatives in the expression of $R^{(2)}_{\mu\nu}$. This expression can be drastically simplified with some considerations. The right hand of equation (1.103) is the Einstein tensor of the background metric. This is a coordinate dependent quantity that is composed of both physical degrees of freedom and coordinate ones. We can get rid of the coordinate degree of freedom with the TT Gauge condition, and this implies that $\partial^\mu h_{\mu\nu} = 0$. Imposing this condition in $R^{(2)}_{\mu\nu}$ we have:

$$\begin{aligned}R^{(2)}_{\mu\nu} = \frac{1}{2}\Big[&\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu}\\ &+h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\alpha\nu} + \partial^\beta h^\alpha_\nu\partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu\partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu}\\ &+\cancel{\partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu}} - \cancel{\partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu}} - \cancel{\frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu}} + \cancel{\frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu}}\\ &+\cancel{\frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}}\Big].\end{aligned} \tag{1.104}$$

Some element have been cancelled using the two conditions $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$. From equation (1.98), we see that we need the average $\left\langle R^{(2)}_{\mu\nu}\right\rangle$. While calculating the average, we can integrate by part and discard the boundary terms. In this way, all the remaining elements of $R^{(2)}_{\mu\nu}$, except the first, depends on either $\partial^\mu h_{\mu\nu}$, h or $\Box h_{\mu\nu}$, that are all zero in the TT gauge. So, we are left with

$$\left\langle R^{(2)}_{\mu\nu} \right\rangle = -\frac{1}{4} \left\langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \right\rangle , \tag{1.105}$$

while $\left\langle R^{(2)} \right\rangle$ vanishes upon integration by parts. So we finally obtain an expression for $t_{\mu\nu}$

$$t_{\mu\nu} = \frac{1}{32} \left\langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \right\rangle . \tag{1.106}$$

Where the tensor $h_{\mu\nu}$ is the Transverse traceless tensor $h^{TT}_{\mu\nu}$. The element $t^{00}$is:

$$t^{00} = \frac{1}{32} \left\langle \partial_0 h^{TT}_{ij} \partial_0 h^{TT}_{ij} \right\rangle = \frac{1}{32} \left\langle \dot{h}^{TT}_{ij} \dot{h}^{TT}_{ij} \right\rangle = \frac{1}{16} \left\langle \dot{h}^2_+ + \dot{h}^2_x \right\rangle . \tag{1.107}$$

1.2.6 Energy Flux

Having calculated the Energy-Momentum tensor carried by Gravitational Waves, we can now compute the energy flux. Starting from the conservation of the Energy-Momentum tensor $\partial_\mu t^{\mu\nu} = 0$, we can write

$$\int_V d^3x \left[\partial_0 t^{00} + \partial_i t^{0i} \right] = 0. \tag{1.108}$$

The GW energy inside the volume V is:

$$E_V = \int_V d^{3x} t^{00}, \tag{1.109}$$

we can calculate the time derivative

$$\frac{dE_V}{dt} = \partial_0 \int_V d^3x t^{00} = \int_V d^3x \partial_0 t^{00} \tag{1.110}$$

From equation (1.108) we can see that $\int_V d^3x \partial_0 t^{00} = - \int_V d^3x \partial_i t^{0i}$, so we have

$$\frac{dE_V}{dt} = - \int_V d^3x \partial_i t^{0i} = - \int_S dA n_i t^{0i}. \tag{1.111}$$

Let's consider a spherical surface at a large distance r from the source. Its surface element is $dA = r^2 d\Omega$, the normal $\hat{n}$is the radial direction $\hat{r}$ and the element t^{0i} becomes

$$t^{0i} = t^{0r} = \frac{1}{32} \left\langle \partial^0 h^{TT}_{ij} \partial^r h^{TT}_{ij} \right\rangle . \tag{1.112}$$

$$\frac{dE_V}{dt} = - \int d\Omega r^2 t^{0r}. \tag{1.113}$$

A GW propagating radially at large distances from the source has the form

$$h_{ij}^{TT}(t,r) = \frac{1}{r} f_{ij}\left(t-r\right), \tag{1.114}$$

where $(t-r)$ is the retarded time t_{ret}. Therefore

$$\partial_r h_{ij}^{TT}\left(t,r\right) = -\frac{1}{r^2} f_{ij}\left(t-r\right) + \frac{1}{r}\partial_r f_{ij}\left(t-r\right). \tag{1.115}$$

Since the function f depends on $t-r$, we can write

$$\partial_r f_{ij}\left(t-r\right) = -\partial_t f_{ij}\left(t-r\right), \tag{1.116}$$

$$\begin{aligned}\partial_r h_{ij}^{TT}\left(t,r\right) &= -\frac{1}{r^2} f_{ij}\left(t-r\right) - \frac{1}{r}\partial_t f_{ij}\left(t-r\right) = -\frac{1}{r^2} f_{ij}\left(t-r\right) - \partial_t \frac{1}{r} f_{ij}\left(t-r\right)\\ &= -\frac{1}{r^2} f_{ij}\left(t-r\right) - \partial_t h_{ij}^{TT}\left(t,r\right) \implies \end{aligned} \tag{1.117}$$

$$\implies \partial_r h_{ij}^{TT}\left(t,r\right) = -\partial_t h_{ij}^{TT}\left(t,r\right) + O\left(\frac{1}{r^2}\right) = \partial^t h_{ij}^{TT}\left(t,r\right) + O\left(\frac{1}{r^2}\right). \tag{1.118}$$

At large distances, $\partial_r h_{ij}^{TT}\left(t,r\right) = \partial^t h_{ij}^{TT}\left(t,r\right)$, so, from equation (1.112) we see that $t^{0r} = t^{00}$.

So, we can write the energy inside a volume as

$$\frac{dE_V}{dt} = \int_S dA t^{00} = \frac{r^2}{32}\int d\Omega \left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right\rangle = \frac{r^2}{16}\int d\Omega \left\langle \dot{h}_+^2 + \dot{h}_x^2\right\rangle. \tag{1.119}$$

We can also compute the total energy flow through dA between $t=-\infty$ and $t=\infty$

$$\frac{dE_V}{dA} = \frac{1}{16}\int dt \left\langle \dot{h}_+^2 + \dot{h}_x^2\right\rangle. \tag{1.120}$$

1.2.7 Explicit expression of the matrix elements

To compute the energy flux, we need to find the expressions of $\dot{h}_+$ and $\dot{h}_x$. To do this, we start with equation (1.86)

$$h_{ij}^{TT}(t,x) = \frac{2}{r}\left[P_{ik}(\hat{n})P_{ij}(\hat{n}) - \frac{1}{2}P_{kl}(\hat{n})P_{ij}(\hat{n})\right]\frac{d^2 I_{ij}}{dt^2}, \tag{1.121}$$

To continue, we need to understand how the tensor (1.63) works. When the direction of propagation of the GWs is equal to $\hat{z}$, the tensor $P_{ij} = \delta_{ij} - n_i n_j$ becomes the projector on the (x, y) plane.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.122}$$

On an arbitrary matrix A_{kl} we have

$$\Lambda_{ij,kl} A_{kl} = \left[P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right] A_{kl} = (PAP)_{ij} - \frac{1}{2} P_{ij} Tr\,(PA). \tag{1.123}$$

Where

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.124}$$

while $Tr\,(PA) = A_{11} + A_{22}$. Therefore

$$\begin{aligned} \Lambda_{ij,kl} A_{kl} &= \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} - \frac{A_{11} + A_{22}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \\ &= \begin{pmatrix} (A_{11} - A_{22})/2 & A_{12} & 0 \\ A_{21} & -(A_{11} - A_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}. \end{aligned} \tag{1.125}$$

We can apply this to the tensor $\dfrac{d^2 I}{dt^2}$,

$$\Lambda_{ij,kl} \ddot{I}_{kl} = \begin{pmatrix} \left(\ddot{I}_{11} - \ddot{I}_{22}\right)/2 & \ddot{I}_{12} & 0 \\ \ddot{I}_{21} & -\left(\ddot{I}_{11} - \ddot{I}_{22}\right)/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}. \tag{1.126}$$

We can use this expression, along with (1.86), to obtain an expression for $h_+$and h_x,

$$\begin{cases} h_+ = \dfrac{1}{r} \left(\ddot{I}_{11} - \ddot{I}_{22}\right) \\ \\ h_x = \dfrac{2}{r} \ddot{I}_{21} \end{cases} \tag{1.127}$$

These expressions are valid only for a GW propagating in the $\hat{z}$ direction. To obtain a formula for a Wave that propagates in a general $\hat{n}$ direction, we can start defining a second orthogonal coordinate system $(\hat{u}, \hat{v}, \hat{n})$

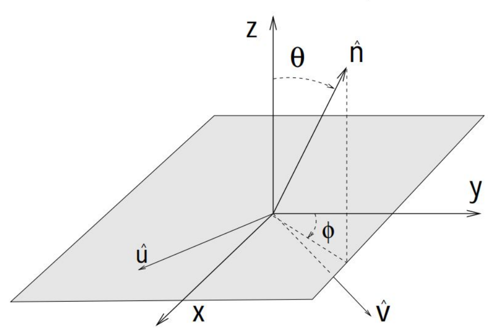


Figure 1.3: Relation between the $(\hat{x}, \hat{y}, \hat{z})$ system and the $(\hat{u}, \hat{v}, \hat{n})$

In the system $(\hat{u}, \hat{v}, \hat{n})$, the elements $h_+$and h_x are

$$\begin{cases} h_+ = \frac{1}{r}\left(\ddot{I}'_{11} - \ddot{I}'_{22}\right) \\ h_x = \frac{2}{r}\ddot{I}'_{21} \end{cases} \tag{1.128}$$

The vector $\hat{n}$, in the $(\hat{x}, \hat{y}, \hat{z})$ frame has coordinates

$$\hat{n} = \left(\sin\left(\theta\right)\sin\left(\phi\right), \sin\left(\theta\right)\cos\left(\phi\right), \cos\left(\theta\right)\right). \tag{1.129}$$

While in the $(\hat{u}, \hat{v}, \hat{n})$ it has components $(0, 0, 1)$. A rotation matrix relates these two $\mathcal{R}$ such that

$$\left(\sin\left(\theta\right)\sin\left(\phi\right), \sin\left(\theta\right)\cos\left(\phi\right), \cos\left(\theta\right)\right) = \mathcal{R}\left(0, 0, 1\right). \tag{1.130}$$

The explicit expression of $\mathcal{R}$ is

$$\mathcal{R} = \begin{pmatrix} \cos\left(\phi\right) & \sin\left(\phi\right) & 0 \\ -\sin\left(\phi\right) & \cos\left(\phi\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta\right) & \sin\left(\theta\right) \\ 0 & -\sin\left(\theta\right) & \cos\left(\theta\right) \end{pmatrix}. \tag{1.131}$$

Similarly, the tensor I has components I_{ij} in the $(\hat{x}, \hat{y}, \hat{z})$ and M'_{ij} in the $(\hat{u}, \hat{v}, \hat{n})$ frame, related by

$$I_{ij} = \mathcal{R}_{ik}\mathcal{R}_{jl}I'_{kl}. \tag{1.132}$$

Using this, we can compute

$$\begin{aligned} h_+\left(t, \theta, \phi\right) = \frac{1}{r}\Big[&\ddot{I}_{11}\left(\cos^2\left(\phi\right) - \sin^2\left(\phi\right)\cos^2\left(\theta\right)\right) + \ddot{I}_{22}\left(\sin^2\left(\phi\right)\right. \\ &\left. - \cos^2\left(\phi\right)\cos^2\left(\theta\right) - \ddot{I}_{33}\sin^2\left(\theta\right) - \ddot{I}_{12}\sin\left(2\phi\right)\left(1 + \cos^2\left(\theta\right)\right)\right. \\ &+ \ddot{I}_{13}\sin\left(\phi\right)\sin\left(2\theta\right) + \ddot{I}_{23}\cos\left(\phi\right)\sin\left(2\theta\right)\Big], \end{aligned} \tag{1.133}$$

$$h_x(t,\theta,\phi) = \frac{1}{r}\Bigg[\left(\ddot{I}_{11} - \ddot{I}_{22}\right)\sin(2\phi)\cos(\theta) + 2\ddot{I}_{12}\cos(2\phi)\cos(\theta) - \\ -2\ddot{I}_{13}\cos(\phi)\cos(\theta) + 2\ddot{I}_{23}\sin(\phi)\sin(\theta)\Bigg]. \tag{1.134}$$

This allows us to compute the angular distribution of the quadrupole radiation, once M_{ij} is given.

1.2.8 Radiated Energy

In this section, we want to derive an expression for the radiated energy and apply it to the system PSR J0737-3039.

We start with the expression of the energy inside a volume 1.119

$$\frac{dE}{dt} = \frac{r^2}{32}\int d\Omega \left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right\rangle, \tag{1.135}$$

from this, we can derive the power radiated per unit solid angle in the quadrupole approximation

$$\left(\frac{dP}{d\Omega}\right)_{quad} = \frac{r^2}{32}\left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right\rangle. \tag{1.136}$$

We can substitute $\dot{h}_{ij}^{TT}$ with their explicit expression

$$\dot{h}_{ij}^{TT} = \frac{2}{r}\Lambda_{ij,kl}(\hat{n})\dddot{I}_{ij} \tag{1.137}$$

,

$$\left(\frac{dP}{d\Omega}\right)_{quad} = \frac{1}{8}\Lambda_{ij,kl}(\hat{n})\Lambda_{ij,kl}(\hat{n})\left\langle \dddot{I}_{ij}\dddot{I}_{ij}\right\rangle. \tag{1.138}$$

We can now use a property of $\Lambda_{ij,kl}$,

$$\Lambda_{ij,kl}\Lambda_{kl,nm} = \Lambda_{ij,mn}. \tag{1.139}$$

And we obtain

$$\left(\frac{dP}{d\Omega}\right)_{quad} = \frac{1}{8}\Lambda_{ij,kl}(\hat{n})\left\langle \dddot{I}_{ij}\dddot{I}_{kl}\right\rangle. \tag{1.140}$$

We can perform the angular integral, observing that the dependence on $\hat{n}$ is only in $\Lambda_{ij,kl}$

$$\int d\Omega\left[\left(\delta_{ik} - n_i n_k\right)\left(\delta_{jl} - n_j n_l\right) - \frac{1}{2}\left(\delta_{ij} - n_i n_j\right)\left(\delta_{kl} - n_k n_l\right)\right] \tag{1.141}$$

We use the following properties

$$\int \frac{d\Omega}{4\pi} n_i n_j = \frac{1}{3}\delta_{ij}, \tag{1.142}$$

$$\int \frac{d\Omega}{4\pi} n_i n_j n_k n_l = \frac{1}{15}\left[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right]. \tag{1.143}$$

Using them we obtain

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}\left[11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{jk}\right]. \tag{1.144}$$

We can now substitute everything in 1.140 and obtain

$$P_{quad} = \frac{1}{5}\left\langle \dddot{I}_{ij} \dddot{I}_{kl} \right\rangle. \tag{1.145}$$

Another formulation for the Radiated Power can be done by subtracting the trace from the tensor I_{ij}

$$I_{ij} = M_{ij} - \frac{1}{3} M_{ij} M_{kk}, \tag{1.146}$$

$$P_{quad} = \frac{1}{5}\left\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}\left(\dddot{M}_{kk}\right)^2 \right\rangle \tag{1.147}$$

1.2.9 Radiated Power for a binary system

In this section, we will apply the formalism that we developed to a binary system made of two compact stars with masses m_1 and m_2 and positions r_1 and r_2. We can simplify this in a one-body problem in the center of mass frame and with mass equal to the reduced mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ subject to an acceleration $\ddot{r} = -\dfrac{m}{r^2}\hat{r}$, where $m = m_1 + m_2$.

The conservation of the angular momentum **L** implies that the orbit lies on a plane. We can introduce polar coordinates (r, ψ) on the plane of the orbit, with the origin in the center of mass. In terms of r and ψ the angular momentum is

$$L = \mu r^2 \dot{\psi}, \tag{1.148}$$

The energy is

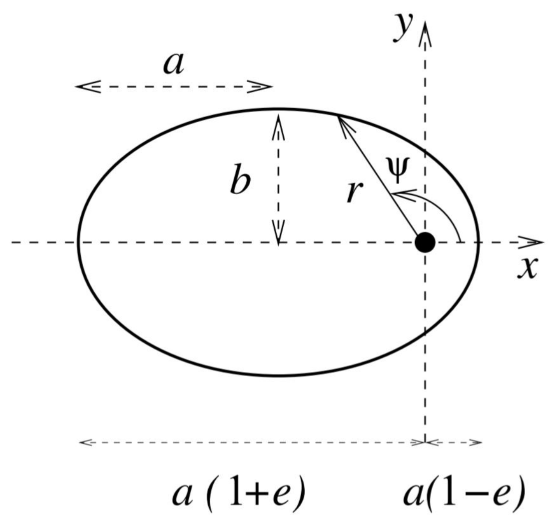
$$E = \frac{1}{2}\mu\left[\dot{r}^2 + r^2\dot{\psi}^2\right] - \frac{\mu m}{r} = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{\mu m}{r}. \tag{1.149}$$

From the two conservation laws, we obtain the equation of the orbit

$$\frac{1}{r} = \frac{1}{R}(1 + e\cos\psi). \tag{1.150}$$

Where R is the length scale and is a constant of motion

$$R = \frac{L^2}{m\mu^2}. \tag{1.151}$$



In this frame, the matrix M is

$$M = \mu r^2 \begin{pmatrix} \cos^2 & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2 \end{pmatrix}. \tag{1.152}$$

We can write M as a function of ψ, using a relation between ψ and r

$$r = \frac{a(1-e^2)}{1+e\cos\psi}, \tag{1.153}$$

$$M = \mu \left(\frac{a(1-e^2)}{1+e\cos\psi}\right)^2 \begin{pmatrix} \cos^2 & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2 \end{pmatrix}. \tag{1.154}$$

When computing the second derivative of the matrix elements, we obtain

$$\dddot{M}_{11} = \beta\left(1+e\cos\psi\right)^2 \left[2\sin 2\psi + 3e\sin\psi\cos^2\psi\right], \tag{1.155}$$

$$\dddot{M}_{22} = \beta\left(1+e\cos\psi\right)^2 \left[-2\sin 2\psi - e\sin\psi\left(1+3\cos^2\psi\right)\right], \tag{1.156}$$

$$\dddot{M}_{12} = \beta\left(1+e\cos\psi\right)^2 \left[-2\cos 2\psi + e\cos\psi\left(1-3\cos^2\psi\right)\right]. \tag{1.157}$$

We can put these expressions in equation 1.147

$$
\begin{aligned}
P(\psi) &= \frac{1}{5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + \dddot{M}_{12}^2 - \frac{1}{3}\left(\dddot{M}_{11} + \dddot{M}_{11}\right)^2\right] \\
&= \frac{2}{5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 3\dddot{M}_{12}^2 - \dddot{M}_{11}\dddot{M}_{22}\right] \\
&= \frac{8}{15}\frac{\mu^2 m^3}{a^5\left(1-e^2\right)^5}\left(1+e\cos\psi\right)^4\left[12\left(1+e\cos\psi\right)^2 + e^2\sin^2\psi\right].
\end{aligned}
\tag{1.158}
$$

We can now perform a time average over a period T

$$P = \frac{1}{T}\int_0^T dt P(\psi), \tag{1.159}$$

when evaluating this integral, we obtain

$$P = \frac{32\mu^2 m^3}{5a^5} f(e), \tag{1.160}$$

where

$$f(e) = \frac{1}{(1-e^2)^{7/2}}\left[1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right]. \tag{1.161}$$

We can compute this formula with the data from PSR J0737-3039. The numerical values are the following [28]

m_1	$1.337M_\odot$
m_2	$1.250M_\odot$
e	0.0878
a	$1.26R_\odot$

Where $M_\odot$ and $R_\odot$ are respectively the mass and radius of the sun.

$$
\begin{cases}
M_\odot = 1,989x10^{30}kg \\
\\
R_\odot = 6,96x10^8 m
\end{cases}
\tag{1.162}
$$

The value for the radiated power we obtain is $2.44x10^{25}W$.

1.3 Gravitation Waves in $f(R)$ theories

The goal of this section is to study the gravitational radiation for an $f(R)$ theory. We start from the action

$$S = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M, \tag{1.163}$$

where $S_M$is the standard matter action.

The first thing to do is to compute the variation of the action with respect to the metric $g_{\mu\nu}$,

$$\begin{aligned}\delta S &= \frac{c^3}{16\pi G}\int d^4x\delta\left[\sqrt{-g}f(R)\right] + \delta S_M \\ &= \frac{c^3}{16\pi G}\int d^4x\left[-\frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu} + \underbrace{\sqrt{-g}f'(R)\delta R}_{*}\right] + \delta S_M,\end{aligned} \tag{1.164}$$

$$\begin{aligned}* &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}f'(R)\delta\left[R_{\mu\nu}g^{\mu\nu}\right] \\ &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \underbrace{\frac{c^3}{16\pi G}\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu}}_{\#},\end{aligned} \tag{1.165}$$

$$\begin{aligned}\# &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}f'(R)\left[g^{\mu\nu}\delta R_{\mu\nu}\right] \\ &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}f'(R)\partial_\sigma\left[\partial^\sigma\left(g_{\mu\nu}\delta g^{\mu\nu}\right) - \partial^\mu\left(g_{\mu\nu}\delta g^{\nu\sigma}\right)\right] \\ &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\partial_\sigma\left(f'(R)\right).\end{aligned} \tag{1.166}$$

Putting everything together, we obtain

$$\begin{aligned}\delta S &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\Bigg[-\frac{1}{2}g_{\mu\nu}f(R) + f'(R)R_{\mu\nu} \\ &+ g_{\mu\nu}\partial_\sigma\partial^\sigma f'(R) - \partial_\mu\partial_\nu f'(R)\Bigg]\delta g^{\mu\nu} + \delta S_M.\end{aligned} \tag{1.167}$$

So, we obtain the following equation of motion

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) - \partial_\mu\partial_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.168}$$

taking the trace, we obtain

$$3\Box_g f'(R) + Rf'(R) - 2f(R) = \frac{8\pi G}{c^4}T. \tag{1.169}$$

We assume the theory

$$f(R) = R + aR^2, \tag{1.170}$$

with $[\alpha] = [R]^{-1}$. We define the first derivative $\phi = f'(R)$ and the scalar field φ by $\phi = 1 + 2a\varphi$.We can substitute everything in equation 1.168

$$\begin{aligned}
\frac{8\pi G}{c^4}T_{\mu\nu} &= (1+2a\varphi)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left[R + aR^2\right] - \nabla_\mu\nabla_\nu(1+2a\varphi) + g_{\mu\nu}\Box_g(1+2a\varphi) \quad (1.171)\\
&= (1+2a\varphi)R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}[1+2aR] + \frac{1}{2}R^2 a g_{\mu\nu} - 2a\nabla_\mu\nabla_\nu\varphi + 2ag_{\mu\nu}\Box_g\varphi\\
\implies& (1+2a\varphi)\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] = \frac{8\pi G}{c^4}T_{\mu\nu} + a\left(2\nabla_\mu\nabla_\nu\varphi - 2g_{\mu\nu}\Box_g\varphi - \frac{1}{2}g_{\mu\nu}R^2\right)\\
\implies& R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{(1+2a\varphi)}\left[\frac{8\pi G}{c^4}T_{\mu\nu} + a\left(2\nabla_\mu\nabla_\nu\varphi - 2g_{\mu\nu}\Box_g\varphi - \frac{1}{2}g_{\mu\nu}\varphi^2\right)\right].
\end{aligned}$$

$$\begin{aligned}
3\Box_g(1+2a\varphi) + R(1+2a\varphi) - 2\left(R + aR^2\right) &= \frac{8\pi G}{c^4}T\\
\implies 6a\Box_g\varphi + \varphi + 2a\varphi^2 - 2\varphi(1+a\varphi) &= \frac{8\pi G}{c^4}T \quad (1.172)\\
\implies 6a\Box_g\varphi = \frac{8\pi G}{c^4}T + \varphi \implies \Box_g\varphi &= \frac{4\pi G}{3ac^4}T + \frac{1}{6a}\varphi.
\end{aligned}$$

We consider a weak perturbation of the Minkowski space-time metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1.173}$$

1.3.1 The expansion coefficients

We define an expansion over $\frac{1}{c}$ for the various elements of the theory[29]:

$$\begin{aligned}
h_{00} &= {}^{(2)}h_{00} + {}^{(4)}h_{00} + O(c^{-6}), \quad (1.174)\\
h_{0i} &= {}^{(3)}h_{0i} + O(c^{-5}),\\
h_{ij} &= {}^{(2)}h_{ij} + O(c^{-4}),
\end{aligned}$$

$$\varphi = {}^{(2)}\varphi + {}^{(4)}\varphi + O(c^{-6}), \tag{1.175}$$

$$\begin{aligned}
T^{00} &= {}^{(-2)}T^{00} + {}^{(0)}T^{00} + O(c^{-2}), \quad (1.176)\\
T^{0i} &= {}^{(-1)}T^{0i} + O(c^{-1}),\\
T^{ij} &= {}^{(0)}T^{ij} + O(c^{-2}).
\end{aligned}$$

The metric tensor, in the weak field limit is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$,

$$g_{\mu\nu} \approx \begin{pmatrix} -1 +^{(2)} h_{tt} + {}^{(4)}h_{tt} & 0 & 0 & {}^{(3)}h_{t\phi} \\ 0 & 1 +^{(2)} h_{rr} & & \\ & & 1 +^{(2)} h_{\theta\theta} & \\ {}^{(3)}h_{t\phi} & & & 1 +^{(2)} h_{\phi\phi} \end{pmatrix}. \tag{1.177}$$

We can now find the approximations of the two functions

$$\begin{cases} \Box_g \varphi = \dfrac{4\pi G}{3ac^4} T + \dfrac{1}{6a}\varphi \\ R_{\mu\nu} - \dfrac{1}{2} g_{\mu\nu} R = \dfrac{1}{(1+2a\varphi)} \left[\dfrac{8\pi G}{c^4} T_{\mu\nu} + a \left(2\nabla_\mu \nabla_\nu \varphi - 2g_{\mu\nu}\Box_g \varphi - \dfrac{1}{2} g_{\mu\nu}\varphi^2 \right) \right] \end{cases} \tag{1.178}$$

We want to approximate the first equation in leading order. We define $\alpha = \dfrac{1}{\sqrt{6a}}$. We can go from $\Box_g \varphi$ to $\nabla^2 \varphi$ because we want to stop to leading order in $\dfrac{1}{c}$, since every time derivative has a $\dfrac{1}{c}$ factor, we can suppress the time derivative. For the same reason, we can approximate φ to the first order $\varphi = {}^{(2)}\varphi$.

$$\nabla^{2(2)}\varphi - \alpha^{2(2)}\varphi = \frac{8\pi G \alpha^2}{c^4} T. \tag{1.179}$$

The last thing to do is to approximate the trace of the Energy-Momentum tensor

$$T = g^{\mu\nu} T_{\mu\nu}, \tag{1.180}$$

using the approximations of $g_{\mu\nu}$ and $T_{\mu\nu}$ written above, and stopping the expansion to c^2 we obtain

$$T \approx -{}^{(-2)}T^{00}. \tag{1.181}$$

So the equation is

$$\nabla^{2(2)}\varphi - \alpha^{2(2)}\varphi = -\frac{8\pi G \alpha^2}{c^4} {}^{(-2)}T^{00}. \tag{1.182}$$

We can now solve this equation with the Green function:

$$\left[\nabla^2 - \alpha^2 \right] G(\vec{r} - \vec{r}') = \delta^{(3)} \left(\vec{r} - \vec{r}' \right), \tag{1.183}$$

where we indicate with r the vector (r, θ, ϕ).
Let's rewrite the equation as

$$\left[\nabla^2 - \alpha^2 \right] G(r) = \delta^{(3)} \left(r \right), \tag{1.184}$$

where, for simplification we wrote $\vec{r'} = 0$.
The Laplacian operator in spherical coordinates reduces to the r component:

$$\nabla^2 G(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right). \tag{1.185}$$

So, when considering the homogeneous solution, equation 1.184 becomes

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) - \alpha^2 G(r) &= 0 \\ \implies \frac{d}{dr}\left(r^2\frac{dG}{dr}\right) - r^2\alpha^2 G(r) &= 0 \\ \frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} - \alpha^2 G(r) &= 0. \end{aligned} \tag{1.186}$$

The general solution to this differential equation is

$$G(r) = A\frac{e^{\alpha r}}{r} + B\frac{e^{-\alpha r}}{r}. \tag{1.187}$$

To find the constant A and B, we will study the function for $r \to 0$ and $r \to \infty$.
For $r \to \infty$, we want the solution not to diverge, so $A = 0$.
Now, we need to normalize the function

$$\int_{\mathbb{R}^3} \left[\nabla^2 - \alpha^2\right] G(r) dr = \int_{\mathbb{R}^3} \delta^{(3)}\left(r\right) dr = 1, \tag{1.188}$$

from this, it is possible to show that

$$B = \frac{1}{4\pi}, \tag{1.189}$$

So, the Green function is

$$G = \frac{e^{-\alpha r}}{4\pi r} = \frac{e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi\left|\boldsymbol{x}-\boldsymbol{x}'\right|}. \tag{1.190}$$

We can now find ${}^{(2)}\varphi$ as the convolution of the Green function with the source

$$\begin{aligned} {}^{(2)}\varphi &= -\frac{8\pi G\alpha^2}{c^4}\int d^3x' \frac{e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi\left|\boldsymbol{x}-\boldsymbol{x}'\right|}{}^{(-2)}T^{00} = \\ &= -\frac{G\alpha^2}{c^4}\int d^3x' \frac{e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{\left|\boldsymbol{x}-\boldsymbol{x}'\right|}{}^{(-2)}T^{00} = \frac{1}{c^2}V(x,t). \end{aligned} \tag{1.191}$$

Where we define the potential

$$V(x,t) = -\frac{G\alpha^2}{c^2}\int d^3x' \frac{e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{\left|\boldsymbol{x}-\boldsymbol{x}'\right|}{}^{(-2)}T^{00}. \tag{1.192}$$

Now, we can work on the (t,t) element of the second equation of (1.178). The first thing to do is rewrite the equation in a form that will be proved to be useful

$$\begin{aligned}
R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R &= \frac{1}{(1+2a\varphi)}\left[\frac{8\pi G}{c^4}T_{\mu\nu}+a\left(2\partial_\mu\partial_\nu\varphi-2g_{\mu\nu}\left(\frac{4\pi}{3a}T+\frac{1}{6a}\varphi\right)-\frac{1}{2}g_{\mu\nu}\varphi^2\right)\right]\\
&= \frac{1}{(1+2a\varphi)}\left[\frac{8\pi G}{c^4}T_{\mu\nu}-\frac{8\pi}{3}g_{\mu\nu}T-\frac{1}{3}g_{\mu\nu}\varphi+a\left(2\nabla_\mu\nabla_\nu\varphi-\frac{1}{2}g_{\mu\nu}\varphi^2\right)\right]\\
&= \frac{1}{(1+2a\varphi)}\left[\frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{3}g_{\mu\nu}T\right)-\frac{1}{3}g_{\mu\nu}\varphi+a\left(2\nabla_\mu\nabla_\nu\varphi-\frac{1}{2}g_{\mu\nu}\varphi^2\right)\right]\\
\Longrightarrow\quad R_{\mu\nu} &= \frac{1}{(1+2a\varphi)}\Big[\frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{3}g_{\mu\nu}T\right)-\frac{1}{3}g_{\mu\nu}\varphi+a\left(2\nabla_\mu\nabla_\nu\varphi-\frac{1}{2}g_{\mu\nu}\varphi^2\right)\\
&\quad+\frac{1}{2}g_{\mu\nu}\left(1+2a\varphi\right)R\Big]\\
&= \frac{1}{(1+2a\varphi)}\left[\frac{8\pi G}{c^4}\left(T_{\mu\nu}-\frac{1}{3}g_{\mu\nu}T\right)+\frac{1}{6}g_{\mu\nu}\varphi+a\left(2\nabla_\mu\nabla_\nu\varphi+\frac{1}{2}g_{\mu\nu}\varphi^2\right)\right],
\end{aligned}$$

Where in the last steps, we use the fact that $\varphi = R$. The t,t element of this equation is

$$R_{tt}=\frac{1}{(1+2a\varphi)}\left[\frac{8\pi G}{c^4}\left(T_{tt}-\frac{1}{3}g_{tt}T\right)+\frac{1}{6}g_{tt}\varphi+a\left(2\nabla_t\nabla_t\varphi+\frac{1}{2}g_{tt}\varphi^2\right)\right]. \tag{1.193}$$

To perform this approximation, we have to derive some elements. The first one is R_{tt}. Using the approximation written at the beginning of the section to the second order in c, it is

$$R_{tt}\approx-\frac{1}{2}\nabla^{2(2)}h_{tt}. \tag{1.194}$$

Then, we have to find the second covariant derivative.

$$\nabla_t\nabla_t\varphi=\nabla_t\left(\partial_t\varphi\right)=\partial_t^2\varphi-\Gamma^\lambda_{tt}\varphi.$$

The Christoffel Symbol can be computed with the same approximations. Putting everything together, we obtain

$$\nabla^{2(2)}h_{tt}=-\frac{32\pi G}{3c^4}{}^{(-2)}T^{00}+\frac{1}{3}{}^{(-2)}\varphi. \tag{1.195}$$

We can solve this equation, too, with the Green function.
The Green function for the Laplacian is

$$G\left(\boldsymbol{x}-\boldsymbol{x}'\right)=-\frac{1}{4\pi\left|\boldsymbol{x}-\boldsymbol{x}'\right|}. \tag{1.196}$$

When convoluting with the source, we obtain

$$
\begin{aligned}
{}^{(2)}h_{00}(x,y,z,t) &= \frac{8G}{3c^4}\int d^3x' \frac{{}^{(-2)}T^{00}}{|\boldsymbol{x}-\boldsymbol{x}'|} - \frac{1}{12\pi c^2}\int d^3x' \frac{V(x',t)}{|\boldsymbol{x}-\boldsymbol{x}'|} \\
&= \frac{1}{c^2}\left[2U(x,t) - W(x,t)\right], \qquad (1.197)
\end{aligned}
$$

where we define

$$
U(x,t) = \frac{4G}{3c^2}\int d^3x' \frac{{}^{(-2)}T^{00}}{|\boldsymbol{x}-\boldsymbol{x}'|}, \qquad (1.198)
$$

$$
W(x,t) = \frac{1}{12\pi}\int d^3x' \frac{V(x',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}. \qquad (1.199)
$$

So, to summarize, the three potential are

$$
V(x,t) = -\frac{G\alpha^2}{c^2}\int d^3x' \frac{e^{-\alpha|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}{}^{(-2)}T^{00} \qquad (1.200)
$$

$$
U(x,t) = \frac{4G}{3c^2}\int d^3x' \frac{{}^{(-2)}T^{00}}{|\boldsymbol{x}-\boldsymbol{x}'|}, \qquad (1.201)
$$

$$
W(x,t) = \frac{1}{12\pi}\int d^3x' \frac{V(x',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}. \qquad (1.202)
$$

1.3.2 Gravitational Radiation

With the potentials we derived, we are now able to find an expression for φ. Let's start with the introduction of the tensor $\bar{h}_{\mu\nu}$,

$$
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} - 2a\eta_{\mu\nu}\varphi, \qquad (1.203)
$$

and impose the following gauge conditions

$$
\nabla^\nu \bar{h}_{\mu\nu} = 0. \qquad (1.204)
$$

Up to linear order in $h_{\mu\nu}$ and φ, equation 1.172 becomes

$$
\Box_\eta \bar{h}_{\mu\nu} = 16\pi T_{\mu\nu}. \qquad (1.205)
$$

We can solve this with the green function

$$
\Box G\left(t,x,t',x'\right) = \delta^{(3)}\left(x-x'\right)\delta\left(t-t'\right). \qquad (1.206)
$$

The Green function associated with the wave operator is

$$
G\left(t,x,t',x'\right) = \frac{\delta\left(t' - \left[t - |x-x'|\right]\right)}{|x-x'|}. \qquad (1.207)
$$

So, the solution is the convolution of the Green function and the source

$$\bar{h}_{\mu\nu} = 16\pi \int d^4xG\left(t,x,t',x'\right)T_{\mu\nu} = 4\int d^3x' \frac{T_{\mu\nu}\left(x',t-|x-x'|\right)}{|x-x'|}. \tag{1.208}$$

We are interested in studying the spacial elements of $\bar{h}_{\mu\nu}$, to do this we study the spacial elements of $T^{\mu\nu}$, from equation

$$2T^{ij} = \partial_0^2\left[T^{00}x^ix^j\right] - \partial_k\partial_l\left[T^{ij}x^kx^l\right] + 2\partial_k\left[x^iT^{kj} + x^jT^{ik}\right], \tag{1.209}$$

when performing the three-dimensional spatial integration, we obtain

$$\begin{aligned}\int d^3x 2T^{ij} &= \partial_0^2\int d^3x\left[T^{00}x^ix^j\right] - \int d^3x\partial_k\partial_l\left[T^{ij}x^kx^l\right] \\ &+ 2\int d^3x\partial_k\left[x^iT^{kj}+x^jT^{ik}\right],\end{aligned} \tag{1.210}$$

we can drop the surface terms and are left with

$$\int d^3xT^{ij} = \frac{1}{2}\partial_0^2\int d^3x\left[T^{00}x^ix^j\right]. \tag{1.211}$$

If we assume $|x| \gg |x'|$,from equation 1.208, when studying the spacial elements we obtain

$$\bar{h}_{ij} = 4\frac{1}{|x|}\int d^3xT_{ij}\left(x',t-|x|\right) = \frac{2}{|x|}\partial_0^2\int d^3x'\left[T^{00}x'^ix'^j\right]. \tag{1.212}$$

The first equation of 1.178 can be approximated as

$$\Box_\eta\varphi - \alpha^2\varphi = \frac{8\pi G\alpha^2}{c^4}S, \tag{1.213}$$

Where S in the source T is extended to the quadratic terms in the perturbations. These are expressed in terms of the Newtonian and Post Newtonian Potentials U, V, W

$$S = T\left[1+\frac{1}{c^2}\left(3W+\frac{2}{3\alpha^2}V\right)\right] + \frac{1}{8\pi G}\left[\frac{1}{3\alpha^4}\left(\nabla V\right)^2 + UV + VW\right]. \tag{1.214}$$

Equation 1.213 can be solved with the Green function for the Klein-Gordon equation

$$\left(\Box_\eta - \alpha^2\right)G = \delta^{(4)}\left(x-x'\right). \tag{1.215}$$

We start considering the equation without the mass term

$$\Box_\eta G_0 = \delta^{(4)}\left(x-x'\right), \tag{1.216}$$

The solution is known

$$G_0(t,x) = -\frac{1}{4\pi}\frac{\delta\left(t - |\boldsymbol{x}|/c\right)}{|\boldsymbol{x}|}, \tag{1.217}$$

Here, we have considered $x \gg x'$.
For the solution with the mass term, we start with

$$\left(\Box_\eta - \alpha^2\right) G_m = \delta^{(4)}\left(x - x'\right) \tag{1.218}$$

and we write the Fourier transformation of G

$$G_m = \frac{1}{(4\pi)^4}\int d^4k\tilde{G}(p)e^{ikx}, \tag{1.219}$$

$$\begin{aligned} &\frac{1}{(2\pi)^4}\int d^4k\tilde{G}(k)\left(\Box_\eta - \alpha^2\right)e^{ikx} = \frac{1}{(2\pi)^4}\int d^4ke^{ikx} \\ \implies \quad &\tilde{G}(k)\left(k_0^2 - \boldsymbol{k}^2 - \alpha^2\right) = 1 \\ \implies \quad &\tilde{G}(k) = \frac{1}{k_0^2 - \boldsymbol{k}^2 - \alpha^2} \\ \implies \quad &G_m = \int \frac{d^3k}{(2\pi)^3}\int \frac{dk_0}{2\pi}\frac{e^{ikx}}{k_0^2 - \boldsymbol{k}^2 - \alpha^2}. \end{aligned} \tag{1.220}$$

We can solve the integral in dk_0 with the Residue theorem

$$\int \frac{dk_0}{2\pi}\frac{e^{-ik_0t}}{k_0^2 - \boldsymbol{k}^2 - \alpha^2} \implies k_0 = \pm\sqrt{\boldsymbol{k}^2 + \alpha^2} = \pm\omega, \tag{1.221}$$

$$Res_\omega = \lim_{k_0 \to \omega}\left[k_0 - \omega\right]\frac{e^{-ik_0t}}{\left(k_0 - \omega\right)\left(k_0 + \omega\right)} = \frac{e^{-i\omega t}}{2\omega}, \tag{1.222}$$

$$Res_\omega = \lim_{k_0 \to -\omega}\left[k_0 + \omega\right]\frac{e^{ik_0t}}{\left(k_0 - \omega\right)\left(k_0 + \omega\right)} = -\frac{e^{i\omega t}}{2\omega}. \tag{1.223}$$

This means that we will have both a retarded and advanced Green function. We will only consider the retarded Green function for the causality of the theory

$$G_{ret} = \theta(t)\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\boldsymbol{x}}}{2\omega}e^{-i\omega t}. \tag{1.224}$$

We can now perform a coordinate change to use polar coordinates $d^3k = k^2\sin(\theta)\,dkd\theta d\phi$. We can start performing the integration in θ

$$\int_0^{2\pi} d\phi \int_0^{\pi} e^{ikr\cos(\theta)}\sin(\theta)\,d\theta = 2\pi\left[2\int_0^1 e^{ikr\cos(\theta)}d\left(\cos(\theta)\right)\right] = 4\pi\frac{\sin(kr)}{kr}. \tag{1.225}$$

So, the whole integral becomes

$$\frac{1}{(2\pi)^2}\int_0^\infty dk\frac{\sin(kr)}{r} \tag{1.226}$$

We will now use the relation $\sqrt{k^2+\alpha^2}=\omega$ and do a change of variables

$$k^2=\sqrt{\omega^2-\alpha^2} \implies dk=\frac{\omega d\omega}{\sqrt{\omega^2-\alpha^2}}. \tag{1.227}$$

Putting everything in the Green function, we obtain

$$G_{ret}=\frac{\theta(t)}{(2\pi)^2}\int_\alpha^\infty d\omega\frac{\omega e^{-i\omega t}}{\sqrt{\omega^2-\alpha^2}}\frac{\sin\left(r\sqrt{\omega^2-\alpha^2}\right)}{r\sqrt{\omega^2-\alpha^2}}. \tag{1.228}$$

We can use the Bessel function of first order $J_1$to rewrite

$$J_1\left(r\sqrt{\omega^2-\alpha^2}\right)=\frac{\sin\left(r\sqrt{\omega^2-\alpha^2}\right)}{r\sqrt{\omega^2-\alpha^2}}, \tag{1.229}$$

$$G_{ret}=\frac{\theta(t)}{(2\pi)^2}\int_\alpha^\infty d\omega\frac{\omega e^{-i\omega t}}{\sqrt{\omega^2-\alpha^2}}J_1\left(r\sqrt{\omega^2-\alpha^2}\right). \tag{1.230}$$

This can be solved analytically by obtaining

$$G_{ret}=\frac{\theta(t-r/c)}{4\pi}\frac{\alpha J_1\left(\alpha\sqrt{t^2-r^2/c^2}\right)}{\sqrt{t^2-r^2/c^2}} \tag{1.231}$$

We can write $r=|\boldsymbol{x}|$ and sum this result with G_0 obtaining

$$G(t,\boldsymbol{x})=-\frac{1}{4\pi}\left[\frac{\delta\left(t-|\boldsymbol{x}|/c\right)}{|\boldsymbol{x}|}-\theta(t-|\boldsymbol{x}|/c)\frac{\alpha J_1\left(\alpha\sqrt{t^2-|\boldsymbol{x}|^2/c^2}\right)}{\sqrt{t^2-|\boldsymbol{x}|^2/c^2}}\right]. \tag{1.232}$$

Now we can find a solution of 1.213 as the convolution of the source with the Green's function

$$\begin{gathered}\varphi(\boldsymbol{x},t)=-\frac{2\pi G\alpha^2}{c^4}\int dt'\int d^3x'\left[\frac{\delta\left(t'-|\boldsymbol{x}-\boldsymbol{x'}|/c\right)}{|\boldsymbol{x'}|}-\right.\\ \left.\theta(t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)\frac{\alpha J_1\left(\alpha\sqrt{(t-t')^2-|\boldsymbol{x}-\boldsymbol{x'}|^2/c^2}\right)}{\sqrt{(t-t')^2-|\boldsymbol{x}-\boldsymbol{x'}|^2/c^2}}S(t',\boldsymbol{x'})\right.\end{gathered} \tag{1.233}$$

We can define a retarded time and change the variable of integration

$$t' = t - \frac{|\boldsymbol{x}-\boldsymbol{x'}|}{c}\sqrt{1+\frac{s^2}{\alpha^2\,|\boldsymbol{x}-\boldsymbol{x'}|^2}}, \tag{1.234}$$

$$dt' = -\frac{1}{\alpha^2\,|\boldsymbol{x}-\boldsymbol{x'}|}\frac{s}{\sqrt{1+\frac{s^2}{\alpha^2\,|\boldsymbol{x}-\boldsymbol{x'}|^2}}}ds, \tag{1.235}$$

$$\varphi(\boldsymbol{x},t) = \frac{G}{3ac^4}\int_{\mathbb{R}} ds\,[J_1(s)\theta(s)-\delta(s)]\int_{\mathbb{R}^3} d^3x' \frac{S\left(t-\frac{|\boldsymbol{x}-\boldsymbol{x'}|}{c}\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}},\boldsymbol{x'}\right)}{|\boldsymbol{x}-\boldsymbol{x'}|\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}}}. \tag{1.236}$$

Let's focus on the second integrand and compute it far away from the source. We can write

$$\begin{aligned} &\frac{S\left(t-\frac{|\boldsymbol{x}-\boldsymbol{x'}|}{c}\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}},\boldsymbol{x'}\right)}{|\boldsymbol{x}-\boldsymbol{x'}|\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}}} \\ &= \int_{\mathbb{R}^3} \frac{S\left(t-\frac{|\boldsymbol{x}-\boldsymbol{x'}|}{c}\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}},\boldsymbol{x'}\right)}{|\boldsymbol{x}-\boldsymbol{x'}|\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}}}\delta\left(\boldsymbol{y}-\boldsymbol{x'}\right)d^3y \\ &= \int_{\mathbb{R}^3} g(\boldsymbol{x},\boldsymbol{x'},\boldsymbol{y})\delta\left(\boldsymbol{y}-\boldsymbol{x'}\right)d^3y \end{aligned} \tag{1.237}$$

$\boldsymbol{x'}$ lies within the near zone, so we can treat it as a small vector and express g as a Taylor expansion around the origin in the variable $\boldsymbol{x'}$ [30]

$$g(\boldsymbol{x},\boldsymbol{x'},\boldsymbol{y}) = g(\boldsymbol{x},0,\boldsymbol{y}) + \frac{\partial g}{\partial x'^i}x'^i + \frac{1}{2}\frac{\partial^2 g}{\partial x'^i\partial x'^j}x'^i x'^j + \dots . \tag{1.238}$$

All the derivatives are evaluated at $\boldsymbol{x'} = 0$ and since g depends on $\boldsymbol{x'}$ only through the combination $|\boldsymbol{x}-\boldsymbol{x'}|$ we can swap the derivatives in x'^i with derivatives in x^i

$$\frac{\partial g}{\partial x'^i} = -\frac{\partial g}{\partial x^i}, \tag{1.239}$$

$$
\begin{aligned}
g(\boldsymbol{x},\boldsymbol{x'},\boldsymbol{y}) &= g(\boldsymbol{x},0,\boldsymbol{y}) - \frac{\partial g}{\partial x^i}x'^i + \frac{1}{2}\frac{\partial^2 g}{\partial x^i \partial x^j}x'^i x'^j + ... \qquad (1.240)\\
&= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L\partial_L \frac{S\left(t - \frac{|\boldsymbol{x}-\boldsymbol{x'}|}{c}\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}}, \boldsymbol{x'}\right)}{|\boldsymbol{x}-\boldsymbol{x'}|\sqrt{1+\frac{6as^2}{|\boldsymbol{x}-\boldsymbol{x'}|}}}.
\end{aligned}
$$

Since all the derivatives are evaluated in $\boldsymbol{x'}=0$, we can substitute $|\boldsymbol{x}-\boldsymbol{x'}| = |\boldsymbol{x}| \equiv r$

$$
g(\boldsymbol{x},\boldsymbol{x'},\boldsymbol{y}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L\partial_L \frac{S\left(t - \frac{r}{c}\sqrt{1+\frac{6as^2}{r}}, \boldsymbol{x'}\right)}{r\sqrt{1+\frac{6as^2}{r}}}. \tag{1.241}
$$

Putting everything together in equation (1.236) we have

$$
\varphi(\boldsymbol{x},t) = \frac{G}{3ac^4}\int_{\mathbb{R}} ds\,[J_1(s)\theta(s)-\delta(s)]\int_{\mathbb{R}^3} d^3x' \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L\partial_L \frac{S\left(t - \frac{r}{c}\sqrt{1+\frac{6as^2}{r}}, \boldsymbol{x'}\right)}{r\sqrt{1+\frac{6as^2}{r}}}, \tag{1.242}
$$

L is a multi-index $L = j_1 j_2 ....j_l$.
We can define the following functions

$$
\begin{cases}
p(s) = \left(1+\frac{6as^2}{r^2}\right)^{-1/2}, \\
\tau = t - \frac{r}{p(s)c} \\
q(s) = p(s)\,[J_1(s)\theta(s)-\delta(s)]\,.
\end{cases} \tag{1.243}
$$

$$
\varphi(\boldsymbol{x},t) = \frac{G}{3ac^4}\int_{\mathbb{R}} ds\frac{q(s)}{p(s)}\int_{\mathbb{R}^3} d^3x' \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L\partial_L \frac{S\left(\tau,\boldsymbol{x'}\right)}{r}p(s). \tag{1.244}
$$

Up to the second order, we have

$$
\begin{aligned}
\varphi(\boldsymbol{x},t) = \frac{G}{3ac^4}\int_{\mathbb{R}} ds\frac{q(s)}{p(s)}\int_{\mathbb{R}^3} d^3x \Bigg[&\frac{p(s)}{r}S\left(\tau,\boldsymbol{x'}\right) - \frac{\partial}{\partial x^i}\left(\frac{p(s)}{r}S\left(\tau,\boldsymbol{x'}\right)\right)x'^i \\
&+\frac{1}{2}\frac{\partial^2}{\partial x^i\partial x^j}\left(\frac{p(s)}{r}S\left(\tau,\boldsymbol{x'}\right)\right)x'^i x'^j,
\end{aligned} \tag{1.245}
$$

for example, the first derivative can be calculated as

$$\frac{\partial}{\partial x^i}\left(\frac{p(s)}{r}\right) S\left(\tau, \boldsymbol{x'}\right) + \frac{p(s)}{r}\frac{\partial}{\partial x^i} S\left(\tau, \boldsymbol{x'}\right), \tag{1.246}$$

the derivative of S in x^i is

$$\frac{\partial}{\partial x^i} S\left(\tau, \boldsymbol{x'}\right) = \frac{\partial S}{\partial t}\frac{\partial t}{\partial x^i} = \frac{\partial S}{\partial t}\frac{\partial}{\partial x^i}\left(\tau + \frac{r}{p(s)c}\right). \tag{1.247}$$

Going up to the hexadecapole moments, we have

$$\begin{aligned} \varphi(\boldsymbol{x}, t) \quad &\approx \quad \frac{G}{3ac^4 r}\int_{\mathbb{R}} ds q(s)\int_{\mathbb{R}^3} d^3x' \Bigg[1 + F_i(s)x'^i + F_{ij}(s)x'^i x'^j + F_{ijk}(s)x'^i x'^j x'^k \\ &+ \quad F_{ijkl}(s)x'^i x'^j x'^k x'^l\Bigg] S\left(\tau, \boldsymbol{x'}\right), \end{aligned} \tag{1.248}$$

where

$$\begin{aligned} F_i(s) \quad &:= \quad n_i\left[\frac{p^2(s)}{r} + \frac{p(s)}{c}\frac{\partial}{\partial t}\right], \qquad (1.249) \\ F_{ij}(s) \quad &:= \quad n_i n_j\left[\frac{3p^4(s)}{2r^2} + \frac{p^3(s)}{rc}\frac{\partial}{\partial t} + \frac{p^2(s)}{2c^2}\frac{\partial^2}{\partial t^2}\right] - \delta_{ij}\left[\frac{p^2(s)}{2r^2} + \frac{p(s)}{2rc}\frac{\partial}{\partial t}\right], \\ F_{ijk}(s) \quad &:= \quad n_i n_j n_k\left[\frac{5p^6(s)}{2r^3} + \frac{p^5(s)}{2r^2c}\frac{\partial}{\partial t} + \frac{p^4(s)}{2rc^2}\frac{\partial^2}{\partial t^2} + \frac{p^3(s)}{6c^3}\frac{\partial^3}{\partial t^3}\right] \\ &\quad - n_i\delta_{jk}\left[\frac{3p^4(s)}{2r^3} + \frac{p^3(s)}{2r^2c}\frac{\partial}{\partial t} + \frac{p^2(s)}{rc^2}\frac{\partial^2}{\partial t^2}\right], \\ F_{ijkl}(s) \quad &:= \quad n_i n_j n_k n_l\left[\frac{35p^8(s)}{8r^4} + \frac{35p^7(s)}{8r^3c}\frac{\partial}{\partial t} + \frac{15p^6(s)}{8r^2c^2}\frac{\partial^2}{\partial t^2} - \frac{p^5(s)}{12rc^3}\frac{\partial^3}{\partial t^3} + \frac{p^4(s)}{24c^4}\frac{\partial^4}{\partial t^4}\right] \\ &\quad - n_i n_j\delta_{kl}\left[\frac{15p^6(s)}{4r^4} + \frac{15p^5(s)}{4r^3c}\frac{\partial}{\partial t} + \frac{3p^4(s)}{2r^2c^2}\frac{\partial^2}{\partial t^2} - \frac{p^3(s)}{12rc^3}\frac{\partial^3}{\partial t^3}\right] \\ &\quad - \delta_{ij}\delta_{kl}\left[\frac{3p^4(s)}{8r^4} + \frac{3p^3(s)}{8r^3c}\frac{\partial}{\partial t} + \frac{p^2(s)}{8r^2c^2}\frac{\partial^2}{\partial t^2}\right]. \end{aligned}$$

Now, we will drop the quadratic post-Newtonian terms in the perturbation fields. This lead to $T_{\mu\nu}$ being the main contribution to the source

$$S = \eta_{\mu\nu}T^{\mu\nu}. \tag{1.250}$$

If we consider a perfect non-viscous fluid with mass density ρ pressure $\mathcal{P}$and velocity field $\boldsymbol{v} = (v_1, v_2, v_3)$, we have

$$\begin{cases} T^{00}\left(t, \boldsymbol{x}\right) = c^2\left[\rho\left(t, \boldsymbol{x}\right) + O\left(c^{-2}\right)\right], \\ T^{0i}\left(t, \boldsymbol{x}\right) = c\left[\rho\left(t, \boldsymbol{x}\right) v^i\left(t, \boldsymbol{x}\right) + O\left(c^{-2}\right)\right], \\ T^{ij}\left(t, \boldsymbol{x}\right) = \rho\left(t, \boldsymbol{x}\right) v^i\left(t, \boldsymbol{x}\right) v^j\left(t, \boldsymbol{x}\right) + \mathcal{P}\left(t, \boldsymbol{x}\right)\delta_{ij} + O\left(c^{-2}\right). \end{cases} \tag{1.251}$$

We can now define the following quantities, which are the momenta of the Energy-Momentum tensor

$$\begin{cases} M^{I_n} = \frac{1}{c^2}\int_{\mathbb{R}^3} d^3x T^{00}(t,\boldsymbol{x})x^{I_n}, \\ S^{ijI_n} = \int_{\mathbb{R}^3} d^3x T^{ij}(t,\boldsymbol{x})x^{I_n}, \end{cases} \tag{1.252}$$

And then the quantities

$$\begin{cases} \mathcal{M}^{I_n}_{klm}(t) = \int_{\mathbb{R}} ds\; q(s)\frac{p^k(s)}{r^l c^m}\frac{\partial^m}{\partial t^m}M^{I_n}(\tau), \\ \mathcal{S}^{ijI_n}_{klm}(t) = \int_{\mathbb{R}} ds\; q(s)\frac{p^k(s)}{r^l c^m}\frac{\partial^m}{\partial t^m}S^{ijI_n}(\tau). \end{cases} \tag{1.253}$$

We can rewrite equation 1.248 with these quantities. For example, if we take in exam only the first non-identical element

$$\begin{aligned} \varphi(\boldsymbol{x},t) &\approx \frac{G}{3ac^4r}\int_{\mathbb{R}} dsq(s)\int_{\mathbb{R}^3} d^3x'\left[1+n_i\left(\frac{p^2(s)}{r}+\frac{p(s)}{c}\frac{\partial}{\partial t}\right)x'^i\right]S\left(\tau,\boldsymbol{x'}\right) \\ &= \frac{G}{3ac^4r}\int_{\mathbb{R}} ds\; q(s)\int_{\mathbb{R}^3} d^3x'\; S\left(\tau,\boldsymbol{x'}\right)+\frac{G}{3ac^4r}\left[\int_{\mathbb{R}} ds\; q(s)n_i\frac{p^2(s)}{r}\right. \\ &+ \left.\int_{\mathbb{R}} ds\; q(s)\; n_i\frac{p(s)}{c}\frac{\partial}{\partial t}\right]\int_{\mathbb{R}^3} d^3x'\; x'^i S\left(\tau,\boldsymbol{x'}\right). \end{aligned} \tag{1.254}$$

Using the quantities 1.253 we have no

$$\begin{aligned} \varphi(\boldsymbol{x},t) &= \frac{G}{3ac^4r}\left[\mathcal{M}(t)+n_i\left(\mathcal{M}^i_{210}(t)+\mathcal{M}^i_{101}(t)\right)\right] \\ &= \frac{G}{3ac^4r}\left[\mathcal{M}(t)+n_i\mathcal{D}^i(t)\right]. \end{aligned} \tag{1.255}$$

1.3.3 Energy-Momentum complex

The last step we want to discuss is the computation of the energy flux, similarly to what we did in GWs so that we can later confront the two results. The total power of the source is [31]

$$P = \frac{G}{3c}\left\langle \dot{\mathcal{M}}^2+\frac{1}{3}\dot{\mathcal{D}}^i\mathcal{D}^j\right\rangle. \tag{1.256}$$

The system we will study is again PSR J0737-3039. We have to make some approximations to understand the energy loss by the emission of gravitational radiation. First, we consider the total mass to change on a time scale much larger than the orbital period. Hence, the monopole contribution $\dot{\mathcal{M}} = 0$. So we are left with

$$P = \frac{G}{9c}\left\langle \dot{\mathcal{D}}^i\mathcal{D}^j\right\rangle = \frac{G}{9c}\left\langle \dot{\mathcal{D}}^i\dot{\mathcal{D}}^j\right\rangle \tag{1.257}$$

We now need to calculate the exact expression of $\dot{\mathcal{D}}^i$for a two-body system.

We choose coordinates such that the motion is restricted to the (x_1, x_2)plane, with $x_3 = 0$.

The mass density can be written as

$$\begin{aligned}\rho\left(\boldsymbol{x}\right) &= \delta\left(x_3\right)\left[m_1\delta\left(x_1-\frac{d}{2}\cos\left(\omega t\right)\right)\delta\left(x_2-\frac{d}{2}\sin\left(\omega t\right)\right)+\right.\\&+ \left.m_2\delta\left(x_1+\frac{d}{2}\cos\left(\omega t\right)\right)\delta\left(x_2+\frac{d}{2}\sin\left(\omega t\right)\right)\right] \qquad (1.258)\end{aligned}$$

We can now calculate $M_1(t)$ and $M_2(t)$, $M_3(t)$ is zero because we are on the (x_1, x_2) plane.

$$\begin{aligned}M_1(t) = \frac{1}{c^2}\int_{\mathbb{R}^3} d^3x T^{00}(t,\boldsymbol{x})x_1 = \int_{\mathbb{R}^3} d^3x &\left[m_1\delta\left(x_1-\frac{d}{2}\cos\left(\omega t\right)\right)\delta\left(x_2-\frac{d}{2}\sin\left(\omega t\right)\right)\right.\\&\left.+m_2\delta\left(x_1+\frac{d}{2}\cos\left(\omega t\right)\right)\delta\left(x_2+\frac{d}{2}\sin\left(\omega t\right)\right)\right]x_1\\= \int_{\mathbb{R}} dx_3\delta\left(x_3\right)&\left[m_1\int_{\mathbb{R}} dx_2\delta\left(x_2-\frac{d}{2}\sin\left(\omega t\right)\right)\int_{\mathbb{R}} dx_1x_1\delta\left(x_1-\frac{d}{2}\cos\left(\omega t\right)\right)\right.\\&\left.+m_2\int_{\mathbb{R}} dx_2\delta\left(x_2+\frac{d}{2}\sin\left(\omega t\right)\right)\int_{\mathbb{R}} dx_1x_1\delta\left(x_1+\frac{d}{2}\cos\left(\omega t\right)\right)\right]\\= \frac{d}{2}\left(m_1-m_2\right)\cos\left(\omega t\right). &\qquad (1.259)\end{aligned}$$

Similarly, we can calculate

$$\begin{aligned}M_2(t) &= \frac{1}{c^2}\int_{\mathbb{R}^3} d^3x T^{00}(t,\boldsymbol{x})x_2 = \frac{d}{2}\left(m_1-m_2\right)\sin\left(\omega t\right) = \frac{d}{2}\left(m_1-m_2\right)\cos\left[\omega t-\frac{\pi}{2}\right]\\&= M_1\left(t-\frac{\pi}{2\omega}\right). \qquad (1.260)\end{aligned}$$

With these we can move to $\mathcal{M}^1_{klm}(t)$ and $\mathcal{M}^2_{klm}(t)$. For the dipole moment we only need $\mathcal{M}^i_{210}(t)$ and $\mathcal{M}^i_{101}(t)$

$$\begin{cases} \mathcal{M}^1_{210}(t) = \dfrac{d}{2}\,(m_1 - m_2) \int_{\mathbb{R}} ds\; q(s) \dfrac{p^2(s)}{r} \cos\left(\omega t\right), \\ \\ \mathcal{M}^1_{101}(t) = \dfrac{d}{2}\,(m_1 - m_2) \int_{\mathbb{R}} ds\; q(s) \dfrac{p(s)}{c} \dfrac{\partial}{\partial t} \cos\left(\omega t\right), \\ \\ \mathcal{M}^2_{210}(t) = \mathcal{M}^1_{210}(t - \dfrac{\pi}{2\omega}) = \dfrac{d}{2}\,(m_1 - m_2) \int_{\mathbb{R}} ds\; q(s) \dfrac{p^2(s)}{r} \cos\left[\omega\left(t - \dfrac{\pi}{2\omega}\right)\right], \\ \\ \mathcal{M}^2_{101}(t) = \mathcal{M}^1_{101}(t - \dfrac{\pi}{2\omega}) = \dfrac{d}{2}\,(m_1 - m_2) \int_{\mathbb{R}} ds\; q(s) \dfrac{p(s)}{c} \dfrac{\partial}{\partial t} \cos\left[\omega\left(t - \dfrac{\pi}{2\omega}\right)\right]. \end{cases} \tag{1.261}$$

To calculate equation 1.257 we need the time average of $\dot{\mathcal{D}}^i$ over one period $\mathcal{T} = \dfrac{2\pi}{\omega}$. This mean that the time average over $\dot{\mathcal{D}}^1(t)$ and $\dot{\mathcal{D}}^2(t) = \dot{\mathcal{D}}^1\left(t - \dfrac{\pi}{2\omega}\right)$ have the same contribution.

$$P = \frac{G\omega}{9\pi c} \int_0^{\mathcal{T}} dt \left[\left(\dot{\mathcal{M}}^1_{210}(t)\right)^2 + \left(\dot{\mathcal{M}}^1_{101}(t)\right)^2 + 2\dot{\mathcal{M}}^1_{210}(t)\dot{\mathcal{M}}^1_{101}(t) \right]. \tag{1.262}$$

–2– Results

In section (1.1.5), we have listed all the limitations of Arthur G. Suvurov's approach. Our next goal is to see if, with these limitations, we can find other results for different metrics and theories.

2.1 Numerical solution

2.1.1 $f(X) = X + \alpha X^2 + \frac{\alpha^2 X^3}{4}$

We want to see if the metric 1.15 is also a solution for a different theory

$$f(X) = X + \alpha X^2 + \frac{\alpha^2 X^3}{4}. \tag{2.1}$$

Let's start by studying the first derivative of the function and looking for an X_0 that is both a zero and a local extremum

$$f'(X) = 1 + 2\alpha X + \frac{3\alpha^2 X^2}{4}, \tag{2.2}$$

the condition $f(X_0) = f'(X_0) = 0$ is satisfied for $X_0 = -\dfrac{2}{\alpha}$.
We then need to calculate

$$F(\phi)\, R + V(\phi) - \omega(\phi)\, \nabla_\alpha \phi \nabla^\alpha \phi = X_0 = -\frac{2}{\alpha}. \tag{2.3}$$

We start by using the Brans-Dicke condition $F(\phi) = \phi$, $V(\phi) = 0$, $\omega(\phi) = \phi^{-1}$

$$\phi R - \frac{1}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = -\frac{2}{\alpha}. \tag{2.4}$$

As we did before, we can consider the scalar field time and azimuth independent $\phi \equiv \phi(r, \theta)$.
So the only metric elements we need are g^{rr} and $g^{\theta\theta}$. Using equation 1.21 and 2.3 we have:

$$\begin{aligned} &-\frac{2M^3\epsilon}{r^3\Sigma}\phi(r,\theta) - \frac{1}{\phi(r,\theta)}\left[\frac{\Delta(r)}{\Sigma}(\partial_r\phi)^2 + \frac{1}{\Sigma}(\partial_\theta\phi)^2\right] = -\frac{2}{\alpha} \\ \implies &\frac{2M^3\epsilon}{r^3}\phi^2(r,\theta) + \Delta(r)(\partial_r\phi)^2 + (\partial_\theta\phi)^2 - \frac{2}{\alpha}\Sigma\phi(r,\theta) = 0. \end{aligned} \tag{2.5}$$

This partial differential equation is not analytical, but we can solve it numerically by imposing the boundary condition:

$$\phi(r, 0) = \phi(r, \pi). \tag{2.6}$$

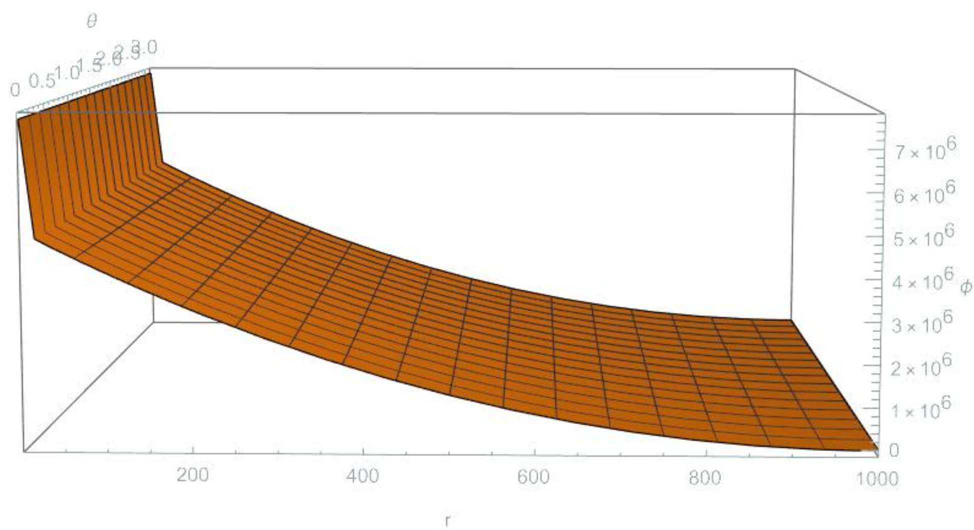


Figure 2.1: Radial $\phi(r,\theta)$ solution for $M = 1$, $a = 0.9$, $\epsilon = -0.2$,$\alpha = 0.1$

2.1.2 Johannsen Metric

We want to understand if it is possible to use the same approach to verify if a different metric is a solution for a specific theory.
In this case, we will use the generalized Kerr metric presented by Tim Johannsen [32], and we want to understand if it is a solution for the scalar-tensor theory:

$$A = k\int d^4x\sqrt{-g}f\left(F\left(\phi\right)R + V\left(\phi\right) - \omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha\phi\right) = k\int d^4x\sqrt{-g}f\left(X\right), \tag{2.7}$$

Where

$$f(X) = X^{1+\delta}. \tag{2.8}$$

The equations of motions with respect to the metric and the scalar field are:

$$\begin{cases} \dfrac{\delta A}{\delta \phi} = f'(X)\left[2\omega(\phi)\Box\phi + \dfrac{d\omega(\phi)}{d\phi}\nabla_\alpha\phi\nabla^\alpha\phi + R\dfrac{dF(\phi)}{d\phi} + \dfrac{dV(\phi)}{d\phi}\right] \\ +2\omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha f'(X) = 0, \\ \\ \dfrac{\delta A}{\delta g^{\mu\nu}} = F(\phi)f'(X)R_{\mu\nu} - \dfrac{f(X)}{2}g_{\mu\nu} + g_{\mu\nu}\Box\left[F(\phi)f'(X)\right] - \nabla_\mu\nabla_\nu\left[F(\phi)f'(X)\right] \\ -\omega(\phi)f'(X)\nabla_\mu\nabla_\nu\phi = 0. \end{cases} \tag{2.9}$$

They are both satisfied for $f(X_0) = f'(X_0) = 0$, that is true for $X_0 = 0$.

$$F\left(\phi\right)R + V\left(\phi\right) - \omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha\phi = 0. \tag{2.10}$$

We will use the Brans-Dicke conditions $F(\phi) = \phi$, $\omega(\phi) = \phi^{-1}$, $V(\phi) = 0$.

$$R\phi - \frac{1}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = 0. \tag{2.11}$$

Generalized Metric

The generalized metric presented by Tim Johannsen is obtained with four deviating functions that modify the Kerr metric:

$$\begin{cases} g_{tt} = -\dfrac{\tilde{\Sigma}\left[\Delta - a^2 A_2(r)^2\sin^2(\theta)\right]}{\left[\left(r^2+a^2\right)A_1(r) - a^2A_2(r)\sin^2(\theta)\right]^2}, \\ \\ g_{t\phi} = -\dfrac{a\left[\left(r^2+a^2\right)A_1(r)A_2(r) - \Delta\right]\tilde{\Sigma}\sin^2(\theta)}{\left[\left(r^2+a^2\right)A_1(r) - a^2A_2(r)\sin^2(\theta)\right]^2}, \\ \\ g_{rr} = \dfrac{\tilde{\Sigma}}{\Delta A_5(r)}, \\ \\ g_{\theta\theta} = \tilde{\Sigma}, \\ \\ g_{\phi\phi} = \dfrac{\left[\left(r^2+a^2\right)A_1^2(r) - a^2\Delta\sin^2(\theta)\right]\tilde{\Sigma}\sin^2(\theta)}{\left[\left(r^2+a^2\right)A_1(r) - a^2A_2(r)\sin^2(\theta)\right]^2}, \end{cases} \tag{2.12}$$

where

$$\Delta = r^2 - 2Mr + a^2, \tag{2.13}$$

$$\tilde{\Sigma} = \underbrace{\Sigma}_{r^2+a^2\cos^2(\theta)} + f(r), \tag{2.14}$$

$$A_1(r) = 1 + \sum_{n=3}^{\infty} \alpha_{1n} \left[\frac{M}{r}\right]^n, \tag{2.15}$$

$$A_2(r) = 1 + \sum_{n=2}^{\infty} \alpha_{2n} \left[\frac{M}{r}\right]^n, \tag{2.16}$$

$$A_5(r) = 1 + \sum_{n=2}^{\infty} \alpha_{5n} \left[\frac{M}{r}\right]^n. \tag{2.17}$$

This metric is asymptotically flat and reduces to the Kerr metric when all the deviating functions vanish.
At the lowest order of the deviation functions, the metric depends on four parameters in addition to the mass$M$ and the spin a: α_{13},α_{22}, α_{52}, ϵ_3.
However, for simplicity, we will consider two examples with a single deviation function:

2.1.3 $A_1(r)$

Metric elements

We consider the function $A_1(r)$ in this case. The condition is:

$$\begin{cases} f(r) = g(\theta) = 0, \\ \\ A_i = 1 \qquad\qquad \forall i \neq 1, \end{cases} \tag{2.18}$$

while

$$A_1(r) = 1 + \sum_{n=3}^{\infty} \alpha_{1n} \left[\frac{M}{r}\right]^n. \tag{2.19}$$

We will consider the firm terms of the series:

$$A_1(r) = 1 + \frac{\alpha_{13} M^3}{r^3}. \tag{2.20}$$

The metric elements are the following:

$$\begin{cases} g_{tt} = -\dfrac{\Sigma\left[\Delta - a^2\sin^2(\theta)\right]}{\left[\left(r^2+a^2\right)A_1(r) - a^2\sin^2(\theta)\right]^2}, \\[2ex] g_{t\phi} = g_{\phi t} = -\dfrac{a\left[\left(r^2+a^2\right)A_1(r) - \Delta\right]\Sigma\sin^2(\theta)}{\left[\left(r^2+a^2\right)A_1(r) - a^2\sin^2(\theta)\right]^2}, \\[2ex] g_{rr} = \dfrac{\Sigma}{\Delta}, \\[2ex] g_{\theta\theta} = \Sigma, \\[2ex] g_{\phi\phi} = \dfrac{\left[\left(r^2+a^2\right)A_1^2(r) - a^2\Delta\sin^2(\theta)\right]\Sigma\sin^2(\theta)}{\left[\left(r^2+a^2\right)A_1(r) - a^2\sin^2(\theta)\right]^2}. \end{cases} \tag{2.21}$$

We can solve (2.11) assuming ϕ to be time and azimuth independent. We will need the metric element g^{rr} and $g^{\theta\theta}$:

$$\begin{cases} g^{rr} = \dfrac{\Delta}{\Sigma}, \\[2ex] g^{\theta\theta} = \dfrac{1}{\Sigma}. \end{cases} \tag{2.22}$$

So (2.11) becomes:

$$R\phi - \frac{1}{\phi}\left[\frac{\Delta}{\Sigma}\left(\partial_r\phi\right)^2 + \frac{1}{\Sigma}\left(\partial_\theta\phi\right)^2\right] = 0. \tag{2.23}$$

The solution we obtain is

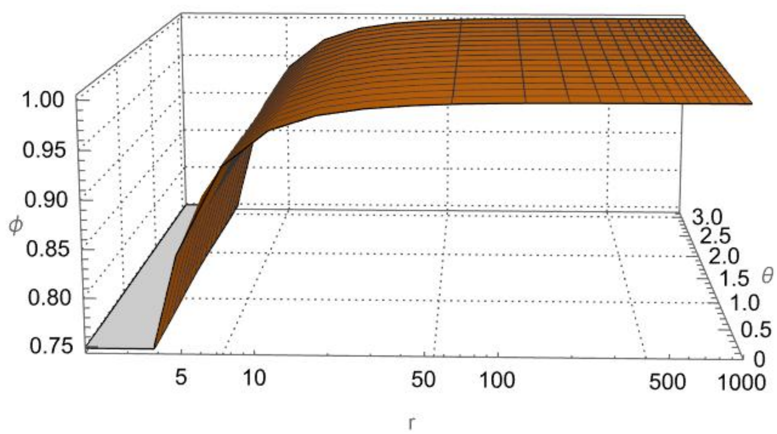


Figure 2.2: $\phi(r,\theta)$ solution for $M = 1$,$a = 0.9$,$\alpha = .1$

2.1.4 $A_5(r)$

In this case, we have:

$$\begin{cases} f(r) = g(\theta) = 0, \\ \\ A_i = 1 \qquad\qquad\qquad \forall i \neq 5. \end{cases} \tag{2.24}$$

$$A_1(r) = 1 + \sum_{n=3}^{\infty} \alpha_{1n} \left[\frac{M}{r}\right]^n . \tag{2.25}$$

The metric is:

$$\begin{cases} g_{tt} = -\left(1 - \dfrac{2Mr}{\Sigma}\right), \\ g_{t\phi} = -\dfrac{2Mar\sin^2(\theta)}{\Sigma}, \\ g_{rr} = \dfrac{\Sigma}{\Delta A_5(r)}, \\ g_{\theta\theta} = \dfrac{\Sigma}{\Delta}, \\ g_{\phi\phi} = \left[r^2 + a^2 + \dfrac{2Ma^2 r\sin^2(\theta)}{\Sigma}\right]\sin^2(\theta). \end{cases} \tag{2.26}$$

The inverse metric elements are:

$$\begin{cases} \mathrm{g}^{\mathrm{tt}} = \frac{4\Sigma}{Mr\cos(2\theta)+7Mr-4\Sigma}, \\ \mathrm{g}^{\mathrm{t}\varphi} = \mathrm{g}^{\varphi\mathrm{t}} = \frac{2\Sigma}{aMr\cos(2\theta)+7aMr-4a\Sigma}, \\ \mathrm{g}^{\mathrm{rr}} = \frac{A_5(r)\Delta(r)}{\Sigma}, \\ \mathrm{g}^{\theta\theta} = \frac{\Delta(r)}{\Sigma}, \\ \mathrm{g}^{\varphi\varphi} = \frac{\Sigma\csc^2(\theta)(\Sigma-2Mr)}{a^2Mr\left(Mr\sin^2(\theta)-4Mr+2\Sigma\right)}. \end{cases} \tag{2.27}$$

We will again understand if the metric (2.26) is a solution for the theory (2.8). The condition $f(X_0) = f'(X_0) = 0$ is satisfied for $X_0 = 0$

We suppose that the field ϕ time and azimuth is independent. $\phi \equiv \phi(r,\theta)$. Using the Brans-Dicke condition, we are left with:

$$\begin{aligned} 0 &= \phi(r,\theta)R + \frac{1}{\phi(r,\theta)}\left[(\partial_r\phi)^2 g^{rr} + (\partial_\theta\phi)^2 g^{\theta\theta}\right] \\ &= \phi(r,\theta)R + \frac{1}{\phi(r,\theta)}\left[\frac{A_5(r)\Delta(r)}{\Sigma}(\partial_r\phi)^2 + \frac{\Delta(r)}{\Sigma}(\partial_\theta\phi)^2\right]. \end{aligned} \tag{2.28}$$

Ricci Scalar

The Ricci Scalar obtained with Mathematica is:

$$
\begin{aligned}
R = & \frac{-2M^2\alpha}{r^3\left(a^2+2r^2+a^2\cos\left[2\theta\right]\right)^3}\left\{3a^4M+4a^2\left(2M-3r\right)r^2\right. \\
& \left. +8\left(3M-r\right)r^4+4a^2[a^2M+r^2(2M+r)]\cos[2\theta]+a^4M\cos[4\theta]\right\},
\end{aligned}
\tag{2.29}
$$

We can write the denominator as:

$$
a^2+2r^2+a^2\cos\left[2\theta\right]=a^2+2r^2+2a^2\cos^2\left[\theta\right]-a^2=2\Sigma, \tag{2.30}
$$

So the Scalar Curvature is:

$$
\begin{aligned}
R = & -\frac{M^2\alpha}{4r^3\Sigma^3}\left\{3a^4M+4a^2\left(2M-3r\right)r^2+8\left(3M-r\right)r^4\right. \\
& \left. +4a^2[a^2M+r^2(2M+r)]\cos[2\theta]+a^4M\cos[4\theta]\right\}.
\end{aligned}
\tag{2.31}
$$

At this point, we can now solve numerically (2.29):

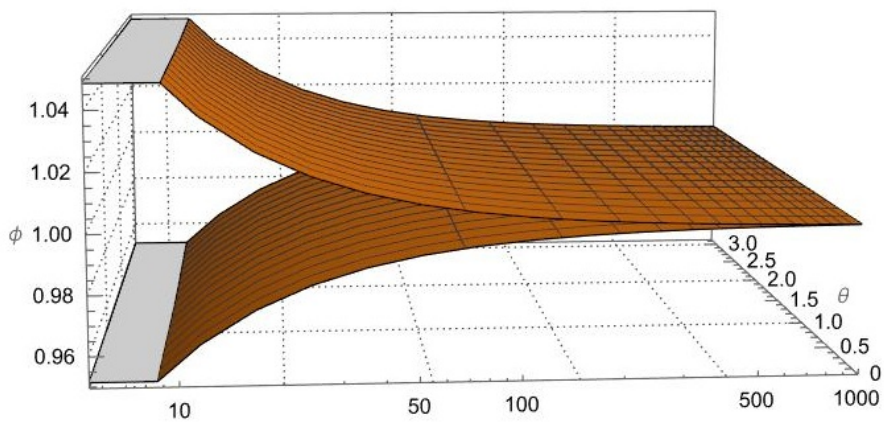


Figure 2.3: $\phi(r,\theta)$ solution for $M=1$,$a=0.9$,$\alpha=-0.1$

2.2 Analytical solution

In this section, we will derive an analytical solution for the metric

$$
\begin{aligned}
ds^2 = & \frac{a^2\sin^2\left(\theta\right)-\Delta}{\Sigma}dt^2-\frac{2a\sin^2\left(\theta\right)\left(a^2+r^2-\Delta\right)}{\Sigma}dtd\varphi \\
& +\frac{\Sigma}{\Delta}dr^2+\Sigma d\theta^2+\frac{\left(a^2+r^2\right)^2-a^2\sin^2\left(\theta\right)\Delta}{\csc\left(\theta\right)\Sigma}d\varphi^2,
\end{aligned}
\tag{2.32}
$$

with the theory

$$f(X) = X + \alpha X^2 + \frac{\alpha^2 X^3}{4}. \tag{2.33}$$

As we have seen before, we want to try to resolve equation 2.3. The difference is that we will try with a set of conditions slightly different than the Brans-Dicke

$$\begin{gathered} F(\phi) R + V(\phi) - \omega(\phi)\left[\frac{\Delta(r)}{\Sigma}(\partial_r \phi)^2 + \frac{1}{\Sigma}(\partial_\theta \phi)^2\right] = -\frac{2}{\alpha} \qquad (2.34) \\ \implies -\frac{2M^3\epsilon}{r^3} F(\phi) + \left[\frac{2}{\alpha} + V(\phi)\right]\Sigma - \omega(\phi)\left[\Delta(r)(\partial_r\phi)^2 + (\partial_\theta\phi)^2\right] = 0. \end{gathered}$$

Let's try imposing again $F(\phi) = \phi$ and $\omega(\phi) = \phi^{-1}$, but leaving $V(\phi) \neq 0$:

$$\begin{aligned} & -\frac{2M^3\epsilon}{r^3}\phi + \left[\frac{2}{\alpha} + V(\phi)\right]\Sigma - \frac{1}{\phi}\left[\Delta(r)(\partial_r\phi)^2 + (\partial_\theta\phi)^2\right] \\ = & -\frac{2M^3\epsilon}{r^3} + \left[\frac{2}{\alpha} + V(\phi)\right]\frac{\Sigma}{\phi} - \frac{1}{\phi^2}\left[\Delta(r)(\partial_r\phi)^2 + (\partial_\theta\phi)^2\right] = 0. \end{aligned} \tag{2.35}$$

We can now impose the following condition on the potential $V(\phi)$

$$V(\phi) = \frac{2}{\alpha}(k\phi - 1). \tag{2.36}$$

Where:

$$k : dim\left[\frac{k}{\alpha}\right] = 0. \tag{2.37}$$

We are left with:

$$\begin{aligned} & -\frac{2M^3\epsilon}{r^3} + \left[\frac{2}{\alpha} + \frac{2}{\alpha}(k\phi - 1)\right]\frac{\Sigma}{\phi} - \frac{1}{\phi^2}\left[\Delta(r)(\partial_r\phi)^2 + (\partial_\theta\phi)^2\right] \\ = & -\frac{2M^3\epsilon}{r^3} + \left[\frac{2}{\alpha} + \frac{2}{\alpha}k\phi - \frac{2}{\alpha}\right]\frac{\Sigma}{\phi} - \frac{1}{\phi^2}\left[\Delta(r)(\partial_r\phi)^2 + (\partial_\theta\phi)^2\right] = 0. \end{aligned} \tag{2.38}$$

This equation can be solved by separation of variables:

$$\phi(r, \theta) = R(r)\Theta(\theta), \tag{2.39}$$

$$-\frac{2M^3\epsilon}{r^3} + \frac{2}{\alpha}k\cancel{R(r)\Theta(\theta)}\frac{\Sigma}{\cancel{R(r)\Theta(\theta)}} - \frac{1}{[R(r)\Theta(\theta)]^2}\left[\Delta(r)\Theta(\theta)(\partial_r R)^2 + R(r)(\partial_\theta\Theta)^2\right] = 0 \tag{2.40}$$

Using the relation $\Sigma = r^2 + a^2\cos^2(\theta)$, we can solve them individually

$$\begin{cases} \dfrac{2M^3\epsilon}{r^3} - \dfrac{2k}{\alpha}r^2 + \dfrac{\Delta(r)}{R^2(r)}\left[\partial_r R\right]^2 = C^2, \\ \\ \dfrac{2k}{\alpha}a^2\cos^2(\theta) - \dfrac{\left[\partial_\theta \Theta\right]^2}{\Theta^2(\theta)} = -C^2. \end{cases} \tag{2.41}$$

We can solve analytically the second equation. For $C = 0$,we have:

$$\frac{\left[\partial_\theta \Theta\right]^2}{\Theta^2(\theta)} = \frac{2k}{\alpha}a^2\cos^2(\theta). \tag{2.42}$$

We can use the ansatz:

$$\Theta(\theta) = \exp\left[a\sqrt{\frac{2k}{\alpha}}\sin[\theta]\right], \tag{2.43}$$

the first derivative is

$$\Theta'(\theta) = -a\sqrt{\frac{2k}{\alpha}}\cos[\theta]\exp\left[a\sqrt{\frac{2k}{\alpha}}\sin[\theta]\right]. \tag{2.44}$$

So equation (2.42) is satisfied.

This also satisfied the boundary condition $\Theta(0) = \Theta(\pi)$. Now, we can work on the radial part. Let's call $\dfrac{k}{\alpha} = \beta$

$$\frac{2M^3\epsilon}{r^3} - 2\beta r^2 + \frac{\Delta(r)}{R^2(r)}\left[\partial_r R\right]^2 = 0, \tag{2.45}$$

$$\frac{\partial_r R}{R(r)} = \partial_r \ln(R), \tag{2.46}$$

$$\frac{2M^3\epsilon}{r^3} - 2\beta r^2 + \left[r^2 + 2Mr + a^2 + \epsilon\frac{M^3}{r}\right]\left[\partial_r \ln(R)\right]^2 = 0, \tag{2.47}$$

$$\left[r^2 + 2Mr + a^2 + \epsilon\frac{M^3}{r}\right]\left[\partial_r \ln(R)\right]^2 = 2\beta r^2 - \frac{2M^3\epsilon}{r^3} \tag{2.48}$$

$$\begin{aligned}
\implies \quad & \frac{1}{r}\left[r^3 + 2Mr^2 + a^2 r + \epsilon M^3\right]\left[\partial_r \ln(R)\right]^2 = \frac{1}{r^3}\left[2\beta r^5 - 2M^3\epsilon\right] \\
\implies \quad & \left[\partial_r \ln(R)\right]^2 = \frac{1}{r^2}\frac{2\beta r^5 - 2M^3\epsilon}{r^3 + 2Mr^2 + a^2 r + \epsilon M^3} \\
\implies \quad & \frac{d}{dr}\ln(R) = \frac{1}{r}\sqrt{\frac{2\beta r^5 - 2M^3\epsilon}{r^3 + 2Mr^2 + a^2 r + \epsilon M^3}} \\
\implies \quad & \ln(R) = \int dr\frac{1}{r}\sqrt{\frac{2\beta r^5 - 2M^3\epsilon}{r^3 + 2Mr^2 + a^2 r + \epsilon M^3}}.
\end{aligned}$$

Up to leading order, we can solve this integral away from the source

$$ln(R) = \int dr \sqrt{\frac{2\beta r^5}{r^5}}. \tag{2.49}$$

From this, we obtain an analytical expression of $R(r)$

$$R(r) = Ae^{\sqrt{2\beta}r}. \tag{2.50}$$

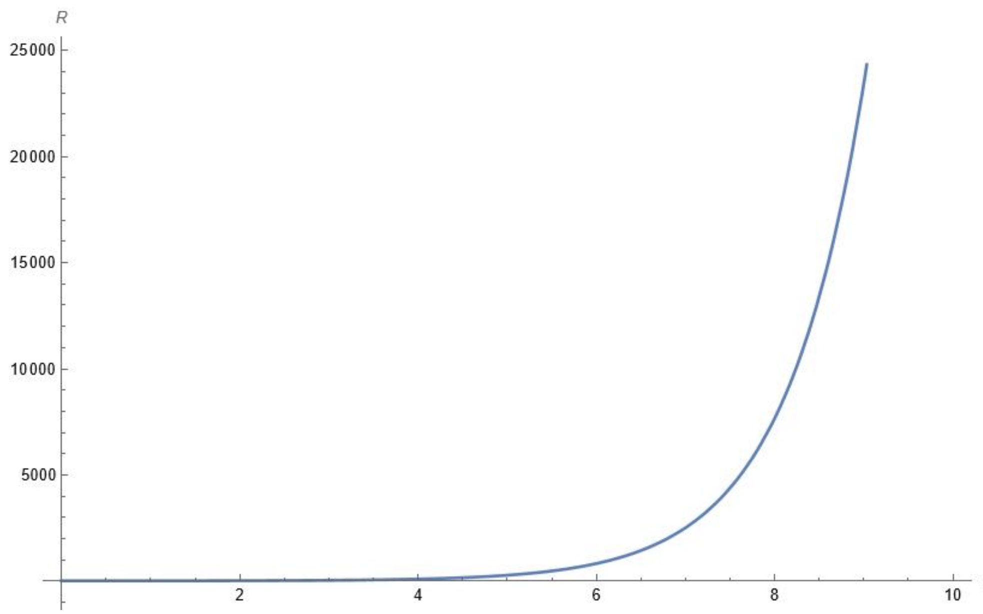


Figure 2.4: Radial ϕ solution for $A = 1$,$\beta = 1/0.8$

As we can see, the scalar field we obtain diverges when r goes to infinity, and this tells us that the metric (2.32) is not a solution for the theory (2.33)

2.2.1 Is it possible to compute the quadrupole formula?

We can try to linearize the equations of motion of the theory and then try to compute the quadrupole formula. The equations of motion are

$$\begin{aligned} 0 &= F(\phi)f'(X)R_{\mu\nu} - \frac{f(X)}{2}g_{\mu\nu} + g_{\mu\nu}\Box\left[F(\phi)f'(X)\right] - \nabla_\mu\nabla_\nu\left[F(\phi)f'(X)\right] \\ &- \omega(\phi)f'(X)\nabla_\mu\phi\nabla_\nu\phi, \end{aligned} \tag{2.51}$$

$$\begin{aligned} 0 &= f'(X)\left[2\omega(\phi)\Box\phi + \frac{d\omega(\phi)}{d\phi}\nabla_\alpha\phi\nabla^\alpha\phi + R\frac{dF(\phi)}{d\phi} + \frac{dV(\phi)}{d\phi}\right] \\ &+ 2\omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha f'(X). \end{aligned} \tag{2.52}$$

We will study the case with a source $T_{\mu\nu}$, and instead of the tensor element, it's easier to study the trace

$$\frac{8\pi G}{c^4}T = F(\phi)f'(X)R - 4f(X) + 3\Box\left[F(\phi)f'(X)\right] - \omega(\phi)f'(X)\nabla_\alpha\phi\nabla^\alpha\phi. \quad (2.53)$$

The next step is to impose the Brans-Dicke conditions

$$\frac{8\pi G}{c^4}T = \phi f'(X)R - 4f(X) + 3\Box\left[\phi f'(X)\right] - \frac{1}{\phi}f'(X)\nabla_\alpha\phi\nabla^\alpha\phi. \quad (2.54)$$

To linearize we need to take the first order of the theory $f(X)$ and the derivative $f'(X)$

$$f'(X) \approx 1 + 2\alpha X, \quad (2.55)$$
$$f(X) \approx X. \quad (2.56)$$

So we obtain

$$\frac{8\pi G}{c^4}T = \phi\left(1 + 2\alpha X\right)R - 4X + 3\Box\left[\phi + 2\alpha\phi X\right] - \frac{\nabla_\alpha\phi\nabla^\alpha\phi}{\phi}\left[1 + 2\alpha X\right], \quad (2.57)$$

where X is

$$X = \phi R + \frac{2}{\alpha}\left(k\phi - 1\right) - \frac{1}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi, \quad (2.58)$$

The linearization of this theory leads to some problems, in particular when substituting $\phi = \phi_0 + \delta\phi$, with $\delta\phi = const$, we obtain a problem that is not possible to solve with the Green function. Possible solutions to this problem can be studied in the future.

2.3 Application to Finsler Gravity

Finsler geometry is a straightforward generalization of Riemannian geometry[33, 34]. Instead of deriving the geometry of a manifold from a Riemannian metric g, its Levi-Civita connection and the corresponding induced length measure $F_g(X) = \sqrt{g(X,X)}$ for vectors X, the geometry is derived from a general 1-homogeneous length measure called the Finsler function and its Cartan non-linear connection.

$$ds^2 = f\left(x^i, dx^i\right). \quad (2.59)$$

In this theory, the Ricci curvature is described as the Riemann Curvature's trace on each tangent space.

The Finsler structure on a manifold M is defined as the function $\mathcal{F} : TM \to [0, \infty[$, which satisfies the below properties

1. $\mathcal{F}$ is a smooth function on the $TM/\{0\}$

2. $\mathcal{F}(x, cy) = c\mathcal{F}(x, y)$ for all $c > 0$

3. Strong Convexity

For the Finsler manifold, the geodesic equation is [35]

$$\frac{d^2x^\nu}{d\tau^2} + 2\mathcal{G}^\nu(x, y) = 0, \tag{2.60}$$

where

$$\mathcal{G}^\nu = \frac{1}{4} g^{\nu\omega} \left[\frac{\partial^2 \mathcal{F}^2}{\partial x^k \partial x^\omega} y^k - \frac{\partial \mathcal{F}^2}{\partial x^\omega} \right]. \tag{2.61}$$

The geometric invariant in the Finsler geometry is the Ricci scalar, and has the following form

$$R = R^\mu_\mu = \frac{1}{\mathcal{F}^2} \left[2\frac{\partial \mathcal{G}^\mu}{\partial x^\mu} - y^\nu \frac{\partial^2 \mathcal{G}^\mu}{\partial x^\mu \partial x^\nu} + 2\mathcal{G}^\nu \frac{\partial^2 \mathcal{G}^\mu}{\partial y^\nu \partial y^\mu} - \frac{\partial \mathcal{G}^\mu}{\partial y^\nu} \frac{\partial \mathcal{G}^\nu}{\partial y^\mu} \right]. \tag{2.62}$$

For our example, we will assume R to be constant and opposite to the cosmological constant Λ

$$R = -\Lambda. \tag{2.63}$$

2.3.1 Inverse problem of gravity

We want to see if a Finsler metric with the Ricci Scalar $R = -\Lambda$ is a solution for the theory $f(X) = X^2$

$$S = \int d^4x \sqrt{-g} X^2, \tag{2.64}$$

where X is always

$$X = F(\phi)R + V(\phi) - \omega(\phi)\nabla_\alpha\phi\nabla^\alpha\phi. \tag{2.65}$$

As we have seen, the equations of motion are

1. $F(\phi)f'(X)R_{\mu\nu} - \frac{f(X)}{2} g_{\mu\nu} + g_{\mu\nu}\Box\left[F(\phi)f'(X)\right]$
   $- \nabla_\mu\nabla_\nu\left[F(\phi)f'(X)\right] - \omega(\phi)f'(X)\nabla_\mu\nabla_\nu\phi = 0,$

2. $f'(X)\left[2\omega(\phi)\Box\phi + \frac{d\omega(\phi)}{d\phi}\nabla_\alpha\phi\nabla^\alpha\phi + R\frac{dF(\phi)}{d\phi} + \frac{dV(\phi)}{d\phi}\right]$
   $+ 2\omega\left(\phi\right)\nabla_\alpha\phi\nabla^\alpha f'(X) = 0.$

For a theory like $f(X) = X^2$,these are satisfied when $X = 0$

$$F(\phi)R + V(\phi) - \omega(\phi)\nabla_\alpha\phi\nabla^\alpha\phi = 0, \tag{2.66}$$

with the Brans-Dicke conditions, it becomes

$$\phi R - \frac{1}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi = 0. \tag{2.67}$$

In our previous examples, we asked the scalar field ϕ to counterbalance the scalar curvature.
Here, we have both a trivial and a non-trivial solution.
Since the scalar field R is constant, if we ask ϕ to be constant too, we have

$$\phi\Lambda = 0. \tag{2.68}$$

That is true when $\phi = 0$. Otherwise, we have

$$0 = \phi^2 + \frac{1}{\Lambda}\nabla_\alpha\phi\nabla^\alpha\phi = \phi^2 + \frac{1}{\Lambda}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi. \tag{2.69}$$

We will assume the field ϕ to be time and azimuth independent

$$\phi \equiv \phi(r, \theta). \tag{2.70}$$

The metric is spherically symmetric and has the following expression

$$g_{\mu\nu} = \begin{pmatrix} \mathcal{B}(r) & 0 & 0 & 0 \\ 0 & -\mathcal{A}(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}, \tag{2.71}$$

$$g^{\mu\nu} = \begin{pmatrix} \mathcal{B}^{-1}(r) & 0 & 0 & 0 \\ 0 & -\mathcal{A}^{-1}(r) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2(\theta)} \end{pmatrix}. \tag{2.72}$$

Where

$$\mathcal{A}(r) = \left(\lambda - \frac{\Lambda}{3}r^2 - \frac{2GM}{a\lambda r}\right)^{-1}, \tag{2.73}$$

$$\mathcal{B}(r,t) = \alpha(t)\left(a\lambda - \frac{a\Lambda}{3}r^2 - \frac{2GM}{\lambda r}\right). \tag{2.74}$$

Where a is an integral constant, λ is the flag curvature while we will put $\alpha(t) = 1$, in this way the metric will be time independent.

We can explicit equation 2.69 as

$$\begin{aligned}
0 =&\phi^2 + \frac{1}{\Lambda}\left[g^{rr}\left(\partial_r\phi\right)^2 + g^{\theta\theta}\left(\partial_\theta\phi\right)^2\right] = 0 = \phi^2 - \frac{1}{\Lambda}\left[\mathcal{A}(r)\left(\partial_r\phi\right)^2 + \frac{1}{r^2}\left(\partial_\theta\phi\right)^2\right] \\
=&\phi^2 - \frac{1}{\Lambda}\left[\left(\lambda - \frac{\Lambda}{3}r^2 - \frac{2GM}{a\lambda r}\right)\left(\partial_r\phi\right)^2 + \frac{1}{r^2}\left(\partial_\theta\phi\right)^2\right] \\
=&\Lambda - \frac{1}{\phi^2}\left[r^2\left(\lambda - \frac{\Lambda}{3}r^2 - \frac{2GM}{a\lambda r}\right)\left(\partial_r\phi\right)^2 + \left(\partial_\theta\phi\right)^2\right],
\end{aligned} \tag{2.75}$$

we can rewrite the field $\phi(r,\theta)$ as

$$\phi(r,\theta) = R(r)\Theta(\theta), \tag{2.76}$$

$$0 = \Lambda - r^2\left(\lambda - \frac{\Lambda}{3}r^2 - \frac{2GM}{a\lambda r}\right)\left(\frac{\partial_r R}{R(r)}\right)^2 - \left(\frac{\partial_\theta\Theta}{\Theta(\theta)}\right)^2. \tag{2.77}$$

It is then possible to separate the variables

$$\begin{cases} \left(\dfrac{\partial_\theta\Theta}{\Theta(\theta)}\right)^2 = C, \\ \\ \Lambda r^2 - r^2\left(\lambda - \dfrac{\Lambda}{3}r^2 - \dfrac{2GM}{a\lambda r}\right)\left(\dfrac{\partial_r R}{R(r)}\right)^2 = -C. \end{cases} \tag{2.78}$$

Let's start with the first one

$$\left(\frac{\partial_\theta\Theta}{\Theta(\theta)}\right)^2 = C \implies \partial_\theta \ln(\Theta) = \sqrt{C} \implies \ln(\Theta) = \sqrt{C}\theta + const \implies \Theta(\theta) = Ae^{\sqrt{C}\theta}. \tag{2.79}$$

The boundary condition to impose is

$$\Theta(0) = \Theta(\pi), \tag{2.80}$$

$$\Theta(0) = A = Ae^{\sqrt{C}\pi} \implies C = 0. \tag{2.81}$$

Now, we can study the radial part

$$\begin{aligned}
0 = \Lambda r^2 - r^2\left(\lambda - \frac{\Lambda}{3}r^2 - \frac{2GM}{a\lambda r}\right)\left(\frac{\partial_r R}{R(r)}\right)^2 \\
\implies \Lambda = \left(\lambda - \frac{\Lambda}{3}r^2 - \frac{2GM}{a\lambda r}\right)\left(\frac{\partial_r R}{R(r)}\right)^2 \\
\implies \left(\frac{\partial_r R}{R(r)}\right)^2 = \frac{\Lambda}{\lambda - \frac{\Lambda}{3}r^2 - \frac{2GM}{a\lambda r}}
\end{aligned} \tag{2.82}$$

So, the final equation we have is

$$\left(\frac{R'(r)}{R(r)}\right)^2 + \frac{\Lambda}{\frac{\Lambda}{3}r^2 + \frac{2GM}{a\lambda r} - \lambda} = 0 \tag{2.83}$$

We can solve this problem numerically, and we can obtain

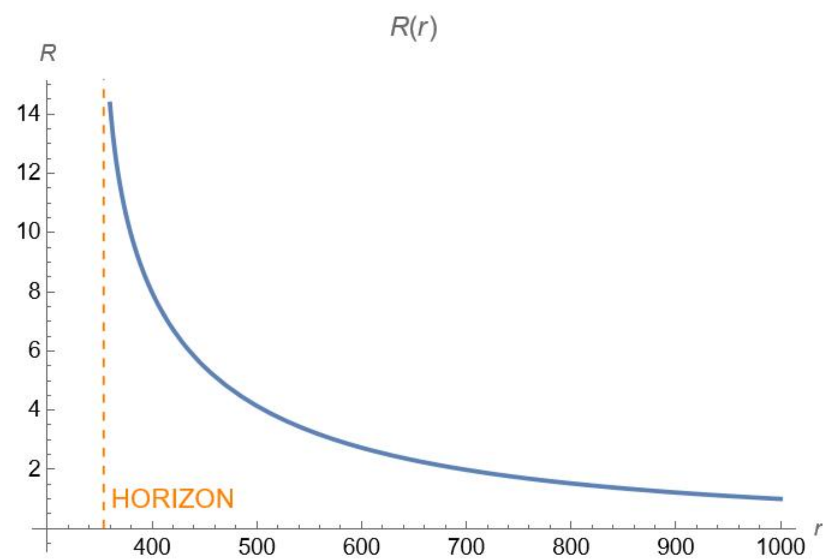


Figure 2.5: Radial ϕ solution for $G = 1$, $M = 1$, $\lambda = 0.8$,$\Lambda = 0.19x10^{-4}$,$a = 397, 35$

As we can see from this plot, the scalar field looks physically suitable, asymptotically, it goes to 1

$$\lim_{r\to\infty} R\left(r\right) = 1, \tag{2.84}$$

and it encounters a horizon for $r = 353, 81m$. This is in agreement with what is written in the article "Black Hole Solutions with Constant Ricci Scalar in a Model of Finsler Gravity" [35], it is important to notice that the horizon we found is not an event horizon, but a cosmological horizon.

What is the Cosmological Horizon?

The cosmological horizon, also known as the particle horizon, is a concept in cosmology that marks the maximum distance from which light has had time to travel to the observer in the age of the universe[36].
It delineates the boundary of the observable universe. Anything beyond this horizon is not observable because the light from such regions hasn't had enough time to reach us since the beginning of the universe.

To determine the cosmological horizon, one must consider the universe's age and the speed of light. Mathematically, it can be derived by integrating the speed of light over the universe's age, taking into account the universe's expansion. The formula is given by:

$$d_H = c\int_0^{t_0} dt\frac{1}{a(t)}, \tag{2.85}$$

where

- d_H is the cosmological horizon distance,
- c is the speed of light,
- t_0 is the current age of the universe,
- $a(t)$ is the scale factor of the universe at time t.

2.3.2 Quadrupole formula

We will now linearize the equations of motion to compute the then quadrupole formula. We will study the trace of the equations of motion, with the presence of a source

$$\begin{aligned}
\frac{8\pi G}{c^4}T =& F(\phi)f'(X)R - 2f(X) + 3\Box_g\left[F(\phi)f'(X)\right] - \omega(\phi)f'(X)\nabla_\alpha\phi\nabla^\alpha\phi \qquad (2.86)\\
=& 2X\phi R - 2X^2 + 6\Box_g\left[\phi X\right] - \frac{2X}{\phi}\nabla_\alpha\nabla^\alpha\phi\\
=& -2\Lambda\phi\left[-\Lambda\phi - \frac{1}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi\right] - 2\left[-\Lambda\phi - \frac{1}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi\right]^2 + 6\Box_g\left[-\Lambda\phi^2 - \nabla_\alpha\phi\nabla^\alpha\phi\right]\\
&- 2\left[-\Lambda - \frac{1}{\phi^2}\nabla_\alpha\phi\nabla^\alpha\phi\right]\nabla_\alpha\phi\nabla^\alpha\phi\\
=& 2\Lambda^2\phi^2 + 2\Lambda\nabla_\alpha\phi\nabla^\alpha\phi - 2\Lambda^2\phi^2 - \frac{2}{\phi^2}\left(\nabla_\alpha\phi\nabla^\alpha\phi\right)^2 - 4\Lambda\nabla_\alpha\phi\nabla^\alpha\phi - 6\Box_\eta\left[\Lambda\phi^2 + \nabla_\alpha\phi\nabla^\alpha\phi\right]\\
&+ 2\Lambda\nabla_\alpha\phi\nabla^\alpha\phi + \frac{2}{\phi^2}\left(\nabla_\alpha\phi\nabla^\alpha\phi\right)^2 = -6\Box_\eta\left[\Lambda\phi^2 + \nabla_\alpha\phi\nabla^\alpha\phi\right]
\end{aligned}$$

So we have

$$\Box_\eta\left[\Lambda\phi^2 + \nabla_\alpha\phi\nabla^\alpha\phi\right] = -\frac{4\pi G}{3c^4}T. \tag{2.87}$$

Linearization

To obtain a quadrupole formula for this problem, we need to linearize this equation, with the assumption

$$\phi = \phi_0 + \delta\phi, \tag{2.88}$$

where $\phi_0 = const$.

$$\phi^2 \sim \phi_0^2 + 2\phi_0\delta\phi, \tag{2.89}$$

$$\nabla_\alpha\phi = \nabla_\alpha\delta\phi_0. \tag{2.90}$$

With this assumption, equation 2.87 becomes

$$\Box_\eta\left[\Lambda\left(\phi_0^2 + 2\phi_0\delta\phi\right) + \nabla_\alpha\delta\phi_0\nabla^\alpha\delta\phi_0\right] = -\frac{4\pi G}{3c^4}T. \tag{2.91}$$

We can discard the term $\nabla_\alpha\delta\phi_0\nabla^\alpha\delta\phi_0$ because it's of a higher order

$$-\frac{4\pi G}{3c^4}T = \Box_\eta\left[\Lambda\phi_0^2 + 2\Lambda\phi_0\delta\phi\right] = 2\Lambda\phi_0\Box_\eta\delta\phi. \tag{2.92}$$

We can then solve the following equation

$$\Box_\eta\delta\phi = -\frac{1}{\Lambda\phi_0}\frac{2\pi G}{3c^4}T. \tag{2.93}$$

This is the same situation found in the derivation of the quadrupole formula in $f(R)$ theories, the solution is

$$\delta\phi = \frac{G}{6\Lambda\phi_0 c^4}\int_{\mathbb{R}^3} d^3x'\frac{T\left(x', t - |x - x'|/c\right)}{|x - x'|}, \tag{2.94}$$

let's focus on $\dfrac{T\left(x', t - |x - x'|/c\right)}{|x - x'|}$. This can be written as

$$\int_{\mathbb{R}^3} d^3y\frac{T\left(x', t - |x - x'|/c\right)}{|x - x'|}\delta^{(3)}(y - x') = \int_{\mathbb{R}^3} g(x, x', y)\delta^{(3)}(y - x'). \tag{2.95}$$

Similarly to what we did before, we can expand

$$\begin{aligned} g(x, x', y) &= g(x, 0, y) + \frac{\partial g}{\partial x'^i}x'^i + \frac{1}{2}\frac{\partial^2 g}{\partial x'^i\partial x'^j}x'^i x'^j + ... \\ &= g(x, 0, y)\frac{\partial g}{\partial x^i}x'^i + \frac{1}{2}\frac{\partial^2 g}{\partial x^i\partial x^j}x'^i x'^j + ... \\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L\partial_L\frac{T\left(x', t - |x - x'|/c\right)}{|x - x'|}. \end{aligned} \tag{2.96}$$

All the derivatives are evaluated at $x' = 0$ so we can substitute $|x| = r$

$$g(x, x', y) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L\partial_L\frac{T\left(x', t - r/c\right)}{r}, \tag{2.97}$$

so we can write $\delta\phi$ as

$$\delta\phi = \frac{G}{6\Lambda\phi_0 c^4}\int_{\mathbb{R}^3} d^3x' \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x'^L \partial_L \frac{T(x', t-r/c)}{r} \tag{2.98}$$

we can rename

$$\tau \equiv t - r/c, \tag{2.99}$$

up to the second order, we have

$$\delta\phi = \frac{G}{6\Lambda\phi_0 c^4}\int_{\mathbb{R}^3} d^3x' \left[\frac{T(x',\tau)}{r} - x'^i \frac{\partial}{\partial x^i}\frac{T(x',\tau)}{r} + \frac{x'^i x'^j}{2}\frac{\partial^2}{\partial x^i \partial x^j}\frac{T(x',\tau)}{r}\right]. \tag{2.100}$$

Let's study the first derivative using the notation $\frac{\partial}{\partial x^i} = \partial_i$,

$$\begin{aligned}
\partial_i \frac{T(x',\tau)}{r} &= T(x',\tau)\,\partial_i \frac{1}{r} + \frac{1}{r}\partial_i T(x',\tau) \qquad (2.101)\\
&= -\frac{n_i}{r^2}T(x',\tau) + \frac{1}{r}\partial_t T(x',\tau)\,\partial_i \underbrace{(\tau + r/c)}_{t}\\
&= -\frac{n_i}{r^2}T(x',\tau) + \frac{1}{rc}\partial_t T(x',\tau)\,\partial_t T(x',\tau)\,\partial_i r\\
&= -\frac{n_i}{r^2}T(x',\tau) + \frac{1}{rc}\partial_t T(x',\tau)\,n_i\\
&= \frac{n_i}{r}\left[-\frac{T(x',\tau)}{r} + \frac{1}{c}\partial_t T(x',\tau)\right] = \frac{n_i}{r}\left[\frac{1}{c}\partial_t - \frac{1}{r}\right]T(x',\tau).
\end{aligned}$$

It's now possible to express $\delta\phi$ explicitly up to the first order

$$\begin{aligned}
\delta\phi &= \frac{G}{6\Lambda\phi_0 c^4}\int_{\mathbb{R}^3} d^3x' \left[\frac{1}{r} - x'^i \frac{n_i}{r}\left[\frac{1}{c}\partial_t - \frac{1}{r}\right]\right]T(x',\tau) \qquad (2.102)\\
&= \frac{G}{6\Lambda\phi_0 c^4 r}\int_{\mathbb{R}^3} d^3x' \left[1 - n_i\left(\frac{1}{c}\partial_t - \frac{1}{r}\right)x'^i\right]T(x',\tau).
\end{aligned}$$

We can define the momenta of the Energy-Momentum tensor

$$M^{I_n} = \frac{1}{c^2}\int_{\mathbb{R}^3} d^3x T^{00}(t,x) x^{I_n}, \tag{2.103}$$

and the quantities

$$\mathcal{M}^{I_n}_{ij}(t) = \frac{1}{r^i c^j}\frac{\partial^j}{\partial t^j}M^{I_n}. \tag{2.104}$$

We will define

$$\mathcal{M}(t) \equiv \mathcal{M}_{00}(t). \tag{2.105}$$

So when substituting into eq. (2.103) we obtain

$$\delta\phi = \frac{G}{6\Lambda\phi_0 c^2 r}\left[\mathcal{M}(t) - n_i\left(\mathcal{M}^i_{01}(t) - \mathcal{M}^i_{10}(t)\right)\right] = \frac{G}{6\Lambda\phi_0 c^2 r}\left[\mathcal{M}(t) - n_i\mathcal{D}^i\right]. \quad (2.106)$$

We can again study a two-body system fixed on the (x_1, x_2) plane. We can again write the mass density as

$$\begin{aligned}\rho(\boldsymbol{x}) &= \delta(x_3)\left[m_1\delta\left(x_1 - \frac{d}{2}\cos(\omega t)\right)\delta\left(x_2 - \frac{d}{2}\sin(\omega t)\right)\right.\\ &+ \left. m_2\delta\left(x_1 + \frac{d}{2}\cos(\omega t)\right)\delta\left(x_2 + \frac{d}{2}\sin(\omega t)\right)\right] \quad (2.107)\end{aligned}$$

and compute the momenta

$$M_1(t) = \frac{d}{2}(m_1 - m_2)\cos(\omega t), \quad (2.108)$$

$$M_2(t) = M_1\left(t - \frac{\pi}{2\omega}\right). \quad (2.109)$$

Now we can compute $\mathcal{M}_{ij}$

$$\begin{cases}\mathcal{M}^1_{10}(t) = \dfrac{d}{2r}(m_1 - m_2)\cos(\omega t), \\[2ex] \mathcal{M}^1_{01}(t) = \dfrac{d}{2c}(m_1 - m_2)\dfrac{d}{dt}\cos(\omega t), \\[2ex] \mathcal{M}^2_{10}(t) = \mathcal{M}^1_{10}(t - \dfrac{\pi}{2\omega}) = \dfrac{d}{2r}(m_1 - m_2)\cos\left(\omega\left(t - \dfrac{\pi}{2\omega}\right)\right), \\[2ex] \mathcal{M}^2_{01}(t) = \mathcal{M}^1_{01}(t - \dfrac{\pi}{2\omega}) = \dfrac{d}{2c}(m_1 - m_2)\dfrac{d}{dt}\cos\left(\omega\left(t - \dfrac{\pi}{2\omega}\right)\right).\end{cases} \quad (2.110)$$

With the same consideration made for the Radiated Power in $f(R)$ theories, we obtain the same results of equation 1.262

$$P = \frac{G\omega}{9\pi c}\int_0^{\mathcal{T}} dt\left[\left(\dot{\mathcal{M}}^1_{10}(t)\right)^2 + \left(\dot{\mathcal{M}}^1_{01}(t)\right)^2 - 2\dot{\mathcal{M}}^1_{10}(t)\dot{\mathcal{M}}^1_{01}(t)\right]. \quad (2.111)$$

–3–
Discussion and future works

CONTENTS: **3.1 Inverse problem**. **3.2 Radiated Power and Gravitational Waves** . **3.3 Criticalities and future perspectives.**.

In this thesis, we discussed the complete background to study an inverse problem of gravity and test it with Gravitational Waves.
Now we will summarize what we have done, discuss the results obtained, the criticalities and also possible future improvement

3.1 Inverse problem

The quest to extend general relativity (GR) arises from its limitations in describing phenomena at the quantum scale and certain cosmological observations. While GR has been extraordinarily successful in explaining large-scale structures and passing numerous precision tests, its incompatibility with quantum mechanics and the inability to fully account for dark energy and dark matter indicate the need for a more comprehensive theory of gravity. This thesis has explored various modifications to GR, focusing mainly on Scalar Tensor $f(X)$ theories and testing them with Gravitational Waves trying to present a complete description of gravitational phenomena.
From the Review of the article "A family of solution to the inverse problem: Building a theory around a metric" we were able to understand that it's possible to solve the inverse problem, this can be extremely useful, in particular when paired with cosmological observation, because it gives us the possibility to build a parametric metric with astrophysical data, and directly check if the metric we built is a solution for a specific theory.
We have then presented new solutions to the inverse problem, both numerical and analytical. Most of them are impossible to compute analytically, and we can only plot them numerically using two variables. Studying these plot, we were able to

understand what is the set of conditions to obtain a "good" scalar field ϕ, it has to diverge near a metric singularity and has to satisfy the condition

$$\lim_{r \to \infty} \phi = 1. \tag{3.1}$$

3.2 Radiated Power and Gravitational Waves

A significant portion of this thesis is dedicated to deriving the quadrupole formula and calculating the radiated power for the pulsar PSR J0737-3039 within both GR and $f(R)$ theories. GWs are ripples in the curvature of space-time that propagate outward from their source. Predicted by Einstein's theory of General Relativity, these waves were first directly detected by the LIGO and Virgo collaborations in 2015

In GR, GWs are generated by the acceleration of massive objects, particularly those in asymmetric configurations such as binary systems. The quadrupole formula is essential for understanding the emission of gravitational waves from such astrophysical sources. It is derived under the weak-field approximation, where the gravitational field is considered a small perturbation on the flat Minkowski space-time.

The quadrupole formula for the power radiated by gravitational waves is given by:

$$P_{quad} = \frac{1}{5} \left\langle \dddot{M}_{ij} \dddot{M}_{ij} - \frac{1}{3} \left(\dddot{M}_{kk} \right)^2 \right\rangle . \tag{3.2}$$

The pulsar PSR J0737-3039 is a highly relativistic double neutron star system, providing an excellent laboratory for testing gravitational theories. In GR, the total power radiated by this system can be derived using the quadrupole formula.

For this system, the components of the quadrupole moment can be expressed in terms of the masses of the stars and their separation. The total energy radiated over time leads to a gradual inspiral of the binary components, which can be observed as a decreasing orbital period.

On the other hand, to understand gravitational wave emission in $f(R)$ theories, we need to derive the equivalent of the quadrupole formula. This involves:

- **Field Equations and Perturbations**: The modified field equations are derived by varying the action with respect to the metric. These equations include additional terms involving $f\ '\ (\ R\ )$ f ′ (R) and $f\ '\ '\ (\ R\ )$ f ′′ (R), where primes denote derivatives with respect to R R.

- **Linearizing the Field Equations**: Under the weak-field approximation, the field equations are linearized around a background metric. This leads to a wave equation for the perturbations, similar to the GR case but with additional terms from the $f\ (\ R\ )$ f(R) function.

- **Modified Quadrupole Formula**: The presence of the additional terms alters the radiative properties of the gravitational waves.

By comparing the radiated power and the orbital decay rate of binary systems like PSR J0737-3039 in both GR and $f(R)$ theories, we can test the validity of these modified theories. Precise measurements can reveal discrepancies that may support or constrain $f(R)$ models.

3.3 Criticalities and future perspectives.

The quest to extend general relativity (GR) arises from its limitations in describing phenomena at the quantum scale and certain cosmological observations. While GR has been extraordinarily successful in explaining large-scale structures and passing numerous precision tests, its incompatibility with quantum mechanics and inability to fully account for dark energy and dark matter indicate the need for a more comprehensive theory of gravity. This thesis has explored various modifications to GR, including $f(R)$ gravity, Scalar-Tensor theories, and higher-dimensional theories, each offering potential pathways to a unified and complete description of gravitational phenomena.
The main criticalities of this problem are two:
The first one is the strong limitation of the algorithm to the inverse problem. The most prominent one is the class of theories that are eligible to be studied with this method, which is very limited. A possible future implementation could be tuning the algorithm to allow a higher class of theories to be tested.
The second prominent criticality is the possibility of linearizing the equation of motion for scalar-tensor theories. In the cases presented in this thesis, we could always analytically find a Green function suitable to describe the scalar field ϕ. However, this is not a general solution since many theories present equations that are too complex to be solved with this method.

Future steps in this field can be to further expand the class of theories that are eligible for the inverse problem and trying to obtain a a general algorithm that, given a metric, can find its theory and derive the quadrupole formula.

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